Converse Bounds on Probabilistic Distillation of Quantum Channel Resources

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Probabilistic protocols can enlarge the set of feasible resource transformations by conditioning on successful outcomes, but in many practical setting there are fundamental limitations on the success probability and the error incurred in such transformation. We derive complementary one-shot converse bounds and their regularized asymptotic rate bounds. Successful outcomes are represented by branches of free superinstruments. While projective robustness yields an accuracy constraint independent of the success probability, generalized robustness directly bounds the probability itself. In particular, we establish the explicit relationship between the success probability and the error of the protocol using robustness measure. For a flagged erasure channel, postselection recovers the identity channel exactly. In this case, generalized robustness remains finite although projective robustness is infinite. Restricting the framework to preparation channels recovers state distillation and gives explicit bounds for coherence distillation from noisy coherent states. Together, these examples distinguish channel-level behavior from the state-theoretic case obtained by restricting the framework to preparation channels.

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I. INTRODUCTION

Quantum resource theories (QRTs) describe physical properties that become useful under operational restrictions and provide a framework for quantifying and manipulating them [1]. Examples include entanglement [2, 3], coherence [4, 5], magic [6–9], and thermodynamic nonequilibrium resources [10]. In a given resource theory, the free objects and free operations determine which physical processes are allowed. Resource monotones provide quantitative constraints on the possible transformations.

Quantum dynamics can themselves be resourceful: a channel may generate coherence, distribute entanglement, or implement a non-Clifford operation beyond the reach of free dynamics. These capabilities motivate dynamical resource theories, in which channels serve as resource objects and superchannels describe physically admissible transformations of channels [11–15]. This framework therefore extends resource theory beyond quantifying channel resources to converting, concentrating, and distilling them under free dynamical transformations.

Distillation is a central operational task in resource theories. Its goal is to convert many copies or uses of noisy, weakly resourceful objects into fewer copies or uses of a higher-quality target resource. Distillation has a long history, from early work on entanglement concentration and purification [16–18] to PPT-assisted protocols [19–21], and has since been extended to coherence [22, 23], magic states [6, 24, 25], and general resource theories [26]. For quantum channels, deterministic distillation under free superchannels has yielded fundamental converse bounds based on robustness- and weight-type monotones [15, 27, 28]. Projective robustness has also been introduced for channels and measurements and shown to have direct operational significance [29].

However, deterministic settings exclude probabilistic protocols, whose distinct success and failure branches require only the normalized output conditioned on success to approximate the target. In state resource theories, this relaxation enlarges the set of achievable transformations, as demonstrated by stochastic entanglement transformations [30, 31]. Subsequent studies have used projective robustness to constrain probabilistic state convertibility [32, 33] and have shown that probabilistic protocols can overcome certain entropic limitations in asymptotic state transformations [34]. Stochastic approximate transformations have also been studied in entanglement theory and general state resource theories [35]. Related probabilistic channel-simulation protocols have been studied in magic resource theory [36]. Regula [27] already studied probabilistic channel purification using worst-case conditional channel fidelity and potentially input-dependent success probabilities. Here, the successful output is a scalar multiple of a channel, each branch preserves free channels after normalization, and approximation is measured by normalized-Choi fidelity.

Conditioning on success raises two operational questions: the accuracy of the conditional output and the probability of obtaining it. We derive converse bounds using projective robustness and generalized robustness. For pure normalized Choi targets, finite input projective robustness and target free fidelity strictly between zero and one give a probability-independent error lower

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bound. Generalized robustness gives an explicit probability–error relation when the relevant quantities are finite. Regularization gives corresponding asymptotic rate bounds; the generalized-robustness rate bound additionally requires $n^{-1}\log(1/p_n) \to 0$.

Theorems 1 and 2 establish conditional-error and success-probability bounds by explicitly accounting for successful-branch normalization. Proposition 1 introduces a binary-test optimization for mixed Choi targets, while Corollary 1 gives a pure-target probability–error relation without requiring finite input projective robustness. The proofs build on the robustness principles of Ye et al. [29], Regula [32, 33], and Regula and Takagi [27].

The applications include heralded identity-channel recovery from a flagged erasure channel, for which generalized robustness gives a finite but non-tight probability bound despite infinite projective robustness. We also evaluate the mixed-target bound for depolarizing channels with finite projective robustness and show that it is attained by retaining the input channel. Finally, restricting the framework to preparation channels recovers state distillation and gives explicit coherence bounds.

The remainder of this paper is organized as follows. Sec. II introduces the framework of dynamical resource theories, free superinstruments, the probabilistic channel-distillation task, and the two robustness measures. Sec. III establishes the one-shot bounds on the conditional error and success probability. Sec. IV derives the corresponding asymptotic rate bounds. Sec. V applies the results to erasure, depolarizing, and preparation channels and compares the latter with probabilistic state transformations. Sec. VI concludes the paper.

II. PRELIMINARIES AND FRAMEWORK

A. Free dynamical transformations

For finite-dimensional systems A_0 and A_1, let $\mathrm{CPTP}(A_0 \to A_1)$ denote the set of quantum channels from A_0 to A_1. A dynamical resource theory specifies a nonempty, convex, and closed subset $\mathfrak{F}(A_0 \to A_1) \subseteq \mathrm{CPTP}(A_0 \to A_1)$ of free channels [13, 14, 37–39].

Transformations between quantum channels are described by superchannels, which are physically admissible maps from channels to channels [11, 40]. A superchannel is free if it maps every free input channel to a free output channel. A quantum superinstrument $\{\Theta_x\}_x$ consists of sub-superchannels whose sum $\sum_x \Theta_x$ is a deterministic superchannel [41]. We call the superinstrument free if its sum is a free superchannel and each branch maps every free input channel to a free channel after normalization. The branch condition requires that, for every $\mathcal{L} \in \mathfrak{F}(A_0 \to A_1)$ and every outcome x, there exist $q_x(\mathcal{L}) \in [0,1]$ and $\widetilde{\mathcal{L}}_x \in \mathfrak{F}(B_0 \to B_1)$ such that

$$\Theta_x(\mathcal{L}) = q_x(\mathcal{L})\,\widetilde{\mathcal{L}}_x. \tag{1}$$

For simplicity of notation, we omit input and output system labels whenever they are clear from context. If $q_x(\mathcal{L}) = 0$, outcome x does not occur. Therefore, postselection cannot map a free input channel to a resourceful normalized output channel.

B. Choi representation and probabilistic distillation

For a channel $\mathcal{N} \in \mathrm{CPTP}(A_0 \to A_1)$, define its Choi operator and the unnormalized maximally entangled vector by

$$J_{\mathcal{N}} := (\mathrm{id}_{A_0} \otimes \mathcal{N})\left(|\Phi\rangle\langle\Phi|_{A_0A_0'}\right), \qquad |\Phi\rangle_{A_0A_0'} := \sum_i |i\rangle_{A_0}|i\rangle_{A_0'}, \tag{2}$$

where $A_0' \simeq A_0$ and $\mathcal{N}$ acts on A_0' [42, 43]. With this convention,

$$\mathrm{Tr}_{A_1} J_{\mathcal{N}} = I_{A_0}, \qquad \widetilde{J}_{\mathcal{N}} := \frac{J_{\mathcal{N}}}{d_{A_0}}, \qquad d_{A_0} := \dim \mathcal{H}_{A_0}. \tag{3}$$

All inequalities between Choi operators below are understood in the positive semidefinite order.

We quantify approximation using the squared Uhlmann fidelity [44, 45],

$$F(\rho,\sigma) := \left\| \sqrt{\rho}\,\sqrt{\sigma} \right\|_1^2. \tag{4}$$

When σ is pure, the fidelity reduces to $F(\rho,\sigma) = \mathrm{Tr}(\rho\sigma)$.

Let $\mathcal{N} \in \mathrm{CPTP}(A_0 \to A_1)$ be the input channel and $\mathcal{M} \in \mathrm{CPTP}(B_0 \to B_1)$ the target channel. An n-to-m probabilistic distillation protocol is a successful branch Θ_s of a free superinstrument such that

$$\Theta_s\left(\mathcal{N}^{\otimes n}\right) = p\,\widetilde{\mathcal{M}}, \qquad 0 < p \le 1, \tag{5}$$

where $\widetilde{\mathcal{M}} \in \mathrm{CPTP}(B_0^{\otimes m} \to B_1^{\otimes m})$ is the conditional output channel and p is the success probability.

The proportionality condition in Eq. (5) is equivalently expressed by the Choi marginal condition

$$\mathrm{Tr}_{B_1^{\otimes m}} J_{\Theta_s(\mathcal{N}^{\otimes n})} = p I_{B_0^{\otimes m}}. \tag{6}$$

This condition makes p independent of the state transmitted through the conditional channel.

Input-independent success is appropriate when a heralded device must act on unknown inputs, including inputs entangled with a reference. A constant p makes conditioning equivalent to a fixed linear CPTP map, whereas input-dependent normalization generally does not define a channel. Examples include transmission with an input-independent erasure flag, as in Sec. V A, and heralded gates whose successful map is proportional to a unitary channel. In these tasks, the flag reports success without preferentially selecting particular input states. This assumption specifies our scope; protocols with input-dependent success require a different conditional description.

The normalized conditional output is $\widetilde{\mathcal{M}} = \Theta_s(\mathcal{N}^{\otimes n})/p$. The protocol has error at most $\varepsilon \in [0, 1]$ if

$$F\left(\widetilde{J}_{\widetilde{\mathcal{M}}}, \widetilde{J}_{\mathcal{M}^{\otimes m}}\right) \geq 1 - \varepsilon. \tag{7}$$

The normalized Choi state is the reference–output state obtained by sending one half of a normalized maximally entangled probe through the successful branch and conditioning on success. Thus Eq. (7) quantifies the conditional accuracy of this entangled probe. For a pure target Choi state, projection onto the target accepts with conditional probability at least $1 - \varepsilon$, giving a joint success-and-acceptance probability of at least $p(1 - \varepsilon)$. For any target, the Fuchs–van de Graaf inequality [46] ensures that every measurement-event probability on the reference–output system differs from its target value by at most $\sqrt{\varepsilon}$. Here ε describes conditional process infidelity, whereas p describes protocol success. This criterion, including its asymptotic use below, concerns normalized-Choi approximation and does not by itself guarantee vanishing worst-case channel error as the input dimension grows.

C. Robustness measures

The two resource monotones used in our bounds are defined via semidefinite inequalities relating the Choi operator of a channel to those of free channels. Projective robustness was introduced to constrain probabilistic resource transformations and was later extended to channel and measurement resource theories [29, 32, 33]. For a channel $\mathcal{N}$ and a free set $\mathfrak{F}$, it is defined by

$$\Omega_{\mathfrak{F}}(\mathcal{N}) := \inf_{\mathcal{L} \in \mathfrak{F}} \inf \left\{ \lambda\mu : J_{\mathcal{N}} \leq \lambda J_{\mathcal{L}},\ J_{\mathcal{L}} \leq \mu J_{\mathcal{N}} \right\}. \tag{8}$$

Here the infimum is taken over $\lambda, \mu \geq 0$. The two operator inequalities compare $\mathcal{N}$ with the same free channel in both directions and will be used to bound the accuracy of a normalized conditional output.

For $\varepsilon \in [0, 1]$, the smoothed projective robustness is defined as

$$\Omega^{\varepsilon}_{\mathfrak{F}}(\mathcal{M}) := \inf_{\mathcal{K}:\, F\left(\widetilde{J}_{\mathcal{K}}, \widetilde{J}_{\mathcal{M}}\right) \geq 1-\varepsilon} \Omega_{\mathfrak{F}}(\mathcal{K}). \tag{9}$$

Here $\mathcal{K}$ ranges over channels with the same input and output spaces as $\mathcal{M}$.

The generalized robustness requires only the inequality $J_{\mathcal{N}} \leq \lambda J_{\mathcal{L}}$ for a free channel $\mathcal{L}$ [27]:

$$\Lambda_{\mathfrak{F}}(\mathcal{N}) := \inf_{\mathcal{L} \in \mathfrak{F}} \inf \left\{ \lambda \geq 0 : J_{\mathcal{N}} \leq \lambda J_{\mathcal{L}} \right\}. \tag{10}$$

It is the infimum of the scaling factors for which the inequality holds. Its dependence on the normalization of a successful branch allows it to bound the success probability whenever the relevant robustness quantities are finite.

Its smoothed form is defined analogously as

$$\Lambda^{\varepsilon}_{\mathfrak{F}}(\mathcal{M}) := \inf_{\mathcal{K}:\, F\left(\widetilde{J}_{\mathcal{K}}, \widetilde{J}_{\mathcal{M}}\right) \geq 1-\varepsilon} \Lambda_{\mathfrak{F}}(\mathcal{K}). \tag{11}$$

For a target channel $\mathcal{M}$ with a pure normalized Choi state, we also define the free fidelity

$$F_{\mathfrak{F}}(\mathcal{M}) := \max_{\mathcal{L} \in \mathfrak{F}} \mathrm{Tr}\left[\widetilde{J}_{\mathcal{M}} \widetilde{J}_{\mathcal{L}}\right]. \tag{12}$$

This is the largest normalized-Choi overlap between $\mathcal{M}$ and a free channel. It yields an explicit form of the projective-robustness error bound.

III. ONE-SHOT BOUNDS FOR PROBABILISTIC CHANNEL DISTILLATION

This section presents two one-shot converse bounds for probabilistic channel distillation.

A. Error bound from projective robustness

Theorem 1 extends the projective-robustness argument to successful channel branches: normalization changes the order coefficients to $\lambda q/p$ and $\mu p/q$, where p and q are the resource and free-output success probabilities. Their product remains $\lambda\mu$; the pure-target error bound then follows by the established projection argument [29, 32, 33].

Theorem 1. *Let $\mathfrak{F}$ be a convex set of free channels. Suppose that a probabilistic free protocol transforms $\mathcal{N}^{\otimes n}$ into a conditional output channel $\widetilde{\mathcal{M}}$ with success probability $p > 0$. For some $\varepsilon \in [0,1]$, assume that $F\left(\widetilde{J}_{\widetilde{\mathcal{M}}}, \widetilde{J}_{\mathcal{M}^{\otimes m}}\right) \geq 1-\varepsilon$. Then*

$$\Omega^{\varepsilon}_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right) \leq \Omega_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right). \tag{13}$$

If, in addition, $\widetilde{J}_{\mathcal{M}^{\otimes m}}$ is pure and $0 < F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right) < 1$, $\Omega_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right) < \infty$, then

$$\varepsilon \geq \left(\frac{F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}{1-F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}\Omega_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right)+1\right)^{-1}. \tag{14}$$

Proof. Let Θ_s denote the successful branch of the probabilistic free protocol. By assumption, $\Theta_s(\mathcal{N}^{\otimes n}) = p\widetilde{\mathcal{M}}$ with $p > 0$.

For any $\mathcal{L} \in \mathfrak{F}$ and $\lambda, \mu \geq 0$ satisfying $J_{\mathcal{N}^{\otimes n}} \leq \lambda J_{\mathcal{L}}$ and $J_{\mathcal{L}} \leq \mu J_{\mathcal{N}^{\otimes n}}$, complete positivity of Θ_s gives

$$J_{\Theta_s(\mathcal{N}^{\otimes n})} \leq \lambda J_{\Theta_s(\mathcal{L})}, \qquad J_{\Theta_s(\mathcal{L})} \leq \mu J_{\Theta_s(\mathcal{N}^{\otimes n})}. \tag{15}$$

Since the successful branch is free, $\Theta_s(\mathcal{L}) = q\widetilde{\mathcal{L}}$ for some $q \in [0,1]$, where $\widetilde{\mathcal{L}} \in \mathfrak{F}$ whenever $q > 0$. The first inequality above and $p > 0$ imply $q > 0$. Hence

$$J_{\widetilde{\mathcal{M}}} \leq \lambda\frac{q}{p}J_{\widetilde{\mathcal{L}}}, \qquad J_{\widetilde{\mathcal{L}}} \leq \mu\frac{p}{q}J_{\widetilde{\mathcal{M}}}. \tag{16}$$

The product of the two coefficients is $\lambda\mu$. Taking the infimum over all feasible $\mathcal{L}, \lambda, \mu$ and using the definition of the smoothed projective robustness therefore gives

$$\Omega^{\varepsilon}_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right) \leq \Omega_{\mathfrak{F}}(\widetilde{\mathcal{M}}) \leq \Omega_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right). \tag{17}$$

This proves the first claim.

For the second claim, suppose that $\widetilde{J}_{\mathcal{M}^{\otimes m}} = |\psi\rangle\langle\psi|$ is pure. By assumption,

$$0 < F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right) < 1, \qquad \Omega_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right) < \infty. \tag{18}$$

If $\varepsilon = 1$, the claimed bound is immediate. If $\varepsilon = 0$, the first claim gives $\Omega_{\mathfrak{F}}(\mathcal{M}^{\otimes m}) \leq \Omega_{\mathfrak{F}}(\mathcal{N}^{\otimes n}) < \infty$. This is impossible. Indeed, $J_{\mathcal{L}} \leq \mu J_{\mathcal{M}^{\otimes m}}$ would force the normalized Choi operator of the free channel $\mathcal{L}$ to have no component orthogonal to $|\psi\rangle$. Since this operator has unit trace, it must equal $|\psi\rangle\langle\psi|$. The target channel would then be free, giving $F_{\mathfrak{F}}(\mathcal{M}^{\otimes m}) = 1$, contrary to the assumption. It remains to consider $0 < \varepsilon < 1$.

Let $\mathcal{K}$ be any channel satisfying $\mathrm{Tr}[\widetilde{J}_{\mathcal{K}}\widetilde{J}_{\mathcal{M}^{\otimes m}}] \geq 1-\varepsilon$. If $\Omega_{\mathfrak{F}}(\mathcal{K}) = \infty$, the required lower bound is immediate. We may therefore assume that $\Omega_{\mathfrak{F}}(\mathcal{K}) < \infty$.

Choose any $\mathcal{L} \in \mathfrak{F}$ and feasible λ, μ satisfying $J_{\mathcal{K}} \leq \lambda J_{\mathcal{L}}$ and $J_{\mathcal{L}} \leq \mu J_{\mathcal{K}}$. Define $f := \mathrm{Tr}[\widetilde{J}_{\mathcal{L}}\widetilde{J}_{\mathcal{M}^{\otimes m}}]$, so that $f \leq F_{\mathfrak{F}}(\mathcal{M}^{\otimes m})$. Projecting the two operator inequalities onto $|\psi\rangle$ and its orthogonal complement, respectively, gives

$$1-\varepsilon \leq \lambda f \leq \lambda F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right), \qquad 1-f \leq \mu\varepsilon. \tag{19}$$

Since $(1-f)/f$ decreases with f and $f \leq F_{\mathfrak{F}}(\mathcal{M}^{\otimes m})$, these inequalities imply

$$\lambda\mu \geq \frac{(1-\varepsilon)\left[1-F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)\right]}{\varepsilon F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}. \tag{20}$$

Taking the infimum first over all feasible $\mathcal{L}, \lambda, \mu$ and then over all admissible $\mathcal{K}$ yields

$$\Omega_{\mathfrak{F}}^{\varepsilon}\left(\mathcal{M}^{\otimes m}\right) \geq \frac{(1-\varepsilon)\left[1-F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)\right]}{\varepsilon F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}. \tag{21}$$

Combining this inequality with the first claim and rearranging gives

$$\varepsilon \geq \left(\frac{F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}{1-F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}\Omega_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right)+1\right)^{-1}, \tag{22}$$

which is the required bound. □

For a mixed Choi target, the pure-projector argument does not directly apply, and its pure-state components need not satisfy the channel marginal condition. Proposition 1 instead combines fidelity monotonicity under a binary test with the projective order inequalities and optimizes over effects $0 \leq T \leq I$.

Let $\tau := \widetilde{J}_{\mathcal{M}^{\otimes m}}$, $Q := \Omega_{\mathfrak{F}}(\mathcal{N}^{\otimes n}) < \infty$. and $[z]_+ := \max\{z, 0\}$ for any real number z. Under the assumptions of Theorem 1, the probabilistic free transformation under consideration obeys the projective-robustness monotonicity bound

$$\Omega_{\mathfrak{F}}^{\varepsilon}(\mathcal{M}^{\otimes m}) \leq Q. \tag{23}$$

For an effect $0 \leq T \leq I$, define

$$a_T := \mathrm{Tr}(T\tau), \qquad c_T := \max_{\mathcal{L}\in\mathfrak{F}} \mathrm{Tr}\left(T\widetilde{J}_{\mathcal{L}}\right). \tag{24}$$

Proposition 1 (General mixed-target error bound)**.** *Under the assumptions of Theorem 1, for every effect T satisfying $0 \leq T \leq I, 0 < c_T < 1$, define $x_T := \frac{Qc_T}{1-c_T+Qc_T}$. Then*

$$\varepsilon \geq \left[\sqrt{a_T(1-x_T)} - \sqrt{(1-a_T)x_T}\right]_+^2. \tag{25}$$

Consequently,

$$\varepsilon \geq \sup_{\substack{0\leq T\leq I \\ 0<c_T<1}} \left[\sqrt{a_T(1-x_T)} - \sqrt{(1-a_T)x_T}\right]_+^2, \qquad x_T = \frac{Qc_T}{1-c_T+Qc_T}. \tag{26}$$

For a pure target, choosing $T = \widetilde{J}_{\mathcal{M}^{\otimes m}}$ gives $a_T = 1$ and $c_T = F_{\mathfrak{F}}(\mathcal{M}^{\otimes m})$, recovering Eq. (14). See the proof in the Appendix A.

B. Success-probability bound from generalized robustness

For generalized robustness, normalization of the successful branch leaves a factor of $1/p$ in the defining operator inequality, yielding the probability bound below.

Theorem 2. *Let $\mathfrak{F}$ be a convex set of free channels. Suppose that a probabilistic free protocol transforms $\mathcal{N}^{\otimes n}$ into a conditional output channel $\widetilde{\mathcal{M}}$ with success probability $p > 0$. For some $\varepsilon \in [0, 1]$, assume that $F\left(\widetilde{J}_{\widetilde{\mathcal{M}}}, \widetilde{J}_{\mathcal{M}^{\otimes m}}\right) \geq 1-\varepsilon$.*

If $\Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right) < \infty, \Lambda_{\mathfrak{F}}^{\varepsilon}\left(\mathcal{M}^{\otimes m}\right) < \infty$, then

$$p \leq \frac{\Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right)}{\Lambda_{\mathfrak{F}}^{\varepsilon}\left(\mathcal{M}^{\otimes m}\right)}. \tag{27}$$

Proof. Let Θ_s denote the successful branch of the probabilistic free protocol. By assumption, $\Theta_s(\mathcal{N}^{\otimes n}) = p\widetilde{\mathcal{M}}$ with $p > 0$.

Choose any $\mathcal{L} \in \mathfrak{F}$ and $\lambda \geq 0$ satisfying $J_{\mathcal{N}^{\otimes n}} \leq \lambda J_{\mathcal{L}}$. Complete positivity of Θ_s gives $J_{\Theta_s(\mathcal{N}^{\otimes n})} \leq \lambda J_{\Theta_s(\mathcal{L})}$. Since the successful branch is free, we may write $\Theta_s(\mathcal{L}) = q\widetilde{\mathcal{L}}$ with $q \in [0, 1]$ and $\widetilde{\mathcal{L}} \in \mathfrak{F}$. Substitution gives

$$pJ_{\widetilde{\mathcal{M}}} \leq \lambda q J_{\widetilde{\mathcal{L}}}. \tag{28}$$

Since $p > 0$, this inequality implies $q > 0$. Dividing by p and applying the definition of generalized robustness gives

$$J_{\widetilde{\mathcal{M}}} \leq \frac{\lambda q}{p} J_{\widetilde{\mathcal{L}}}, \qquad \Lambda_{\mathfrak{F}}(\widetilde{\mathcal{M}}) \leq \frac{\lambda q}{p} \leq \frac{\lambda}{p}. \tag{29}$$

This also shows that $\Lambda_{\mathfrak{F}}(\widetilde{\mathcal{M}})$ is finite. For normalized channels, every feasible scaling factor is at least one.

The last inequality implies $p \le \lambda/\Lambda_{\mathfrak{F}}(\widetilde{\mathcal{M}})$. Taking the infimum over all feasible $\mathcal{L}$ and λ gives

$$p \le \frac{\Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right)}{\Lambda_{\mathfrak{F}}(\widetilde{\mathcal{M}})} \le \frac{\Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right)}{\Lambda^{\varepsilon}_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}, \tag{30}$$

where the second inequality follows from the fidelity condition and the definition of the smoothed generalized robustness. □

The proof uses the robustness principles of Regula and Takagi [27], with successful-branch normalization accounted for explicitly. For pure Choi targets, evaluating the underlying operator inequality on the target projector gives Corollary 1.

Corollary 1. *Under the assumptions of Theorem 2, suppose additionally that $\widetilde{J}_{\mathcal{M}^{\otimes m}}$ is pure and*

$$0 < F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right) \le 1.$$

Then, for $0 \le \varepsilon < 1$,

$$p \le \min\left\{1,\ \frac{\Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right) F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}{1-\varepsilon}\right\}. \tag{31}$$

Equivalently, for a prescribed success probability $p > 0$,

$$\varepsilon \ge \max\left\{0,\ 1 - \frac{\Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right) F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}{p}\right\}. \tag{32}$$

Proof. Let $\mathcal{K}$ satisfy the smoothing condition in Eq. (11). For any $\mathcal{L} \in \mathfrak{F}$ and $\lambda \ge 0$ such that $J_{\mathcal{K}} \le \lambda J_{\mathcal{L}}$, purity of the target gives

$$1 - \varepsilon \le \mathrm{Tr}\left[\widetilde{J}_{\mathcal{K}}\widetilde{J}_{\mathcal{M}^{\otimes m}}\right] \le \lambda F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right). \tag{33}$$

Every feasible scaling factor also satisfies $\lambda \ge 1$. Taking the infimum over $\mathcal{L}$, λ, and $\mathcal{K}$ therefore yields

$$\Lambda^{\varepsilon}_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right) \ge \max\left\{1,\ \frac{1-\varepsilon}{F_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right)}\right\}. \tag{34}$$

Combining this inequality with Theorem 2 and $p \le 1$ gives Eq. (31). Rearranging the same inequality gives Eq. (32). □

The proofs identify several possible sources of slack. For projective robustness, these arise from the order inequalities and the fidelity estimate used to obtain the error bound. In the depolarizing example of Sec. V A, the binary measurement preserves the fidelity between the isotropic Choi states, and the identity superchannel attains the resulting bound.

For generalized robustness, the proof first gives $p\Lambda_{\mathfrak{F}}(\widetilde{\mathcal{M}}) \le \lambda q$, where q is the success probability on the free comparison channel. Replacing q by one and replacing the conditional-output robustness by the smoothed target robustness can weaken the bound. Saturation therefore depends on these estimates and on the existence of an allowed physical branch. The erasure example below identifies the loss caused by $q \le 1$. These examples establish tightness for a specific mixed-target family and explain a non-tight probability bound; they do not provide a general attainability criterion or establish achievable asymptotic rates.

IV. ASYMPTOTIC RATE BOUNDS

We now consider sequences of probabilistic free protocols that transform $\mathcal{N}^{\otimes n}$ into approximations of $\mathcal{M}^{\otimes m_n}$ with vanishing error. The asymptotic rate measures the number of target copies obtained per input copy.

We apply the standard regularization method [10, 27, 29, 47, 48] to the one-shot bounds, controlling the vanishing smoothing error along the output-copy sequence. For generalized robustness, the term $n^{-1}\log(1/p_n)$ requires an additional condition on the success probabilities.

Theorem 3. *Let $\mathfrak{F}$ be a convex set of free channels. Define*

$$\omega^{\mathrm{in}}_{\mathfrak{F}}(\mathcal{N}) := \limsup_{n\to\infty} \frac{1}{n} \log \Omega_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right), \tag{35}$$

$$\omega^{\mathrm{out}}_{\mathfrak{F}}(\mathcal{M}) := \lim_{\varepsilon\to 0} \liminf_{m\to\infty} \frac{1}{m} \log \Omega^{\varepsilon}_{\mathfrak{F}}\left(\mathcal{M}^{\otimes m}\right). \tag{36}$$

Assume that $0 < \omega_{\widetilde{\mathfrak{F}}}^{\mathrm{out}}(\mathcal{M}) < \infty$ *and* $\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{in}}(\mathcal{N}) < \infty$. *Suppose that a sequence of probabilistic free protocols transforms* $\mathcal{N}^{\otimes n}$ *into approximations of* $\mathcal{M}^{\otimes m_n}$ *with success probabilities* $p_n > 0$, *where* $m_n \to \infty$ *and* $\varepsilon_n \to 0$. *Then every achievable rate* $R := \liminf_{n\to\infty} m_n/n$ *satisfies*

$$R \le \frac{\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{in}}(\mathcal{N})}{\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{out}}(\mathcal{M})}. \tag{37}$$

Proof. For each n, the protocol transforms $\mathcal{N}^{\otimes n}$ into a channel $\widetilde{\mathcal{M}}_n$ with nonzero success probability and $F(\widetilde{J}_{\widetilde{\mathcal{M}}_n}, \widetilde{J}_{\mathcal{M}^{\otimes m_n}}) \ge 1 - \varepsilon_n$. Since $\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{in}}(\mathcal{N}) < \infty$, the input projective robustness is finite for all sufficiently large n. Discarding finitely many terms does not affect the asymptotic limits.

For all such n, Theorem 1 gives $\Omega_{\widetilde{\mathfrak{F}}}^{\varepsilon_n}(\mathcal{M}^{\otimes m_n}) \le \Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N}^{\otimes n})$. Taking logarithms yields

$$\frac{m_n}{n}\frac{1}{m_n}\log \Omega_{\widetilde{\mathfrak{F}}}^{\varepsilon_n}\left(\mathcal{M}^{\otimes m_n}\right) \le \frac{1}{n}\log \Omega_{\widetilde{\mathfrak{F}}}\left(\mathcal{N}^{\otimes n}\right). \tag{38}$$

Fix $\delta > 0$. Since $\varepsilon_n \to 0$, we have $\varepsilon_n \le \delta$ for all sufficiently large n. The smoothed projective robustness is nonincreasing in its smoothing parameter, and hence $\Omega_{\widetilde{\mathfrak{F}}}^{\varepsilon_n}(\mathcal{M}^{\otimes m_n}) \ge \Omega_{\widetilde{\mathfrak{F}}}^{\delta}(\mathcal{M}^{\otimes m_n})$ for such n. Therefore,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{m_n}\log \Omega_{\widetilde{\mathfrak{F}}}^{\varepsilon_n}\left(\mathcal{M}^{\otimes m_n}\right) \\ \ge \liminf_{m\to\infty} \frac{1}{m}\log \Omega_{\widetilde{\mathfrak{F}}}^{\delta}\left(\mathcal{M}^{\otimes m}\right).\end{aligned} \tag{39}$$

This inequality combines smoothing monotonicity with the fact that, since $m_n \to \infty$, the lower limit along $\{m_n\}$ is no smaller than the lower limit over all m.

Taking $\delta \to 0^+$ and using the definition of $\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{out}}(\mathcal{M})$ gives

$$\liminf_{n\to\infty} \frac{1}{m_n}\log \Omega_{\widetilde{\mathfrak{F}}}^{\varepsilon_n}\left(\mathcal{M}^{\otimes m_n}\right) \ge \omega_{\widetilde{\mathfrak{F}}}^{\mathrm{out}}(\mathcal{M}). \tag{40}$$

Both factors on the left-hand side of the finite-n inequality are nonnegative because $\Omega_{\widetilde{\mathfrak{F}}} \ge 1$. Applying the product inequality for nonnegative sequences and then using the definition of $\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{in}}(\mathcal{N})$ yields

$$\begin{aligned} R\,\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{out}}(\mathcal{M}) &\le \liminf_{n\to\infty}\left[\frac{m_n}{n}\frac{1}{m_n}\log \Omega_{\widetilde{\mathfrak{F}}}^{\varepsilon_n}\left(\mathcal{M}^{\otimes m_n}\right)\right] \\ &\le \limsup_{n\to\infty}\frac{1}{n}\log \Omega_{\widetilde{\mathfrak{F}}}\left(\mathcal{N}^{\otimes n}\right) \\ &= \omega_{\widetilde{\mathfrak{F}}}^{\mathrm{in}}(\mathcal{N}). \end{aligned} \tag{41}$$

Since $\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{out}}(\mathcal{M}) > 0$, division gives

$$R \le \frac{\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{in}}(\mathcal{N})}{\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{out}}(\mathcal{M})}. \tag{42}$$

□

If the free set is closed under tensor products, the input exponent in Theorem 3 admits a single-copy upper bound.

Corollary 2. *Assume that* $\mathfrak{F}$ *is closed under tensor products,* $0 < \omega_{\widetilde{\mathfrak{F}}}^{\mathrm{out}}(\mathcal{M}) < \infty$, *and* $\Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N}) < \infty$. *Then every achievable probabilistic distillation rate from* $\mathcal{N}$ *to* $\mathcal{M}$ *satisfies*

$$R \le \frac{\log \Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N})}{\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{out}}(\mathcal{M})}. \tag{43}$$

Proof. Closure of $\mathfrak{F}$ under tensor products makes the projective robustness submultiplicative:

$$\Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N}_1 \otimes \mathcal{N}_2) \le \Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N}_1)\Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N}_2). \tag{44}$$

Iterating this inequality gives $\Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N}^{\otimes n}) \le \Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N})^n$. Hence

$$\omega_{\widetilde{\mathfrak{F}}}^{\mathrm{in}}(\mathcal{N}) = \limsup_{n\to\infty}\frac{1}{n}\log \Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N}^{\otimes n}) \le \log \Omega_{\widetilde{\mathfrak{F}}}(\mathcal{N}). \tag{45}$$

Substituting this inequality into Theorem 3 proves the claim. □

When the success probabilities do not decay exponentially, namely, when $n^{-1}\log(1/p_n) \to 0$, Theorem 2 yields an analogous asymptotic rate bound.

Corollary 3. *Let $\mathfrak{F}$ be a convex set of free channels. Define*

$$\lambda_{\mathfrak{F}}^{\mathrm{in}}(\mathcal{N}) := \limsup_{n\to\infty} \frac{1}{n} \log \Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right), \tag{46}$$

$$\lambda_{\mathfrak{F}}^{\mathrm{out}}(\mathcal{M}) := \lim_{\varepsilon\to 0} \liminf_{m\to\infty} \frac{1}{m} \log \Lambda_{\mathfrak{F}}^{\varepsilon}\left(\mathcal{M}^{\otimes m}\right). \tag{47}$$

Assume that $0 < \lambda_{\mathfrak{F}}^{\mathrm{out}}(\mathcal{M}) < \infty$ and $\lambda_{\mathfrak{F}}^{\mathrm{in}}(\mathcal{N}) < \infty$. Suppose that a sequence of probabilistic free protocols has output copy numbers m_n, errors $\varepsilon_n \to 0$, and success probabilities $p_n > 0$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{p_n} = 0. \tag{48}$$

Then every achievable rate $R := \liminf_{n\to\infty} m_n/n$ satisfies

$$R \le \frac{\lambda_{\mathfrak{F}}^{\mathrm{in}}(\mathcal{N})}{\lambda_{\mathfrak{F}}^{\mathrm{out}}(\mathcal{M})}. \tag{49}$$

Proof. If $R = 0$, the bound is immediate. Suppose therefore that $R > 0$, which implies $m_n \to \infty$. Since $\lambda_{\mathfrak{F}}^{\mathrm{in}}(\mathcal{N}) < \infty$, the input generalized robustness is finite for all sufficiently large n. The successful-branch argument in the proof of Theorem 2 then shows that the normalized conditional output also has finite generalized robustness. Thus Theorem 2 applies for all sufficiently large n.

Discarding finitely many terms does not affect the asymptotic limits. For every remaining n, Theorem 2 gives

$$\Lambda_{\mathfrak{F}}^{\varepsilon_n}\left(\mathcal{M}^{\otimes m_n}\right) \le \frac{\Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right)}{p_n}. \tag{50}$$

Taking logarithms and dividing by n yields

$$\frac{m_n}{n}\frac{1}{m_n} \log \Lambda_{\mathfrak{F}}^{\varepsilon_n}\left(\mathcal{M}^{\otimes m_n}\right) \le \frac{1}{n} \log \Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right) + \frac{1}{n} \log \frac{1}{p_n}. \tag{51}$$

The smoothed generalized robustness is nonincreasing in its smoothing parameter. Therefore, the δ-argument used in the proof of Theorem 3, with $\Lambda_{\mathfrak{F}}$ in place of $\Omega_{\mathfrak{F}}$, gives

$$\liminf_{n\to\infty} \frac{1}{m_n} \log \Lambda_{\mathfrak{F}}^{\varepsilon_n}\left(\mathcal{M}^{\otimes m_n}\right) \ge \lambda_{\mathfrak{F}}^{\mathrm{out}}(\mathcal{M}). \tag{52}$$

Using the product inequality for nonnegative sequences and the assumption $n^{-1}\log(1/p_n) \to 0$, we obtain

$$\begin{aligned} R\,\lambda_{\mathfrak{F}}^{\mathrm{out}}(\mathcal{M}) &\le \liminf_{n\to\infty} \left[\frac{m_n}{n}\frac{1}{m_n} \log \Lambda_{\mathfrak{F}}^{\varepsilon_n}\left(\mathcal{M}^{\otimes m_n}\right)\right] \\ &\le \limsup_{n\to\infty} \left[\frac{1}{n} \log \Lambda_{\mathfrak{F}}\left(\mathcal{N}^{\otimes n}\right) + \frac{1}{n} \log \frac{1}{p_n}\right] \\ &= \lambda_{\mathfrak{F}}^{\mathrm{in}}(\mathcal{N}). \end{aligned} \tag{53}$$

Since $\lambda_{\mathfrak{F}}^{\mathrm{out}}(\mathcal{M}) > 0$, division gives the claimed rate bound. □

If p_n decays exponentially, the term $n^{-1}\log(1/p_n)$ does not vanish and contributes to the asymptotic rate bound.

V. EXAMPLES AND COMPARISON WITH STATE-TRANSFORMATION BOUNDS

We consider erasure-channel recovery and mixed-target depolarizing-channel conversion, followed by preparation-channel coherence distillation and comparison with state-transformation bounds.

A. Communication-channel examples

Erasure-channel recovery. Consider the resource theory of quantum communication in which replacer channels are free [49]. Let A be a d-dimensional input system with $d > 1$, let $B \simeq A$, and set

$$B_e := B \oplus \mathrm{span}\{|e\rangle\}, \tag{54}$$

where the erasure flag $|e\rangle$ is orthogonal to B. A replacer channel from A to an output system C has the form $\mathcal{R}_\sigma^{A\to C}(\rho) = \mathrm{Tr}(\rho)\sigma_C$ and hence carries no information about its input. In this example, a probabilistic branch is free if it maps every replacer channel to a subnormalized replacer channel, as required by Eq. (1).

The resource channel is the d-dimensional quantum erasure channel [50],

$$\mathcal{E}_\eta^{A\to B_e}(\rho) = (1-\eta)\rho_B + \eta\,\mathrm{Tr}(\rho)\,|e\rangle\langle e|, \qquad 0 \le \eta \le 1, \tag{55}$$

where the identification $A \simeq B$ is understood. The target is the identity channel $\mathrm{id}_d^{A\to B}$. Since id_d must preserve arbitrary inputs, including those entangled with a reference system, the transformation $\mathcal{E}_\eta \to \mathrm{id}_d$ is a channel transformation rather than a state-preparation task.

Let P_B be the projector from B_e onto the transmitted subspace B. The successful post-processing map and the corresponding sub-superchannel are

$$Q_s(X) = P_B X P_B, \qquad \Theta_s(\mathcal{L}) = Q_s \circ \mathcal{L}. \tag{56}$$

This branch distinguishes an erasure from a successful transmission without resolving the state inside B. Its action on the erasure channel is

$$\Theta_s(\mathcal{E}_\eta) = (1-\eta)\mathrm{id}_d, \qquad p_{\mathrm{succ}} = 1-\eta. \tag{57}$$

Indeed, $\mathrm{Tr}_B J_{\Theta_s(\mathcal{E}_\eta)} = (1-\eta)I_A$, so the success probability is independent of the input state. Thus, for $0 \le \eta < 1$, the normalized conditional output is exactly id_d; for $\eta = 1$, the successful branch never occurs.

The branch Θ_s is free. For every replacer channel $\mathcal{R}_\sigma^{A\to B_e}$,

$$\Theta_s(\mathcal{R}_\sigma) = r\,\mathcal{R}_{\widetilde{\sigma}}, \qquad r = \mathrm{Tr}(P_B\sigma), \qquad \widetilde{\sigma} = \frac{P_B\sigma P_B}{r} \quad (r > 0). \tag{58}$$

If $r = 0$, the output is the zero map and $\widetilde{\sigma}$ may be chosen arbitrarily. Hence the normalized output, whenever it exists, is again a replacer channel. To complete the protocol, fix a state τ_B and define

$$Q_f(X) = \mathrm{Tr}[(I_{B_e} - P_B)\,X]\,\tau_B. \tag{59}$$

The sum $Q_s + Q_f$ is trace preserving, and both branches map replacer channels to subnormalized replacer channels. They therefore form a free superinstrument. In particular, $\Theta_f(\mathcal{E}_\eta) = \eta\mathcal{R}_\tau$.

To prove optimality for every allowed free successful branch, write $\mathcal{E}_\eta = (1-\eta)\mathcal{V} + \eta\mathcal{R}_{|e\rangle\langle e|}$, where $\mathcal{V}$ is the noiseless embedding. If $\Theta_s(\mathcal{E}_\eta) = p\mathrm{id}_d$, the free-branch condition and positivity imply

$$pJ_{\mathrm{id}_d} = (1-\eta)J_{\Theta_s(\mathcal{V})} + \eta q J_{\mathcal{R}_\sigma}, \qquad q \ge 0.$$

For $0 < \eta < 1$, both positive terms must lie in the rank-one support of J_{id_d}. Since a nonzero replacer Choi operator has rank $d\,\mathrm{rank}\,\sigma \ge d > 1$, $q = 0$. Thus $\Theta_s(\mathcal{V}) = [p/(1-\eta)]\mathrm{id}_d$, and trace nonincrease gives $p \le 1-\eta$, attained by flag measurement. At $\eta = 0$, $p \le 1$; at $\eta = 1$, the input is free and exact recovery is impossible. Hence $p_{\mathrm{opt}} = 1-\eta$ throughout.

We now evaluate the generalized robustness. Define

$$|\phi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle_A|i\rangle_B, \qquad \Phi_d = |\phi_d\rangle\langle\phi_d|. \tag{60}$$

The normalized Choi states of the erasure channel and a replacer channel are

$$\widetilde{J}_{\mathcal{E}_\eta} = (1-\eta)\Phi_d + \eta\,\frac{I_A}{d}\otimes|e\rangle\langle e|, \tag{61}$$

$$\widetilde{J}_{\mathcal{R}_\sigma} = \frac{I_A}{d}\otimes\sigma. \tag{62}$$

Suppose that $J_{\mathcal{E}_\eta} \le \lambda J_{\mathcal{R}_\sigma}$ and let $a = \mathrm{Tr}(P_B\sigma)$. Evaluating this operator inequality on $|\phi_d\rangle$ and on the erasure sector gives

$$\lambda \ge \max\left\{\frac{d^2(1-\eta)}{a}, \frac{\eta}{1-a}\right\}. \tag{63}$$

Both terms are attained by

$$\sigma_a = a\,\frac{I_B}{d} + (1-a)\,|e\rangle\langle e|\,. \tag{64}$$

For $0 < \eta < 1$, minimizing the right-hand side of Eq. (63) gives

$$a = \frac{d^2(1-\eta)}{d^2(1-\eta)+\eta}, \qquad \Lambda_{\mathfrak{F}}(\mathcal{E}_\eta) = d^2(1-\eta)+\eta. \tag{65}$$

The same expression follows directly from the definition at $\eta = 0$ and $\eta = 1$.

The normalized Choi state of id_d is Φ_d, and hence $F_{\mathfrak{F}}(\mathrm{id}_d) = 1/d^2$. Equivalently, $\Lambda_{\mathfrak{F}}(\mathrm{id}_d) = d^2$. Corollary 1 therefore gives, for $0 \le \eta < 1$ and $0 \le \varepsilon < 1$,

$$p \le \min\left\{1,\ \frac{1-\eta+\eta/d^2}{1-\varepsilon}\right\}. \tag{66}$$

For exact recovery, $\varepsilon = 0$, and Eq. (66) reduces to $p \le 1-\eta+\eta/d^2$. The flag-measurement protocol achieves the optimal exact probability $p_{\mathrm{opt}} = 1-\eta$. The gap η/d^2 between the generalized-robustness bound and the optimal probability vanishes as d increases.

For $0 < \eta < 1$, the flag-measurement protocol succeeds on the optimal free comparison channel $\mathcal{R}_{\sigma_a}$ with probability $q = a$, where a is given in Eq. (65). Retaining this factor in the proof of Theorem 2 gives $p \le \lambda a/d^2 = 1-\eta$, with $\lambda = d^2(1-\eta)+\eta$. Replacing q by one instead gives $p \le 1-\eta+\eta/d^2$. Thus, for this protocol and comparison channel, the slack arises from discarding q; optimality over all allowed branches follows independently from the support argument above.

This example also separates the roles of the two robustness measures. For $0 \le \eta < 1$, the projective robustness of $\mathcal{E}_\eta$ is infinite. Any replacer Choi operator that dominates the maximally entangled term must be nonzero in every direction of $A \otimes B$. By contrast, within this sector, $J_{\mathcal{E}_\eta}$ has no component in directions orthogonal to $|\phi_d\rangle$. The same replacer Choi operator therefore cannot be bounded above by any finite multiple of $J_{\mathcal{E}_\eta}$. Hence Theorem 1 gives no nontrivial constraint for this flagged transformation, whereas Corollary 1 provides the finite probability–error relation in Eq. (66).

Mixed-target depolarizing-channel conversion. To complement the erasure example, which has infinite projective robustness and a pure target Choi state, consider qubit depolarizing channels in the same resource theory

$$\mathcal{D}_u(\rho) = u\rho + (1-u)\mathrm{Tr}(\rho)\frac{I_2}{2}, \qquad \widetilde{J}_{\mathcal{D}_u} = u\Phi_2 + (1-u)\frac{I_4}{4}, \quad 0 < u < 1. \tag{67}$$

These channels retain input dependence and have full-rank mixed Choi states. Twirling both projective order inequalities by $U^* \otimes U$ leaves $\widetilde{J}_{\mathcal{D}_u}$ invariant and sends every free comparison state $(I_2/2)\otimes\sigma$ to $I_4/4$. The largest and smallest Choi eigenvalues therefore give

$$\Omega_{\mathfrak{F}}(\mathcal{D}_u) = \frac{1+3u}{1-u} < \infty. \tag{68}$$

For a one-use conversion $\mathcal{D}_s \to \mathcal{D}_t$ with $0 < s < t < 1$, choose $T = \Phi_2$. Then $c_T = 1/4$, $a_T = (1+3t)/4 =: a_t$, and $x_T = (1+3s)/4 =: a_s$. Proposition 1 gives, for every allowed successful branch with $p > 0$,

$$\varepsilon \ge \left[\sqrt{a_t(1-a_s)} - \sqrt{(1-a_t)a_s}\right]^2 = 1 - F(\widetilde{J}_{\mathcal{D}_s}, \widetilde{J}_{\mathcal{D}_t}). \tag{69}$$

The equality follows from the spectra of the two isotropic Choi states. The free identity superchannel $\Theta_{\mathrm{id}}(\mathcal{L}) = \mathcal{L}$ attains this error at $p = 1$; an independent success flag realizes branches $p\Theta_{\mathrm{id}}$ and $(1-p)\Theta_{\mathrm{id}}$ at any $0 < p < 1$, with the same conditional output. Thus the bound is tight for this family and postselection cannot improve the one-use accuracy. For $s = 1/2$ and $t = 3/4$, the input projective robustness is 5 and both the converse and achievable error are $\varepsilon_{\mathrm{opt}} \simeq 0.0440$. This is a mixed-target converse benchmark, rather than an example of successful noise reduction.

B. Probabilistic coherence distillation through preparation channels

The preparation-channel example concerns the resource theory of coherence. Unlike the communication setting considered above, not every replacer channel is free: a replacer channel is free here only if its output state is incoherent. A channel that prepares a coherent state is therefore resourceful.

A state ρ can be embedded into the channel framework through the preparation channel

$$\mathcal{P}_\rho : \mathbb{C} \to A, \qquad \mathcal{P}_\rho(1) = \rho, \qquad \widetilde{J}_{\mathcal{P}_\rho} = \rho. \tag{70}$$

It follows that $F(\mathcal{P}_\rho, \mathcal{P}_\sigma) = F(\rho, \sigma)$ and that the operator inequality $J_{\mathcal{P}_\rho} \le \lambda J_{\mathcal{P}_\sigma}$ is equivalent to $\rho \le \lambda\sigma$. Hence the corresponding monotone expressions and converse bounds coincide with the state-level ones.

We take the free states to be those diagonal in a fixed reference basis [4, 22, 51, 52]. The input state is the noisy coherent qubit state

$$\rho_\gamma = (1-\gamma)\,|+\rangle\langle+| + \gamma\frac{I}{2}, \qquad |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad 0 < \gamma \le 1. \tag{71}$$

For n copies, the input channel is $\mathcal{N}^{\otimes n} = \mathcal{P}_{\rho_\gamma}^{\otimes n}$.

The target is the preparation channel of the D-dimensional maximally coherent state

$$\Psi_D = |\Psi_D\rangle\langle\Psi_D|, \qquad |\Psi_D\rangle = \frac{1}{\sqrt{D}}\sum_{i=1}^{D}|i\rangle, \qquad D = 2^m. \tag{72}$$

Under the standard product-basis identification, $\Psi_D \simeq \Psi_2^{\otimes m}$, and hence $\mathcal{M}^{\otimes m} = \mathcal{P}_{\Psi_2}^{\otimes m} \simeq \mathcal{P}_{\Psi_D}$.

Since the normalized Choi states of the preparation channels are the states themselves, Theorems 1 and 2 reduce to state-resource bounds. For the input family,

$$\Omega_{\mathfrak{F}}\left(\mathcal{P}_{\rho_\gamma}^{\otimes n}\right) = \Omega_{\mathfrak{F}}\left(\rho_\gamma^{\otimes n}\right) = \left(\frac{2-\gamma}{\gamma}\right)^n. \tag{73}$$

To verify this equality, write $\rho_\gamma = a\,|+\rangle\langle+| + b\,|-\rangle\langle-|$, where $a = (2-\gamma)/2$, $b = \gamma/2$, and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. Let $\delta_n = I/2^n$ be the maximally mixed incoherent state. Since the largest and smallest eigenvalues of $\rho_\gamma^{\otimes n}$ are a^n and b^n, respectively,

$$\rho_\gamma^{\otimes n} \le (2-\gamma)^n\delta_n, \qquad \delta_n \le \gamma^{-n}\rho_\gamma^{\otimes n}. \tag{74}$$

These inequalities give $\Omega_{\mathfrak{F}}(\rho_\gamma^{\otimes n}) \le [(2-\gamma)/\gamma]^n$.

For the reverse inequality, let σ_n be any incoherent n-qubit state satisfying $\rho_\gamma^{\otimes n} \le \lambda\sigma_n$ and $\sigma_n \le \mu\rho_\gamma^{\otimes n}$. Because every incoherent σ_n is diagonal in the computational basis,

$$\langle+|^{\otimes n}\,\sigma_n\,|+\rangle^{\otimes n} = \langle-|^{\otimes n}\,\sigma_n\,|-\rangle^{\otimes n} = 2^{-n}. \tag{75}$$

Taking the expectation value of the first operator inequality in $|+\rangle^{\otimes n}$ and that of the second in $|-\rangle^{\otimes n}$ yields

$$\lambda \ge (2-\gamma)^n, \qquad \mu \ge \gamma^{-n}. \tag{76}$$

Thus $\lambda\mu \ge [(2-\gamma)/\gamma]^n$ for every feasible σ_n, λ, μ, proving Eq. (73).

The free fidelity of the maximally coherent target is

$$F_{\mathfrak{F}}(\mathcal{P}_{\Psi_D}) = F_{\mathfrak{F}}(\Psi_D) = \max_{\sigma\in\mathfrak{F}}\mathrm{Tr}(\Psi_D\sigma) = \frac{1}{D}. \tag{77}$$

Substitution into Theorem 1 gives

$$\varepsilon \ge \frac{D-1}{\left(\frac{2-\gamma}{\gamma}\right)^n + D - 1}. \tag{78}$$

For $D = 2^m$, the numerator is $2^m - 1$. In particular, for the single-qubit target Ψ_2,

$$\varepsilon \ge \left[\left(\frac{2-\gamma}{\gamma}\right)^n + 1\right]^{-1}. \tag{79}$$

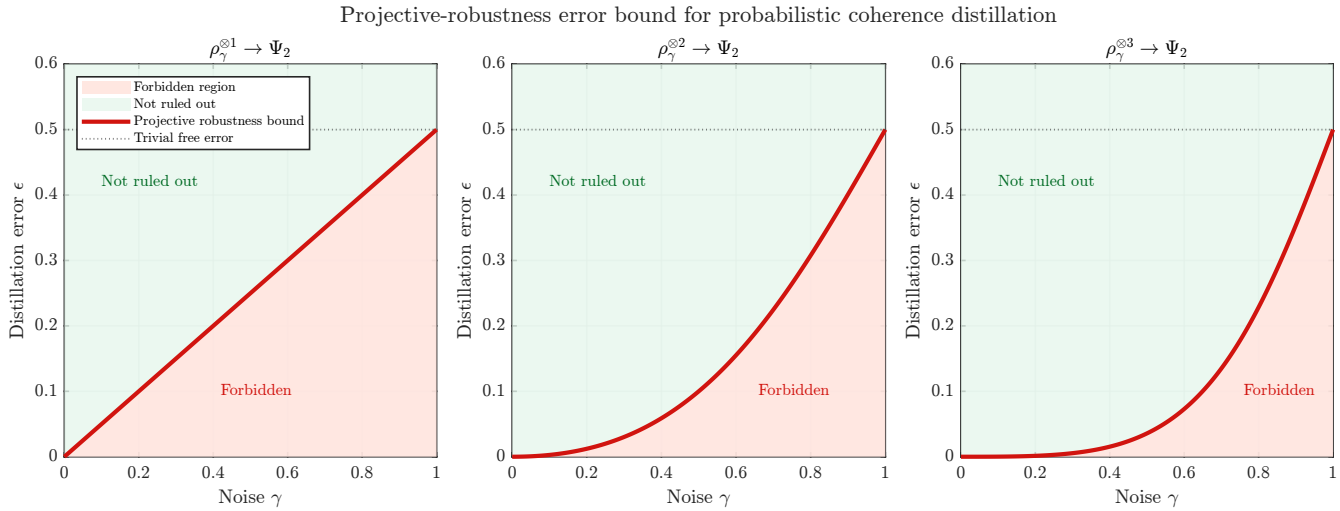


FIG. 1. Lower bounds on the error of probabilistic coherence distillation from $\rho_\gamma^{\otimes n}$ to the single-qubit maximally coherent state Ψ_2, obtained from projective robustness. Each curve corresponds to the indicated number n of input copies. Points below a curve are excluded by Theorem 1; points above it are not excluded by this theorem alone and are not necessarily achievable. The dotted line marks the limiting free-state error $1/2$.

This is a converse bound and need not be achievable. At fixed n and m, it increases with γ because $\Omega_{\mathfrak{F}}(\rho_\gamma^{\otimes n})$ decreases as the input becomes noisier. Figure 1 shows the single-qubit bound for several values of n. As $\gamma \to 1$, the input becomes incoherent and the bound approaches $1/2$, equal to one minus the maximal overlap between an incoherent state and Ψ_2.

Generalized robustness provides a complementary constraint on the success probability. For the input family,

$$\Lambda_{\mathfrak{F}}\left(\mathcal{P}_{\rho_\gamma}^{\otimes n}\right) = \Lambda_{\mathfrak{F}}\left(\rho_\gamma^{\otimes n}\right) = (2-\gamma)^n. \tag{80}$$

To see this, taking the expectation value of $\rho_\gamma^{\otimes n} \le \lambda\sigma_n$ in $|+\rangle^{\otimes n}$ gives $\lambda \ge (2-\gamma)^n$ for every incoherent state σ_n. Equality is attained by $\sigma_n = \delta_n$, proving the stated expression.

Since Ψ_D is pure and $F_{\mathfrak{F}}(\Psi_D) = 1/D$, Corollary 1 gives

$$p \le \min\left\{1, \frac{(2-\gamma)^n}{D(1-\varepsilon)}\right\}, \qquad 0 \le \varepsilon < 1. \tag{81}$$

For exact distillation, $\varepsilon = 0$, and the bound reduces to $p \le \min\{1, (2-\gamma)^n/D\}$, which is the exact-target specialization of Theorem 2. This necessary condition does not imply that exact distillation is possible. For $0 < \gamma \le 1$ and finite n, Theorem 1 independently rules out exact distillation to Ψ_D with any nonzero success probability. For the approximate single-qubit target,

$$p \le \min\left\{1, \frac{(2-\gamma)^n}{2(1-\varepsilon)}\right\}.$$

Figure 2 shows the success-probability constraint from Corollary 1 for two input copies.

For the two-copy transformation $\rho_\gamma^{\otimes 2} \to \Psi_4 \simeq \Psi_2^{\otimes 2}$, the two constraints can be displayed simultaneously. Substituting $n = 2$ and $D = 4$ into the error bound from Theorem 1 and solving for γ shows that a nonzero success probability is possible only if

$$\gamma \le \gamma_{\mathrm{T1}}(\varepsilon) := \frac{2}{1 + \sqrt{3(1-\varepsilon)/\varepsilon}}, \qquad 0 < \varepsilon < 1. \tag{82}$$

For the same target, Corollary 1 gives

$$p \le \min\left\{1, \frac{(2-\gamma)^2}{4(1-\varepsilon)}\right\}. \tag{83}$$

Figure 3 combines the necessary conditions from Theorem 1 and Corollary 1.

We finally evaluate the asymptotic bounds for the same preparation-channel family. Let Ψ_2 be the elementary target, so that m output copies correspond to Ψ_{2^m}. Since

$$\Omega_{\mathfrak{F}}\left(\rho_\gamma^{\otimes n}\right) = \left(\frac{2-\gamma}{\gamma}\right)^n, \tag{84}$$

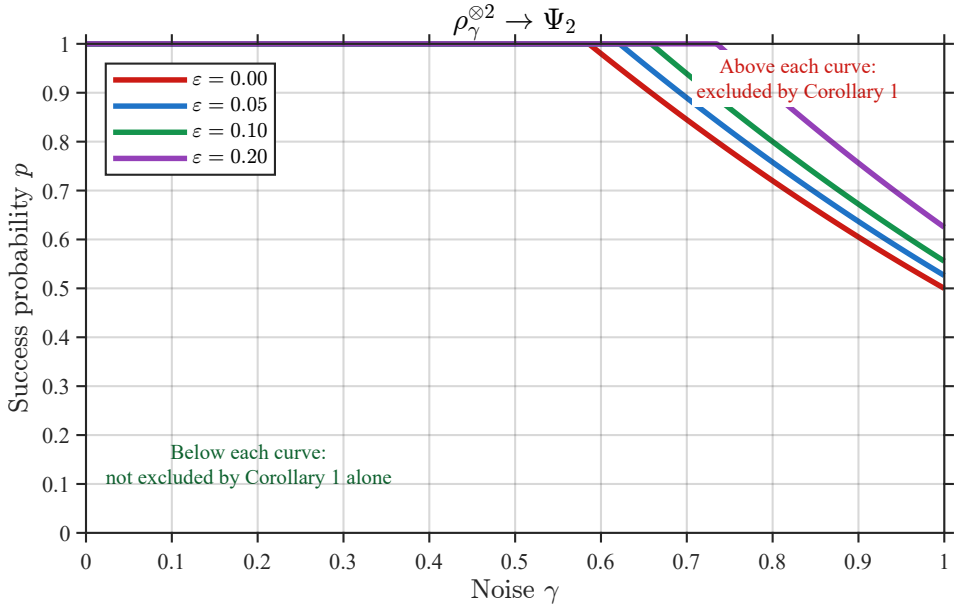


FIG. 2. Success-probability upper bounds from Corollary 1 for approximate probabilistic coherence distillation from $\rho_\gamma^{\otimes 2}$ to Ψ_2. Each curve corresponds to the indicated error tolerance ε. Points above a curve are excluded by Corollary 1, whereas points below it merely satisfy this particular necessary condition and are not necessarily achievable. The independent error constraint from Theorem 1 must also be satisfied but is not plotted here; neither condition nor their conjunction is sufficient to establish protocol existence.

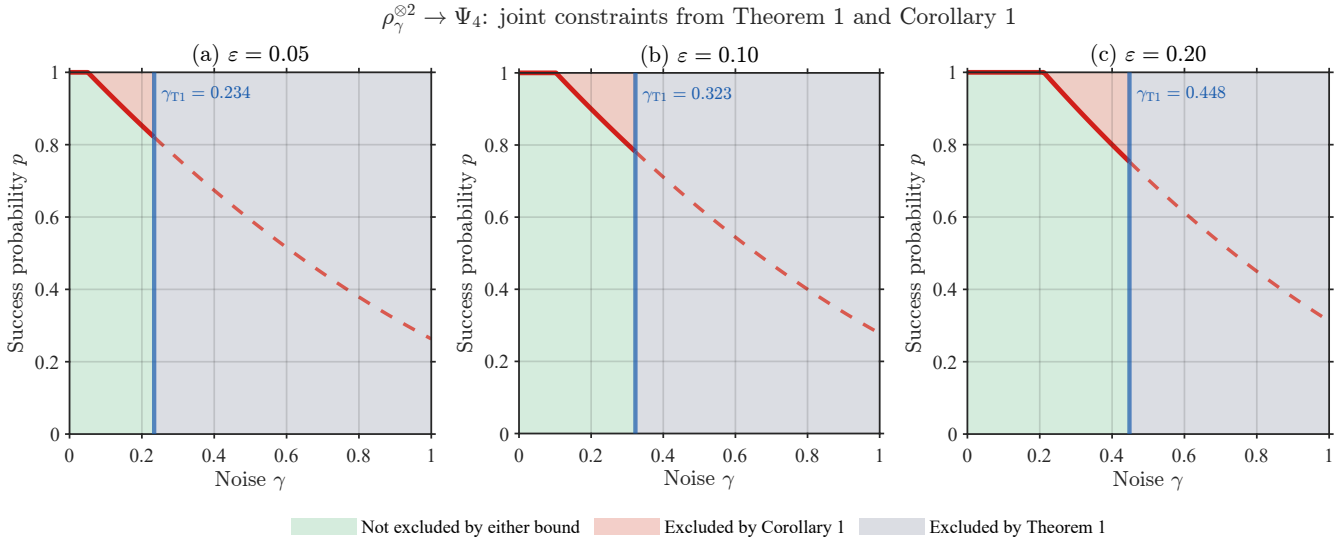


FIG. 3. Joint necessary conditions for approximate coherence distillation from $\rho_\gamma^{\otimes 2}$ to $\Psi_4 \simeq \Psi_2^{\otimes 2}$. The panels correspond to $\varepsilon = 0.05, 0.10, 0.20$. The blue vertical line marks $\gamma_{\mathrm{T1}}(\varepsilon)$; Theorem 1 excludes every nonzero success probability to its right. To its left, Corollary 1 excludes the region above the red curve. The green region satisfies these two independent necessary conditions, whose conjunction does not establish the existence of a protocol.

the input exponent defined in Sec. IV is

$$\omega_\delta^{\mathrm{in}}(\rho_\gamma) = \log \frac{2-\gamma}{\gamma}. \tag{85}$$

The smoothed projective robustness of the target can also be obtained explicitly. For $0 < \varepsilon < 1 - 1/D$, define

$$\tau_{D,\varepsilon} := (1-\varepsilon)\Psi_D + \frac{\varepsilon}{D-1}(I - \Psi_D). \tag{86}$$

This state satisfies $F(\tau_{D,\varepsilon}, \Psi_D) = 1 - \varepsilon$. With $\delta_D := I/D$, its largest and smallest eigenvalues give

$$\tau_{D,\varepsilon} \le D(1-\varepsilon)\delta_D, \qquad \delta_D \le \frac{D-1}{D\varepsilon}\tau_{D,\varepsilon}. \tag{87}$$

It follows that

$$\Omega^{\varepsilon}_{\widetilde{\delta}}(\Psi_D) \le \frac{(1-\varepsilon)(D-1)}{\varepsilon}. \tag{88}$$

For the converse, let τ satisfy $F(\tau, \Psi_D) \ge 1 - \varepsilon$, and suppose that an incoherent state σ obeys $\tau \le \lambda\sigma$ and $\sigma \le \mu\tau$. Every such σ satisfies

$$\mathrm{Tr}(\Psi_D\sigma) = \frac{1}{D}, \qquad \mathrm{Tr}[(I - \Psi_D)\sigma] = \frac{D-1}{D}. \tag{89}$$

Applying these two functionals to the corresponding operator inequalities gives

$$\lambda \ge D(1-\varepsilon), \qquad \mu \ge \frac{D-1}{D\varepsilon}, \tag{90}$$

where the second inequality follows from $\mathrm{Tr}[(I - \Psi_D)\tau] \le \varepsilon$. Hence

$$\Omega^{\varepsilon}_{\widetilde{\delta}}(\Psi_D) = \frac{(1-\varepsilon)(D-1)}{\varepsilon}. \tag{91}$$

Setting $D = 2^m$ and applying the definition of the output exponent yields

$$\omega^{\mathrm{out}}_{\widetilde{\delta}}(\Psi_2) = \log 2. \tag{92}$$

Theorem 3 therefore gives

$$R \le \frac{\log\left((2-\gamma)/\gamma\right)}{\log 2} = \log_2 \frac{2-\gamma}{\gamma}. \tag{93}$$

We next evaluate the generalized-robustness exponents. The input satisfies

$$\Lambda_{\widetilde{\delta}}\left(\rho_\gamma^{\otimes n}\right) = (2-\gamma)^n, \tag{94}$$

and therefore $\lambda^{\mathrm{in}}_{\widetilde{\delta}}(\rho_\gamma) = \log(2-\gamma)$.

For the target, the lower bound derived above is attained by $\tau = \tau_{D,\varepsilon}$ and $\sigma = \delta_D$, giving

$$\Lambda^{\varepsilon}_{\widetilde{\delta}}(\Psi_D) = D(1-\varepsilon), \qquad 0 < \varepsilon < 1 - \frac{1}{D}. \tag{95}$$

For $D = 2^m$ and every fixed $0 < \varepsilon < 1$, the condition $\varepsilon < 1 - 2^{-m}$ holds for all sufficiently large m. Therefore,

$$\liminf_{m\to\infty} \frac{1}{m} \log \Lambda^{\varepsilon}_{\widetilde{\delta}}(\Psi_{2^m}) = \log 2 \tag{96}$$

for every fixed $0 < \varepsilon < 1$. Taking $\varepsilon \to 0^+$ then gives

$$\lambda^{\mathrm{out}}_{\widetilde{\delta}}(\Psi_2) = \log 2. \tag{97}$$

Under the additional assumption

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{p_n} = 0, \tag{98}$$

the generalized-robustness rate bound yields

$$R \le \frac{\log(2-\gamma)}{\log 2} = \log_2(2-\gamma). \tag{99}$$

Thus this bound applies when the success probabilities do not decay exponentially. For $0 < \gamma < 1$, it is tighter than the projective-robustness rate bound, but it relies on this additional condition on the success probabilities.

C. Comparison with previous state-transformation bounds

Probabilistic transformations of states have been studied in several resource theories. For pure-state entanglement, stochastic local operations can realize transformations that are impossible deterministically, with optimal success probabilities constrained by entanglement monotones [30, 31, 53]. Projective robustness extends this type of constraint to general state resource theories [32, 33]. Under the preparation-channel restriction, Theorem 1 recovers the projective-robustness state bound, while Theorem 2 and Corollary 1 recover the robustness-based probability–error relation discussed in Refs. [27, 33].

Approximate and asymptotic probabilistic state transformations have also been investigated. Some entropy-based restrictions on deterministic asymptotic transformations can be removed by relaxing the requirement on the success probability [34], while joint constraints on fidelity and success probability have been derived for stochastic approximate transformations in entanglement theory and general state resource theories [35]. Other related results concern universal conclusive entanglement concentration [54] and the asymptotic undistillability of bound magic states assisted by correlating catalysts in odd-prime dimensions [55]. Although these works differ in their settings and assumptions, they all treat states as the resource objects to be transformed.

The present framework instead acts directly on quantum channels through successful branches of free superinstruments. The erasure-channel transformation in Subsec. V A cannot be reduced to a state-preparation task: its conditional output must transmit every input state and preserve correlations with an external reference. Preparation channels identify the special case in which the channel framework reduces to a state resource theory.

Table I compares the resource objects and admissibility conditions in the state and channel settings.

TABLE I. Comparison of probabilistic state-distillation bounds with the present channel framework.

Comparison	Previous state frameworks [32, 33]	Present channel framework
Objects and applications	Probabilistic resource-state conversion	Probabilistic resource-channel conversion, including erasure recovery and depolarizing-channel conversion
Conditional output	An approximate target state obtained from a specified resource state	A fixed channel acting on arbitrary, possibly reference-entangled inputs; success probability is independent of the transmitted input
Physical admissibility	Constraints of free state operations	Constraints of free superinstrument branches and the conditional channel's Choi marginal condition
Relation to state tasks	Corresponding state-distillation bounds	State quantities and bounds are recovered under the preparation-channel restriction

This distinction also separates the present task from optimizing entanglement distribution through a noisy channel [56]. In the latter problem, the channel is used to distribute a state, whereas here the channel itself is transformed by a probabilistic higher-order map. Tight projective-robustness bounds are known in several state resource theories, including particular affine theories [32, 33], but no analogous general tightness condition is currently known for channel transformations.

VI. CONCLUSION

We have derived one-shot and asymptotic converse bounds for probabilistic quantum channel distillation using successful branches of free superinstruments. For pure Choi targets with free fidelity strictly between zero and one and finite input projective robustness, the projective bound gives an explicit error lower bound for every nonzero success probability. When the relevant generalized robustness quantities are finite, the generalized bound limits the success probability and yields an explicit probability–error relation for pure Choi targets.

Regularizing the two robustness measures gives two asymptotic rate bounds. The projective-robustness bound applies to sequences of protocols with nonzero success probabilities, without imposing a condition on their numerical values. The generalized-robustness bound additionally assumes $n^{-1}\log(1/p_n) \to 0$. For the noisy coherent-state preparation channels, the two bounds become $R \le \log_2[(2-\gamma)/\gamma]$ and $R \le \log_2(2-\gamma)$, respectively. The second bound is tighter for $0 < \gamma < 1$, but it requires the success probabilities not to decay exponentially.

The erasure-channel example illustrates a genuinely channel-level phenomenon: postselection recovers the identity channel exactly even though the projective robustness is infinite, while generalized robustness still gives a finite constraint on the success probability. In contrast, the preparation-channel example instead uses the standard identification of a state ρ with the channel $\mathcal{P}_\rho$, for which $\widetilde{J}_{\mathcal{P}_\rho} = \rho$. Under this identification, the channel quantities and converse bounds reduce exactly to their state counterparts, leading to explicit coherence-distillation bounds. For depolarizing channels, the mixed-target bound is attained by retaining the input, providing a tight one-use benchmark.

The derivations rely only on the following conditions: the deterministic superchannel maps free channels to free channels, and each successful branch maps a free channel to a subnormalized free channel. They therefore apply to other dynamical resource theories satisfying the assumptions in Sec. II. The converse bounds obtained here are necessary conditions but are not generally known to be tight. In the erasure-channel example, the flag-measurement protocol attains the optimal exact-distillation probability $p_{\mathrm{opt}} = 1-\eta$, whereas the generalized-robustness bound $p \le 1-\eta+\eta/d^2$ is finite but not tight. Therefore, determining the tightness and attainability of these channel-transformation bounds, particularly for non-preparation channels, is an interesting direction for future investigation.

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Appendix A: The proof of Proposition 1

Let

$$\kappa := \widetilde{J}_{\mathcal{K}} \tag{A1}$$

be any normalized Choi state satisfying

$$F(\kappa,\tau) \ge 1-\varepsilon, \tag{A2}$$

and put

$$a := a_T, \qquad k := \mathrm{Tr}(T\kappa). \tag{A3}$$

Consider the binary measurement channel

$$Q_T(X) := \mathrm{Tr}(TX)|0\rangle\langle 0| + \mathrm{Tr}((I-T)X)|1\rangle\langle 1|. \tag{A4}$$

Fidelity is monotone nondecreasing under completely positive trace-preserving maps. Therefore,

$$1-\varepsilon \le F(\kappa,\tau) \tag{A5}$$
$$\le F(Q_T(\kappa), Q_T(\tau)) \tag{A6}$$
$$= \left(\sqrt{ak} + \sqrt{(1-a)(1-k)}\right)^2. \tag{A7}$$

Write

$$a = \cos^2\alpha, \qquad k = \cos^2\beta, \qquad \alpha,\beta \in [0,\pi/2]. \tag{A8}$$

The last line of Eq. (A7) is $\cos^2(\alpha-\beta)$. Hence

$$|\alpha-\beta| \le \arcsin\sqrt{\varepsilon}. \tag{A9}$$

The smallest admissible value of $k = \cos^2\beta$ is obtained by making β as large as Eq. (A9) permits, with $\beta \le \pi/2$. It follows that

$$k \ge k_-(a,\varepsilon) := \left[\sqrt{a(1-\varepsilon)} - \sqrt{(1-a)\varepsilon}\right]_+^2. \tag{A10}$$

Let

$$\sigma := \widetilde{J}_{\mathcal{L}} \tag{A11}$$

be a free normalized Choi state, and set

$$f := \mathrm{Tr}(T\sigma). \tag{A12}$$

Suppose that $\lambda,\mu \ge 0$ satisfy the two projective order inequalities

$$\kappa \le \lambda\sigma, \qquad \sigma \le \mu\kappa. \tag{A13}$$

Testing the first inequality against T and the second against $I - T$ gives

$$k \le \lambda f, \qquad 1 - f \le \mu(1 - k). \tag{A14}$$

For $0 < f < 1$ and $0 < k < 1$, it follows that

$$\lambda\mu \ge \frac{k}{1-k}\,\frac{1-f}{f}. \tag{A15}$$

The endpoint cases follow by continuity whenever the order pair is feasible.

By Eq. (24), $f \le c_T$. Since $(1 - f)/f$ is decreasing in f, $k/(1 - k)$ is increasing in k, and $k \ge k_-(a_T, \varepsilon)$, Eq. (A15) gives

$$\lambda\mu \ge \frac{k_-(a_T,\varepsilon)}{1-k_-(a_T,\varepsilon)}\,\frac{1-c_T}{c_T}. \tag{A16}$$

Taking the infimum over all free comparison states and all feasible order witnesses, and then over all normalized Choi states in the fidelity ball (A2), yields

$$\Omega^{\varepsilon}_{\widetilde{\diamond}}(\mathcal{M}^{\otimes m}) \ge \frac{k_-(a_T,\varepsilon)}{1-k_-(a_T,\varepsilon)}\,\frac{1-c_T}{c_T}. \tag{A17}$$

Combining Eqs. (23) and (A17), we obtain

$$\frac{k_-(a_T,\varepsilon)}{1-k_-(a_T,\varepsilon)}\,\frac{1-c_T}{c_T} \le Q. \tag{A18}$$

Solving Eq. (A18) for k_- gives

$$k_-(a_T,\varepsilon) \le \frac{Qc_T}{1-c_T+Qc_T} = x_T. \tag{A19}$$

If $a_T \le x_T$, this effect gives only the trivial lower bound $\varepsilon \ge 0$. Suppose now that $a_T > x_T$. Write

$$a_T = \cos^2\alpha, \qquad x_T = \cos^2\gamma, \qquad 0 \le \alpha < \gamma \le \frac{\pi}{2}. \tag{A20}$$

The inequality $k_-(a_T, \varepsilon) \le x_T$ is equivalent to

$$\arcsin\sqrt{\varepsilon} \ge \gamma - \alpha. \tag{A21}$$

Thus

$$\varepsilon \ge \sin^2(\gamma - \alpha) \tag{A22}$$

$$= \left(\sqrt{a_T(1-x_T)} - \sqrt{(1-a_T)x_T}\right)^2. \tag{A23}$$

The cases $a_T \le x_T$ and $a_T > x_T$ combine into the positive-part formula (25). Since that bound holds for every admissible effect T, taking the supremum proves Eq. (26).

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