Metric completion of the Bender–Brody–Müller Hamiltonian: dilation spectrum and missing eigenstates*

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Abstract

The Bender–Brody–Müller (BBM) Hamiltonian was proposed as a non-Hermitian Hilbert–Pólya operator. We analyze the Hilbert completion induced, on the standard half-line core $C_c^\infty(0,\infty) \subset L^2(\mathbb{R}_+)$, by BBM's explicit candidate metric $\hat{\eta} = \sin^2(\hat{p}/2) = \Delta^\dagger\Delta/4$. The bounded positive form $\eta_0 = \Delta^\dagger\Delta$ has trivial kernel but is not coercive. Completing the core in the norm $\|\psi\|_{\eta_0} = \|\Delta\psi\|$ gives a Hilbert space canonically unitarily equivalent to $L^2(\mathbb{R}_+)$; its free self-adjoint realization is the dilation generator, with simple, purely absolutely continuous spectrum $\mathbb{R}$. No bounded sandwich $\Delta^\dagger h(D)\Delta$ is boundedly invertible. For the symmetric restriction transported from the stated initial core, the closure has deficiency indices (∞,∞) and its adjoint has every real point as an eigenvalue of infinite multiplicity, whereas the free extension is purely continuous. These domain-dependent statements are separated from the realization-independent conclusion about BBM's specified candidate functions: $\Delta\psi_z = x^{-z}$, so for $\mathrm{Re}\, z = 1/2$ they do not belong to this completed space. Thus changing the self-adjoint extension within this completion cannot make the original BBM functions and boundary condition produce point-spectrum Riemann-zero states. Other non-L^2, rigged, distributional, or unbounded-metric formulations are outside the theorem.

Keywords: pseudo-Hermitian quantum mechanics; Bender–Brody–Müller Hamiltonian; metric completion; dilation operator

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The Hilbert–Pólya idea asks for a self-adjoint operator whose spectrum gives the imaginary parts of the nontrivial zeros of $\zeta(s)$. It is one of the reasons why random-matrix statistics, xp Hamiltonians, trace formulas, and de Branges spaces keep reappearing around the Riemann hypothesis [1–5]. The usual difficulty is not only to write a formal Hamiltonian with the right scaling behavior. One must also identify a Hilbert space, a domain, a self-adjoint realization, and a mechanism that turns a continuum of scale transformations into a discrete sequence.

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In 2017 Bender, Brody, and Müller (BBM) proposed a particularly sharp non-Hermitian candidate [6]. In the notation used below, their formal Hamiltonian is

$$H_{\mathrm{BBM}} = (1 - e^{-i\hat{p}})^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}) = \Delta^{-1} D \Delta, \tag{1}$$

where

$$(\hat{S}\psi)(x) = \psi(x-1)\mathbf{1}_{x>1}, \qquad \Delta = 1 - \hat{S}, \tag{2}$$

on $L^2(\mathbb{R}_+)$, and

$$D = \hat{x}\hat{p} + \hat{p}\hat{x} = -i(2x\partial_x + 1) \tag{3}$$

is the generator of dilations. The proposed physical interpretation was that a suitable pseudo-Hermitian Hilbert space would make (1) a real-spectrum Hamiltonian with eigenvalues proportional to the Riemann ordinates: for $z = \frac{1}{2} + i\gamma$, the BBM normalization gives $E = i(2z-1) = -2\gamma$. This is a natural physical move: in pseudo-Hermitian quantum mechanics a non-Hermitian Hamiltonian can represent ordinary unitary physics if there is a positive, invertible metric operator η for which $H^\dagger \eta = \eta H$ [7]. The metric, however, plays a structural role: it fixes the topology of the physical state space.

The proposal immediately raised a precise set of structural questions. Bellissard showed that $\hat{p}$ has no self-adjoint extension on the half-line, that the BBM eigenfunctions are not L^2 states on the critical line, and that a weighted-space cure destroys the critical line [8]. Moxley observed that a formal state redefinition conjugates the BBM eigenvalue equation to a dilation equation [9]. BBM replied that ordinary L^2 nonnormalizability is not decisive in a pseudo-Hermitian or biorthogonal formulation: normalization must instead be tested in the proposed physical metric or pairing [10,11]. Bender and Brody later left the momentum-space formulation of the eigenvalue problem as a challenging open problem [12]. Recent Hilbert–Pólya Hamiltonians use different operators and similarity structures and do not address the BBM shift metric [13].

Bellissard's objection and the obstruction proved here concern different Hilbert norms. Bellissard showed that the BBM functions ψ_z are not vectors of the ambient $L^2(\mathbb{R}_+)$ space on the critical line. We do not infer our conclusion from that fact. BBM explicitly proposed $\hat{\eta} = \sin^2(\hat{p}/2)$ as the metric associated with their shift similarity; on the half-line its quadratic form is $\eta_0 = \Delta^\dagger \Delta/4$. On the standard core the corresponding norm is $\|\psi\|_{\eta_0} = \|\Delta\psi\|/2$. Because this form is noncoercive, its completion has a weaker topology and is a priori different from the ambient L^2 space. Membership must therefore be checked anew. For the BBM functions the Hurwitz recurrence gives $\Delta\psi_z = x^{-z}$, which is not square-integrable when $\mathrm{Re}\, z = 1/2$. This completion-level obstruction does not follow from Bellissard's ordinary-L^2 nonmembership result.

The relevance of this completion is proposal-specific, not a claim of uniqueness. Completing $C_c^\infty(0,\infty)$ in the quadratic-form norm is the direct Hilbert-space implementation of BBM's explicit bounded positive form within the standard half-line L^2 realization [14,15]. It answers their response to Bellissard for the metric they wrote down: replacing the ordinary norm by η_0-normalizability still does not admit their stated candidate functions. A positive metric with unbounded inverse need not define an equivalent Hilbert topology, and its completion may change which candidate states exist, as in exactly solvable non-Hermitian models with the same obstruction [16]. We do not claim that η_0 is the unique possible metric, nor do we classify non-L^2, rigged-Hilbert-space, distributional, or unbounded-metric formulations.

For the free transported realization the result is the dilation continuum. More generally, the completion admits many self-adjoint realizations, but none can contain the BBM candidate eigenfunctions as Hilbert-space vectors on the critical line.

The result has two layers. The BBM-specific layer determines the completed state space and its operator realizations: the completion is canonically $L^2(\mathbb{R}_+)$ in the coordinate $\Phi = \Delta\psi$, the free transported realization is the self-adjoint dilation generator, and the original BBM eigenfunctions are absent because their shift differences are the non-L^2 power functions x^{-z} on the critical line. Two further statements hold independently of the zeta-function proposal. First, the loss of coercivity is universal within the bounded-sandwich class: no $\Delta^\dagger h(D)\Delta$ with bounded h is boundedly invertible. Second, the closure of the symmetric restriction transported from the stated initial core has deficiency indices (∞, ∞); its free extension is purely continuous, whereas its adjoint has every real point as an eigenvalue of infinite multiplicity. The latter indices are properties of this initial-domain realization, not of the dilation differential expression alone. The proof below follows this order.

The natural metric. The similarity form (1) canonically selects the positive form

$$\eta_0 = \Delta^\dagger\Delta = 2I - \hat{S} - \hat{S}^\dagger. \tag{4}$$

This is the precise half-line meaning of the physics shorthand $2(1 - \cos\hat{p})$. On the test domain

$$\mathcal{D}_0 = C_c^\infty(0, \infty) \tag{5}$$

the shift preserves $\mathcal{D}_0$. Indeed, if ψ is smooth and compactly supported away from the origin, then $\psi(x-1)\mathbf{1}_{x>1}$ is again smooth and compactly supported away from the origin. Hence $\Delta\psi \in \mathcal{D}_0$. The similarity notation suggests

$$H_{\mathrm{BBM}}^\dagger \eta_0 = \eta_0 H_{\mathrm{BBM}} \qquad \text{(formal)}. \tag{6}$$

Before an operator domain is chosen, neither side is an operator identity. Its rigorous content on the test space is the symmetry of D on $\Delta\mathcal{D}_0$; the resulting densely defined operator is $T = \overline{D}|_{\Delta\mathcal{D}_0}$ below. The first question is whether the positive form defines a Hilbert topology equivalent to the ambient one. In the usual bounded-metric setting it must be coercive. The BBM form fails precisely there.

Extend functions on $\mathbb{R}_+$ by zero to $\mathbb{R}$. With the Fourier convention

$$(\mathcal{F}\psi)(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx}\psi(x)\, dx,$$

Paley–Wiener gives the unitary identification

$$\mathcal{F} : L^2(\mathbb{R}_+) \simeq H^2(\mathbb{C}^-). \tag{7}$$

In this representation Δ is boundary multiplication by $1 - e^{-ik}$. The quadratic form of (4) has boundary symbol

$$\eta_0(k) = |1 - e^{-ik}|^2 = 2(1 - \cos k). \tag{8}$$

That is, $\langle\psi, \eta_0\psi\rangle = \int_{\mathbb{R}} \eta_0(k)|\mathcal{F}\psi(k)|^2\, dk$. Its zeros at $2\pi\mathbb{Z}$ do not give L^2 null vectors, but they do give soft directions of the metric. This is the first indication that the metric topology is weaker than the ambient one.

Physically, Δ differentiates against a unit shift, so slowly varying wave packets—Fourier-concentrated near the zeros of $1 - e^{-ik}$—have arbitrarily small metric length. The metric therefore changes the topology; its spectral realizations must be analyzed separately.

Metric degeneracy. For $\varepsilon > 0$ set

$$\phi_\varepsilon(k) = \frac{\sqrt{\varepsilon/\pi}}{k - 2\pi - i\varepsilon}. \tag{9}$$

The pole lies in $\mathbb{C}^+$, hence $\phi_\varepsilon \in H^2(\mathbb{C}^-)$, and $\|\phi_\varepsilon\|_{L^2(\mathbb{R})} = 1$. A residue calculation gives

$$\begin{aligned}\langle \phi_\varepsilon, \eta_0 \phi_\varepsilon \rangle &= \frac{2\varepsilon}{\pi} \int_{\mathbb{R}} \frac{1 - \cos u}{u^2 + \varepsilon^2}\, du \\ &= 2(1 - e^{-\varepsilon}) \longrightarrow 0.\end{aligned} \tag{10}$$

Since the Fourier images of $C_c^\infty(0, \infty)$ are dense in $H^2(\mathbb{C}^-)$ and the form is bounded above by 4, the same zero infimum is obtained on the original BBM test domain:

$$\inf_{\psi \in \mathcal{D}_0,\, \|\psi\|=1} \langle \psi, \eta_0 \psi \rangle = 0. \tag{11}$$

Thus the BBM metric is positive and has trivial kernel, but it is not bounded below. This is stronger than saying that a formal differential expression has a domain problem. The physical metric topology selected by the BBM similarity transform is itself noncoercive. The symbol and the degeneracy of the form are shown in Fig. 1.

The Cauchy kernel stays inside the half-line Hardy space: the probe is a legitimate Paley–Wiener state, normalized in L^2, whose metric length goes to zero; no full-line plane waves, which lie outside the Hardy space, are used.

It is useful to separate this from an eigenvector statement. The zero set $2\pi\mathbb{Z}$ has measure zero, so the multiplier on the ambient $L^2(\mathbb{R}, dk)$ has no nonzero kernel. The degeneracy is form-level: the form assigns arbitrarily small metric length to normalized states.

A universal obstruction. One may try to insert an additional observable in the dilation representation. Let h be bounded and measurable and define the metric sandwich

$$\eta_h = \Delta^\dagger h(D) \Delta. \tag{12}$$

This includes the positive case $h \geq 0$ and the natural indefinite attempt $h = \mathrm{sgn}$ in the Mellin spectral variable. The repair never produces a boundedly invertible metric.

The physical obstruction is a symmetry mismatch. The degeneracy of $\Delta^\dagger \Delta$ is localized in momentum: the symbol $|1 - e^{-ik}|^2$ vanishes at $k \in 2\pi\mathbb{Z}$. A multiplier $h(D)$, however, belongs to the spectral algebra of scale transformations. Momentum and scale cannot be diagonalized together. With our normalization of $D = xp + px$ one has

$$[D, p] = 2ip, \tag{13}$$

or $[D/2, p] = ip$ for the half-dilation generator. Thus a bounded function of D cannot cancel the momentum zeros of Δ pointwise: in the momentum representation it acts across scales rather than near the set $2\pi\mathbb{Z}$. Cancelling them pointwise would require an unbounded operation in the wrong spectral variable—not a bounded metric deformation.

Indeed choose $\chi \in C_c^\infty(1, 2)$ with $\|\chi\| = 1$ and set

$$\psi_n(x) = n^{-1/2} \chi(x/n). \tag{14}$$

Then $\|\psi_n\| = 1$, while translation continuity gives

$$\|\Delta\psi_n\|_{L^2(\mathbb{R}_+)} = \|\psi_n - \hat{S}\psi_n\| \longrightarrow 0. \tag{15}$$

Since $h(D)$ and Δ are bounded,

$$\|\eta_h\psi_n\| \leq \|\Delta^\dagger\|\,\|h(D)\|\,\|\Delta\psi_n\| \longrightarrow 0. \tag{16}$$

Hence η_h is not bounded below on the unit sphere and cannot have a bounded inverse. In particular, the indefinite form $\Delta^\dagger \mathrm{sgn}(D)\Delta$ is not a classical Krein or Pontryagin metric with a bounded fundamental symmetry [17]. This also lies outside the usual bounded-invertible metric setting of pseudo-Hermitian quantum mechanics [7].

This obstruction is universal: it is independent of the choice of h. Even for h bounded away from zero, the outer factors $\Delta^\dagger$ and Δ still make broad slowly varying states almost invisible. The obstruction is therefore not caused by the sign, phase, or positivity of the inserted observable. It is caused by the shift-difference factor that BBM needs in order to connect the Hurwitz recurrence to the dilation generator.

For the indefinite choice $h = \mathrm{sgn}$, one can say more. The operator

$$\eta_{\mathrm{sgn}} = \Delta^\dagger \mathrm{sgn}(D)\Delta \tag{17}$$

is self-adjoint and indefinite, because $\Delta\mathcal{D}_0$ is dense and $\mathrm{sgn}(D)$ has nonzero positive and negative spectral subspaces. The same broad states give $0 \in \mathrm{spec}(\eta_{\mathrm{sgn}})$. But 0 is not an eigenvalue: if $\eta_{\mathrm{sgn}}\psi = 0$, density gives $\ker\Delta^\dagger = \{0\}$, hence $\mathrm{sgn}(D)\Delta\psi = 0$, and since $\mathrm{sgn}(D)^2 = I$, $\Delta\psi = 0$. The equation $\Delta\psi = 0$ says $\psi(x) = \psi(x-1)$ for $x > 1$ and $\psi = 0$ on $(0, 1)$, so $\psi = 0$. Thus 0 lies in the continuous spectrum of the would-be Krein metric. This is exactly the regime where the usual indefinite-metric spectral arguments lose their main hypothesis.

The L^2-based completion. We now complete the standard test space in the norm defined by (4):

$$\|\psi\|_{\eta_0} = \langle\psi, \eta_0\psi\rangle^{1/2} = \|\Delta\psi\|_{L^2(\mathbb{R}_+)}, \qquad \psi \in \mathcal{D}_0. \tag{18}$$

Let $\widetilde{\mathcal{H}}_{\eta_0}$ be this completion. The map $\Delta : \mathcal{D}_0 \to L^2(\mathbb{R}_+)$ is an isometry from the η_0 norm to the ordinary L^2 norm. Its image is dense. To see this, if $g \perp \Delta\mathcal{D}_0$, then $\Delta^\dagger g = 0$, i.e.

$$g(x) = g(x+1) \quad \text{a.e. on } \mathbb{R}_+. \tag{19}$$

The only $L^2(\mathbb{R}_+)$ function with this one-periodicity is zero, since otherwise

$$\begin{aligned}\int_0^\infty |g(x)|^2\,dx &= \sum_{n=0}^{\infty}\int_0^1 |g(x+n)|^2\,dx \\ &= \sum_{n=0}^{\infty}\int_0^1 |g(x)|^2\,dx\end{aligned} \tag{20}$$

would diverge. Thus $\Delta\mathcal{D}_0$ is dense, and the isometry extends to a unitary map

$$\overline{\Delta} : \widetilde{\mathcal{H}}_{\eta_0} \longrightarrow L^2(\mathbb{R}_+). \tag{21}$$

Thus this η_0 completion is canonically $L^2(\mathbb{R}_+)$, with coordinate map given by the shift difference Δ. This determines the completed state space, but not a self-adjoint realization of the formal Hamiltonian.

In physical language, the completed theory has already chosen its observable wavefunction:

$$\Phi = \overline{\Delta}\psi = \Delta\psi. \tag{22}$$

Once the state is written as Φ, the similarity reduces the formal action to that of the dilation expression. It still leaves the operator domain open. The free choice below makes rigorous the reduction to the Berry–Keating Hamiltonian that BBM noted when conjugating with $\rho = \Delta$ [6].

Let $\overline{D}$ be the self-adjoint closure of D on $C_c^\infty(0,\infty)$. Choose the free transported realization

$$\widetilde{H}_{\mathrm{BBM}} = \overline{\Delta}^{-1}\overline{D\Delta}, \qquad \mathrm{Dom}(\widetilde{H}_{\mathrm{BBM}}) = \overline{\Delta}^{-1}\,\mathrm{Dom}(\overline{D}). \tag{23}$$

For every $\psi \in \mathcal{D}_0$ one has $\Delta\psi \in \mathcal{D}_0$, hence

$$\overline{\Delta}\,\widetilde{H}_{\mathrm{BBM}}\psi = D\Delta\psi. \tag{24}$$

Thus (23) is one rigorous realization of the formal expression $\Delta^{-1}D\Delta$ in the metric completion, without requiring Δ^{-1} to be a bounded operator on all of L^2.

For this choice, the formal cancellation becomes rigorous. On the original L^2 space Δ^{-1} is not bounded and everywhere defined; between the completed space and L^2, however, $\overline{\Delta}^{-1}$ is unitary. Equation (23) therefore has the spectrum of $\overline{D}$.

Spectrum of the free realization. Under the Mellin transform,

$$(\mathcal{M}D\mathcal{M}^{-1}f)(\tau) = 2\tau f(\tau). \tag{25}$$

Therefore $\overline{D}$ has purely absolutely continuous spectrum $\mathbb{R}$ with multiplicity one. Since (23) is a unitary equivalence,

$$\mathrm{spec}(\widetilde{H}_{\mathrm{BBM}}) = \mathbb{R}, \qquad \mathrm{spec}_{\mathrm{pp}}(\widetilde{H}_{\mathrm{BBM}}) = \varnothing, \tag{26}$$

and the spectrum is purely absolutely continuous with multiplicity one. This realization has no isolated levels.

The adjoint's real point spectrum. The same construction exhibits a spectral phenomenon of interest beyond the BBM problem. Let $T_0 = \overline{D}|_{\Delta\mathcal{D}_0}$ and $A_0 = \overline{T_0}$. Under the unitary (21), T_0 is the initial symmetric operator obtained by transporting the formal BBM rule on $\mathcal{D}_0$. It is not the minimal differential realization of D: the operator $D|_{\mathcal{D}_0}$ is essentially self-adjoint with closure $\overline{D}$, whereas $\Delta\mathcal{D}_0$ is not a graph core for $\overline{D}$. The Supplemental Material proves this distinction and shows that A_0 has deficiency indices (∞,∞) and that $\ker(A_0^* - \lambda)$ is infinite-dimensional for every $\lambda \in \mathbb{R}$. The key step is that $g \in \ker(A_0^* - \mu)$ exactly when $(D-\mu)g$ is a period-one distribution; explicit L^2 solutions produce both deficiency spaces and the real adjoint eigenspaces. The free operator $\overline{D}$ is one self-adjoint extension of A_0 and is purely continuous. Thus the deficiency indices depend on the standard initial BBM domain, while the metric completion by itself does not select the free extension. Other extensions that agree with the initial action on $\mathcal{D}_0$ may introduce point spectrum, as in related xp models [5,18].

The missing eigenstates. What decides the Hilbert–Pólya question is not which realization one prefers, but whether the BBM eigenstates live in the completed space at all. They do not. For $\psi_z(x) = -\zeta(z, x+1)$ the Hurwitz identity $\zeta(z,x) - \zeta(z,x+1) = x^{-z}$ gives

$$(\Delta\psi_z)(x) = x^{-z}, \qquad x > 1. \tag{27}$$

On the critical line $|x^{-z}| = x^{-1/2}$, so $\Delta\psi_z \notin L^2(\mathbb{R}_+)$: the candidate eigenfunction has no representative in $\widetilde{\mathcal{H}}_{\eta_0}$, whose elements are exactly the states with L^2 shift-difference. This is stronger than Bellissard's ordinary-L^2 observation in a different direction: nonmembership in the weaker η_0 completion had to be established independently and follows from the shift difference, not from $\psi_z \notin L^2$. It rules out the specified functions ψ_z together with the boundary condition $\psi_z(0) = 0$ for every self-adjoint realization that extends the initial BBM action on $\mathcal{D}_0$ in this completed space. Other extensions may have unrelated point spectrum, but they cannot restore these missing vectors.

Remark on de Branges-space realizations. The present L^2-based analysis neither excludes nor constructs an unbounded-metric, rigged-Hilbert-space, or de Branges realization. A de Branges space would replace the present state-space topology by a reproducing-kernel Hilbert space of entire functions and would require a Hermite–Biehler structure that makes the relevant multiplication operator symmetric or self-adjoint [3,19]. For the free transported choice this is a separate construction, not another self-adjoint extension inside $\widetilde{\mathcal{H}}_{\eta_0}$. The immediate candidate $E(z) = \xi(\frac{1}{2} + iz)$ satisfies $E^{\#} = E$, so it fails the strict Hermite–Biehler inequality; this rules out only that direct choice and does not classify other entire functions or de Branges spaces [4]. The Supplemental Material gives the short calculation and states the remaining scope.

This locates the new content relative to the earlier comments [8–10,12], which identified the similarity to the dilation generator and raised the associated domain questions. What the present analysis adds is the completion generated by BBM's metric, the nonuniqueness of the operator realization on that completion, and the failure of the original BBM candidate states to survive it.

The physics message is that the metric fixes a topology, not a Hamiltonian domain. BBM's form is noncoercive, and its completion from the standard core is unitarily $L^2(\mathbb{R}_+)$ in the coordinate $\Delta\psi$. Choosing the free dilation realization gives a simple absolutely continuous spectrum, whereas the symmetric restriction transported from the initial core has infinitely many self-adjoint extensions. That freedom cannot restore the proposed ψ_z, because they are absent from the state space. The theorem therefore excludes the original BBM candidate-function/boundary-condition mechanism only in the completion generated from $C_c^\infty(0,\infty)$ by the bounded positive form $\Delta^\dagger\Delta$. It does not exclude other Hilbert–Pólya operators or alternative non-L^2, rigged, distributional, or unbounded-metric formulations. Changing only the self-adjoint extension inside this same completion cannot restore the missing functions; a route outside the theorem must change the state-space topology or the candidate eigenfunction ansatz and establish its spectral claims independently. The same quasi-invertibility issue occurs in exactly solvable non-Hermitian models [16]; here it is forced by the BBM shift factor, and no bounded insertion $h(D)$ repairs it. Complete proofs are given in the Supplemental Material.

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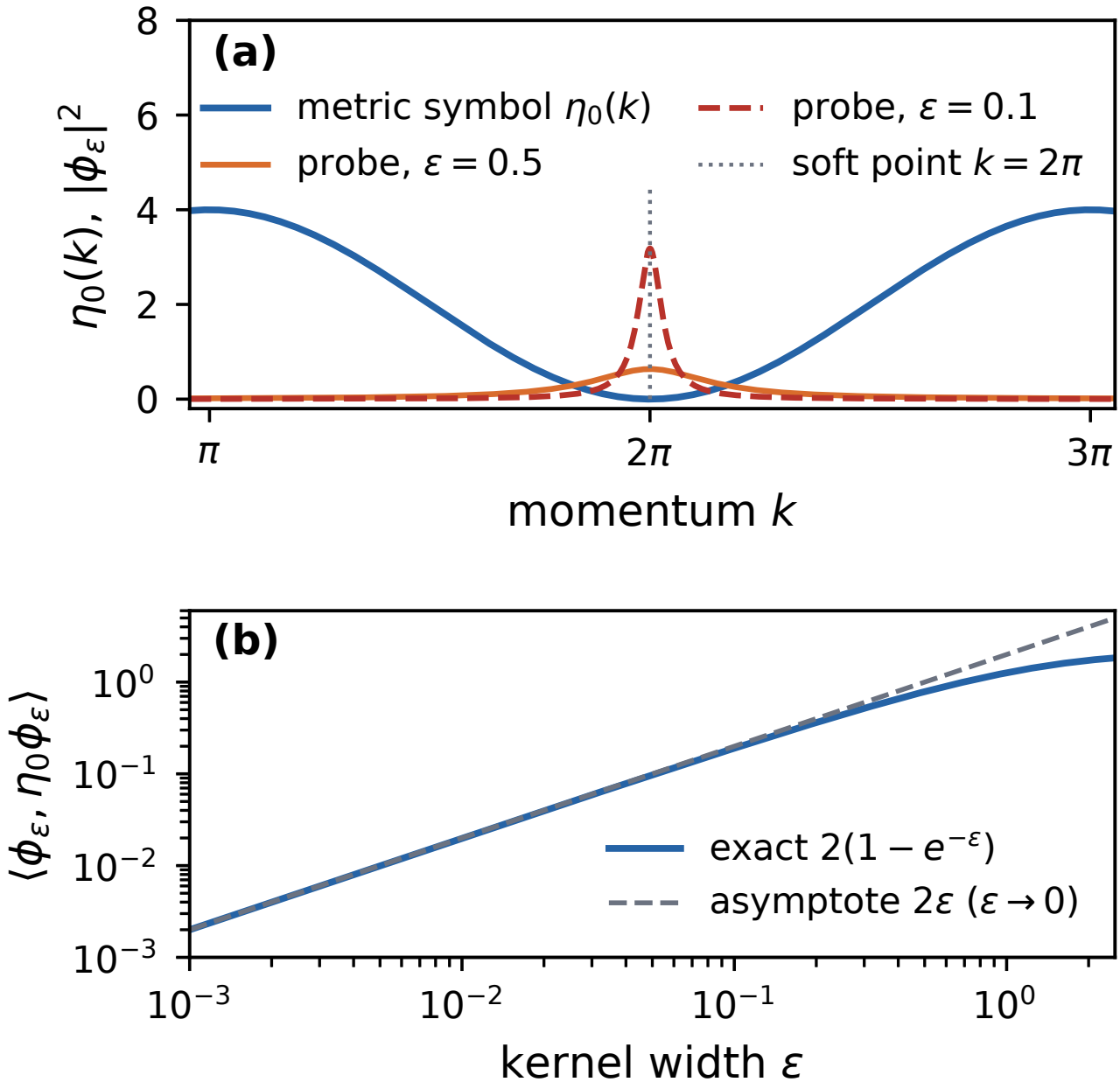


Figure 1: Metric degeneracy of the BBM form $\eta_0 = \Delta^\dagger \Delta$ in the Hardy representation. (a) Local window $k \in [\pi, 3\pi]$ around the soft point $k = 2\pi$: the boundary symbol $\eta_0(k) = 2(1 - \cos k)$ vanishes quadratically at $k = 2\pi\mathbb{Z}$, while the normalized Cauchy-kernel probes $|\phi_\varepsilon|^2$ concentrate at the soft point as ε decreases. (b) The squared η_0-norm of the unit-L^2-norm probe, $\langle \phi_\varepsilon, \eta_0 \phi_\varepsilon \rangle = 2(1 - e^{-\varepsilon})$ (exact residue evaluation), vanishes linearly in the small-width limit, $2(1 - e^{-\varepsilon}) = 2\varepsilon + O(\varepsilon^2)$ as $\varepsilon \to 0$: normalized states can carry arbitrarily small metric length, i.e. η_0 is not coercive.