New entanglement-assistant and asymmetric quantum codes from quasi-cyclic codes of index two

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Abstract

In this paper, we systematically investigate the construction of two important classes of quantum error-correcting codes, namely entanglement-assisted quantum error-correcting codes and asymmetric quantum error-correcting codes, from a generalized class of index-2 quasi-cyclic codes. For entanglement-assisted quantum error-correcting codes, we establish a systematic construction framework and derive sufficient conditions for obtaining maximal-entanglement entanglement-assisted quantum error-correcting codes, from which two new classes of such codes are obtained. For asymmetric quantum error-correcting codes, we derive necessary and sufficient conditions for the inclusion relation between any two index-2 QC codes and use them to establish a systematic construction framework. Several new entanglement-assisted quantum error-correcting codes and asymmetric quantum error-correcting codes with parameters improving upon previously reported results are obtained as applications of the proposed constructions.

Keywords: Quasi-cyclic codes, Entanglement-assisted quantum error-correcting codes, Asymmetric quantum error-correcting codes, Gilbert-Varshamov bound

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1. Introduction

Quantum information carried by physical quantum systems is inevitably affected by environmental noise, decoherence, and imperfect physical control during transmission and computation [1,2]. Quantum error-correcting codes (QECCs) are fundamental tools for protecting quantum information against these physical errors [3,4]. More importantly, QECCs provide a fundamental basis for fault-tolerant quantum computation. The fault-tolerance theorem shows that reliable quantum computation can be achieved when the physical error rates is below a certain threshold [5]. Recent experiments have demonstrated repeated rounds of quantum error correction and have shown that increasing the code distance can suppress logical errors [2,6]. These advances highlight the importance of developing QECCs with strong error-correcting capability for reliable quantum information processing.

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The physical resources and noise characteristics of practical quantum systems have motivated the study of extended classes of QECCs. From the perspective of physical resources, pre-shared entanglement between the sender and receiver can serve as an auxiliary quantum resource to assist the reliable transmission of quantum information over noisy quantum channels and provide additional flexibility in the use of communication and entanglement resources [7,8]. In standard QECC constructions, the underlying classical codes are required to satisfy self-orthogonality or dual-containing properties. However, classical codes that satisfy these conditions are limited, which restricts the range and performance of the resulting QECCs. Entanglement-assisted quantum error-correcting codes (EAQECCs) were introduced to relax this restriction by allowing the sender and receiver to share a certain number of entangled pairs before transmission. Compared with standard QECCs, EAQECCs consume additional entanglement resources but, in return, remove the restrictions imposed by the self-orthogonality or dual-containing conditions in standard QECC constructions, thereby allowing a broader class of classical linear codes to be used and providing greater flexibility in the trade-offs among coding rate, minimum distance, and entanglement consumption. From an application perspective, EAQECCs are therefore particularly attractive in scenarios where entanglement can be established in advance and exploited to improve coding flexibility and error-correction performance. From the perspective of noise characteristics, bit-flip and phase-flip errors may occur with different probabilities in physical quantum systems [9,10]. Such biased noise motivates asymmetric quantum error-correcting codes (AQECCs), which provide different error-correcting capabilities for these two types of errors [9,11]. Recent experiments on noise-biased cat qubits have further demonstrated significantly different bit-flip and phase-flip error rates and the correction of dominant phase-flip errors [12]. Taken together, EAQECCs and AQECCs are two physically motivated classes of QECCs arising from distinct features of practical quantum systems.

Constructing QECCs with good parameters is an important goal in quantum coding theory, and classical codes, such as cyclic, constacyclic, and quasi-cyclic (QC) codes, serve as important tools for constructing QECCs. QC codes are an important generalization of cyclic codes and possess rich algebraic structures while offering greater flexibility in code design [13,14]. In recent years, QC codes with various generator structures have been widely studied for QECC construction, among which index-2 QC codes have received particular attention [15–22]. Early studies mainly focused on index-2 QC codes with specific generator structures and constructed QECCs based on the corresponding self-orthogonality or dual-containing conditions. More recently, refs. [23,24] studied arbitrary index-2 QC codes from a more general perspective and systematically investigated their applications to QECC construction. In particular, ref. [24] gave a polynomial representation of the generating form and proved that every index-2 QC code admits the proposed generating form, thereby providing a unified structural description of index-2 QC codes. Based on this framework, the authors also constructed many QECCs with improved parameters. This unified description provides a structural basis for systematically studying the construction of EAQECCs and AQECCs from arbitrary index-2 QC codes.

Similar to QECCs, constructing EAQECCs and AQECCs with good parameters is also an important problem in related research. However, compared with traditional QECCs, relatively comprehensive and systematic parameter records are still lacking for these two types of quantum codes. For maximal-entanglement EAQECCs, the parameters of quaternary Hermitian linear complementary dual (LCD) codes provide a useful basis for comparison, since their code parameters can be directly translated into those of the corresponding binary

maximal-entanglement EAQECCs. In this direction, Araya and Harada [25] investigated the largest minimum weights of quaternary Hermitian LCD codes for various lengths and dimensions, providing useful parameter benchmarks for the corresponding maximal-entanglement EAQECCs. Many EAQECCs with good parameters have been constructed from classical codes such as cyclic, constacyclic, QC and quasi-twisted codes [26–39]. The Hull dimension of classical codes plays an important role in determining EAQECC parameters. However, existing EAQECC constructions from QC or quasi-twisted codes mainly focus on specific classes of index-2 QC or quasi-twisted codes. In [40], the authors investigated the LCD properties of arbitrary index-2 QC codes and systematically characterized the Hermitian LCD case. Since Hermitian LCD codes can be used to construct maximal-entanglement EAQECCs, these results provide a basis for such constructions from arbitrary index-2 QC codes. However, this study primarily addresses the maximal-entanglement case and assumes $\gcd(q, m) = 1$, thereby limiting the range of code lengths considered.

Classical linear codes, especially cyclic and constacyclic codes, have also been widely used to construct AQECCs with good parameters [41–46]. Nevertheless, QC constructions of AQECCs over small finite fields remain limited. In 2021, Lv et al. [47] investigated two classes of QC codes with special generator structures and proposed two explicit AQECC constructions, obtaining several AQECCs with good parameters. These results motivate further study of QC constructions of AQECCs over small finite fields, which are of particular interest for practical quantum systems. However, existing studies on constructing AQECCs from QC codes mainly focus on QC codes with specific generator structures.

Taken together, existing QC constructions of EAQECCs and AQECCs are mainly limited to specific subclasses or additional conditions. A general, systematic framework for constructing both classes of quantum codes from arbitrary index-2 QC codes remains lacking.

Motivated by the above studies, we focus on a generalized class of index-2 QC codes in [24] and systematically investigate their application in the construction of EAQECCs and AQECCs. The remainder of this paper is organized as follows. Section 2 reviews the basic definitions and notation. Section 3 presents the generating form of the index-2 QC codes considered in this work, together with the generating forms of their Hermitian and Euclidean dual codes. For EAQECCs, we develop a systematic construction from index-2 QC codes and further derive sufficient conditions under which such codes yield maximal-entanglement EAQECCs, leading to two classes of maximal-entanglement EAQECCs. For AQECCs, we first establish necessary and sufficient conditions for the inclusion relation between two index-2 QC codes and then develop a systematic construction of AQECCs from such codes. Illustrative examples are provided for both constructions. Section 4 provides several new EAQECCs and AQECCs with good parameters obtained from the proposed constructions. Finally, Section 5 summarizes the work.

2. Preliminaries

This section reviews the basic definitions of EAQECCs and AQECCs, their relationships with classical codes and the fundamental concepts of QC codes.

Let $q = p^r$, where p is prime and $r \geq 1$, and let F_q be a finite field with q elements. A linear code C over F_q with parameters $[n, k]_q$ is a k-dimensional subspace of F_q^n. The Hamming distance between two vectors in F_q^n is the number of coordinates in which they differ. The minimum distance of C is the smallest Hamming

distance between any pair of distinct codewords in C. For vectors $u = (u_0, \ldots, u_{n-1})$ and $v = (v_0, \ldots, v_{n-1})$ in F_q^n, the Euclidean inner product is defined as $[u, v] = \sum_{i=0}^{n-1} u_i v_i$. The Euclidean dual of a linear code $C \subseteq F_q^n$ is defined as $C^{\perp} = \{v \in F_q^n \mid [u, v] = 0,\ \forall u \in C\}$. For vectors $u = (u_0, \ldots, u_{n-1})$ and $v = (v_0, \ldots, v_{n-1})$ in $F_{q^2}^n$, the Hermitian inner product is defined as $[u, v]_h = \sum_{i=0}^{n-1} u_i v_i^q$. The Hermitian dual of a linear code $C \subseteq F_{q^2}^n$ is defined as $C^{\perp_h} = \{\, v \in F_{q^2}^n \mid [u, v]_h = 0,\ \forall u \in C \,\}$. Furthermore, the Hermitian hull of C is defined as

$$\mathrm{Hull}_h(C) = C \cap C^{\perp_h}.$$

2.1. Relationship between EAQECCs and classical linear codes

EAQECCs generalize the framework of standard QECCs by allowing the use of pre-shared entanglement (ebits) between the sender and receiver. An EAQECC Q over the finite field F_q is denoted by $[[n, k, d; c]]_q$, where n denotes the number of physical qudits, k is the number of logical qudits, d is the minimum distance, and c is the number of ebits. The code Q uses c shared entangled pairs to encode k logical qudits into n physical qudits and is capable of correcting up to $\lfloor \frac{d-1}{2} \rfloor$ errors. When $c = 0$, Q reduces to a standard QECC. When $c = n - k$, Q is referred to as a maximal-entanglement EAQECC.

One of the relationships between EAQECCs and classical codes is as follows:

Theorem 1 [48] **(Corollary 2)** Let C be an $[n, k, d]_{q^2}$ classical linear code over F_{q^2} with parity-check matrix H. Then, C can derive an EAQECC with parameters

$$[[n,\ 2k - n + c,\ d;\ c]]_q,$$

where

$$c = \mathrm{rank}(HH^{\dagger})$$

is the required number of ebits.

The number of ebits required by an EAQECC is determined by the Hermitian hull of the corresponding classical linear code.

Theorem 2 [32] **(Proposition 3.2)** Let C be a $[n, k, d]_{q^2}$ classical linear code over F_{q^2} with parity-check matrix H and generator matrix G. Then, $\mathrm{rank}(HH^{\dagger})$ and $\mathrm{rank}(GG^{\dagger})$ are independent of H and G so that

$$\mathrm{rank}(HH^{\dagger}) = n - k - \dim(\mathrm{Hull}_h(C)) = n - k - \dim(\mathrm{Hull}_h(C^{\perp_h})),$$

and

$$\mathrm{rank}(GG^{\dagger}) = k - \dim(\mathrm{Hull}_h(C)) = k - \dim(\mathrm{Hull}_h(C^{\perp_h})).$$

According to Theorem 2, we can directly obtain that $c = \mathrm{rank}\left(HH^{\dagger}\right) = \mathrm{rank}\left(GG^{\dagger}\right) + n - 2k$. We next recall the relationship between maximal-entanglement EAQECCs and Hermitian LCD codes. As a special case of an EAQECC with parameters $[[n, k, d; c]]_q$, a maximal-entanglement EAQECC has parameters $[[n, k, d; n - k]]_q$, namely, it satisfies $c = n - k$. Let C be a classical linear code over F_{q^2}. If C is used to construct a maximal-entanglement EAQECC, according to Theorems 1 and 2, the Hermitian Hull of C should satisfy

$$\dim(\mathrm{Hull}\, h(C)) = 0,$$

By the definition of the Hermitian Hull, this condition is equivalent to

$$C \cap C^{\perp_h} = \{0\}.$$

A linear code satisfying this condition is called a Hermitian LCD code. Hence, we can conclude that maximal-entanglement EAQECCs can be directly obtained from Hermitian LCD codes. For example, a Hermitian LCD code with parameters $[n,k,d]_{q^2}$ gives rise to a maximal-entanglement EAQECC with parameters $[[n,k,d;n-k]]_q$. This relationship provides a direct way to compare the parameters of maximal-entanglement EAQECCs with those obtained from Hermitian LCD codes.

2.2. Relationship between AQECCs and classical linear codes

A QECC with parameters $[[n,k,d]]_q$ can be regarded as a symmetric QECC, since it has the same error-correcting capability against bit-flip and phase-flip errors. In the notation of AQECCs, this corresponds to the special case $[[n,k,d_z/d_x]]_q$ with $d_z = d_x = d$. In contrast, an AQECC with parameters $[[n,k,d_z/d_x]]_q$ distinguishes these two types of errors: it can correct up to $\lfloor (d_x-1)/2 \rfloor$ bit-flip errors and up to $\lfloor (d_z-1)/2 \rfloor$ phase-flip errors. In addition, this code is capable of detecting up to d_x-1 bit-flip errors and d_z-1 phase-flip errors.

Analogous to the construction of QECCs, classical linear codes also provide a basis for deriving AQECCs.

Theorem 3 [41](Lemma 4) Let C_i be a classical code with parameters $[n,k_i,d_i]_q$ for $i \in \{1,2\}$. Assuming that $C_1^{\perp} \subseteq C_2$, an AQECCs with parameters $[[n,k_1+k_2-n,d_z/d_x]]_q$ can be constructed, where $d_x = \min\{\mathrm{wt}\left(C_2 \backslash C_1^{\perp}\right), \mathrm{wt}(C_1 \backslash C_2^{\perp})\}$ and $d_z = \max\left\{\mathrm{wt}\left(C_2 \backslash C_1^{\perp}\right), \mathrm{wt}\left(C_1 \backslash C_2^{\perp}\right)\right\}$.

To evaluate the performance of AQECCs constructed from classical linear codes, their parameters are typically compared with those of fundamental existence bounds. Among these, the asymmetric quantum Gilbert-Varshamov (GV) bound is notable because it offers an existence criterion for AQECCs with given parameters. When the parameters of an AQECC outperform the GV bound, the code is generally regarded as having good error-correcting performance.

Theorem 4 [49](Theorem 2) Suppose that n, k_1, k_2, d_x and d_z are positive integers satisfying $k_1 + k_2 > n$ and

$$\frac{q^{k_1}-q^{n-k_2}}{q^n-1}\sum_{i=1}^{d_x-1}\binom{n}{i}(q-1)^i + \frac{q^{k_2}-q^{n-k_1}}{q^n-1}\sum_{i=1}^{d_z-1}\binom{n}{i}(q-1)^i < 1,$$

then $[[n,k_1+k_2-n,d_z/d_x]]_q$ AQECCs exist.

2.3. QC codes

QC codes generalize cyclic codes. Let $l,m>0$ and $z = (z_{00}, z_{01}, \ldots, z_{0,l-1}, z_{10}, \ldots, z_{1,l-1}, \ldots z_{m-1,0}, \ldots, z_{m-1,l-1}) \in F_q^{lm}$, arranged as m blocks of length l. Define the l-cyclic shift σ^l by

$$\sigma^l(z) = (z_{m-1,0}, \ldots, z_{m-1,l-1}, z_{0,0}, \ldots, z_{0,l-1}, \ldots, z_{m-2,0}, \ldots, z_{m-2,l-1}).$$

A linear code $C \subseteq F_q^{lm}$ is called a QC code of length lm of index l if $T^l(C) = C$.

Let quotient ring $R_m = F_q[x]/(x^m-1)$. A vector $v = (v_0, \ldots, v_{m-1})$ in $\mathbb{F}_q^m$ can be identified with the residue class

$$[v(x)] = [v_0 + v_1 x + v_2 x^2 + \cdots + v_{m-1}x^{m-1}] \in R_m \tag{1}$$

The polynomial representation of codewords of C is obtained via the mapping $\phi : F_q^{lm} \to R_m^l$,

$$\phi(z) = ([z_0(x)], [z_1(x)], \ldots, [z_{l-1}(x)]),$$

where $z_j(x) = \sum_{i=0}^{m-1} z_{ij}x^i \in F_q[x]$. Thus, an index-$l$ QC code of length lm can be identified with an R_m-submodule of R_m^l. If the code C is generated by t elements $\alpha_1, \ldots, \alpha_t \in R_m^l$ such that $C = \left\{ \sum_{i=1}^{t} \alpha_i[k_i(x)] | k_i(x) \in F_q[x] \right\}$, then C is said to be a t-generators QC code.

An upper triangular generating form can be given for any index-2 QC code. For convenience, a polynomial $a(x) \in \mathbb{F}_q[x]$ will simply be denoted by a in what follows.

Theorem 5 [24](Theorem 4) Any index-2 QC code C of length $2m$ over F_q admits the following generating form :

$$\begin{pmatrix} [g] & [af] \\ 0 & [s] \end{pmatrix},$$

where $g, s, a, d, f \in \mathbb{F}_q[x]$ such that

$$g,\ s \mid (x^m - 1),$$
$$d = \gcd(gs,\ x^m - 1),$$
$$gs = ad,$$
$$\partial f < m.$$

The parameters of C are

$$[2m,\ 2m - \partial g - \partial s]_q.$$

Throughout this paper, $\partial f = \deg(f)$ denotes the degree of $f \in F_q[x]$, and $\gcd(h, f)$ and $\mathrm{lcm}(h, f)$ denote the greatest common divisor(gcd) and the least common multiple(lcm) of h and f, respectively. The notation $f \mid h$ indicates that the polynomial f divides h. For a divisor f of $x^m - 1$, we also define the polynomials $\bar{f}$ as $\bar{f} = f(1/x)\ mod\ (x^m - 1)$, f^* as $f^* = (x^m - 1)/f$ and $\tilde{f}$ as $\tilde{f} = f^*(1/x)x^{\partial f^*}$.

3. Construction of two special classes of quantum error-correcting codes.

The following notation will be used in the construction of EAQECCs and AQECCs.

Remark 1 Denote

$$C_q(g, s, f) = (([g],\ [af]),\ (0,\ [s])),$$

where the polynomials in $F_q[x]$ satisfy Theorem 5. Then, $C_q(g, s, f)$ is a index-2 QC code of length $2m$ over F_q.

Theorem 5 implies that the parameters of $C_q(g, s, f)$ are $[2m, 2m - \partial g\ -\partial s]_q$. Moreover, any QC code of index 2, including the index-2 QC codes studied in the literature [15–22], is contained in this class of QC codes and can be represented in the above unified generating form.

Reference [24] presented the polynomial generating form of the Hermitian dual code $C_{q^2}^{\perp_h}(g, s, f)$ of $C_{q^2}(g, s, f)$ over F_{q^2}.

Lemma 1 [24](Proposition 1) Let $C_{q^2}(g, s, f)$ be a QC code of length $2m$ over F_{q^2} with generators $([g], [af])$ and $(0, [s])$. Let $x^m - 1 = bd$, where $d = \gcd(gs, x^m - 1)$. The Hermitian dual $C_{q^2}^{\perp_h}(g, s, f)$ is the QC code with generators $([\tilde{g}^q], 0)$ and $([-\bar{f}^q\bar{b}^q x^{\partial s^*}], [\tilde{s}^q])$, whose parameters are $[2m, \partial g + \partial s]_{q^2}$.

By a straightforward generalization of the above Lemma, we obtain the polynomial generating form of the Euclidean dual $C_q^{\perp}(g,s,f)$ of $C_q(g,s,f)$.

Corollary 1 Let $C_q(g,s,f)$ be a QC code of length $2m$ over F_q with generators $([g],[af])$ and $(0,[s])$. Let $x^m-1=bd$, where $d=\gcd(gs,x^m-1)$. The Euclidean dual $C_q^{\perp}(g,s,f)$ is the QC code with generators $([\tilde{g}],0)$ and $([-\overline{f}\overline{b}x^{\partial s^*}],[\tilde{s}])$, whose parameters are $[2m,\partial g+\partial s]_q$.

Proof. The proof follows the same line as that of Lemma 1 in [24] Proposition 1.

3.1. Construction of EAQECCs

In this section, we present a systematic generator-matrix-based framework for constructing EAQECCs and maximal-entanglement EAQECCs by using the QC codes described in Remark 1 that satisfy Theorem 5.

The generator matrix of $C_{q^2}(g,s,f)$ is provided:

Lemma 2 Let matrix A_1, A_2 and A_3 over F_{q^2} be

$$A_1=\begin{pmatrix}[g]\\ [xg]\\ \vdots\\ [x^{m-\partial g-1}g]\end{pmatrix}, A_2=\begin{pmatrix}[s]\\ [xs]\\ \vdots\\ [x^{m-\partial s-1}s]\end{pmatrix}, A_3=\begin{pmatrix}[af]\\ [xaf]\\ \vdots\\ [x^{m-\partial g-1}af]\end{pmatrix},$$

where the corresponding vectors, determined by their residue class in R_m, are described in Equation (1). The parameters of $C_{q^2}(g,s,f)$ are $[2m,2m-\partial g-\partial s]_{q^2}$. Its generator matrix is

$$G=\begin{pmatrix}A_1 & A_3\\ \mathbf{0} & A_2\end{pmatrix}.$$

We develop a systematic approach for constructing EAQECCs based on the QC codes of index 2.

Theorem 6 Let $C_{q^2}(g,s,f)$ be a QC code with parameters $[2m,\ 2m-\partial g-\partial s,\ d]_{q^2}$. Suppose that G is a generator matrix of $C_{q^2}(g,s,f)$ in Lemma 2. Then, $C_{q^2}(g,s,f)$ can derive two EAQECCs with parameters

$$[[2m,c,d;2\partial g+2\partial s-2m+c]]_q$$

and

$$\left[\left[2m,2\partial g+2\partial s-2m+c,d^{\perp_h};c\right]\right]_q,$$

where $c=\mathrm{rank}(GG^{\dagger})$ and $d^{\perp_h}$ denotes the minimum distance of $C_{q^2}^{\perp_h}(g,s,f)$.

According to Theorem 5, any index-2 QC code can be rewritten in the proposed generating form; these QC codes include the index-2 QC codes studied in [15–22]. Consequently, Theorem 6 provides broader flexibility for constructing EAQECCs with enhanced parameters.

We give an example illustrating the construction of EAQECCs with the assistance of algebraic software Magma [50] in calculating code parameters.

Example 1 Let $q=2$,$m=7$. Choose the polynomials $g=x^4+x^3+x^2+1$, $s=x^6+x^5+x^4+x^3+x^2+x+1$ and $f=1$ in $F_4[x]$. Then a QC code $C_{q^2}(g,s,f)$ over F_4 can be obtained.

Since

$$x^7-1=(x^4+x^3+x^2+1)(x^3+x^2+1),$$
$$x^7-1=(x^6+x^5+x^4+x^3+x^2+x+1)(x+1),$$

we have

$$x^7-1=g(x^3+x^2+1), x^7-1=s(x+1).$$

Thus, the polynomials g, s satisfy the condition $g,s \mid (x^7-1)$ in Theorem 5. Hence, we obtain $g^*=x^3+x^2+1$, $s^*=x+1$. Furthermore, one can compute that

$$gs=x^{10}+x^8+x^7+x^3+x+1, \tilde{g}^q=x^3+x+1, \tilde{s}^q=x+1,$$

Since

$$x^{10}+x^8+x^7+x^3+x+1=(x^7-1)(x^3+x+1),$$

we have $d=\gcd(gs,x^7-1)=x^7-1$. Therefore,

$$a=gs/d=x^3+x+1, b=(x^7-1)/d=1, -\bar{f}^q\bar{b}^q x^{\partial s^*}=x,$$

Since

$$A_1=\begin{pmatrix} 1011100 \\ 0101110 \\ 0010111 \end{pmatrix}, A_2=\begin{pmatrix} 1111111 \end{pmatrix}, A_3=\begin{pmatrix} 1101000 \\ 0110100 \\ 0011010 \end{pmatrix}.$$

we have

$$GG^\dagger=\begin{pmatrix} 1111 \\ 1111 \\ 1111 \\ 1111 \end{pmatrix},$$

Therefore, $c=\mathrm{rank}(GG^\dagger)=1$. By Theorem 6, the code $C_{q^2}(g,s,f)$ yields an EAQECCs $[[14,1,7;7]]_2$, and $C_{q^2}^{\perp_h}(g,s,f)$ yields an EAQECCs $[[14,7,3;1]]_2$.

Based on the systematic framework provided in Theorem 6 for constructing EAQECCs from index-2 QC codes, we further give sufficient conditions under which such QC codes can yield maximal-entanglement EAQECCs, and accordingly propose a systematic generator-matrix-based construction method for maximal-entanglement EAQECCs from index-2 QC codes. Compared with the method in [40], our approach does not require m and q to be coprime, thereby enlarging the range of searchable code lengths and providing a broader search space for obtaining more maximal-entanglement EAQECCs with good parameters.

Theorem 7 Suppose that $C_{q^2}(g,s,f)$ is a QC code with parameters $[2m,2m-\partial g-\partial s,d]_{q^2}$, which has generator matrix G in Lemma 2. The sufficient conditions for the existence of two maximal-entanglement EAQECCs with parameters $[[2m,2m-\partial g-\partial s,d;\partial g+\partial s]]_q$ and $\left[\left[2m,\partial g+\partial s,d^{\perp_h};2m-\partial g-\partial s\right]\right]_q$ are as follows:

(1) $A_2A_2^\dagger$ is a nonsingular matrix;

(2) The matrix P has no zero eigenvalues, where $P=A_1A_1^\dagger+A_3A_3^\dagger-A_3A_2^\dagger(A_2A_2^\dagger)^{-1}A_2A_3^\dagger$.

Proof. The parameters of $C_{q^2}(g,s,f)$ and $C_{q^2}^{\perp_h}(g,s,f)$ is $[2m,2m-\partial g-\partial s,d]_{q^2}$ and $\left[2m,\partial g+\partial s,d^{\perp_h}\right]_{q^2}$. By Theorem 6, we obtain two EAQECCs $[[2m,c,d;2\partial g+2\partial s-2m+c]]_q$ and $[[2m,2\partial g+2\partial s-2m+c,d^{\perp_h};c]]_q$ EAQECCs, where $c=\mathrm{rank}(GG^\dagger)$ and $d^{\perp_h}$ denotes the minimum distance of $C_{q^2}^{\perp_h}(g,s,f)$. We compute the number of entangled ebits c. Note that

$$GG^\dagger = \begin{pmatrix} A_1 & A_3 \\ \mathbf{0} & A_2 \end{pmatrix} \begin{pmatrix} A_1^\dagger & \mathbf{0} \\ A_3^\dagger & A_2^\dagger \end{pmatrix} = \begin{pmatrix} A_1A_1^\dagger + A_3A_3^\dagger & A_3A_2^\dagger \\ A_2A_3^\dagger & A_2A_2^\dagger \end{pmatrix}$$

Since $A_2A_2^\dagger$ is nonsingular by assumption, we define the following matrices:

$$A = \begin{pmatrix} I_{n-k_1} & \mathbf{0} \\ 0 & \left(A_2A_2^\dagger\right)^{-1} \end{pmatrix}, B = \begin{pmatrix} I_{n-k_1} & -A_3A_2^\dagger \\ \mathbf{0} & I_{n-k_2} \end{pmatrix}.$$

Then

$$AGG^\dagger = \begin{pmatrix} A_1A_1^\dagger + A_3A_3^\dagger & A_3A_2^\dagger \\ (A_2A_2^\dagger)^{-1}A_2A_3^\dagger & I_{n-k_2} \end{pmatrix},$$

and

$$BAGG^\dagger = \begin{pmatrix} A_1A_1^\dagger + A_3A_3^\dagger - A_3A_2^\dagger(A_2A_2^\dagger)^{-1}A_2A_3^\dagger & \mathbf{0} \\ (A_2A_2^\dagger)^{-1}A_2A_3^\dagger & I_{n-k_2} \end{pmatrix}.$$

If matrix $P = A_1A_1^\dagger + A_3A_3^\dagger - A_3A_2^\dagger(A_2A_2^\dagger)^{-1}A_2A_3^\dagger$ has no zero eigenvalues, then both P and $BAGG^\dagger$ are full rank. Since A, B are invertible, we have $c = \text{Rank}\left(GG^\dagger\right) = \text{Rank}\left(AGG^\dagger\right) = \text{Rank}\left(BAGG^\dagger\right) = 2m - \partial g - \partial s$. Hence, two maximal-entanglement EAQECCs with the following parameters $[[2m, 2m - \partial g - \partial s, d; \partial g + \partial s]]_q$ and $\left[\left[2m, \partial g + \partial s, d^{\perp_h}; 2m - \partial g - \partial s\right]\right]_q$, respectively, are derived.

The following example illustrates the proposed construction of maximal-entanglement EAQECCs.

Example 2 Let $q = 2$, $m = 7$. Choose the polynomials in $F_2[x]$ as $g = x+1$, $s = x^6+x^5+x^4+x^3+x^2+x+1$, and $f = x^3 + x^2 + wx + w^2$, we can obtain a QC code $C_{q^2}(g, s, f)$ over F_4.

Owing to

$$x^7 - 1 = (x + 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1),$$

therefore,

$$x^7 - 1 = g(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$
$$x^7 - 1 = s(x + 1).$$

Polynomials g, s satisfy the conditions $g, s \mid (x^7 - 1)$ in Theorem 5. Thus, $g^* = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$, $s^* = x + 1$. The following polynomials can be obtained:

$$gs = x^7 - 1.$$
$$\tilde{g}^q = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, \tilde{s}^q = x + 1,$$

therefore, $d = \gcd(gs, x^7 - 1) = x^7 - 1$. We can obtain that

$$a = gs/d = 1, b = (x^7 - 1)/d = 1,$$
$$-\bar{f}^q\bar{b}^q x^{\partial s^*} = x^6 + x^5 + wx + w^2,$$

Then, we verify the conditions given in Theorem 7 for constructing maximal-entanglement EAQECCs from our QC codes. Owing to

$$A_1 = \begin{pmatrix} 1100000 \\ 0110000 \\ 0011000 \\ 0001100 \\ 0000110 \\ 0000011 \end{pmatrix}, A_2 = \begin{pmatrix} 1111111 \end{pmatrix}, A_3 = \begin{pmatrix} \omega^2\omega 11000 \\ 0\omega^2\omega 1100 \\ 00\omega^2\omega 110 \\ 000\omega^2\omega 11 \\ 1000\omega^2\omega 1 \\ 11000\omega^2\omega \end{pmatrix}.$$

We can obtain that

$$P = \begin{pmatrix} 110\omega^2\omega 0 \\ 1110\omega^2\omega \\ 01110\omega^2 \\ \omega 01110 \\ \omega^2\omega 0111 \\ 0\omega^2\omega 011 \end{pmatrix}$$

Since the characteristic polynomial of matrix P is $(x+1)^6$, the matrix P has no zero eigenvalues. By Theorem 7, $C_{q^2}(g,s,f)$ yields a maximal-entanglement EAQECC $[[14,7,6;7]]_2$ and $C_{q^2}^{\perp_h}(g,s,f)$ yields a maximal-entanglement EAQECC $[[14,7,6;7]]_2$.

3.2. Construction of AQECCs

In this section, we derive the necessary and sufficient conditions for the inclusion relation between two index-2 QC codes, which leads to a systematic framework for constructing new classes of AQECCs from QC codes.

Theorem 8 Let $C_q(g',s',f')$ be a QC code of length $2m$ over F_q with generators $([g'],[a'f'])$ and $(0,[s'])$. The corresponding polynomials satisfy: g', $s' \mid (x^m-1)$, $d' = \gcd(g's', x^m-1)$, $g's' = a'd'$ and $\partial f' < m$. The necessary and sufficient conditions of QC codes $C_q(g,s,f)$ be contained in $C_q(g',s',f')$ are

(1) A polynomial u in $\mathbb{F}_q[x]$ exists satisfying $[ug'] = [g]$ and $[s'] \mid [af - ua'f']$,

(2) A polynomial v in $\mathbb{F}_q[x]$ exists satisfying $[s'] \mid [s - v(g')^* a'f']$.

Proof. Assume that the QC codes $C_q(g',s',f')$ satisfy these conditions. According to condition (1), we know that $[ug'] = [g]$, and a polynomial n in $\mathbb{F}_q[x]$ can be found such that:

$$[ns'] = [af - ua'f'].$$

Therefore,

$$\begin{aligned} &[u]([g'],[a'f']) + [n](0,[s']) \\ &= ([ug'],[ug'f']) + (0,[ns']) \\ &= ([ug'],[ua'f' + ns']) \\ &= ([ug'],[ua'f' + af - ua'f']) \\ &= ([g],[af]). \end{aligned}$$

Likewise, based on condition (2), a polynomial m in $\mathbb{F}_q[x]$ can be found such that $[ms'] = [s - (g')^* va'f']$.

Therefore,

$$\begin{aligned}
&[(g')^*v]([g'],[a'f']) + [m](0,[s']) \\
&= ([(g')^*vg'],[(g')^*va'f']) + ([0],[ms']) \\
&= ([(g')^*vg'],[ms' + (g')^*va'f']) \\
&= ([(g')^*vg'],[s - (g')^*va'f' + (g')^*va'f']) \\
&= ([0],[s]).
\end{aligned}$$

Thus, $C_q(g', s', f')$ contains $C_q(g, s, f)$.

Assume that $C_q(g', s', f')$ contains $C_q(g, s, f)$. There exist polynomials u, f_1, n, m in $\mathbb{F}_q[x]$ satisfying:

$$\begin{aligned}
&([g],[af]) \\
&= [u]([g'],[a'f']) + [n](0,[s']) \\
&= ([ug'],[u'a'f']) + (0,[ns']) \\
&= ([ug'],[ua'f' + ns'])
\end{aligned} \tag{2}$$

$$\begin{aligned}
&(0,[s]) \\
&= [f_1]([g'],[a'f']) + [m](0,[s']) \\
&= ([f_1g'],[f_1a'f']) + (0,[ms']) \\
&= ([f_1g'],[f_1a'f' + ms'])
\end{aligned} \tag{3}$$

According to Equation (2), we can conclude that $[ug'] = [g]$ and $[ua'f' + ns'] = [af]$. We can then derive the following:

$$[ns'] = [af - ua'f']$$

Thus, $[s'] \mid [af - ua'f']$.

According to Equation (3), $[f_1g'] = 0$. As a result, a polynomial v in $\mathbb{F}_q[x]$ can be found such that $f_1 = v(g')^*$. Since $[s] = [f_1a'f' + ms']$, we can conclude that $[ms'] = [s - f_1a'f'] = [s - v(g')^*a'f']$. Thus, $[s'] \mid [s - v(g')^*a'f']$. The proof is concluded.

By using the QC codes described in Remark 1 that satisfy Theorem 5, we are able to formulate a systematic framework for constructing AQECCs, leading to new classes of AQECCs.

Theorem 9 For a QC code $C_q(g, s, f)$, its Euclidean dual $C_q^{\perp}(g, s, f)$ is derived in Corollary 1. Let $C_q(g', s', f')$ be a index-2 QC code with generators $([g'],[a'f'])$ and $(0,[s'])$. If the corresponding polynomials of $C_q(g', s', f')$ satisfy the conditions in Theorem 8, an AQECC with parameters

$$[[2m, \partial g + \partial s - \partial g' - \partial s', \geq d_z/ \geq d_x]]_q$$

can be constructed, where

$$\begin{aligned}
d_z &= \max\{\mathrm{wt}(C_q(g', s', f')), \mathrm{wt}(C_q^{\perp}(g, s, f)\}, \\
d_x &= \min\{\mathrm{wt}(C_q(g', s', f')), \mathrm{wt}\left(C_q^{\perp}(g, s, f)\right)\}.
\end{aligned}$$

Proof. According to the Corollary 1, the generators of $C_q^{\perp}(g, s, f)$ are $([\tilde{g}], 0)$ and $([-\overline{f}\overline{b}x^{\partial s^*}], [\tilde{s}])$. Since the corresponding polynomials of QC codes $C_q(g', s', f')$ satisfy the conditions in Theorem 8, we can conclude that $C_q(g, s, f) \subseteq C_q(g', s', f')$. The code $C_q(g', s', f')$ has dimension $2m - \partial g' - \partial s'$.

Using the QC codes $C_q(g,s,f)$ and $C_q(g',s',f')$, an AQECCs with parameters $[[2m, \partial g+\partial s-\partial g'-\partial s', \geq d_z/\geq d_x]]_q$ can be obtained from Theorem 3, where

$$d_z = \max\{\mathrm{wt}(C_q(g',s',f')\backslash C_q^{\perp}(g,s,f)), \mathrm{wt}(C_q(g,s,f)\backslash C_q^{\perp}(g',s',f'))\},$$

and

$$d_x = \min\{\mathrm{wt}(C_q(g',s',f')\backslash C_q^{\perp}(g,s,f)), \mathrm{wt}(C_q(g,s,f)\backslash C_q^{\perp}(g',s',f'))\}.$$

We can conclude that $\mathrm{wt}\left(C_q(g',s',f')\backslash C_q^{\perp}(g,s,f)\right) \geq \mathrm{wt}\left(C_q(g',s',f')\right)$ and $\mathrm{wt}(C_q(g,s,\ f)\backslash C_q^{\perp}(g',s',f')) \geq \mathrm{wt}\left(C_q(g,s,f)\right)$. Thus,

$$d_z = \max\{\mathrm{wt}\left(C_q(g',s',f')\right), \mathrm{wt}\left(C_q(g,s,f)\right)\}$$

and

$$d_x = \min\left\{\mathrm{wt}\left(C_q(g',s',f')\right), \mathrm{wt}\left(C_q(g,s,f)\right)\right\}.$$

In Theorem 8, we characterize the necessary and sufficient conditions for the inclusion relation between two index-2 QC codes. It follows that for any index-2 QC code, the AQECCS constructed via Theorem 3 is contained in the class of AQECCs obtained via Theorem 9. Consequently, the AQECCs constructed in this work contain those derived in [47] from the subclass of index-2 QC codes.

Additionally, the inclusion criterion established in Theorem 8 is both necessary and sufficient. Thus, our criterion is less restrictive than existing criteria and applies to a wider range of QC code pairs satisfying the required conditions for constructing AQECCs. This provides a larger search space for obtaining AQECCs with improved parameters.

The following examples illustrate the proposed construction of AQECCs.

Example 3 Let $q=2$, $m=7$. Choose the polynomials in $\mathbb{F}_2[x]$:

$$g = x^4+x^2+x+1, s = x^4+x^3+x^2+1, f = 1.$$
$$g' = x+1, s' = x^4+x^3+x^2+1, f' = x+1.$$

Polynomials g, s, g', s' satisfy the conditions $g, s \mid (x^7-1)$ and $g', s' \mid (x^7-1)$ in Remark 1. Thus, $g^* = x^3+x+1$, $s^* = x^3+x^2+1$. We can derive the following polynomials:

$$\tilde{g} = x^3+x^2+1, \tilde{s} = x^3+x+1,$$

Owing to

$$x^8+x^7+x+1 = (x^7-1)(x+1),$$
$$x^7-1 = (x^4+x^3+x^2+1)(x^3+x^2+1),$$

therefore, $d = \gcd(gs, x^7-1) = x^7-1$ and $d' = \gcd(g's', x^7-1) = x^4+x^3+x^2+1$. Hence

$$a = gs/d = x+1, b = (x^7-1)/d = 1,$$
$$a' = g's'/d' = x+1, -\overline{f}\overline{b}x^{\partial s^*} = x^3,$$

Then, we verify the conditions of $C_q(g,s,f) \subseteq C_q(g',s',f')$. Owing to

$$[x^4+x^2+x+1] = [(x+1)(x^3+x^2+1)]$$

therefore,

$$[g] = [g'(x^3 + x^2 + 1)], [s] = [s'].$$

There exists polynomials $v = 0$ such that $[s'] \mid [s - v(g')^* a' f']$ (i.e.$[s'] \mid [s]$). There exists polynomials $u = x^3 + x^2 + 1$ such that $[ug'] = [g]$. We can derive that

$$af - ua'f' = x^5 + x^4 + x^3 + x.$$

Owing to

$$[x^5 + x^4 + x^3 + x] = [(x^4 + x^3 + x^2 + 1)(x)],$$

therefore

$$[af - ua'f'] = [s'(x)].$$

We can obtain $[s'] \mid [af - ua'f']$. Thus, the conditions of Theorem 8 are fulfilled by these polynomials.

The parameters of $C_q^{\perp}(g, s, f)$ and $C_q(g', s', f')$ are $[14, 8, 3]_2$ and $[14, 9, 4]_2$. Theorem 9 yields an AQECC with parameters $[[14, 3, \geq 4/ \geq 3]]_2$. If we fix $k_1 = 8$, $k_2 = 9$, $dz = 4$ and $dx = 3$, Theorem 4 shows that the asymmetric quantum GV bound yields a minimum code length of 17 for binary AQECCs. Hence, our AQECCs can break the GV bound.

Example 4 Let $q = 3$, $m = 8$. Choose polynomials in $\mathbb{F}_3[x]$:

$$g = x^4 + 2x^3 + 2x + 2, s = x^6 + x^5 + 2x^4 + 2x^2 + 2x + 1, f = 1.$$
$$g' = x^2 + 2x + 2, s' = x^6 + x^5 + 2x^4 + 2x^2 + 2x + 1,$$
$$f' = x^4 + x^3 + x^2 + 2x + 2.$$

Polynomials g, s, g', s' satisfy the conditions $g, s \mid (x^8 - 1)$ and $g', s' \mid (x^8 - 1)$ in Remark 1. Thus, $g^* = x^4 + x^3 + x^2 + 2x + 1$, $s^* = x^2 + 2x + 2$. We can derive the following polynomials:

$$\tilde{g} = x^4 + 2x^3 + x^2 + x + 1, \tilde{s} = 2x^2 + 2x + 1,$$

Owing to

$$x^{10} + x^8 + 2x^2 + 2 = (x^8 - 1)(x^2 + 1),$$
$$x^8 - 1 = (x^8 - 1)(1),$$

therefore, $d = \gcd(gs, x^8 - 1) = x^8 - 1$ and $d' = \gcd(g's', x^8 - 1) = x^8 - 1$. Hence

$$a = gs/d = x^2 + 1, b = (x^8 - 1)/d = 1,$$
$$a' = g's'/d' = 1, -\overline{f}\overline{b}x^{\partial s^*} = 2x^2,$$

Then, we verify the conditions of $C_q(g, s, f) \subseteq C_q(g', s', f')$. Owing to

$$[x^4 + 2x^3 + 2x + 2] = [(x^2 + 2x + 2)(x^2 + 1)]$$
$$[x^6 + x^5 + 2x^4 + 2x^2 + 2x + 1] = [(x^6 + x^5 + 2x^4 + 2x^2 + 2x + 1)]$$

therefore,

$$[g] = [g'(x^2 + 1)], [s'] = [s].$$

There exists polynomials $v = 0$ such that $[s'] \mid [s - v(g')^* a' f']$ (i.e.$[s'] \mid [s]$). There exists polynomials $u = x^2 + 1$ such that $[ug'] = [g]$. We can derive that

$$af - ua'f' = 2x^6 + 2x^5 + x^4 + x^2 + x + 2.$$

Owing to

$$\begin{aligned}&[2x^6 + 2x^5 + x^4 + x^2 + x + 2]\\&= [(x^6 + x^5 + 2x^4 + 2x^2 + 2x + 1)(2)],\end{aligned}$$

therefore

$$[af - ua'f'] = [s'(2)].$$

We can obtain $[s'] \mid [af - ua'f']$. Thus, the conditions of Theorem 8 are fulfilled by these polynomials.

The parameters of $C_q^{\perp}(g, s, f)$ and $C_q(g', s', f')$ are $[16, 10, 4]_2$ and $[16, 8, 6]_2$. Theorem 9 yields an AQECC with parameters $[[16, 2, \geq 6/ \geq 4]]_2$. If we fix $k_1 = 10$, $k_2 = 8$, $dz = 6$ and $dx = 4$, Theorem 4 shows that the asymmetric quantum GV bound yields a minimum code length of 18 for binary AQECCs. Hence, our AQECCs can break the GV bound.

4. New EAQECCs and AQECCs

In this section, we present the results of the construction of two classes of QECCs obtained from the proposed systematic framework. In the EAQECC part, we present 12 new EAQECCs with good parameters by applying Theorem 7. Although we have checked the well-known Grassl code tables [51], the available records of EAQECCs are still limited and incomplete. Therefore, our EAQECCs are mainly compared with the known codes reported in [25,29,35,39,40]. For comparison with the Hermitian LCD codes reported in [25,40], we convert these Hermitian LCD codes into the corresponding maximal-entanglement EAQECCs according to the correspondence described at the end of Section 2.1, so that their parameters can be compared directly. The obtained codes outperform the corresponding known EAQECCs and, to the best of our knowledge, have not been reported elsewhere.

In the AQECC part, we present 9 new AQECCs with good parameters by applying Theorem 9. Compared with standard QECCs (i.e., symmetric QECCs), the construction of AQECCs over small finite fields remains relatively limited, and a comprehensive code table analogous to the Grassl code tables [51] for QECCs is currently unavailable. Hence, the AQECCs obtained in this work are mainly compared with the symmetric QECCs of the same length and dimension listed in [51], where the correspondence between QECCs and symmetric AQECCs is provided in Section 2.2, as well as with previously reported AQECCs of the same length and dimension and the asymmetric quantum GV bound. An AQECC is referred to as a new code if, compared with a symmetric QECC of the same length and dimension, it has improved error-correcting capability for either bit-flip or phase-flip errors, and its parameters improve on those of previously known AQECCs in the literature and exceed those guaranteed by the asymmetric quantum GV bound.

To save space and facilitate the presentation of the construction details, the associated polynomials are expressed in a compact form using their coefficient sequences arranged in increasing degree order, where exponents denote repeated coefficients.As illustrations, the polynomial $1 + wx^3 + w^2x^5$ over F_4 is denoted by 10^2203, while the polynomials $1 + x + x^3$ over F_2 and $1 + x^3 + 2x^5$ over F_3 are denoted by 1^201 and 10^2102, respectively.

4.1. New EAQECCs

A total of 12 new maximal-entanglement EAQECCs are obtained, and their construction details are listed below.

(1) $[[22,10,9;12]]_2 : g=1,\ s=1^{11}$ and $f=312(21)^2$. $C_{q^2}(g,s,f)$ is $[22,12,6]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[22,10,9]_4$ with generators $\tilde{a}_1=x^{11}-1$, $\tilde{a}_2=1^2$, and $\bar{f}^q\bar{b}^q x^{\partial s^*}=320^3 12^3 02$. By Theorem 7, we obtain the EAQECCs $[[22,12,6;10]]_2$ and $[[22,10,9;12]]_2$. Compared with the $[[22,10,8;12]]_2$ EAQECCs in [25], our EAQECC have better performances.

(2) $[[26,12,10;12]]_2 : g=1,\ s=1^{13}$ and $f=013103(12)^2 21$. $C_{q^2}(g,s,f)$ is $[26,14,7]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[26,12,10]_4$ with generators $\tilde{a}_1=x^{13}-1$, $\tilde{a}_2=1^2$, and $\bar{f}^q\bar{b}^q x^{\partial s^*}=101202^3 3213^2$. By Theorem 7, we obtain the EAQECCs $[[26,14,7;12]]_2$ and $[[26,12,10;14]]_2$. Compared with the $[[26,12,9;14]]_2$ EAQECCs in [25], our EAQECC have better performances.

(3) $[[26,13,9;13]]_2 : g=1^2,\ s=1^{13}$ and $f=2023^3(023)^2$. $C_{q^2}(g,s,f)$ is $[26,13,9]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[26,13,9]_4$ with generators $\tilde{a}_1=1^{13}$, $\tilde{a}_2=1^2$, and $\bar{f}^q\bar{b}^q x^{\partial s^*}=(03)^2 202302^3 3$. By Theorem 7, we obtain the EAQECCs $[[26,13,9;13]]_2$ and $[[26,13,9;13]]_2$. Compared with the $[[26,13,8;13]]_2$ EAQECCs in [25,40], our EAQECC have better performances.

(4) $[[28,12,11;16]]_2 : g=1,\ s=(10)^6 1$ and $f=31201^2 0^2 3^3$. $C_{q^2}(g,s,f)$ is $[28,16,7]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[28,12,11]_4$ with generators $\tilde{a}_1=x^{14}-1$, $\tilde{a}_2=101$, and $\bar{f}^q\bar{b}^q x^{\partial s^*}=1^2 2(022)^2 1^3 21$. By Theorem 7, we obtain the EAQECCs $[[28,16,7;12]]_2$ and $[[28,12,11;16]]_2$. Compared with the $[[28,12,10;16]]_2$ EAQECCs in [25], our EAQECC have better performances.

(5) $[[38,18,12;20]]_2 : g=1,\ s=1^{19}$ and $f=0^2 2^2 32^2 30^2 31320^2 1$. $C_{q^2}(g,s,f)$ is $[38,20,9]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[38,18,12]_4$ with generators $\tilde{a}_1=x^{19}-1$, $\tilde{a}_2=1^2$, and $\bar{f}^q\bar{b}^q x^{\partial s^*}=0^3 1^2 0313^2 202101^2 03$. By Theorem 7, we obtain the EAQECCs $[[38,20,9;18]]_2$ and $[[38,18,12;20]]_2$. Compared with the $[[38,18,11;20]]_2$ EAQECCs in [35], our EAQECC have better performances.

(6) $[[42,21,12;21]]_2 : g=1^2,\ s=(231)^7$ and $f=03010(32)^2 10^2 101203^2$. $C_{q^2}(g,s,f)$ is $[42,21,12]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[42,21,11]_4$ with generators $\tilde{a}_1=1^{21}$, $\tilde{a}_2=31$, and $\bar{f}^q\bar{b}^q x^{\partial s^*}=20^2 132^2 013103^2 202^2 31^2$. By Theorem 7, we obtain the EAQECCs $[[42,21,12;8]]_2$ and $[[42,21,11;6]]_2$. Compared with the $[[43,21,8;22]]_2$ EAQECCs in [29], our EAQECC achieves a higher minimum distance under similar parameters.

(7) $[[44,24,11;20]]_2 : g=1,\ s=1(01)^{10}$ and $f=01^2 3010213020^2 231203$. $C_{q^2}(g,s,f)$ is $[44,24,11]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[44,20,13]_4$ with generators $\tilde{a}_1=x^{22}-1$, $\tilde{a}_2=101$, and $\bar{f}^q\bar{b}^q x^{\partial s^*}=1^2 0201^3 2^2 3^2 01^4 20313$. By Theorem 7, we obtain the EAQECCs $[[44,24,11;8]]_2$ and $[[44,20,13;6]]_2$. Compared with the $[[44,24,10;20]]_2$ EAQECCs in [39], our EAQECC have better performances.

(8) $[[46,23,13;23]]_2 : g=1^2,\ s=1^{23}$ and $f=3^4 01^2 23^3 2132^4 12^2 1$. $C_{q^2}(g,s,f)$ is $[46,23,13]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[46,23,13]_4$ with generators $\tilde{a}_1=1^{23}$, $\tilde{a}_2=1^2$, and $\bar{f}^q\bar{b}^q x^{\partial s^*}=2^2 013^2 13^4 2132^3 31^2 02^2$. By Theorem 7, we obtain the EAQECCs $[[46,23,13;23]]_2$ and $[[46,23,13;23]]_2$. Compared with the $[[46,23,8;23]]_2$ EAQECCs in [40], our EAQECC have better performances.

(9) $[[58,29,15;29]]_2 : g=1^2,\ s=1^{29}$ and $f=1^2 02^2 02^3 301^2 030^2 13023^2 1^2 21$. $C_{q^2}(g,s,f)$ is $[58,29,15]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[58,29,15]_4$ with generators $\tilde{a}_1=1^{29}$, $\tilde{a}_2=1^2$, and $\bar{f}^q\bar{b}^q x^{\partial s^*}=1^2 0^2 131^2 2^2 30210^2 201^2 023^3 03^2$. By Theorem 7, we obtain the EAQECCs $[[58,29,15;29]]_2$ and $[[58,29,15;29]]_2$. Compared with the $[[58,29,8;29]]_2$ EAQECCs in [40], our EAQECC have better performances.

(10) $[[62,31,16;31]]_2 : g = 1^2$, $s = 1^{31}$ and $f = 320^2 1^2 0232^2 0^2 12^2 3202313^3 02^2 32$. $C_{q^2}(g,s,f)$ is $[62,31,16]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[62,31,16]_4$ with generators $\tilde{a}_1 = 1^{31}$, $\tilde{a}_2 = 1^2$, and $\bar{f}^q \bar{b}^q x^{\partial s^*} = 320323^2 02^3 1230323^2 10^2 3^2 2301^2$. By Theorem 7, we obtain the EAQECCs $[[62,31,16;31]]_2$ and $[[62,31,16;31]]_2$. Compared with the $[[62,31,8;31]]_2$ EAQECCs in [40], our EAQECC have better performances.

(11) $[[70,35,17;35]]_2 : g = 1^2$, $s = 1^{35}$ and $f = 023^4 2^2 0^2 202^2 1^2 30^2 301^2 21^2 0102^2 103$. $C_{q^2}(g,s,f)$ is $[70,35,17]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[70,35,17]_4$ with generators $\tilde{a}_1 = 1^{35}$, $\tilde{a}_2 = 1^2$, and $\bar{f}^q \bar{b}^q x^{\partial s^*} = 30^2 2013^2 (01)^2 131^2 020^2 21^2 3^2 030^2 3^2 2^4$. By Theorem 7, we obtain the EAQECCs $[[70,35,17;35]]_2$ and $[[70,35,17;35]]_2$. Compared with the $[[70,35,8;35]]_2$ EAQECCs in [40], our EAQECC have better performances.

(12) $[[74,37,18;37]]_2 : g = 1^2$, $s = 1^{37}$ and $f = 3^3 2^2 020^2 202^3 3^2 03^2 030^2 10121302^3 312$. $C_{q^2}(g,s,f)$ is $[74,37,18]_4$. $C_{q^2}^{\perp_h}(g,s,f)$ is $[74,37,18]_4$ with generators $\tilde{a}_1 = 1^{37}$, $\tilde{a}_2 = 1^2$, and $\bar{f}^q \bar{b}^q x^{\partial s^*} = 2^2 03123^3 02131010^2 20 2^2 02^2 3^3 030^2 303^2 2$. By Theorem 7, we obtain the EAQECCs $[[74,37,18;37]]_2$ and $[[74,37,18;37]]_2$. Compared with the $[[74,37,8;37]]_2$ EAQECCs in [40], our EAQECC have better performances.

Table 1. Code comparisons.

No.	EAQECCs	Compared codes	ref.
1	$[[22,10,9;12]]_2$	$[[22,10,8;12]]_2$	[25]
2	$[[26,12,10;12]]_2$	$[[26,12,9;14]]_2$	[25]
3	$[[26,13,9;13]]_2$	$[[26,13,8;13]]_2$	[25,40]
4	$[[28,12,11;16]]_2$	$[[28,12,10;16]]_2$	[25]
5	$[[38,18,12;20]]_2$	$[[38,18,11;20]]_2$	[35]
6	$[[42,21,12;21]]_2$	$[[43,21,8;22]]_2$	[29]
7	$[[44,24,11;20]]_2$	$[[44,24,10;20]]_2$	[39]
8	$[[46,23,13;23]]_2$	$[[46,23,8;23]]_2$	[40]
9	$[[58,29,15;29]]_2$	$[[58,29,8;29]]_2$	[40]
10	$[[62,31,16;31]]_2$	$[[62,31,8;31]]_2$	[40]
11	$[[70,35,17;35]]_2$	$[[70,35,8;35]]_2$	[40]
12	$[[74,37,18;37]]_2$	$[[74,37,8;37]]_2$	[40]

Table 1 summarizes the new EAQECCs. We contrast these EAQECCs with the EAQECCs in [25,29,35, 39,40], and they all improve the corresponding minimum distance.

4.2. New AQECCs

A total of 9 new AQECCs are obtained, and their construction details are listed below.

(1) $[[10,4,\geq 4/\geq 2]]_2 : g = 1^5, s = x^5 - 1, f = 1, g' = 1, s' = x^5 - 1, f' = 01^2 01$. The QC code $C_q(g',s',f')$ is $[10,5,4]_2$. The Euclidean dual $C_q^{\perp}(g,s,f)$ of $C_q(g,s,f)$ is $[10,9,2]_2$. The binary AQECCs $[[10,4,\geq 4/\geq 2]]_2$ can be obtained from Theorem 9. Compared with the $[[10,4,\geq 3/\geq 2]]_2$ AQECC in [47], our AQECC have better performances. Compared with the $[[10,4,\geq 3/\geq 3]]_2$ symmetric QECC in [51], our AQECC has a stronger capability for correcting phase-flip errors.

(2) $[[18,3,\geq 6/\geq 3]]_2 : g = 10^2 1, s = x^9 - 1, f = 1^2, g' = 1, s' = x^9 - 1, f' = 1^3 0^2 10^2 1$. The QC code $C_q(g',s',f')$ is $[18,9,6]_2$. The Euclidean dual $C_q^{\perp}(g,s,f)$ of $C_q(g,s,f)$ is $[18,12,3]_2$. The binary AQECCs $[[18,3,\geq 6/\geq 3]]_2$ can be obtained from Theorem 9. Compared with the $[[18,3,\geq 4/\geq 3]]_2$ AQECC in [47], our AQECC have better performances. Compared with the $[[18,3,\geq 5/\geq 5]]_2$ symmetric QECC in [51], our AQECC has a stronger capability for correcting phase-flip errors.

(3) $[[30, 14, \geq 8/ \geq 2]]_2 : g = 1^{15}, s = x^{15} - 1, f = 1, g' = 1^2 0^2 1, s' = 1^4 (01)^2 1^2 0^2 1,\ f' = 1(0^3 1^2)^2$. The QC code $C_q(g', s', f')$ is $[30, 15, 8]_2$. The Euclidean dual $C_q^{\perp}(g, s, f)$ of $C_q(g, s, f)$ is $[30, 29, 2]_2$. The binary AQECCs $[[30, 14, \geq 8/ \geq 2]]_2$ can be obtained from Theorem 9. Compared with the $[[30, 14, \geq 7/ \geq 2]]_2$ AQECC in [47], our AQECC have better performances. Compared with the $[[30, 14, \geq 5/ \geq 5]]_2$ symmetric QECC in [51], our AQECC has a stronger capability for correcting phase-flip errors.

(4) $[[34, 9, \geq 8/ \geq 4]]_2 : g = 10^2 1^4 0^2 1, s = x^{17} - 1, f = (10)^2 1^2, g' = 1, s' = x^{17} - 1, f' = 10^5 10^2 1^3 010^2 1$. The QC code $C_q(g', s', f')$ is $[34, 17, 8]_2$. The Euclidean dual $C_q^{\perp}(g, s, f)$ of $C_q(g, s, f)$ is $[34, 26, 4]_2$. The binary AQECCs $[[34, 9, \geq 8/ \geq 4]]_2$ can be obtained from Theorem 9. Compared with the $[[34, 9, \geq 7/ \geq 4]]_2$ AQECC in [47], our AQECC have better performances. Compared with the $[[34, 9, \geq 7/ \geq 7]]_2$ symmetric QECC in [51], our AQECC has a stronger capability for correcting phase-flip errors.

(5) $[[38, 18, \geq 8/ \geq 2]]_2 : g = 1, s = x^{19} - 1, f = 1^4 0^2 101^4 010^2 1, g' = 1, s' = 1^2, f' = 1$. The QC code $C_q(g', s', f')$ is $[38, 37, 2]_2$. The Euclidean dual $C_q^{\perp}(g, s, f)$ of $C_q(g, s, f)$ is $[38, 19, 8]_2$. The binary AQECCs $[[38, 18, \geq 8/ \geq 2]]_2$ can be obtained from Theorem 9. Compared with the $[[38, 18, \geq 7/ \geq 2]]_2$ AQECC in [47], our AQECC have better performances. Compared with the $[[38, 18, \geq 6/ \geq 6]]_2$ symmetric QECC in [51], our AQECC has a stronger capability for correcting phase-flip errors.

(6) $[[20, 5, \geq 7/ \geq 4]]_3 : g = 2^2 121^2, s = x^{10} - 1, f = 2^2 01, g' = 1, s' = x^{10} - 1, f' = 10210121$. The QC code $C_q(g', s', f')$ is $[20, 10, 7]_3$. The Euclidean dual $C_q^{\perp}(g, s, f)$ of $C_q(g, s, f)$ is $[20, 15, 4]_3$. The ternary AQECCs $[[20, 5, \geq 7/ \geq 4]]_3$ can be obtained from Theorem 9. Compared with the $[[20, 5, \geq 6/ \geq 4]]_3$ AQECC in [47], our AQECC have better performances. Compared with the $[[20, 5, \geq 6/ \geq 6]]_3$ symmetric QECC in [51], our AQECC has a stronger capability for correcting phase-flip errors.

(7) $[[26, 0, \geq 13/ \geq 6]]_3 : g = 102^3 01, s = 1^{13}, f = 1^2 0^2 1, g' = 102^3 01, s' = 1^{13}, f' = 1^2 0^2 1$. The QC code $C_q(g', s', f')$ is $[26, 8, 13]_3$. The Euclidean dual $C_q^{\perp}(g, s, f)$ of $C_q(g, s, f)$ is $[26, 18, 6]_3$. The ternary AQECCs $[[26, 0, \geq 13/ \geq 6]]_3$ can be obtained from Theorem 9. Compared with the $[[26, 0, \geq 10/ \geq 6]]_3$ AQECC in [47], our AQECC have better performances. Compared with the $[[26, 0, \geq 10/ \geq 10]]_3$ symmetric QECC in [51], our AQECC has a stronger capability for correcting phase-flip errors.

(8) $[[26, 12, \geq 8/ \geq 2]]_3 : g = 1^{13}, s = x^{13} - 1, f = 1, g' = 1, s' = x^{13} - 1, f' = 01^2 012^3 01021$. The QC code $C_q(g', s', f')$ is $[26, 13, 8]_3$. The Euclidean dual $C_q^{\perp}(g, s, f)$ of $C_q(g, s, f)$ is $[26, 25, 2]_3$. The ternary AQECCs $[[26, 12, \geq 8/ \geq 2]]_3$ can be obtained from Theorem 9. Compared with the $[[26, 12, \geq 7/ \geq 2]]_3$ AQECC in [47], our AQECC have better performances. Compared with the $[[26, 12, \geq 6/ \geq 6]]_3$ symmetric QECC in [51], our AQECC has a stronger capability for correcting phase-flip errors.

(9) $[[28, 13, \geq 9/ \geq 2]]_3 : g = 1^{14}, s = x^{14} - 1, f = 2, g' = 1^2, s' = (21)^7, f' = 0102012^2 02^4$. The QC code $C_q(g', s', f')$ is $[28, 14, 9]_3$. The Euclidean dual $C_q^{\perp}(g, s, f)$ of $C_q(g, s, f)$ is $[28, 27, 2]_3$. The ternary AQECCs $[[28, 13, \geq 9/ \geq 2]]_3$ can be obtained from Theorem 9. Compared with the $[[28, 13, \geq 8/ \geq 2]]_3$ AQECC in [47], our AQECC have better performances. Compared with the $[[28, 13, \geq 6/ \geq 6]]_3$ symmetric QECC in [51], our AQECC has a stronger capability for correcting phase-flip errors.

Table 2 summarizes the new AQECCs. These AQECCs have improved phase-flip error-correcting capability compared with symmetric quantum codes of the same length and dimension. Moreover, their parameters improve on those of previously known codes in [47] and exceed those guaranteed by the GV bound. Furthermore, the code rates of our codes are greater than those satisfying the asymmetric quantum GV bound. As an example,

consider the code having parameters $[[18, 3, \geq 4/ \geq 3]]_2$ listed in the second row of Table 2. Under the same dimension and the same bit-flip and phase-flip error-correcting capabilities, it achieves a markedly higher code rate than the AQECC determined by the asymmetric quantum GV bound.

Table 2. Code comparisons.

No.	q	AQECCs	Codes in [47]	GV bounds
1	2	$[[10, 4, \geq 4/ \geq 2]]_2$	$[[10, 4, \geq 3/ \geq 2]]_2$	$[[14, 4, 4/2]]_2$
2	2	$[[18, 3, \geq 6/ \geq 3]]_2$	$[[18, 3, \geq 4/ \geq 3]]_2$	$[[21, 3, 6/3]]_2$
3	2	$[[30, 14, \geq 8/ \geq 2]]_2$	$[[30, 14, \geq 7/ \geq 2]]_2$	$[[40, 14, 8/2]]_2$
4	2	$[[34, 9, \geq 8/ \geq 4]]_2$	$[[34, 9, \geq 7/ \geq 4]]_2$	$[[42, 9, 8/4]]_2$
5	2	$[[38, 18, \geq 8/ \geq 2]]_2$	$[[38, 18, \geq 7/ \geq 2]]_2$	$[[56, 18, 8/2]]_2$
6	3	$[[20, 5, \geq 7/ \geq 4]]_3$	$[[20, 5, \geq 6/ \geq 4]]_3$	$[[24, 5, \geq 7/ \geq 4]]_3$
7	3	$[[26, 0, \geq 13/ \geq 6]]_3$	$[[26, 0, \geq 10/ \geq 6]]_3$	-
8	3	$[[26, 12, \geq 8/ \geq 2]]_3$	$[[26, 12, \geq 7/ \geq 2]]_3$	$[[32, 12, \geq 8/ \geq 2]]_3$
9	3	$[[28, 13, \geq 9/ \geq 2]]_3$	$[[28, 13, \geq 8/ \geq 2]]_3$	$[[35, 13, \geq 9/ \geq 2]]_3$

5. Conclusion

In this work, we introduce systematic frameworks for constructing entanglement-assisted quantum error-correcting codes and asymmetric quantum error-correcting codes via a suitable class of index-2 quasi-cyclic codes. For entanglement-assisted quantum error-correcting codes, we establish a systematic construction based on index-2 QC codes. Furthermore, we derive sufficient conditions for obtaining maximal-entanglement entanglement-assisted quantum error-correcting codes and, on this basis, obtain two new classes of maximal-entanglement entanglement-assisted quantum error-correcting codes. For asymmetric quantum error-correcting codes, using the same class of index-2 QC codes, we characterize the inclusion relation between any two such codes by establishing necessary and sufficient conditions, and subsequently develop a systematic construction of asymmetric quantum error-correcting codes. The proposed constructions yield 12 EAQECCs whose parameters improve upon previously reported results and 9 new AQECCs with improved parameters. Moreover, all of the constructed AQECCs achieve higher code rates than the corresponding codes permitted by the asymmetric quantum Gilbert-Varshamov bound. These results provide additional quantum error-correcting code candidates with good parameters, together with systematic construction tools, for quantum communication scenarios with available preshared entanglement and for quantum systems characterized by asymmetric noise.

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