A measure for genuine tripartite entanglement

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(Dated: August 31, 2026)

We introduce a single real-valued functional $I(\vec{n}_1, \vec{n}_2)$, built from four three-qubit correlation expectation values, that turns the Greenberger–Horne–Zeilinger (GHZ) algebraic paradox into a *quantitative* witness of genuine tripartite entanglement. We prove that for every three-qubit state ρ and every pair of measurement directions $|I(\vec{n}_1, \vec{n}_2; \rho)| \leq 2$, with the bound saturated if and only if $\vec{n}_1 \perp \vec{n}_2$ and ρ is locally unitarily equivalent to the GHZ state. We obtain a closed-form expression for $I(\hat{x}, \hat{y})$ on the five-parameter Acín canonical family of three-qubit pure states; it depends only on the product $\lambda_0\lambda_4$ and is maximised when $\lambda_0 = \lambda_4 = 1/\sqrt{2}$. For the W state we show that $I(\hat{x}, \hat{y}) = 0$ and that $\max_{\vec{n}_1,\vec{n}_2} |I_{\mathrm{W}}| = 35/27 \approx 1.296$, strictly below the GHZ value. Maximising the underlying correlation structure over *independent* local orthonormal frames on the three parties yields a manifestly local-unitary (LU) invariant quantity $\mathcal{E}_{\mathrm{GHZ}}(\rho) \in [0, 1]$ that equals one if and only if ρ is LU equivalent to the GHZ state, takes the value $35/54 \approx 0.648$ on the W state, and is bounded by $1/2$ on all biseparable and fully separable states; it is therefore a local-frame-independent indicator of GHZ-type genuine tripartite correlation, obtainable from four three-body Pauli correlators alone. We relate $\mathcal{E}_{\mathrm{GHZ}}$ to GHZ fidelity witnesses, analyse its behaviour under local operations and classical communication, and delimit what it does and does not certify about genuine tripartite nonlocality. We carefully delimit which properties are proven and which (notably global convexity and the resulting genuine-multipartite-entanglement witness threshold) are established numerically and remain open analytically. We also outline a generalisation of I to three-qudit systems built from the Heisenberg–Weyl operators, recovering the standard qubit construction when $d = 2$.

PACS numbers: 03.67.Mn, 03.65.Ud, 03.67.Ac, 02.10.Ox

I. INTRODUCTION

The Greenberger–Horne–Zeilinger (GHZ) theorem [1, 2] shows that quantum mechanics conflicts with local realism in a *deterministic* way for three or more parties, requiring no inequalities and no averages over runs. A single algebraic identity relating four products of dichotomic outcomes is enough to produce a contradiction [3–5]. Although the qualitative content of this identity is celebrated, the quantitative question *how strongly does a given state ρ exhibit GHZ-type correlations?* has remained somewhat scattered across the literature, with answers expressed through Mermin polynomials, fidelities with $|\mathrm{GHZ}\rangle$, the three-tangle [6], or convex-roof extensions of various entanglement monotones [7]. Quantitative studies in concrete physical settings—for instance the evolution of the entanglement of three-particle entangled states in the Tavis–Cummings model [8], or entropic entanglement measures for two-qubit mixed states based on the negativity of the quantum conditional entropy [9]—illustrate how strongly the answer depends on the functional one chooses. The purpose of this paper is to formulate and analyse a single functional $I(\vec{n}_1, \vec{n}_2)$ that

(i) exactly reproduces the GHZ algebraic paradox when $\vec{n}_1 = \hat{x}$, $\vec{n}_2 = \hat{y}$;

(ii) satisfies a tight upper bound, $|I| \leq 2$, that is universal across all three-qubit states;

(iii) saturates this bound *only* on the GHZ orbit under local unitaries, thereby providing a witness of GHZ-type genuine tripartite entanglement;

(iv) admits closed-form evaluation on canonical state families (Acín form, W class, biseparable states); and

(v) extends naturally from qubits to qudits.

Position with respect to prior work. Our quantity sits at the intersection of three lines of research. First, in contrast to the standard Mermin polynomial [3–5, 10] and its genuine-tripartite refinement by Svetlichny and Seevinck [11, 12], the functional I is *multiplicative* in the correlators rather than linear; the algebraic identity (3) replaces the inequality. Multiplicative Bell-type quantities have been considered abstractly by Te'eni *et al.* [13] and (in quadratic form) by Uffink [14], but never specialised to the GHZ stabiliser. Second, in contrast to the inequality-free GHZ-vs-W paradoxes of Cabello [15], our construction yields a single LU-invariant *scalar* that orders all three-qubit states on a $[0, 1]$ scale. Third, in contrast to the three-tangle τ_3 [6], GHZ fidelity witnesses [16, 17], the genuine multipartite negativity [18],

and the GME concurrence of Ma *et al.* [19], our $\mathcal{E}_{\rm GHZ}$ is non-zero on *both* the W class and biseparable states, with sharp ordering $\mathcal{E}_{\rm GHZ}(\mathrm{GHZ}) = 1 > \mathcal{E}_{\rm GHZ}(\mathrm{W}) = 35/54 > 1/2 \geq \mathcal{E}_{\rm GHZ}(\mathrm{bisep})$.

Notation. Three qubits live in a Hilbert space $\mathcal{H} = \mathbb{C}^2_A \otimes \mathbb{C}^2_B \otimes \mathbb{C}^2_C$, with state $\rho^{ABC} \in \mathcal{S}(\mathcal{H})$. Marginals are $\rho^{AB} = \mathrm{Tr}_C \rho^{ABC}$, etc. For a unit vector $\vec{n} \in \mathbb{R}^3$, the spin observable on a single qubit is $\sigma_{\vec{n}} = \vec{n} \cdot \boldsymbol{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, with $\sigma_x, \sigma_y, \sigma_z$ the standard Pauli matrices. For three direction labels $\vec{n}_a, \vec{n}_b, \vec{n}_c$ we denote the tripartite local observable

$$\sigma(\vec{n}_a, \vec{n}_b, \vec{n}_c) \;=\; \sigma_{\vec{n}_a} \otimes \sigma_{\vec{n}_b} \otimes \sigma_{\vec{n}_c}, \tag{1}$$

and its expectation $\langle \sigma(\vec{n}_a, \vec{n}_b, \vec{n}_c) \rangle = \mathrm{Tr}[\sigma(\vec{n}_a, \vec{n}_b, \vec{n}_c)\rho^{ABC}]$.

The functional. For two unit vectors $\vec{n}_1, \vec{n}_2 \in \mathbb{R}^3$ we define, using the short-hand $E_{abc} \equiv \langle \sigma(\vec{n}_a, \vec{n}_b, \vec{n}_c) \rangle$,

$$I(\vec{n}_1, \vec{n}_2; \rho) = E_{111} - E_{122}E_{212}E_{221}. \tag{2}$$

The quantity I is a real-valued function of the state ρ and the two directions; only product correlations of the spin observables on each subsystem appear, so I is, in principle, accessible to estimation from purely local Pauli-string measurements, with no shared reference frame between the three parties (see Sec. III for what this does and does not certify).

Local hidden-variable identity. For any deterministic local hidden-variable (LHV) model assigning values $A(\vec{n}), B(\vec{n}), C(\vec{n}) \in \{\pm 1\}$ to the three qubits, every realisation of the underlying parameter λ obeys the algebraic identity

$$\begin{aligned}&\big[A(\vec{n}_1)B(\vec{n}_2)C(\vec{n}_2)\big]\big[A(\vec{n}_2)B(\vec{n}_1)C(\vec{n}_2)\big] \\ &\quad\times\big[A(\vec{n}_2)B(\vec{n}_2)C(\vec{n}_1)\big] = A(\vec{n}_1)B(\vec{n}_1)C(\vec{n}_1),\end{aligned} \tag{3}$$

because each of $A(\vec{n}_2)^2, B(\vec{n}_2)^2, C(\vec{n}_2)^2$ equals $+1$. Hence *whenever the three product-correlation factors in I are simultaneously deterministic at the LHV level*, eq. (2) forces

$$I(\vec{n}_1, \vec{n}_2)\,\big|_{\text{deterministic LHV}} = 0, \tag{4}$$

which is the standard GHZ–Mermin equality [2, 3]. For the special case $\vec{n}_1 = \hat{x}$, $\vec{n}_2 = \hat{y}$ on the GHZ state $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$, quantum mechanics gives $\langle \sigma(\vec{n}_1, \vec{n}_2, \vec{n}_2) \rangle = \langle \sigma(\vec{n}_2, \vec{n}_1, \vec{n}_2) \rangle = \langle \sigma(\vec{n}_2, \vec{n}_2, \vec{n}_1) \rangle = -1$ and $\langle \sigma(\vec{n}_1, \vec{n}_1, \vec{n}_1) \rangle = +1$, so

$$I_{\rm GHZ}(\hat{x}, \hat{y}) = 1 - (-1)^3 = 2, \tag{5}$$

manifestly violating (4). The maximal violation $|I| = 2$ is the algebraic essence of the GHZ paradox.

The remainder of this paper studies the functional I quantitatively: who else can saturate the bound (Sec. III), what value it takes on a generic three-qubit pure state (Sec. IV), how it gives rise to a genuine tripartite entanglement measure (Sec. V), and how it lifts to qudits (Sec. VI).

II. THE FOUR STABILISER-LIKE OPERATORS

It is convenient to package $I(\vec{n}_1, \vec{n}_2)$ in terms of four Hermitian observables on $\mathcal{H}$:

$$\begin{aligned} O_1 &= \sigma_{\vec{n}_1} \otimes \sigma_{\vec{n}_2} \otimes \sigma_{\vec{n}_2}, \\ O_2 &= \sigma_{\vec{n}_2} \otimes \sigma_{\vec{n}_1} \otimes \sigma_{\vec{n}_2}, \\ O_3 &= \sigma_{\vec{n}_2} \otimes \sigma_{\vec{n}_2} \otimes \sigma_{\vec{n}_1}, \\ O_4 &= \sigma_{\vec{n}_1} \otimes \sigma_{\vec{n}_1} \otimes \sigma_{\vec{n}_1}. \end{aligned} \tag{6}$$

With $e_i \equiv \langle O_i \rangle_\rho$ $(i = 1, \ldots, 4)$,

$$I(\vec{n}_1, \vec{n}_2) \;=\; e_4 - e_1e_2e_3. \tag{7}$$

Each O_i is unitary and Hermitian and obeys $O_i^2 = \mathbb{1}$, hence $|e_i| \leq 1$ and $|I| \leq 2$ trivially.

The algebraic structure of these operators is most transparent in terms of the inner and cross products of the two direction vectors, $c \equiv \vec{n}_1 \cdot \vec{n}_2$ and $\boldsymbol{m} \equiv \vec{n}_1 \times \vec{n}_2$. Using $\sigma_{\vec{n}_1}\sigma_{\vec{n}_2} = c\,\mathbb{1} + \mathrm{i}\,\boldsymbol{m} \cdot \boldsymbol{\sigma}$ one finds the following two key identities (proved in App. A):

$$[O_1, O_2] = 2\mathrm{i}\,c\,\big(\boldsymbol{m} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} - \mathbb{1} \otimes \boldsymbol{m} \cdot \boldsymbol{\sigma}\big) \otimes \mathbb{1}, \tag{8}$$

$$O_1O_2O_3 + O_4 = 2c\,\sigma_{\vec{n}_1} \otimes \sigma_{\vec{n}_2} \otimes \sigma_{\vec{n}_1}. \tag{9}$$

Equation (9) is the operator avatar of Mermin's identity (3): the right-hand side vanishes exactly when $\vec{n}_1 \perp \vec{n}_2$, in which case $\{O_1, O_2, O_3, O_4\}$ commute pairwise and satisfy the abelian relation

$$O_1\,O_2\,O_3\,O_4 = -\mathbb{1} \qquad (\text{when } \vec{n}_1 \perp \vec{n}_2). \tag{10}$$

This is precisely the defining relation of the three-qubit GHZ stabiliser group [20, 21] written in the locally rotated Pauli basis $(\sigma_{\vec{n}_1}, \sigma_{\vec{n}_2})$.

III. MAXIMUM VIOLATION: A SHARP BOUND

Our first main result completely characterises the states and directions saturating $|I| = 2$.

Proposition 1 (Maximum violation)**.** *For every three-qubit state ρ and every pair of unit vectors $\vec{n}_1, \vec{n}_2 \in \mathbb{R}^3$ one has*

$$|I(\vec{n}_1, \vec{n}_2; \rho)| \;\leq\; 2. \tag{11}$$

Equality holds if and only if

(i) *$\vec{n}_1 \perp \vec{n}_2$, and*

(ii) *$\rho = |\psi\rangle\langle\psi|$ is pure and a common eigenstate of the four operators $\{O_1, O_2, O_3, O_4\}$ with respective eigenvalues $\epsilon_i = \pm 1$ satisfying $\epsilon_1\epsilon_2\epsilon_3 = -\epsilon_4$.*

The eight such common eigenstates form an orthonormal basis of $\mathcal{H}$, all of whose elements are connected to the standard GHZ state $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ by local unitary transformations.

Proof. Upper bound. Since $|e_i| \le 1$, $|I| = |e_4 - e_1e_2e_3| \le |e_4| + |e_1e_2e_3| \le 2$.

Necessity of $|e_i| = 1$. Saturation $|I| = 2$ requires $|e_1e_2e_3| = 1$ *and* $|e_4| = 1$. In particular each $|e_i| = 1$. For a Hermitian operator O with eigenvalues ± 1 and a normalised state ρ, $|\langle O\rangle_\rho| = 1$ forces ρ to be supported in a single eigenspace of O. Hence ρ lies in the simultaneous ± 1-eigenspace of all four O_i; in particular every pure-state component $|\psi\rangle$ in the support satisfies $O_i|\psi\rangle = \epsilon_i|\psi\rangle$.

Necessity of $\vec{n}_1 \perp \vec{n}_2$. Acting (9) on a common eigenstate yields

$$(\epsilon_1\epsilon_2\epsilon_3 + \epsilon_4)|\psi\rangle = 2c\left(\sigma_{\vec{n}_1} \otimes \sigma_{\vec{n}_2} \otimes \sigma_{\vec{n}_1}\right)|\psi\rangle . \tag{12}$$

The operator on the right is unitary with spectrum $\{\pm 1\}$, so the right-hand side has norm exactly $2|c|$. The left-hand side has norm $|\epsilon_1\epsilon_2\epsilon_3 + \epsilon_4| \in \{0, 2\}$. Two cases arise:

- If $\epsilon_1\epsilon_2\epsilon_3 = \epsilon_4$ then $I = \epsilon_1\epsilon_2\epsilon_3 - \epsilon_4 = 0$, contradicting $|I| = 2$.
- If $\epsilon_1\epsilon_2\epsilon_3 = -\epsilon_4$ then the left-hand side of (12) vanishes, forcing $c = \vec{n}_1 \cdot \vec{n}_2 = 0$.

Hence saturation requires $\vec{n}_1 \perp \vec{n}_2$ and the sign relation $\epsilon_1\epsilon_2\epsilon_3 = -\epsilon_4$, equivalently $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1$, which is consistent with (10).

Achievability and LU-equivalence to GHZ. For $\vec{n}_1 \perp \vec{n}_2$ the four operators in (6) commute and generate an abelian group of order eight, with relation (10). Three of them (say O_1, O_2, O_3) are independent and have a joint orthonormal basis of ± 1 eigenstates [20]; the fourth eigenvalue is then $\epsilon_4 = -\epsilon_1\epsilon_2\epsilon_3$. Hence $I = \epsilon_4 - \epsilon_1\epsilon_2\epsilon_3 = -2\epsilon_1\epsilon_2\epsilon_3 = \pm 2$ on each of the eight common eigenstates.

When $\vec{n}_1 = \hat{x}, \vec{n}_2 = \hat{y}$ this basis is the textbook GHZ basis

$$\tfrac{1}{\sqrt{2}}\big(|ijk\rangle \pm |\bar{i}\bar{j}\bar{k}\rangle\big), \qquad i, j, k \in \{0, 1\}, \tag{13}$$

where $\bar{0} = 1$, $\bar{1} = 0$. All eight states in (13) are LU-equivalent to $|\mathrm{GHZ}\rangle$ via local Pauli flips and phase gates. For arbitrary orthogonal $(\vec{n}_1, \vec{n}_2)$ the joint basis is the image of (13) under the local unitary $U^{\otimes 3}$ that sends $(\hat{x}, \hat{y})$ to $(\vec{n}_1, \vec{n}_2)$, and is therefore again LU-equivalent to $|\mathrm{GHZ}\rangle$. □

Bell-type interpretation. Proposition 1 is more than a Cirel'son-type bound [22]: the LHV identity (4) predicts $I = 0$ whereas quantum mechanics achieves $|I| = 2$ exactly on the GHZ orbit. In other words, $I(\vec{n}_1, \vec{n}_2)$ is an *equality-form* GHZ witness whose violation *measures* how close ρ is, in the sense of stabiliser correlations, to a GHZ-type state. We emphasise that the saturation condition is sharper than what is available for the linear Mermin operator: the closed-form upper bound of Siddiqui and Sazim [23] for the linear functional $M_3 = \langle\sigma_{xxx}\rangle - \langle\sigma_{xyy}\rangle - \langle\sigma_{yxy}\rangle - \langle\sigma_{yyx}\rangle$ involves diagonalisation of state-dependent correlation matrices, whereas our multiplicative bound *factorises* and is achieved on the pure GHZ orbit alone. The complementary "inequality-free" viewpoint of Cabello [15], which produces an all-versus-nothing separator of GHZ and W via incompatible composite observables, parallels but does not subsume the present quantitative witness.

Because I is built from the deterministic local-hidden-variable identity (3), one might read $|I| > 0$ as evidence of nonlocality. This reading is not warranted: eq. (4) is an algebraic identity holding run by run for a *deterministic* value assignment, not a Bell inequality, so $|I| > 0$ is not by itself a violation of a local-realistic bound for general stochastic LHV models with shared randomness, and $I = 0$ does not certify an LHV model. The distinction matters because for three parties entanglement and nonlocality are not interchangeable: by Gisin's theorem every entangled pure two-qubit state violates a Bell inequality [24], but no tripartite counterpart is known, and Zhang *et al.* [25] established the equivalence of genuine multipartite nonlocality, steering and entanglement only under an additional multicopy/network-inflation assumption, leaving the single-copy question open. Throughout, I and $\mathcal{E}_{\mathrm{GHZ}}$ quantify GHZ-type stabiliser correlation—entanglement structure—and not nonlocality; accordingly $\mathcal{E}_{\mathrm{GHZ}}(\rho) > 1/2$ is (conditional, see (P6) below) evidence of genuine tripartite *entanglement*, and we describe $\mathcal{E}_{\mathrm{GHZ}}$ as *local-frame independent* rather than device independent: the parties need share no reference frame, but the quantity is not certified in the Bell sense.

IV. EVALUATION ON CANONICAL PURE STATES

We now evaluate I on prototypical three-qubit pure states.

A. The Acín canonical family

Up to local unitaries, every pure three-qubit state can be brought to the Acín canonical form [26, 27]

$$|\Psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle , \tag{14}$$

with $\lambda_i \ge 0$, $\sum_i \lambda_i^2 = 1$, and $\varphi \in [0, \pi]$. This parametrisation has five independent real parameters.

A direct calculation (App. B) yields

$$\langle\sigma_x^{\otimes 3}\rangle_{|\Psi\rangle} = 2\lambda_0\lambda_4, \tag{15}$$

$$\langle\sigma_x\sigma_y\sigma_y\rangle_{|\Psi\rangle} = \langle\sigma_y\sigma_x\sigma_y\rangle_{|\Psi\rangle} = \langle\sigma_y\sigma_y\sigma_x\rangle_{|\Psi\rangle} = -2\lambda_0\lambda_4, \tag{16}$$

independently of $\lambda_1, \lambda_2, \lambda_3, \varphi$. Substituting in (2) we obtain the closed form

$$I(\hat{x}, \hat{y}; |\Psi\rangle) = 2\lambda_0\lambda_4\big[4(\lambda_0\lambda_4)^2 + 1\big]. \tag{17}$$

Maximising $|I|$ over the simplex $\{\lambda_i^2 : \sum_i \lambda_i^2 = 1,\ \lambda_i \geq 0\}$ reduces to maximising the single product $\mu \equiv \lambda_0\lambda_4$, which by Cauchy–Schwarz is bounded by $\mu \leq 1/2$ with equality iff $\lambda_0 = \lambda_4 = 1/\sqrt{2}$ and all other $\lambda_i = 0$. This is precisely the GHZ state, and

$$|I_{\mathrm{GHZ}}(\hat{x},\hat{y})| = 8\,(\tfrac{1}{2})^3 + 2\cdot\tfrac{1}{2} = 1 + 1 = 2, \tag{18}$$

in agreement with Proposition 1. The function $|I(\hat{x},\hat{y})| = 8\mu^3 + 2\mu$ is monotonically increasing in μ on $[0, 1/2]$ [Fig. 1(a)]. In particular every state with $\lambda_0\lambda_4 = 0$ — including all states whose Acín form has no $|111\rangle$ component — gives $I(\hat{x},\hat{y}) = 0$. This is the case for the W state, see Sec. IV B below.

A further consequence of (17): on the GHZ subfamily $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\varphi = 0$, namely $|\Psi(\beta)\rangle = \cos\beta\,|000\rangle + \sin\beta\,|111\rangle$, one has $\lambda_0\lambda_4 = \frac{1}{2}\sin(2\beta)$ and hence

$$I(\hat{x},\hat{y};|\Psi(\beta)\rangle) = \sin^3(2\beta) + \sin(2\beta), \tag{19}$$

which is determined solely by the Schmidt angle and reaches the GHZ extremum $|I| = 2$ at $\beta = \pi/4$.

Relation to the three-tangle. On the Acín canonical family the Coffman–Kundu–Wootters three-tangle equals $\tau_3 = 4\lambda_0^2\lambda_4^2$ [6]. Equation (17) therefore admits the equivalent form

$$I(\hat{x},\hat{y};|\Psi\rangle) = \sqrt{\tau_3}\,(\tau_3 + 1), \tag{20}$$

exhibiting $|I(\hat{x},\hat{y})|$ as a strictly increasing degree-3 polynomial in $\sqrt{\tau_3}$ on the canonical pure-state slice. However, this identification is *specific to the canonical form and to the direction pair* $(\hat{x},\hat{y})$: on a generic state, neither representative of an LU orbit need be in Acín form, and I depends on $(\vec{n}_1,\vec{n}_2)$. Accordingly, the invariant measure $\mathcal{E}_{\mathrm{GHZ}}$ of Sec. V is *not* a function of τ_3 on arbitrary states: most strikingly, $\tau_3(\mathrm{W}) = 0$ [6] while $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{W}) = 35/54$ (see Sec. V below). This separation reflects that $\mathcal{E}_{\mathrm{GHZ}}$ couples to the entire *stabiliser-correlation* structure of the GHZ class, not merely to the residual hyperdeterminant captured by τ_3 [28].

B. The W state

The W state $|\mathrm{W}\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ is the representative of the W class, which is SLOCC-inequivalent to the GHZ class [29]. Direct computation of all 27 three-qubit Pauli correlators yields

$$\begin{aligned} \langle\sigma_z^{\otimes 3}\rangle_{\mathrm{W}} &= -1, \\ \langle\sigma_a\otimes\sigma_a\otimes\sigma_z\rangle_{\mathrm{W}} &= \langle\sigma_a\otimes\sigma_z\otimes\sigma_a\rangle_{\mathrm{W}} \\ &= \langle\sigma_z\otimes\sigma_a\otimes\sigma_a\rangle_{\mathrm{W}} = \tfrac{2}{3}, \end{aligned} \tag{21}$$

for $a \in \{x, y\}$, all other components vanishing.

Special directions. With $\vec{n}_1 = \hat{x}$, $\vec{n}_2 = \hat{y}$, every correlator appearing in I vanishes by (21); therefore

$$I_{\mathrm{W}}(\hat{x},\hat{y}) = 0. \tag{22}$$

By contrast, with $\vec{n}_1 = \hat{z}$, $\vec{n}_2 = \hat{x}$ (or any $\vec{n}_2$ in the $\hat{x}$-$\hat{y}$ plane, by the $\hat{z}$-rotation symmetry of $|\mathrm{W}\rangle$) one obtains

$$I_{\mathrm{W}}(\hat{z},\hat{x}) = (-1) - \left(\tfrac{2}{3}\right)^3 = -1 - \frac{8}{27} = -\frac{35}{27}. \tag{23}$$

Maximum over directions. For arbitrary orthogonal pairs we set $a_3 \equiv \hat{z}\cdot\vec{n}_1$, $b_3 \equiv \hat{z}\cdot\vec{n}_2$. Using (21) together with $\vec{n}_1\cdot\vec{n}_2 = 0$ and $\vec{n}_2\cdot\vec{n}_2 = 1$ one obtains the explicit formula (derived in App. C)

$$I_{\mathrm{W}}(\vec{n}_1,\vec{n}_2) = a_3\big(2 - 3a_3^2\big) - a_3^3\Big(\tfrac{2}{3} - 3b_3^2\Big)^3. \tag{24}$$

A short Lagrangian analysis (App. C) shows that the global extremum of $|I_{\mathrm{W}}|$ on the constraint surface $a_3^2 + b_3^2 \leq 1$ is attained at $(a_3, b_3) = (\pm 1, 0)$, giving

$$\max_{\vec{n}_1,\vec{n}_2} |I_{\mathrm{W}}(\vec{n}_1,\vec{n}_2)| = \frac{35}{27} \approx 1.296. \tag{25}$$

This is strictly less than the GHZ value 2, reflecting the well-known fact that the W state lies outside the GHZ class [29]. Equation (25) can therefore be regarded as a sharp *quantitative* separator of the two SLOCC classes via I. To the best of our knowledge the value $35/27$ has not previously appeared in the Mermin-functional literature: under the linear Mermin operator, recent analytical work [23] reports $\max\langle M_3\rangle_{\mathrm{W}} \approx 3.046$ (out of an algebraic ceiling of 4), while the present multiplicative functional gives the simpler closed form (25).

A remark on orthogonal versus general settings. It is worth emphasising a structural contrast between the linear and the multiplicative functionals. The Mermin maximum $\max\langle M_3\rangle_{\mathrm{W}} \approx 3.046$ is achieved only over *general* (non-orthogonal) measurement directions; if one restricts the two Mermin settings to be mutually orthogonal on each party, the W state attains exactly $\langle M_3\rangle_{\mathrm{W}} = 3$, and the extra 0.046 requires tilting the settings away from orthogonality. We have verified both numbers numerically (the orthogonal value 3 analytically, the general value 3.046 by multistart optimisation over unconstrained unit vectors). Our functional I behaves oppositely: by Proposition 1 its algebraic ceiling is reached *exactly* at orthogonal settings $\vec{n}_1 \perp \vec{n}_2$, and tilting away from orthogonality can only decrease $|I|$ on the GHZ orbit. Thus the GHZ paradox is most naturally a statement about orthogonal (maximally complementary) local bases, which is precisely the regime singled out by the multiplicative form.

C. Biseparable and product states

For an $A|BC$ biseparable state $\rho = \rho_A \otimes \rho_{BC}$, all four expectations factorise as $\langle\sigma(\vec{n}_a,\vec{n}_b,\vec{n}_c)\rangle = \langle\sigma_{\vec{n}_a}\rangle_A \cdot \langle\sigma_{\vec{n}_b}\otimes\sigma_{\vec{n}_c}\rangle_{BC}$. When $\rho_A = |0\rangle\langle 0|$ and $\rho_{BC} = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, taking $\vec{n}_1 = \hat{z}, \vec{n}_2 = \hat{x}$ yields $e_1 = \langle\sigma_z\rangle\langle\sigma_{xx}\rangle = 1\cdot 1 = 1$, $e_2 = e_3 = 0$, $e_4 = \langle\sigma_z\rangle\langle\sigma_{zz}\rangle =$

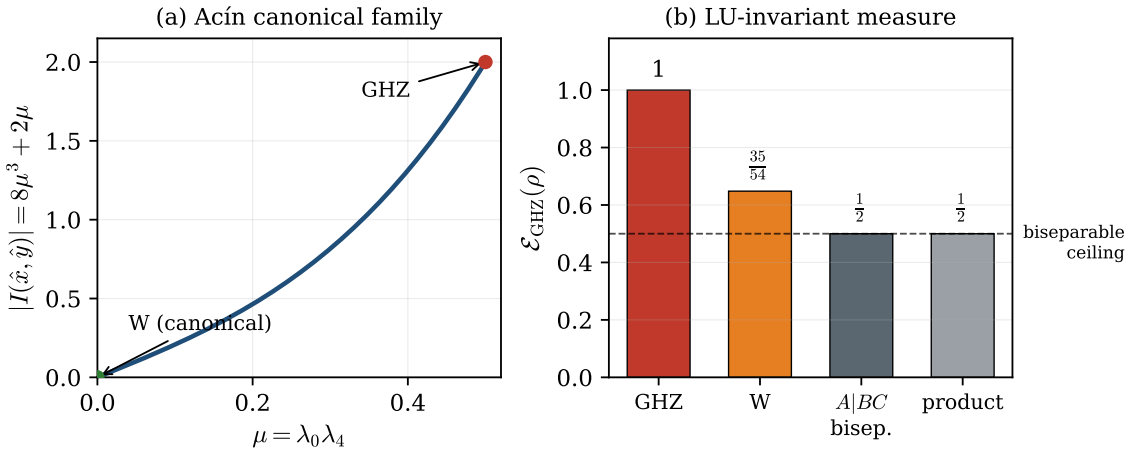


FIG. 1. (a) The functional $|I(\hat{x},\hat{y})|$ evaluated on the Acín canonical family (14) depends only on the product $\mu = \lambda_0\lambda_4$, with the closed form $|I| = 8\mu^3 + 2\mu$ [Eq. (17)]. The maximum value 2 is attained at $\mu = 1/2$, the GHZ state. The W state has $\mu = 0$ in its canonical form, giving $I(\hat{x},\hat{y}) = 0$. (b) The local-unitary invariant measure $\mathcal{E}_{\mathrm{GHZ}}$ of Sec. V, obtained by maximising the correlation structure over *independent* orthonormal frames on the three parties, for four representative classes of three-qubit states. The GHZ state attains the maximal value $\mathcal{E}_{\mathrm{GHZ}} = 1$; the W state achieves $35/54 \approx 0.648$ (the independent-frame optimisation does not exceed the aligned-frame value here, due to the permutation symmetry of $|\mathrm{W}\rangle$); biseparable $A|BC$ states and fully separable states are both bounded by, and numerically attain, 1/2. Values are obtained by multistart Powell optimisation over 120 random initial frame triples and verified to machine precision against the exact values of Sec. V; full numerical details are given in App. D.

$1 \cdot 1 = 1$, so $I = 1$. Numerical optimisation over all directions gives $\sup_{\vec{n}_1 \perp \vec{n}_2} |I_{\mathrm{bisep}}| = 1$ for the shared-frame functional, in agreement with Proposition 1 forbidding saturation by biseparable states. Fully separable (product) states likewise reach at most 1. The corresponding values of the LU-invariant measure $\mathcal{E}_{\mathrm{GHZ}}$ of Sec. V, which optimises over independent local frames and carries an extra factor 1/2, are $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{bisep}) = \mathcal{E}_{\mathrm{GHZ}}(\mathrm{prod}) = 1/2$ [Fig. 1(b)]; see Sec. V and App. D for the derivation.

Random pure states. Sampling Haar-random three-qubit pure states and numerically maximising $|I|$ over directions, we find $\sup|I| < 2$ in every case; random pure states cluster around values 0.4–1.6, consistent with the fact that the GHZ orbit is a measure-zero subset of pure-state space.

V. A GENUINE GHZ-TYPE ENTANGLEMENT MEASURE

The quantitative content of Proposition 1 suggests building a state functional by optimising $|I|$ over measurement directions. Doing this naively, however, runs into an invariance subtlety that we now make explicit and then resolve.

A. The aligned-frame quantity and its invariance defect

The most direct construction reuses the *same* ordered orthonormal pair $(\vec{n}_1, \vec{n}_2)$ on all three parties,

$$\mathcal{E}^{\mathrm{al}}_{\mathrm{GHZ}}(\rho) \;=\; \frac{1}{2} \sup_{\vec{n}_1 \perp \vec{n}_2} \big|I(\vec{n}_1, \vec{n}_2; \rho)\big|, \tag{26}$$

which we call the *aligned-frame* quantity. By Proposition 1, $\mathcal{E}^{\mathrm{al}}_{\mathrm{GHZ}} \in [0,1]$ and $\mathcal{E}^{\mathrm{al}}_{\mathrm{GHZ}}(\rho) = 1$ if and only if ρ is LU equivalent to $|\mathrm{GHZ}\rangle$. It is, however, *not* invariant under general local unitaries. A collective rotation $U^{\otimes 3}$ with $U \in SU(2)$ does leave (26) unchanged, because the induced $R \in SO(3)$ maps the orthonormal pair $(\vec{n}_1, \vec{n}_2) \mapsto (R^{-1}\vec{n}_1, R^{-1}\vec{n}_2)$, again orthonormal, and the supremum runs over all such pairs. Thus $\mathcal{E}^{\mathrm{al}}_{\mathrm{GHZ}}$ is invariant under the diagonal subgroup $\{U \otimes U \otimes U\}$. Under *independent* local unitaries $U_A \otimes U_B \otimes U_C$ the three parties' Pauli bases rotate by three *different* elements of $SO(3)$, which a single shared frame cannot track. Numerically the violation is dramatic: for the GHZ state one finds $\mathcal{E}^{\mathrm{al}}_{\mathrm{GHZ}}(|\mathrm{GHZ}\rangle) = 1$, but under randomly drawn independent local unitaries the same state yields values that scatter between roughly 0.52 and 0.67; the W state's aligned-frame value 0.648 likewise drops to 0.36–0.49.

B. Independent local frames and the invariant measure

The fix is to allow each party its own orthonormal frame. Let $(\hat{a}_1, \hat{a}_2)$, $(\hat{b}_1, \hat{b}_2)$, $(\hat{c}_1, \hat{c}_2)$ be orthonormal pairs on A, B, C, and set

$$I_\star(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2; \rho) = \langle \sigma_{\hat{a}_1} \sigma_{\hat{b}_1} \sigma_{\hat{c}_1} \rangle - \langle \sigma_{\hat{a}_1} \sigma_{\hat{b}_2} \sigma_{\hat{c}_2} \rangle \langle \sigma_{\hat{a}_2} \sigma_{\hat{b}_1} \sigma_{\hat{c}_2} \rangle \langle \sigma_{\hat{a}_2} \sigma_{\hat{b}_2} \sigma_{\hat{c}_1} \rangle, \quad (27)$$

the natural independent-frame analogue of (2) (here $\sigma_{\hat{a}}\sigma_{\hat{b}}\sigma_{\hat{c}}$ abbreviates $\sigma_{\hat{a}} \otimes \sigma_{\hat{b}} \otimes \sigma_{\hat{c}}$). It is governed by the four Hermitian observables

$$\begin{aligned} A_1 &= \sigma_{\hat{a}_1} \otimes \sigma_{\hat{b}_1} \otimes \sigma_{\hat{c}_1}, \quad A_2 = \sigma_{\hat{a}_1} \otimes \sigma_{\hat{b}_2} \otimes \sigma_{\hat{c}_2}, \\ A_3 &= \sigma_{\hat{a}_2} \otimes \sigma_{\hat{b}_1} \otimes \sigma_{\hat{c}_2}, \quad A_4 = \sigma_{\hat{a}_2} \otimes \sigma_{\hat{b}_2} \otimes \sigma_{\hat{c}_1}, \end{aligned} \quad (28)$$

so that $I_\star = \langle A_1 \rangle - \langle A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle$. Because each party contributes either a repeated vector or its orthogonal partner, any two of the A_i differ by an *anticommuting* single-qubit factor on an even number of sites; one checks (App. D) that

$$[A_i, A_j] = 0 \quad (\text{all } i, j), \qquad A_1 A_2 A_3 A_4 = -\mathbb{1}. \quad (29)$$

Define the *local-unitary invariant* measure

$$\mathcal{E}_{\mathrm{GHZ}}(\rho) \;=\; \frac{1}{2} \sup_{\substack{\hat{a}_1 \perp \hat{a}_2,\ \hat{b}_1 \perp \hat{b}_2,\\ \hat{c}_1 \perp \hat{c}_2}} \big| I_\star(\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2; \rho) \big|. \quad (30)$$

Proposition 2 (Identification and LU invariance)**.** *For every three-qubit state ρ, writing $V = U_A \otimes U_B \otimes U_C$,*

$$\mathcal{E}_{\mathrm{GHZ}}(\rho) = \max_{U_A, U_B, U_C \in SU(2)} \mathcal{E}^{\mathrm{al}}_{\mathrm{GHZ}}\big(V \rho V^\dagger\big), \quad (31)$$

and consequently $\mathcal{E}_{\mathrm{GHZ}}$ is invariant under arbitrary local unitaries.

Proof. Conjugating ρ by $U_A \otimes U_B \otimes U_C$ and then evaluating the aligned functional $I(\vec{n}_1, \vec{n}_2)$ replaces each single-qubit observable $\sigma_{\vec{n}}$ on party X by $\sigma_{R_X \vec{n}}$, where $R_X \in SO(3)$ is the rotation associated with U_X. Thus the aligned pair $(\vec{n}_1, \vec{n}_2)$ is seen by the three parties as $(R_A\vec{n}_1, R_A\vec{n}_2)$, $(R_B\vec{n}_1, R_B\vec{n}_2)$, $(R_C\vec{n}_1, R_C\vec{n}_2)$. Since $SO(3)$ acts transitively on ordered orthonormal pairs, as $(U_A, U_B, U_C, \vec{n}_1, \vec{n}_2)$ range over all values these triples of pairs range over *all* triples of independent orthonormal frames in (30). Hence the right-hand side of (31) equals $\mathcal{E}_{\mathrm{GHZ}}(\rho)$. Invariance is immediate: the maximisation over all local unitaries absorbs any further local unitary applied to ρ. □

We can now reinstate, correctly, the saturation theorem at the level of the invariant measure.

Proposition 3 (Genuine GHZ indicator)**.** $\mathcal{E}_{\mathrm{GHZ}}(\rho) \in [0, 1]$ *for every three-qubit state, and $\mathcal{E}_{\mathrm{GHZ}}(\rho) = 1$ if and only if ρ is LU equivalent to $|\mathrm{GHZ}\rangle$.*

Proof. Each A_i is a ± 1-valued Hermitian unitary, so $|\langle A_i \rangle| \le 1$ and $|I_\star| \le |\langle A_1 \rangle| + |\langle A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle| \le 2$, giving $\mathcal{E}_{\mathrm{GHZ}} \le 1$. Saturation $|I_\star| = 2$ forces $|\langle A_i \rangle| = 1$ for all i, hence ρ is supported in a joint ± 1 eigenspace of the four commuting operators (29). These generate an abelian group with three independent generators (App. D), whose eight common eigenspaces are one-dimensional and span $\mathcal{H}$; on each, $A_1 A_2 A_3 A_4 = -\mathbb{1}$ gives $\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 = -1$ and $I_\star = \epsilon_1 - \epsilon_2 \epsilon_3 \epsilon_4 = 2\epsilon_1 = \pm 2$. Thus ρ is the pure projector onto one such common eigenstate. Applying independent local rotations that send $(\hat{a}_1, \hat{a}_2)$, $(\hat{b}_1, \hat{b}_2)$, $(\hat{c}_1, \hat{c}_2)$ each to $(\hat{x}, \hat{y})$ turns $\{A_1, A_2, A_3, A_4\}$ into the standard GHZ–Mermin stabiliser operators $\{\sigma_{xxx}, \sigma_{xyy}, \sigma_{yxy}, \sigma_{yyx}\}$, whose common eigenbasis is the GHZ basis (13). Hence the eigenstate is LU equivalent to $|\mathrm{GHZ}\rangle$, and conversely $|\mathrm{GHZ}\rangle$ attains $\mathcal{E}_{\mathrm{GHZ}} = 1$ at orthogonal frames. □

C. Values on canonical states

GHZ and W. Proposition 3 gives $\mathcal{E}_{\mathrm{GHZ}}(|\mathrm{GHZ}\rangle) = 1$. For the W state the maximisation over independent frames does *not* exceed the aligned-frame value, a consequence of the permutation symmetry of $|\mathrm{W}\rangle$ and its invariance under collective rotations about $\hat{z}$: the optimum is already realised by the aligned configuration $\vec{n}_1 = \hat{z}$, $\vec{n}_2 \in \hat{x}$-$\hat{y}$ of Sec. IV B, so that

$$\mathcal{E}_{\mathrm{GHZ}}(|\mathrm{W}\rangle) = \tfrac{1}{2} \cdot \tfrac{35}{27} = \frac{35}{54} \approx 0.648, \quad (32)$$

which we have confirmed to machine precision by independent-frame optimisation. The strict ordering $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{GHZ}) = 1 > \mathcal{E}_{\mathrm{GHZ}}(\mathrm{W}) = 35/54$ persists.

Product states (exact). For a pure product state the correlation tensor factorises, $\langle \sigma_i \sigma_j \sigma_k \rangle = a_i b_j c_k$ with unit Bloch vectors $\vec{a}, \vec{b}, \vec{c}$. Writing $p_r = \vec{a}\hat{a}_r$, $q_r = \vec{b}\hat{b}_r$, $s_r = \vec{c}\hat{c}_r$ $(r = 1, 2$, $p_1^2 + p_2^2 \le 1$, etc.),

$$I_\star = p_1 q_1 s_1 \big(1 - p_2^2 q_2^2 s_2^2\big), \quad (33)$$

whence $|I_\star| \le |p_1 q_1 s_1| \le 1$, with equality at $\hat{a}_1 = \vec{a}$, $\hat{b}_1 = \vec{b}$, $\hat{c}_1 = \vec{c}$. Therefore

$$\mathcal{E}_{\mathrm{GHZ}}(\text{product}) = \tfrac{1}{2}. \quad (34)$$

Biseparable states. For $A|BC$ biseparable $\rho = \rho_A \otimes \rho_{BC}$ let $u = \vec{a} \cdot \hat{a}_1$, $u' = \vec{a} \cdot \hat{a}_2$ (with $u^2 + u'^2 \le 1$) and let $f_{jk} = \hat{b}_j^{\mathsf{T}} S\, \hat{c}_k$ be the relevant correlators of the two-qubit block, where $S_{jk} = \langle \sigma_j \otimes \sigma_k \rangle_{BC}$. Since every singular value of a two-qubit correlation matrix is at most one, the 2×2 array $[f_{jk}] = B^{\mathsf{T}} S C$ (with $B = [\hat{b}_1, \hat{b}_2]$, $C = [\hat{c}_1, \hat{c}_2]$) has operator norm ≤ 1, so $|f_{jk}| \le 1$ and $f_{11}^2 + f_{12}^2 \le 1$ along each row and column. Then

$$|I_\star| = |u\, f_{11} - u\, u'^2 f_{22} f_{12} f_{21}| \le |u|\,(1 + u'^2) \le \frac{4\sqrt{6}}{9}, \quad (35)$$

the last step maximising $|u|(1+u'^2)$ over $u^2+u'^2\le 1$. Hence

$$\mathcal{E}_{\mathrm{GHZ}}(A|BC) \;\le\; \frac{2\sqrt{6}}{9}\approx 0.544 \;<\; 1, \tag{36}$$

already enough to exclude biseparable states from the maximal value. The triangle bound (36) is, however, not tight: the four two-qubit operators entering (27) include anticommuting pairs (for instance $\sigma_{\hat{b}_1}\sigma_{\hat{c}_1}$ and $\sigma_{\hat{b}_1}\sigma_{\hat{c}_2}$ anticommute, since they share $\hat{b}_1$ but differ by the orthogonal pair $\hat{c}_1\perp\hat{c}_2$), so they cannot be simultaneously ±1. Numerically maximising $\mathcal{E}_{\mathrm{GHZ}}$ over all biseparable $A|BC$ states we find the sharp value

$$\sup_{\rho\,\in\,A|BC}\mathcal{E}_{\mathrm{GHZ}}(\rho)=\tfrac{1}{2}, \tag{37}$$

attained e.g. by $|0\rangle_A\otimes|\Phi^+\rangle_{BC}$ and coinciding with the product-state value (34); the analytic proof of the exact constant $1/2$ remains open. The same $1/2$ holds for the $B|AC$ and $C|AB$ partitions by symmetry.

D. Summary of properties

Proposition 4. *The functional $\mathcal{E}_{\mathrm{GHZ}}$ satisfies:*

(P1) $0\le\mathcal{E}_{\mathrm{GHZ}}(\rho)\le 1$*;*

(P2) $\mathcal{E}_{\mathrm{GHZ}}$ *is invariant under arbitrary local unitaries* (Proposition 2)*;*

(P3) $\mathcal{E}_{\mathrm{GHZ}}(\rho)=1$ *if and only if ρ is LU equivalent to* $|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$ (Proposition 3)*;*

(P4) $\mathcal{E}_{\mathrm{GHZ}}(|\mathrm{W}\rangle)=35/54\approx 0.648$*;*

(P5) $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{product})=1/2$ *exactly ;* $\mathcal{E}_{\mathrm{GHZ}}\le 2\sqrt{6}/9$ *for every biseparable state , with sharp value* $1/2$ (numerical)*;*

(P6) $\mathcal{E}_{\mathrm{GHZ}}$ *is conjectured convex,* $\mathcal{E}_{\mathrm{GHZ}}(\sum_i p_i\rho_i)\le\sum_i p_i\mathcal{E}_{\mathrm{GHZ}}(\rho_i)$ (supported numerically; see below and App. D).

We stress the status of (P6). For *fixed* frames the integrand $|\langle A_1\rangle-\langle A_2\rangle\langle A_3\rangle\langle A_4\rangle|$ is the modulus of a linear term minus a *cubic* term in ρ, and is *not* convex: sampling random pairs of states we find explicit midpoint violations of fixed-frame convexity in roughly 16% of cases. What we do observe is that the *optimised* measure $\mathcal{E}_{\mathrm{GHZ}}$ behaves convexly on every mixture we have tested: along the GHZ–W segment, for example, $\mathcal{E}_{\mathrm{GHZ}}\big(\frac{1}{2}|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|+\frac{1}{2}|\mathrm{W}\rangle\langle\mathrm{W}|\big)\approx 0.374$ lies well below the chord value ≈ 0.824, and dips below $1/2$ for balanced mixtures. A proof (or refutation) of global convexity is left open.

A conditional GME witness. If $\mathcal{E}_{\mathrm{GHZ}}$ is convex (P6), then it is a genuine-multipartite-entanglement witness above the biseparable ceiling: any state with $\mathcal{E}_{\mathrm{GHZ}}(\rho)>1/2$ cannot be written as a mixture of biseparable states across the three bipartitions, since each biseparable pure state obeys $\mathcal{E}_{\mathrm{GHZ}}\le 1/2$ by (37) and convexity would extend the bound to their mixtures. We have not found a counterexample, but absent a convexity proof we state the GME-witness property as conditional.

Comparison with other measures. Unlike the three-tangle τ_3 [6] or the GHZ⊕W-restricted closed forms of Eltschka *et al.* [28], both of which vanish on the W class, $\mathcal{E}_{\mathrm{GHZ}}$ takes nontrivial values on the W class and on biseparable states: it tells GHZ from W (with $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{GHZ})=1>\mathcal{E}_{\mathrm{GHZ}}(\mathrm{W})=35/54$) and from biseparable states (with $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{bisep})\le 1/2<\mathcal{E}_{\mathrm{GHZ}}(\mathrm{W})$). In contrast to GHZ-fidelity-based witnesses $F_{\mathrm{GHZ}}(\rho)=\langle\mathrm{GHZ}|\,\rho\,|\mathrm{GHZ}\rangle$ [16, 17], $\mathcal{E}_{\mathrm{GHZ}}$ is constructed from local Pauli measurements alone and, by Proposition 2, is invariant under arbitrary local unitaries, so it does not depend on a choice of GHZ representative. Compared to the genuine multipartite negativity [18] or the GME-concurrence of Ma *et al.* [19], both of which involve semidefinite-programming or convex-roof constructions, the aligned functional $\mathcal{E}^{\mathrm{al}}_{\mathrm{GHZ}}$ has a closed form on every pure Acín state; the invariant $\mathcal{E}_{\mathrm{GHZ}}$ trades that closed form for an optimisation over three local frames, while remaining local-frame independent in the operational sense that only spin directions, agreed locally, need be measured.

E. Relation to GHZ fidelity witnesses

The most widely used detector of GHZ-type genuine multipartite entanglement is the fidelity witness $W_{\mathrm{GHZ}}=\frac{1}{2}\mathbb{1}-|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$, whose expectation is negative—certifying genuine tripartite entanglement—precisely when $F_{\mathrm{GHZ}}(\rho)=\langle\mathrm{GHZ}|\,\rho\,|\mathrm{GHZ}\rangle>1/2$ [16, 17, 30]. Because the projector onto $|\mathrm{GHZ}\rangle$ is the uniform average of the eight elements of the GHZ stabiliser group,

$$F_{\mathrm{GHZ}}(\rho)=\tfrac{1}{8}\Big[1+\langle\sigma_{zz\mathbb{1}}\rangle+\langle\sigma_{\mathbb{1}zz}\rangle+\langle\sigma_{z\mathbb{1}z}\rangle + \langle\sigma_{xxx}\rangle-\langle\sigma_{xyy}\rangle-\langle\sigma_{yxy}\rangle-\langle\sigma_{yyx}\rangle\Big], \tag{38}$$

the fidelity is assembled from exactly the same laboratory data as $I(\hat{x},\hat{y})$: the four Mermin correlators together with the three two-body zz correlators. The two quantities nevertheless differ in three structural respects, with different physical readings. *(i) Linear versus multiplicative.* F_{GHZ} is linear in ρ, hence automatically convex, which is what makes the threshold $1/2$ a rigorous witness. Our I is a linear term minus a *cubic* one, and the convexity of $\mathcal{E}_{\mathrm{GHZ}}$ is supported only numerically (property (P6)). What the multiplicative form buys in exchange is that the GHZ paradox enters as an *equality* (3), which yields the sharp saturation statement of Proposition 1: $|I|=2$ holds *if and only if* the state lies on the GHZ orbit, whereas $F_{\mathrm{GHZ}}>1/2$ is a sufficient but not necessary criterion. *(ii) Two-body content.* Equation (38)

mixes two-body (zz) and three-body correlators, whereas I uses three-body correlators only, so $\mathcal{E}_{\mathrm{GHZ}}$ is blind to bipartite correlations and can separate genuine three-body correlation from residual two-body crosstalk in a device. *(iii) Reference-frame dependence.* F_{GHZ} is defined relative to a chosen representative $|\mathrm{GHZ}\rangle$ and changes under independent local unitaries; a frame-free version requires a maximisation over $SU(2)^{\otimes 3}$ of the same numerical cost as (30). By Proposition 2, $\mathcal{E}_{\mathrm{GHZ}}$ is LU invariant by construction.

The resulting physical interpretations are complementary: F_{GHZ} measures proximity in Hilbert space to a nominated target state, while $\mathcal{E}_{\mathrm{GHZ}}$ measures how strongly the state realises the GHZ algebraic paradox, and the induced orderings differ accordingly. Maximised over local unitaries, the GHZ fidelity of the W state is $3/4$ (verified numerically to 10^{-6}; note that this maximised quantity is not itself a witness, since the threshold $1/2$ applies to a fixed target state), against $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{W}) = 35/54 \approx 0.648$. In summary, the fidelity witness is the sharper tool when a rigorous yes/no certificate with a proven threshold is wanted; $\mathcal{E}_{\mathrm{GHZ}}$ is the more informative one when the task is to grade states on a representative-free, purely three-body scale, requiring only four measurement settings rather than seven correlators.

F. Behaviour under local operations and classical communication

We now make precise in what sense $\mathcal{E}_{\mathrm{GHZ}}$ is, and is not, compatible with the LOCC paradigm. By Proposition 2, $\mathcal{E}_{\mathrm{GHZ}}$ is exactly invariant under the reversible elements of LOCC (the local unitaries), and under relabelling of the parties, since (30) optimises over all three frames independently.

$\mathcal{E}_{\mathrm{GHZ}}$ is, however, not an entanglement monotone. This is not merely unproven—it fails, and for a structural reason. A pure product state has $\mathcal{E}_{\mathrm{GHZ}} = 1/2$ by (34), whereas the maximally mixed state $\rho = \mathbb{1}/8$ has all correlators zero and hence $\mathcal{E}_{\mathrm{GHZ}} = 0$. Local preparation, a legitimate LOCC operation, maps $\mathbb{1}/8$ to $|000\rangle\langle 000|$ and thereby *raises* $\mathcal{E}_{\mathrm{GHZ}}$ from 0 to $1/2$. The obstruction is that $\mathcal{E}_{\mathrm{GHZ}}$ does not vanish on separable states: the value $1/2$ is a correlation offset carried by every product state, not a residue of entanglement. The natural repair is to subtract the biseparable ceiling,

$$G(\rho) \;=\; \max\{0,\; 2\mathcal{E}_{\mathrm{GHZ}}(\rho) - 1\} \;\in\; [0,1], \tag{39}$$

which vanishes on every pure product and pure biseparable state, equals 1 exactly on the GHZ orbit (Proposition 3), and takes the value $G(|\mathrm{W}\rangle) = 2 \cdot \frac{35}{54} - 1 = \frac{8}{27} \approx 0.296$; being built from $\mathcal{E}_{\mathrm{GHZ}}$, G is LU invariant. Whether its convex roof is non-increasing on average under LOCC remains an open question, left for future work.

One monotonicity statement can be made exactly. Let Λ_q be the single-qubit depolarising channel, applied independently to each of the three qubits. Every three-body correlator is then scaled by q^3, so that

$$\mathcal{E}_{\mathrm{GHZ}}\big(\Lambda_q^{\otimes 3}(\rho)\big) \;\le\; q^3 \qquad \text{for every } \rho. \tag{40}$$

For the GHZ state the value is attained in closed form,

$$\mathcal{E}_{\mathrm{GHZ}}\big(\Lambda_q^{\otimes 3}(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|)\big) = \tfrac{1}{2}\big(q^3 + q^9\big), \tag{41}$$

confirmed numerically to 10^{-5} over $q \in \{0.6, \ldots, 1\}$. Thus $\mathcal{E}_{\mathrm{GHZ}}$ degrades smoothly and monotonically under local depolarising noise, and (41) can be inverted to extract an effective local-noise parameter from the four measured correlators.

VI. EXTENSION TO THREE-QUDIT SYSTEMS

We now sketch a generalisation of I from qubits ($d = 2$) to qudits ($d \ge 2$). Two natural strategies exist: (a) replace the spin observables $\sigma_{\vec{n}}$ by the higher-spin generalisations $S_{\vec{n}} = \vec{n} \cdot \boldsymbol{S}$ acting on the symmetric subspace, or (b) replace them by the unitary Heisenberg–Weyl operators [31–34]. We adopt the latter, which preserves the algebraic clarity of Sec. II and is directly compatible with the qudit GHZ paradox of Cerf, Massar, and Pironio [32] and its multisetting refinement by Ryu *et al.* [34].

Heisenberg–Weyl operators. On a single qudit $\mathbb{C}^d$, with $\omega = \mathrm{e}^{2\pi \mathrm{i}/d}$, define

$$X\,|j\rangle = |j+1 \bmod d\rangle\,, \qquad Z\,|j\rangle = \omega^j\,|j\rangle\,. \tag{42}$$

For $\boldsymbol{p} = (p, q) \in \mathbb{Z}_d^2$, the Weyl operator is

$$W(\boldsymbol{p}) = \omega^{-pq/2}\, X^p Z^q. \tag{43}$$

The $W(\boldsymbol{p})$ are unitary (not Hermitian for $d > 2$) and obey

$$W(\boldsymbol{p})W(\boldsymbol{q}) = \omega^{\langle \boldsymbol{p},\boldsymbol{q}\rangle} W(\boldsymbol{q})W(\boldsymbol{p}), \tag{44}$$

$$\langle \boldsymbol{p}, \boldsymbol{q}\rangle \equiv p_1 q_2 - p_2 q_1 \pmod d, \tag{45}$$

together with $W(\boldsymbol{p})^d = \mathbb{1}$ for odd d. For $d = 2$, $W(0,0) = \mathbb{1}$, $W(1,0) = \sigma_x$, $W(0,1) = \sigma_z$, and $W(1,1) \propto \sigma_y$, so the Pauli operators are recovered.

Three-qudit GHZ state. The qudit GHZ state is $|\mathrm{GHZ}_d\rangle = d^{-1/2} \sum_{j=0}^{d-1} |jjj\rangle$. It satisfies the perfect-correlation identities $X^{\otimes 3}\,|\mathrm{GHZ}_d\rangle = |\mathrm{GHZ}_d\rangle$ and $(Z^{q_1} \otimes Z^{q_2} \otimes Z^{q_3})\,|\mathrm{GHZ}_d\rangle = |\mathrm{GHZ}_d\rangle$ when $q_1 + q_2 + q_3 \equiv 0 \pmod d$. The qudit Mermin paradox [32, 33] exploits these algebraic relations to produce a complex-valued analogue of the dichotomic GHZ identity.

Qudit functional. Fix two pairs $\boldsymbol{g}_1, \boldsymbol{g}_2 \in \mathbb{Z}_d^2$ and define

$$\begin{aligned} G_1 &= W(\boldsymbol{g}_1) \otimes W(\boldsymbol{g}_2) \otimes W(\boldsymbol{g}_2), \\ G_2 &= W(\boldsymbol{g}_2) \otimes W(\boldsymbol{g}_1) \otimes W(\boldsymbol{g}_2), \\ G_3 &= W(\boldsymbol{g}_2) \otimes W(\boldsymbol{g}_2) \otimes W(\boldsymbol{g}_1), \\ G_4 &= W(\boldsymbol{g}_1) \otimes W(\boldsymbol{g}_1) \otimes W(\boldsymbol{g}_1). \end{aligned} \tag{46}$$

A short calculation analogous to (9) gives

$$G_1 G_2 G_3 = \omega^{2\langle \boldsymbol{g}_1, \boldsymbol{g}_2 \rangle} G_4 \tag{47}$$

when $\boldsymbol{g}_2$ is twice-applied (using $W(\boldsymbol{p})^2 = \omega^{-pq} W(2\boldsymbol{p})$ for the Heisenberg–Weyl group; here $W(\boldsymbol{g}_2)^2 = \mathbb{1}$ in the case $2\boldsymbol{g}_2 \equiv 0 \bmod d$ that arises naturally for $d = 2$). Define the qudit functional

$$I_d(\boldsymbol{g}_1, \boldsymbol{g}_2) \;=\; \langle G_4 \rangle - \omega^{2\langle \boldsymbol{g}_1, \boldsymbol{g}_2 \rangle} \langle G_1 \rangle \langle G_2 \rangle \langle G_3 \rangle. \tag{48}$$

For $d = 2$, with $\boldsymbol{g}_1 = (1, 0)$ (so $W(\boldsymbol{g}_1) = \sigma_x$) and $\boldsymbol{g}_2 = (0, 1)$ (so $W(\boldsymbol{g}_2) = \sigma_z$), the symplectic form is $\langle \boldsymbol{g}_1, \boldsymbol{g}_2 \rangle = 1$, $\omega = -1$, and (48) reduces (after relabelling $\sigma_z \leftrightarrow \sigma_y$) to the original $I(\hat{x}, \hat{y})$.

Bound and saturation. Each $\langle G_i \rangle$ is a complex number with $|\langle G_i \rangle| \le 1$, hence $|I_d| \le 2$. Saturation occurs at the qudit GHZ state when $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ generate a maximally non-commutative subgroup of the single-qudit Heisenberg–Weyl group; the proof is structurally identical to that of Proposition 1 once one reads "orthogonality" as "maximal symplectic non-commutation" $\langle \boldsymbol{g}_1, \boldsymbol{g}_2 \rangle \neq 0 \bmod d$. We omit the detailed argument; see [32, 33] for parallel constructions.

Spin-S alternative. For experimental settings in which spin-S measurements are natural (e.g. atomic ensembles), one may instead replace $\sigma_{\vec{n}}/2$ by the projection $S_{\vec{n}} = \vec{n} \cdot \boldsymbol{S}$ rescaled to have spectrum in $[-1, 1]$ via a normalisation factor $1/S$. All qualitative statements survive, but eigenvalues are no longer dichotomic and Proposition 1 becomes a strict inequality with saturation only in the $S \to \infty$ classical limit.

VII. EXPERIMENTAL ACCESSIBILITY AND POTENTIAL APPLICATIONS

The economy of $\mathcal{E}_{\mathrm{GHZ}}$ in the laboratory is twofold. For a fixed frame triple, evaluating $I_\star$ requires the expectation values of only the four commuting operators (28)—four measurement settings, against 27 for a full three-qubit correlation-tensor determination and 63 real parameters for full state tomography. Since (30) is a supremum, any frame choice already yields a valid lower bound, so in practice a near-optimal triple can be fixed from a coarse pre-scan. Because $\mathcal{E}_{\mathrm{GHZ}}$ is invariant under independent local unitaries, the three parties need not share a reference frame, which is directly relevant to photonic links with drifting polarisation frames (free-space, satellite or long-fibre channels) and to separately calibrated ion-trap or superconducting registers: a fidelity witness referred to a fixed $|\mathrm{GHZ}\rangle$ representative degrades under such drift even when the entanglement is intact, while $\mathcal{E}_{\mathrm{GHZ}}$ does not. The noise bounds (40)–(41) further turn $\mathcal{E}_{\mathrm{GHZ}}$ into a compact diagnostic for the effective local depolarising parameter, and the stabiliser correlations it measures are precisely the resource consumed by GHZ-based protocols such as quantum secret sharing and conference key agreement.

VIII. CONCLUSION

We have introduced a real-valued functional $I(\vec{n}_1, \vec{n}_2)$ on three-qubit states, built from four locally measurable Pauli correlations, that simultaneously

- reproduces the GHZ algebraic paradox at $\vec{n}_1 \perp \vec{n}_2$,
- obeys a tight quantum bound $|I| \le 2$ that is saturated only on the GHZ orbit (Proposition 1),
- admits a closed-form expression on the Acín canonical family showing that $I(\hat{x}, \hat{y})$ depends only on $\lambda_0 \lambda_4$ [Eq. (17)],
- gives a sharp distinction between the GHZ and W classes, with $\max |I_{\mathrm{W}}| = 35/27$ [Eq. (25)],
- induces, after maximisation over independent local frames, a manifestly LU-invariant measure $\mathcal{E}_{\mathrm{GHZ}} \in [0, 1]$ that equals one exactly on the GHZ orbit, takes the value $35/54$ on the W class, and is bounded by $1/2$ on biseparable and product states (Sec. V),
- generalises naturally to three qudits via the Heisenberg–Weyl operators [Eq. (48)].

A number of natural extensions remain open. First, while $\mathcal{E}_{\mathrm{GHZ}}$ is LU-invariant by construction, we have not settled whether it is globally convex; our numerical evidence is consistent with convexity, and a proof would immediately upgrade $\mathcal{E}_{\mathrm{GHZ}}$ to a genuine-multipartite-entanglement witness above the biseparable ceiling $1/2$ and motivate the convex-roof monotone $\mathcal{E}^{\mathrm{cr}}_{\mathrm{GHZ}}$. Establishing monotonicity under the full SLOCC hierarchy would promote it to a bona fide entanglement monotone. Second, the qudit construction in (48) merits independent quantitative study: in particular, the qudit analogue of the closed-form Acín evaluation in (17) would shed light on the high-dimensional analogue of the $\lambda_0 \lambda_4$ "GHZ slot." Third, $I(\vec{n}_1, \vec{n}_2)$ generalises immediately to N-qubit GHZ-Mermin functionals [4, 5, 10]; an N-party version of the entanglement measure $\mathcal{E}_{\mathrm{GHZ}}$ would furnish a uniform indicator of GHZ-type genuine multipartite entanglement, with maximal value $2^{N-1}/2$ on the N-qubit GHZ state. Experimental implementation requires only Pauli-string correlation measurements, well within reach of current photonic and trapped-ion platforms [35, 36].

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (Grants 12475020 and 92565111), Quantum Science and Technology-National Science and Technology Major Project (2021ZD0301701), and the National Key Research and Development Program of China (2023YFC2205802).

Appendix A: Operator identities of Section II

Using $\sigma_{\vec{n}_1}\sigma_{\vec{n}_2} = c\,\mathbb{1} + \mathrm{i}\,\boldsymbol{m}\cdot\boldsymbol{\sigma}$ with $c = \vec{n}_1\cdot\vec{n}_2$, $\boldsymbol{m} = \vec{n}_1\times\vec{n}_2$,

$$O_1O_2 = \sigma_{\vec{n}_1}\sigma_{\vec{n}_2}\otimes\sigma_{\vec{n}_2}\sigma_{\vec{n}_1}\otimes\mathbb{1} \tag{A1}$$
$$= (c+\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{\sigma})\otimes(c-\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{\sigma})\otimes\mathbb{1}, \tag{A2}$$
$$O_2O_1 = (c-\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{\sigma})\otimes(c+\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{\sigma})\otimes\mathbb{1}, \tag{A3}$$

so that $[O_1,O_2] = 2\mathrm{i}c\,(\boldsymbol{m}\cdot\boldsymbol{\sigma}\otimes\mathbb{1} - \mathbb{1}\otimes\boldsymbol{m}\cdot\boldsymbol{\sigma})\otimes\mathbb{1}$, which is (8).

For (9), one computes

$$O_1O_2O_3 = \sigma_{\vec{n}_1}\sigma_{\vec{n}_2}^2\otimes\sigma_{\vec{n}_2}\sigma_{\vec{n}_1}\sigma_{\vec{n}_2}\otimes\sigma_{\vec{n}_2}^2\sigma_{\vec{n}_1} \tag{A4}$$
$$= \sigma_{\vec{n}_1}\otimes\left(\sigma_{\vec{n}_2}\sigma_{\vec{n}_1}\sigma_{\vec{n}_2}\right)\otimes\sigma_{\vec{n}_1}. \tag{A5}$$

The middle factor satisfies $\sigma_{\vec{n}_2}\sigma_{\vec{n}_1}\sigma_{\vec{n}_2} = 2c\,\sigma_{\vec{n}_2} - \sigma_{\vec{n}_1}$, since $\{\sigma_{\vec{n}_1},\sigma_{\vec{n}_2}\} = 2c\mathbb{1}$. Substituting,

$$O_1O_2O_3 = 2c\,\sigma_{\vec{n}_1}\otimes\sigma_{\vec{n}_2}\otimes\sigma_{\vec{n}_1} - O_4, \tag{A6}$$

which is (9).

Appendix B: Closed form for the Acín family

For $|\Psi\rangle$ as in (14), the only $\sigma_x^{\otimes3}$ matrix elements between basis states present in $|\Psi\rangle$ that survive are $\langle000|\,\sigma_x^{\otimes3}\,|111\rangle = 1$ and its conjugate. Hence $\langle\sigma_x^{\otimes3}\rangle_\Psi = 2\lambda_0\lambda_4$. For $\sigma_x\sigma_y\sigma_y$, $\sigma_x\otimes\sigma_y\otimes\sigma_y\,|000\rangle = -|111\rangle$ and $\sigma_x\otimes\sigma_y\otimes\sigma_y\,|111\rangle = -|000\rangle$; the only contributing matrix elements are again $\langle000|\cdots|111\rangle = -1$ and $\langle111|\cdots|000\rangle = -1$, so $\langle\sigma_x\sigma_y\sigma_y\rangle_\Psi = -2\lambda_0\lambda_4$. The remaining basis states $|100\rangle, |101\rangle, |110\rangle$ contribute zero to all four expectations because the action of any tensor product of σ_x,σ_y that flips the first qubit cannot return any of $\{|000\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$ to itself with nonzero coefficient. ∎

Appendix C: Maximum of $|I_{\mathrm{W}}|$ over directions

Reduction to two scalar variables. Using (21),

$$\langle\sigma_{\vec{n}_1}\otimes\sigma_{\vec{n}_2}\otimes\sigma_{\vec{n}_2}\rangle_{\mathrm{W}} = \sum_{i,j,k} n_i^{(1)}n_j^{(2)}n_k^{(2)}\langle\sigma_i\otimes\sigma_j\otimes\sigma_k\rangle_{\mathrm{W}} \tag{C1}$$
$$= \tfrac{2}{3}\left[2b_3(\vec{n}_1\cdot\vec{n}_2) - 2a_3b_3^2 + a_3(1-b_3^2)\right] - a_3b_3^2, \tag{C2}$$

where $a_3 = \hat{z}\cdot\vec{n}_1$, $b_3 = \hat{z}\cdot\vec{n}_2$, and $n^{(1)}, n^{(2)}$ are the cartesian components of $\vec{n}_1,\vec{n}_2$. When $\vec{n}_1\cdot\vec{n}_2 = 0$, the first square-bracket term simplifies giving

$$\langle\sigma_{\vec{n}_1}\otimes\sigma_{\vec{n}_2}\otimes\sigma_{\vec{n}_2}\rangle_{\mathrm{W}} = a_3\left(\tfrac{2}{3} - 3b_3^2\right). \tag{C3}$$

By the permutation symmetry of $|\mathrm{W}\rangle$, the other two "one-$\vec{n}_1$, two-$\vec{n}_2$" correlators take the same value. Similarly,

$$\langle\sigma_{\vec{n}_1}^{\otimes3}\rangle_{\mathrm{W}} = a_3(2-3a_3^2), \tag{C4}$$

yielding (24).

Optimisation. Let $f(u,v) = u^3(\frac{2}{3}-3v^2)^3 - u(2-3u^2)$ with constraints $u^2+v^2\le1$, $u,v\in[-1,1]$. Critical points in the interior: $\partial_v f = -18u^3v(\frac{2}{3}-3v^2)^2$ vanishes at $v=0$, $u=0$, or $v^2=2/9$. The branch $v=0$ reduces f to $\frac{89}{27}u^3 - 2u$, maximised at the boundary $u=\pm1$ giving $|f| = 35/27$. The branch $v^2=2/9$ gives $f = -u(2-3u^2)$, maximised under $u^2\le7/9$ at $u^2=2/9$ with $|f| = 4\sqrt{2}/9 \approx 0.629 < 35/27$. On the boundary $u^2+v^2=1$, $f(u) = u^3(3u^2-7/3)^3 - u(2-3u^2)$ is again maximised at $u=\pm1$, $v=0$. Hence $\max|f| = 35/27$, attained at $\vec{n}_1 = \pm\hat{z}$, $\vec{n}_2$ in the $\hat{x}$-$\hat{y}$ plane. ∎

Appendix D: Independent-frame operators and convexity

Commutation and the stabiliser relation. We verify (29) for the operators (28). Two tensor products of single-qubit Hermitian unitaries commute when they anticommute at an even number of sites and anticommute when they do so at an odd number. At each party the two vectors appearing across $\{A_1,A_2,A_3,A_4\}$ are either equal (so the single-qubit factors commute) or orthogonal (so, by $\sigma_{\hat{u}}\sigma_{\hat{v}} = -\sigma_{\hat{v}}\sigma_{\hat{u}}$ for $\hat{u}\perp\hat{v}$, they anticommute). Listing the per-site relations: each pair (A_i,A_j) differs by orthogonal factors at exactly two of the three sites — e.g. $A_1 = (\hat{a}_1,\hat{b}_1,\hat{c}_1)$ and $A_2 = (\hat{a}_1,\hat{b}_2,\hat{c}_2)$ agree on A and are orthogonal on B and C. Two anticommuting sites give an even count, so all six pairs commute. For the product, reading the four operators site by site,

$$\text{site } A: \sigma_{\hat{a}_1}\sigma_{\hat{a}_1}\sigma_{\hat{a}_2}\sigma_{\hat{a}_2} = \mathbb{1},$$
$$\text{site } B: \sigma_{\hat{b}_1}\sigma_{\hat{b}_2}\sigma_{\hat{b}_1}\sigma_{\hat{b}_2} = (\sigma_{\hat{b}_1}\sigma_{\hat{b}_2})^2 = -\mathbb{1},$$
$$\text{site } C: \sigma_{\hat{c}_1}\sigma_{\hat{c}_2}\sigma_{\hat{c}_2}\sigma_{\hat{c}_1} = \mathbb{1},$$

using $\sigma_{\hat{u}}^2 = \mathbb{1}$ and $(\sigma_{\hat{b}_1}\sigma_{\hat{b}_2})^2 = (\mathrm{i}\,\hat{b}_3\cdot\boldsymbol{\sigma})^2 = -\mathbb{1}$ for the orthonormal pair $\hat{b}_1\perp\hat{b}_2$. Hence $A_1A_2A_3A_4 = \mathbb{1}\otimes(-\mathbb{1})\otimes\mathbb{1} = -\mathbb{1}$. Finally $A_1A_2A_3 = -A_4$ (the same computation without the fourth factor), so only three of the four operators are independent; the abelian group they generate has order eight and its common eigenspaces are the eight one-dimensional joint eigenspaces used in Proposition 3.

Failure of fixed-frame convexity. Write $a_i(\rho) = \langle A_i\rangle = \mathrm{Tr}(A_i\rho)$, each linear in ρ. For *fixed* frames the integrand

$$g(\rho) \equiv \left|a_1(\rho) - a_2(\rho)\,a_3(\rho)\,a_4(\rho)\right| \tag{D1}$$

is the modulus of a linear term minus a *cubic* monomial in the components of ρ. A modulus of a non-affine function need not be convex, and indeed it is not here: sampling pairs of random three-qubit states ρ_1,ρ_2 and fixed random frames, the midpoint inequality $g\left(\frac{1}{2}\rho_1+\frac{1}{2}\rho_2\right) \le \frac{1}{2}g(\rho_1)+\frac{1}{2}g(\rho_2)$ is *violated* in a sizeable fraction (≈ 482 of 3000 trials, i.e. about 16%).

Numerical convexity of the optimised measure. The optimised measure $\mathcal{E}_{\mathrm{GHZ}}$ of (30) is a different object: for

each ρ the frames are chosen to maximise $|I_\star|$. Empirically $\mathcal{E}_{\mathrm{GHZ}}$ is convex on every mixture we have examined. On the GHZ–W line $\rho_t = (1-t)\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + t\,|\mathrm{W}\rangle\langle\mathrm{W}|$ the measure lies below the chord throughout, with $\mathcal{E}_{\mathrm{GHZ}}(\rho_{1/2}) \approx 0.374$ against a chord value $\frac{1}{2}(1)+\frac{1}{2}(35/54) \approx 0.824$; balanced mixtures fall below the biseparable ceiling 1/2. Random two-component mixtures of Haar pure states show no convexity violation within numerical tolerance. A general proof or a counterexample is left as an open problem. ■

Numerical methods. All computations were carried out in double precision. Haar-random pure states on $\mathbb{C}^8$ were generated as normalised complex Gaussian vectors (16 i.i.d. $\mathcal{N}(0,1)$ real components; this induces exactly the Haar measure), random mixed states from the Ginibre ensemble $\rho = GG^\dagger/\mathrm{Tr}(GG^\dagger)$ ($k = 8$ gives the Hilbert–Schmidt measure), and random local unitaries Haar-uniformly on $U(2)$ (phase-corrected QR; the LU-scatter test of Sec. V A used 200 independent triples). Each ordered orthonormal pair is parametrised by three angles, so a frame triple is a point of $[0, 2\pi)^9$ and orthonormality holds by construction. The supremum in (30) was evaluated with Powell's derivative-free method (`scipy.optimize.minimize`, `method='Powell'`), tolerances `xtol` $= 10^{-9}$, `ftol` $= 10^{-11}$, from 120 random restarts per state; the exact values $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{GHZ}) = 1$, $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{W}) = 35/54$ and $\mathcal{E}_{\mathrm{GHZ}}(\mathrm{product}) = \mathcal{E}_{\mathrm{GHZ}}(A|BC) = 1/2$ are reproduced to better than 10^{-6}, and increasing the restarts from 40 to 120 changes the optimum by less than 10^{-6}. The fixed-frame convexity test (midpoint inequality, absolute tolerance 10^{-12}) failed in 482 of 3000 trials, i.e. 16.1% (independent seeds give 15.8%–16.4%), while $\mathcal{E}_{\mathrm{GHZ}}$ itself showed no convexity violation along the GHZ–W segment (step 0.02) or on 2000 random two-component mixtures, within the 10^{-6} optimisation tolerance—evidence for, not a proof of, (P6). The biseparable supremum (37) used 500 outer restarts, and the maximal GHZ fidelity of the W state quoted in Sec. V E (3/4) and the depolarising formula (41) were verified to 10^{-6} and 10^{-5}, respectively.

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