Spatial optical solitons in nonlocal nonlinear media: from the Snyder-Mitchell model to long-range interactions in liquid-crystal and photorefractive media

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Spatial optical solitons in nonlocal nonlinear media are important objects for studying the formation, stable propagation, and interaction of localized waves in nonlinear optics. Different from local nonlinear media, nonlocal response makes the refractive-index change of the medium depend on the optical-field distribution within a certain spatial range. As a result, the formation conditions, stability, and propagation dynamics of solitons can be greatly changed. This review focuses on the theoretical basis and research progress of nonlocal spatial optical solitons. It first introduces the general description of nonlocal nonlinear response, the strongly nonlocal limit, and the Snyder-Mitchell model, and analyzes their physical meanings and applicable conditions. Then, three typical systems, including liquid crystals, photorefractive crystals, and thermal nonlinear media, are discussed. The formation types, propagation features, and control methods of spatial optical solitons under different nonlocal mechanisms are summarized. Recent representative progress in long-range interactions, new localized wave structures, and controllable transport is further reviewed. The main feature of this review is that it follows the main line of strongly nonlocal theory and connects the common physical mechanisms with the specific response features of different media. It shows the important role of nonlocality in stabilizing complex localized waves, controlling soliton interactions, and realizing programmable optical-field transport. In contrast to earlier reviews that mainly emphasize an early model or one material platform, we compare liquid-crystal, photorefractive, and thermal media within the same strongly nonlocal framework, and relate the Snyder-Mitchell approximation to material anisotropy, boundaries, and competing nonlinearities. The comparison indicates that nonlocality can stabilize complex solitons and enable reconfigurable transport, although response speed, loss, boundary sensitivity, and integration remain important challenges.

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1. Introduction

Nonlinear localized waves are an important topic in nonlinear optics. Among

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them, spatial optical solitons are one of the most representative forms, being self-trapped localized wave packets formed by the balance between diffraction and nonlinearity.[1,2] Different from local nonlinear media, many real optical media have clear nonlocal nonlinear responses. This means that the refractive-index change at one point is determined not only by the optical intensity at that point, but also by the optical-field distribution in the surrounding region.[3,4] Nonlocal response can change the formation conditions, stability, interactions, and propagation dynamics of spatial optical solitons. It shows special physical properties that do not exist in local media, especially in the stability of high-dimensional localized waves, long-range interactions between solitons, and controllable transport.[5,6] Under strongly nonlocal conditions, the Snyder-Mitchell model provides a clear and physically meaningful theoretical framework for understanding accessible solitons and related self-trapped states.[7] In recent years, with the continuous progress of experimental and theoretical studies in nonlocal nonlinear media such as liquid crystals and photorefractive crystals, research on spatial optical solitons in strongly nonlocal media is no longer limited to early ideal models. It has further expanded to systematic studies of real material response features,[8] competing nonlinear effects,[9] new localized wave structures,[10] and controllable propagation and transport.[11] Therefore, this topic remains an important research direction in nonlinear localized waves.

Over the past two decades, studies on spatial optical solitons in nonlocal nonlinear media have developed continuously, and important progress has been made in theoretical analysis, numerical simulation, and experimental observation.[12,13] Early studies mainly focused on the construction of nonlocal models and the analysis of their basic propagation properties. A relatively systematic theoretical understanding was formed around the formation mechanism, propagation behavior, stability, and higher-order localized states of accessible solitons.[5,7,14] With the further development of nonlocal nonlinear media, related studies have gradually moved from idealized models to material systems closer to real physical processes.[15,16] Studies in typical nonlocal media, such as liquid crystals and photorefractive crystals, show that strongly nonlocal spatial optical solitons are not only important objects for studying

soliton formation, stability, and propagation, but also have important research value in long-range interactions, two-dimensional soliton states, and controllable transport.[17,18] At the same time, many existing reviews mainly focus on the early theoretical basis of nonlocal solitons, or discuss one specific type of medium. A systematic summary is still needed for the common physical mechanisms, recent representative progress, and future development trends of strongly nonlocal behavior in different nonlocal media. Therefore, it is necessary to summarize the study of spatial optical solitons based on strongly nonlocal theory and recent progress in typical nonlocal media.

Although rich research results have been obtained on nonlocal spatial optical solitons,[19,20] it is still necessary to further organize this field in a systematic way, considering current research progress and the focus of existing reviews. Existing studies and reviews on nonlocal spatial optical solitons mainly focus on theoretical models based on the nonlocal nonlinear Schrödinger equation, the formation and propagation properties of solitons, and the propagation laws and interaction features in typical nonlocal media.[21] In comparison, the applicable conditions of the strongly nonlocal approximation and the relationship between the Snyder-Mitchell model and real material responses still need further discussion and comparison. In fact, with the continuous progress of recent research, the study of nonlocal spatial optical solitons is no longer limited to basic soliton solutions and their stability analysis, but pays more attention to related studies in real media.[22,23] These studies show that the description of spatial optical soliton behavior in real media by early ideal models still needs to be discussed together with specific material properties. Therefore, it is necessary to start from strongly nonlocal theory and combine representative studies in typical nonlocal media, so as to further summarize the related work on spatial optical solitons and clarify the main research topics, common physical mechanisms, and development trends in different material systems.

Previous general reviews established the nonlocal nonlinear Schrödinger framework, whereas many material-oriented reviews concentrated on one platform. The present review addresses three remaining gaps: a direct comparison of

liquid-crystal, photorefractive, and thermal responses; a clear separation between the ideal Snyder-Mitchell model and real material limitations; and a joint summary of recent structured-beam localization, long-range interaction, and controllable transport.

Table 1. Comparison of the scope of earlier reviews and the present review.

Review scope	Typical focus	Gap addressed in this review
General nonlocal theory	Models, basic soliton formation, and stability	Links the strongly nonlocal model to real boundaries, anisotropy, and competing effects
Single-medium reviews	Liquid crystals or another individual platform	Directly compares liquid-crystal, photorefractive, and thermal mechanisms
Recent topic-specific studies	One structured beam, interaction, or control method	Combines structured localization, long-range interaction, stability, and transport

This review mainly discusses spatial optical solitons in nonlocal nonlinear media. It focuses on their theoretical basis, propagation properties in typical material systems, and recent progress. In connection with studies under strongly nonlocal response conditions, it discusses the applicable conditions, propagation behavior, and related progress of the Snyder-Mitchell model in real material systems. On this basis, this review further combines studies in typical nonlocal media such as liquid crystals and photorefractive crystals, and analyzes the formation, propagation, interaction, and control of spatial optical solitons in different material systems. It also discusses several topics that have attracted continuous attention in recent years, including long-range interactions, new localized wave structures, and controllable transport. The structure of this review is arranged as follows. Section 2 introduces the theoretical basis of spatial optical solitons in nonlocal media, with emphasis on the general description of nonlocal nonlinear response, the strongly nonlocal limit, and the Snyder-Mitchell model. Section 3 discusses the main research progress of nonlocal spatial optical solitons in typical nonlocal media systems. Section 4 further summarizes recent progress in long-range interactions, new localized wave structures, and controllable transport. Section 5 summarizes the whole review and gives an

outlook on the development of this field.

2. Theoretical basis of spatial optical solitons in nonlocal media

Before discussing spatial optical solitons in specific media, it is necessary to first clarify the basic physical picture and mathematical description of nonlocal nonlinear response, and then distinguish it from local response. Nonlocality determines the spatial range of the refractive-index change induced by the optical beam, and directly influences the formation conditions, stability, and interaction behavior of solitons. Therefore, this section starts from the general model of nonlocal nonlinear response, and then discusses the strongly nonlocal limit and the Snyder-Mitchell model, providing a theoretical basis for the following analysis of spatial optical solitons in different material systems.

2.1. General description of nonlocal nonlinear response

Under the paraxial approximation, for a spatial optical beam propagating along the z direction, its slowly varying envelope $\varphi(x,y,z)$ satisfies the following propagation equation

$$i\frac{\partial\varphi}{\partial z}+\frac{1}{2k}\nabla_{\perp}^{2}\varphi+k\frac{\Delta n}{n_0}\varphi=0 \tag{1}$$

where $k=n_0k_0$ is the wave number in the medium, $k_0=2\pi/\lambda$ is the wave number in vacuum, n_0 is the linear refractive index of the medium, $\nabla_{\perp}^{2}=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the transverse Laplacian operator, and Δn denotes the refractive-index change caused by the nonlinear effect.[24–27] In Eq. (1), the second term describes the transverse spreading of the beam caused by diffraction during propagation, while the third term represents the modulation of beam propagation by the nonlinear response of the medium.

For local Kerr-type nonlinear media, the refractive-index change usually depends only on the local optical intensity, namely

$$\Delta n(\boldsymbol{r},z)=n_2\,|\,\varphi(\boldsymbol{r},z)\,|^2 \tag{2}$$

where $\boldsymbol{r}=(x,y)$. Substituting Eq. (2) into Eq. (1) gives the standard two-dimensional nonlinear Schrödinger equation. However, in many real material systems, such as liquid crystals, photorefractive crystals, and thermal nonlinear media, the nonlinear response often shows clear nonlocal features. This means that the refractive-index change at one point in the medium is related not only to the optical intensity at that point, but also to the optical-field distribution in the surrounding region.[28,29] In this case, Δn and the optical intensity distribution $I(\boldsymbol{r},z)=|\varphi(\boldsymbol{r},z)|^2$ usually satisfy a convolution relation

$$\Delta n(\boldsymbol{r},z)=n_2\int R(\boldsymbol{r}-\boldsymbol{r}')I(\boldsymbol{r}',z)d\boldsymbol{r}' \tag{3}$$

where $R(\boldsymbol{r})$ is the nonlocal response function, which describes the spatial response of the medium to the optical intensity distribution in the neighboring region. If $R(\boldsymbol{r})=\gamma\delta(\boldsymbol{r})$, Eq. (3) reduces to the local response form. If $R(\boldsymbol{r})$ is spread over a finite spatial range, the system shows nonlocal nonlinearity. Substituting Eq. (3) into Eq. (1), one obtains the general nonlocal nonlinear Schrödinger equation

$$i\frac{\partial\varphi}{\partial z}+\frac{1}{2k}\nabla_{\perp}^2\varphi+k\frac{n_2}{n_0}\varphi\int R(\boldsymbol{r}-\boldsymbol{r}')\,|\varphi(\boldsymbol{r}',z)|^2\,d\boldsymbol{r}'=0. \tag{4}$$

Equation (4) gives a general theoretical model for the propagation of nonlocal spatial optical solitons.[30] Compared with the local model, the nonlinear term is no longer determined simply by the local amplitude, but by the spatial average of the optical intensity distribution under the response function. Therefore, the width, spatial shape, and symmetry of the response function directly influence the existence condition, propagation stability, and interaction behavior of nonlocal solitons.

To measure the strength of the nonlocal effect, the characteristic width w_{m} of the response function and the beam width w_0 are usually introduced, and the nonlocality parameter is defined as $\sigma=w_{\mathrm{m}}/w_0$. When $\sigma\ll1$, the width of the response function is much smaller than the beam width, and the system corresponds to the local case. When $\sigma\approx1$, the width of the response function is comparable to the beam width, and the system corresponds to the general or medium nonlocal case. When

$\sigma \gg 1$, the width of the response function is much larger than the beam width, and the system corresponds to the strongly nonlocal case. Thus, the three regimes are written unambiguously as $\sigma << 1$ (local), σ approximately 1 (moderately nonlocal), and $\sigma >> 1$ (strongly nonlocal), respectively.

It can be seen that the essence of a nonlocal nonlinear response is as follows: the refractive-index change in the medium is no longer determined only by the local optical intensity, but by the optical-field distribution within a certain spatial range. Figure 1 gives a direct physical comparison. In the local case, the spatial response function approaches a Dirac function. The induced refractive-index change at a given point is determined only by the light intensity at that point; therefore, $\Delta n(x)$ and $I(x)$ have the same spatial profile. In the nonlocal case, the response function has a finite spatial width, and the refractive-index change at a given point also contains contributions from the light intensity in the surrounding region. For a smooth positive

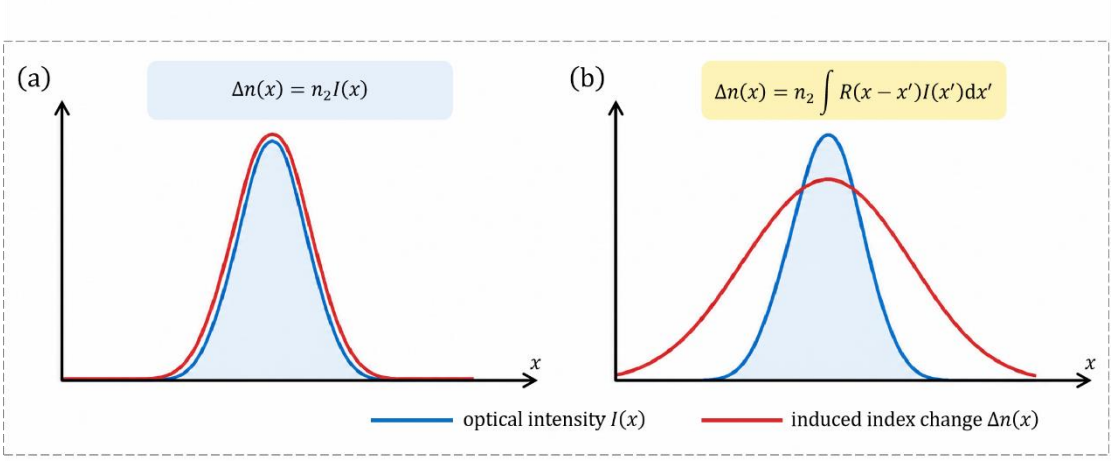


Fig. 1. Schematic comparison of local and nonlocal nonlinear responses.
(a) Local nonlinearity; (b) nonlocal nonlinearity

response kernel, this spatial averaging makes $\Delta n(x)$ broader than $I(x)$.[30,123] This feature makes the evolution of spatial optical solitons in nonlocal media very different from that in local media. It also provides an important theoretical basis for studying the stability of high-dimensional localized states, long-range interactions between solitons, and equivalent potential models under strongly nonlocal conditions.

2.2. Strongly nonlocal limit and its physical features

In the nonlocal nonlinear model, the nonlinear refractive-index change of the medium is usually determined by the convolution between the optical intensity distribution and the response function. When the characteristic width of the response function is much larger than the transverse size of the beam, the beam only occupies a small region near the center of the response function. In this case, the response function changes very slowly within the spatial range occupied by the beam, so the medium can be regarded as being in the strongly nonlocal regime. To further build a propagation model under strongly nonlocal conditions, one can start from the general nonlocal nonlinear Schrödinger equation. For a differentiable response function, when its width is much larger than the beam width, the convolution kernel $R(\mathbf{x}-\mathbf{x}')$ can be expanded to second order near $\mathbf{x}'=0$

$$R(\mathbf{x}-\mathbf{x}')\approx R_0+\frac{1}{2}R_0''(\mathbf{x}-\mathbf{x}')^2 \tag{5}$$

where $R_0=R(0)$, R_0'' is the second-order derivative of the response function at the origin, and $\mathbf{x}$ is the transverse spatial coordinate vector. Since the response function takes its maximum value at the center, $R_0''<0$. By defining $\gamma=-R_0''>0$, γ is a positive quantity.[30] In this way, the second-order expanded response function shows a parabolic distribution with a peak at the center and a slow decrease in the transverse direction. Substituting Eq. (5) into the general nonlocal nonlinear Schrödinger equation, and assuming that the beam is axially symmetric with respect to the propagation direction, the approximate propagation equation under strongly nonlocal conditions can be obtained

$$i\frac{\partial\varphi}{\partial z}+\mu\nabla_\perp^2\varphi-\frac{1}{2}\rho\gamma P_0 r^2\varphi+\rho R_0 P_0\varphi-\frac{1}{2}\rho\gamma\varphi\int|\varphi(r',z)|^2\,r'^2d^D\mathrm{x}'=0 \tag{6}$$

Here, $\mu=1/(2k)$, $\rho=k\eta$, where η is a medium-related constant, P_0 is the input power, $r=|\mathbf{x}|$, and the spatial dimension is $D=1$ or $D=2$, corresponding to one-dimensional or two-dimensional cases. From Eq. (6), it can be seen that under

strongly nonlocal conditions, the original convolution-type nonlocal response can be approximately transformed into a parabolic effective potential proportional to r^2. The remaining terms mainly contribute to the overall phase change of the beam. More specifically, the third term in Eq. (6) represents the self-induced parabolic refractive-index potential near the beam center. This is the main mechanism that provides transverse confinement for the beam and maintains soliton propagation. The fourth and fifth terms mainly affect the phase of the beam, but do not change its transverse width or amplitude distribution. Furthermore, when the response width of the medium is large enough and the transverse size of the beam can be neglected, the fifth term can be further ignored. Then the equation becomes

$$i\frac{\partial \varphi}{\partial z}+\mu\nabla_{\perp}^{2}\varphi-\frac{1}{2}\rho\gamma P_0 r^2\varphi+\rho R_0 P_0\varphi=0 \tag{7}$$

By introducing the phase transformation $\varphi(r,z)=\psi(r,z)\exp(i\rho R_0 P_0 z)$, Eq. (7) can be further reduced to

$$i\frac{\partial \psi}{\partial z}+\mu\nabla_{\perp}^{2}\psi-\frac{1}{2}\rho\gamma P_0 r^2\psi=0 \tag{8}$$

Equation (8) shows that, under the strongly nonlocal approximation, the propagation behavior of a beam in a nonlocal medium is mathematically equivalent to that of a linear system confined by a parabolic potential.

In the strongly nonlocal limit, if the nonlocal response function satisfies the differentiability condition, the nonlocal nonlinear Schrödinger equation can be further reduced to the Snyder-Mitchell model. This means that, under the strongly nonlocal approximation, the beam propagation described by a differentiable response-function model is basically equivalent to that described by the Snyder-Mitchell model. However, the nonlinear response mechanisms in real media are very different, so the specific form of the response function $R(\boldsymbol{r})$ may also be different. For convenience in analysis, a Gaussian-type response function is often first used as a typical approximation

$$R(\boldsymbol{r}) = \frac{1}{\pi w_m^2} \exp\left(-\frac{|\boldsymbol{r}|^2}{w_m^2} \right) \tag{9}$$

Here, $R(\boldsymbol{r})$ satisfies the normalization condition $\int R(\boldsymbol{r}) \mathrm{d}\boldsymbol{r} = 1$, and can well describe the basic feature that the nonlocal response decreases with spatial distance.

2.3. Snyder-Mitchell model and its applicability

Under strongly nonlocal conditions, the response range of the medium to the optical field is much larger than the transverse size of the beam. Therefore, the refractive-index change is no longer determined only by the local optical intensity, but by the optical-field distribution of the beam over a larger spatial range. In this case, the spatial beam can be regarded as propagating in a refractive-index potential well induced by itself. The Snyder-Mitchell model is built on this theoretical basis. It transforms the beam propagation problem under strongly nonlocal conditions into a mode-evolution problem in a parabolic effective potential. In this way, it provides a clear theoretical framework for studying stable soliton states and related bound states.[14,30–38]

In the Snyder-Mitchell model, the square of the refractive index near the beam center can be approximately written as

$$n^2(r, P) = n_0^2 - r^2 \alpha^2(P) \tag{10}$$

where n_0 is the linear refractive index of the medium, r is the transverse radial distance, P is the total beam power, and $\alpha(P)$ is a power-dependent parameter. Equation (10) shows that, under strongly nonlocal conditions, the refractive-index distribution induced by the beam can be approximately regarded as a parabolic potential well near the beam center, and its curvature is determined by the beam power. By writing the optical field in the slowly varying envelope form $E = \varphi(r, z) e^{ikz}$, where $k = 2\pi n_0 / \lambda$ and λ is the vacuum wavelength, and substituting Eq. (10) into Eq. (1), one obtains

$$2ik \frac{\partial \varphi}{\partial z} + \nabla_\perp^2 \varphi - \frac{k^2 \alpha^2(P)}{n_0^2} r^2 \varphi = 0 \tag{11}$$

Equation (11) is the basic propagation equation of the Snyder-Mitchell model. It clearly shows that, under the strongly nonlocal approximation, beam propagation can be approximately regarded as an evolution process in a parabolic effective potential. To analyze the steady propagation of solitons, one can let $\varphi(r,z)=u(r)e^{i\beta z}$. Substituting this expression into Eq. (11), one obtains

$$\nabla_{\perp}^{2}u-\frac{k^{2}\alpha^{2}(P)}{n_{0}^{2}}r^{2}u-2k\beta u=0 \tag{12}$$

Equation (12) shows that, in the Snyder-Mitchell model, strongly nonlocal spatial optical solitons can be understood as localized solutions in a parabolic effective potential. Therefore, the Snyder-Mitchell model can give a unified description of fundamental modes and higher-order modes under strongly nonlocal conditions. This is one of its important theoretical values.

This parabolic effective potential also establishes a direct connection between the Snyder-Mitchell model and the quantum harmonic oscillator. Under the usual analogy between the paraxial wave equation and the Schrödinger equation, the propagation distance plays the role of time, and Eq. (12) has the same eigenvalue structure as the equation for a two-dimensional harmonic oscillator. Consequently, the parabolic Snyder-Mitchell potential acts as a continuous fractional Fourier transformer. The fractional order increases with propagation distance, and the input field periodically passes through its Fourier-transformed distribution and then revives. Symmetric modes can recover after a shorter period, whereas asymmetric or structured beams may exhibit inversion or rotation.[124–126]

It should be noted that the Snyder-Mitchell model is essentially an ideal approximation in the strongly nonlocal limit, and it is not applicable to all nonlocal media. Usually, the Snyder-Mitchell approximation requires four main conditions: the response function is smooth near its center; the response width is much larger than the beam width; boundary distortion is negligible in the beam region; and the induced index profile is approximately parabolic. In real nonlocal media, boundary conditions, material anisotropy, finite response width, and competing nonlinear effects may all

make the propagation behavior deviate from this ideal model. These ideal assumptions and their common deviations in real nonlocal media are summarized in Table 2. Therefore, the Snyder-Mitchell model is more suitable as a basic theory for understanding the propagation laws of strongly nonlocal spatial optical solitons, rather than as a universal accurate model.

Table 2. Ideal Snyder-Mitchell assumptions and common deviations in real media.

Ideal Snyder-Mitchell model	Real nonlocal media
Response width much larger than beam width	Finite response width can change the mode and critical power
Smooth, parabolic response near the center	Non-parabolic or oscillatory kernels modify trajectories and stability
Negligible boundary distortion near the beam	Finite samples shift the potential center and bend the beam
Isotropic, single focusing mechanism	Anisotropy, loss, saturation, and competing nonlinearities may coexist

3. Spatial optical solitons in typical nonlocal media systems

In the above theoretical framework, nonlocal response is usually described by the response function, the strongly nonlocal approximation, and the effective-potential model. However, in real media, nonlocality is also affected by material properties, boundary conditions, and external-field control methods. Therefore, it is necessary to discuss typical material systems. Liquid crystals, photorefractive crystals, and thermal nonlinear media correspond to different nonlocal mechanisms, including molecular reorientation, charge-carrier transport, and thermal diffusion. They provide important platforms for studying the formation, stability, and control of nonlocal spatial optical solitons.

3.1. Spatial optical solitons in liquid-crystal media

Liquid crystal is one of the typical media for studying nonlocal spatial optical solitons, and their nonlinear response mainly comes from the molecular reorientation process. This response usually shows clear spatial nonlocality, anisotropy, and surface

anchoring effects. Therefore, under certain conditions, spatial optical solitons in liquid crystals can show propagation features close to those predicted by the strongly nonlocal theory. However, because the real response mechanism is more complex, especially when reorientational and thermal nonlinearities coexist or compete, their propagation behavior often deviates from the ideal Snyder-Mitchell model.[9] Thus, liquid-crystal media are often used not only to study the basic laws of nonlocal spatial optical solitons, but also to understand whether ideal theoretical models can still apply in real media and what practical factors may influence them. In recent years, research on spatial optical solitons in liquid-crystal media has no longer been limited to the formation and stable propagation of nematicons. It has further extended to areas such as the control of propagation trajectories, self-trapped propagation of structured light beams, and the manipulation of complex localized waves.

3.1.1. Self-localized propagation of structured light beams in liquid crystals

The formation of spatial optical solitons in nematic liquid crystals usually depends on the nonlinear response caused by molecular reorientation. For an extraordinary-polarized beam, the optical field can change the average orientation of nematic liquid-crystal molecules through optical torque, thus changing the refractive index distribution. When the self-focusing effect produced by this refractive index change balances the diffraction of the beam, the beam can form a stable self-trapped propagation state, namely a nematicon.[39] Compared with spatial optical solitons in

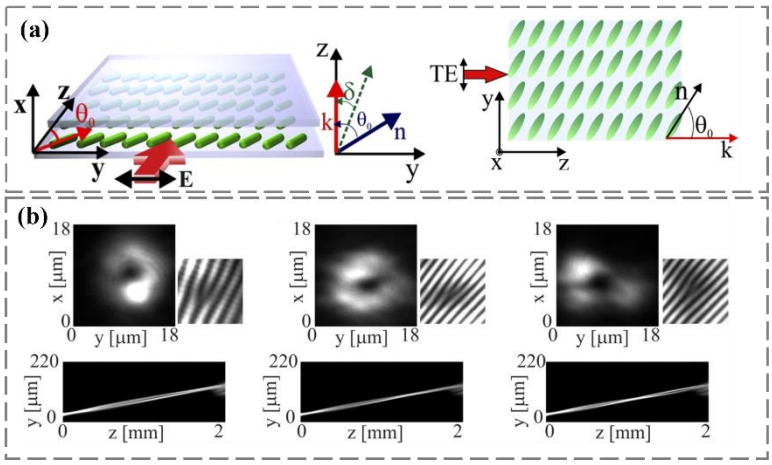


Fig. 2. Generation and stable propagation of vortex nematicons in nematic liquid crystals. (a) Schematic diagram of the nematic liquid-crystal sample and the excitation of the optical beam; (b) output intensity patterns, interferograms, and propagation evolution of vortex nematicons in nematic liquid crystals under different input powers. Reproduced with permission from Ref.

local Kerr media, an important feature of nematicons is that their nonlinear response has clear spatial nonlocality. The reorientation of nematic liquid-crystal molecules occurs not only in the region with high optical intensity, but also extends to the surrounding region through elastic interactions. Therefore, the refractive index distribution induced by the beam usually has a wider spatial range than the optical intensity distribution. This broad refractive index response helps improve the stability of two-dimensional localized waves and provides a physical basis for phenomena such as long-range interactions and optically induced waveguides.

In recent years, research on nematicons in nematic liquid crystals has gradually moved from the self-trapped propagation of traditional Gaussian beams to structured

light beams carrying orbital angular momentum. Laudyn et al. showed that stable vortex solitons can also be generated in nematic liquid crystals without applying an external electric or magnetic field, as shown in Fig. 2(a). In this study, an extraordinary-polarized vortex beam was launched into an undoped nematic liquid crystal. The results showed that the nonlocal response of the nematic liquid crystal could suppress the splitting effect that is commonly seen for vortex beams in self-focusing media. As a result, the ring-shaped beam carrying a topological charge could keep stable self-trapped propagation, as shown in Fig. 2(b). The study also showed that the stability of vortex solitons is related not only to the nonlocal response, but also to factors such as boundary conditions, beam size, and material birefringence.

In addition to molecular reorientation nonlinearity, photothermal nonlinearity in

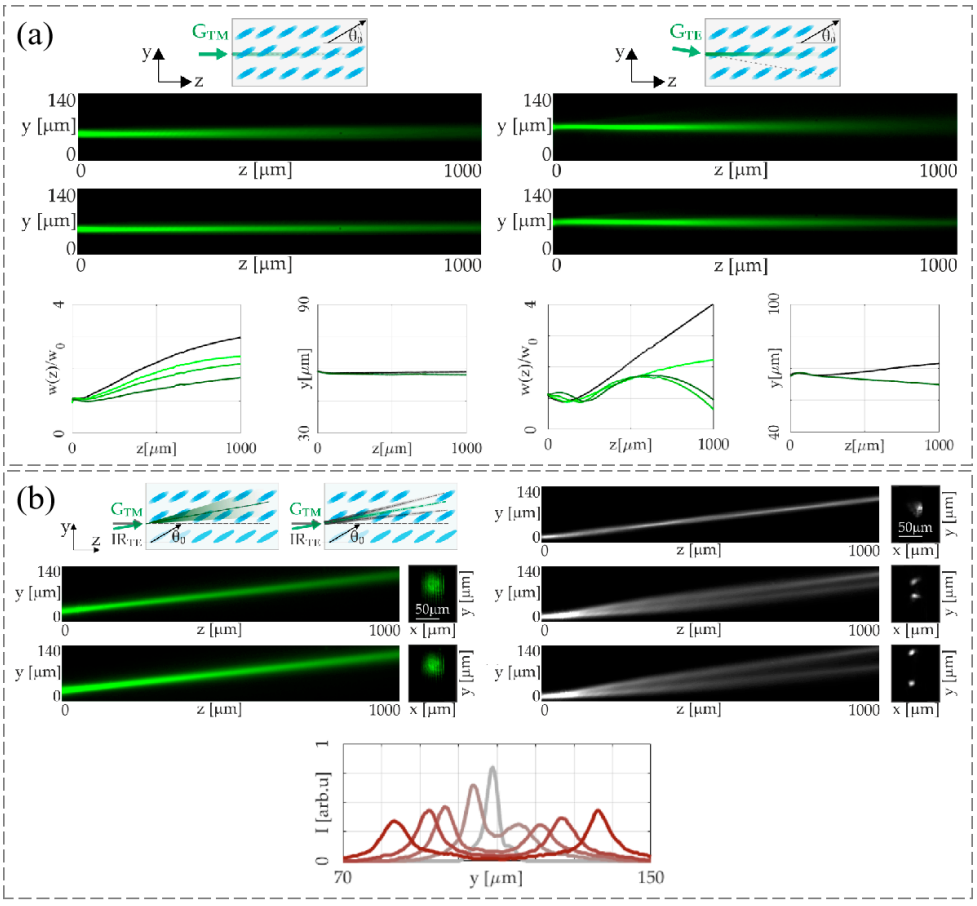


Fig. 3. Spatial self-localized propagation involving photothermal nonlinearity in dye-doped nematic liquid crystals. (a) Power-dependent propagation of a 532nm beam in Sudan Blue doped 6CHBT nematic liquid crystal. The left and right parts correspond to the coupling geometry, low-power and high-power propagation images, and the changes in beam width and propagation trajectory with propagation distance under TM-polarized and TE-polarized conditions, respectively; (b) co-propagation of an infrared TE beam and a green visible TM beam in dye-doped nematic liquid crystal. Reproduced with permission from Ref. [42].

dye-doped nematic liquid crystals can also produce spatially nonlocal refractive-index modulation and support the stable formation of vortex solitons. Kwaśny et al. realized the self-trapped propagation of vortex beams in dye-doped nematic liquid crystals by using photothermal nonlocal nonlinearity, and stable vortex solitons were formed.[41] Figure 3 shows the influence of photothermal nonlinearity in dye-doped nematic liquid crystals on spatial self-localized propagation and soliton interactions. In Fig. 3(a), for the TM-polarized beam, the molecular reorientation effect is weak, and the propagation is mainly dominated by the thermal response. As the input power increases, the beam width becomes clearly smaller, while the change in the propagation trajectory is small. This indicates that thermally induced refractive-index modulation mainly leads to beam compression. In comparison, the TE-polarized beam is affected by both the reorientation effect and the thermal effect. When the power increases, it not only shows a self-localization tendency, but also has an obvious transverse shift. Figure 3(b) further shows the soliton interaction when two nonlinear mechanisms act together. In the experiment, the infrared TE beam forms a reorientation-based soliton, while the green TM beam forms a thermal soliton because of stronger absorption. At low power, the thermal effect is weak, and the propagation of the infrared soliton is almost unchanged. As the power of the green beam increases, the temperature field becomes stronger, changing the effective refractive index and the walk-off angle of the infrared beam. In the reported geometry, the absorbed visible TM beam mainly writes a thermal lens and produces beam compression, whereas the infrared TE beam is governed mainly by positive reorientational self-focusing. For the extraordinary TE component in the chosen alignment, heating gives a negative index contribution that competes with reorientation and changes the walk-off; weak competition gives a transverse shift, while stronger imbalance can split the guided soliton.

3.1.2. Soliton interactions and propagation-path control in nematic liquid crystals

In nematic liquid crystals, the nonlocal response can not only support stable

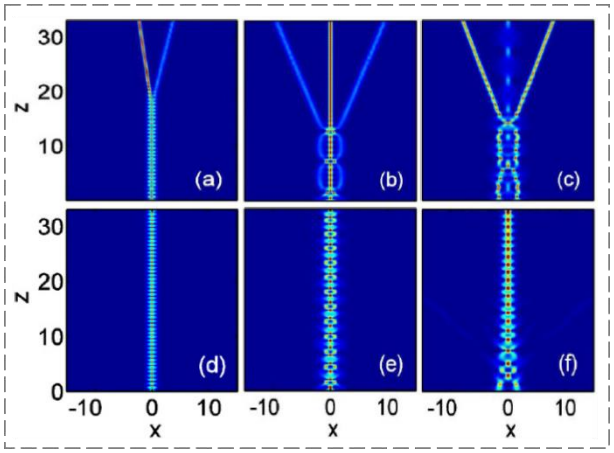


Fig. 4. Trajectories of interacting Airy beams and Gaussian beams in nematic liquid crystals. Reproduced with permission from Ref. [44].

spatial optical solitons, but also influence the attraction and repulsion between several light beams. Compared with local nonlinear media, the molecular reorientation response in nematic liquid crystals has a larger spatial action range. The refractive-index change induced by a single beam can extend into the propagation region of nearby beams. As a result, it can change the refractive-index distribution around these beams and lead to long-range interactions between them.[43] Therefore, soliton interactions in nematic liquid crystals are determined not only by the near-field intensity distribution between the beams, but also by factors such as beam separation, phase difference, nonlocal response strength, and externally applied control light fields.

Shen et al. studied the anomalous interactions between Airy beams in nematic liquid crystals through numerical simulations. They also proposed a method to control the interactions between Airy beams by using a fundamental Gaussian beam, as shown in Fig. 4. This study showed that, in nematic liquid crystals, the attraction between two in-phase Airy beams can be weakened by introducing an in-phase Gaussian beam, and can even be changed into repulsion. When the beam separation is large, an out-of-phase Gaussian beam can enhance the attraction between two Airy beams. Under the action of the fundamental Gaussian beam, the optical field distribution between the two Airy beams changes. This further changes the nonlocal

refractive-index modulation in the nematic liquid crystal, and thus realizes the control of Airy beam interactions.

Besides controlling soliton interactions through auxiliary light fields, the spatial modulation of liquid-crystal molecular orientation also provides an effective way to control the propagation paths of nematicons.[45,46] Zhang et al. designed a liquid-crystal cell with a comb-shaped electrode structure. Under an applied electric

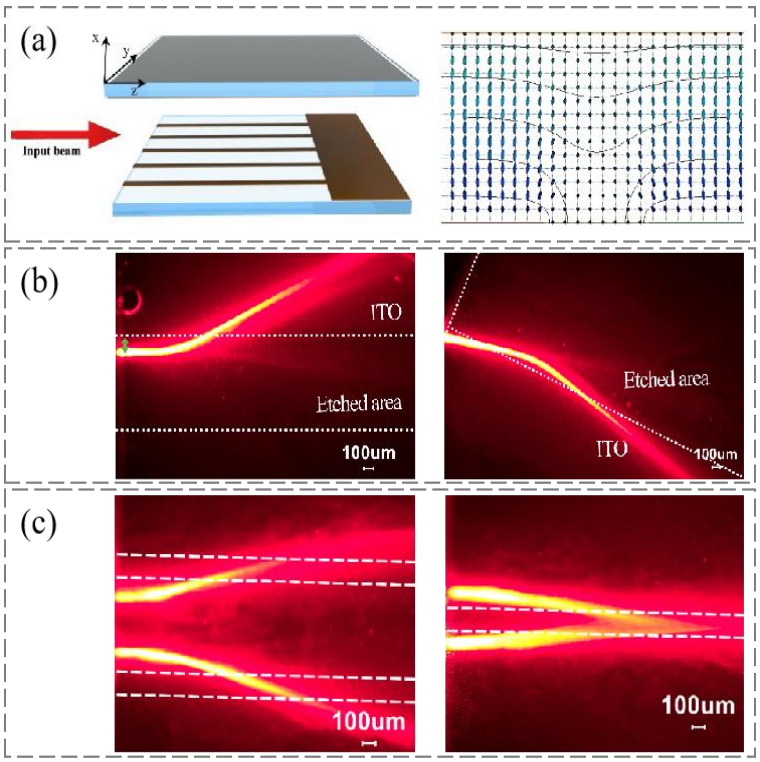


Fig. 5. Control of nematicon propagation paths by liquid-crystal molecular orientation arrays. (a) Structure of the liquid-crystal cell with comb-shaped electrodes and the liquid-crystal molecular orientation array formed under an applied voltage; (b) deflection of a single nematicon in different orientation regions and its reflection behavior at the orientation-discontinuity interface; (c) different deflection combinations of nematicon pairs in the orientation array, including outward deflection of two beams and inward deflection of two beams. Reproduced with permission from Ref. [47].

field, a programmable array of liquid-crystal molecular orientations was formed, as shown in Fig. 5(a). Since different orientation regions correspond to different effective refractive-index distributions, the incident beam can be deflected at different angles in

different regions. This study further showed that the reflection behavior of the beam at the abrupt interface of liquid-crystal molecular orientation can be controlled by changing the incident angle, as shown in Fig. 5(b). For nematicon pairs, different incident positions and liquid-crystal orientation regions can lead to different deflection combinations, including single-beam deflection, outward deflection of two beams, and inward deflection of two beams.

In the study of spatial optical solitons in nematic liquid crystals, Gaussian beams are the most common research object. Under the nonlinear effect caused by molecular reorientation, they can form nematicons. In recent years, the research focus has gradually extended from this problem to more general structured light beams. For example, an experimental study in 2024 on the propagation of Airy beams in nematic liquid crystals showed that finite-energy Airy beams can undergo a continuous evolution process as the nonlinear effect increases. They can evolve from self-accelerating propagation to partial localization, and finally to off-shooting solitons (OSS). At the same time, the propagation direction of OSS is influenced by the optical power and the walk-off effect of extraordinary light. These examples show that nonlocal media can give complex wavefronts new localized behaviors. At the same time, the research focus has gradually shifted from “whether solitons can be formed” to “how their propagation paths can be controlled”. The programmable distribution of liquid-crystal molecular orientation can realize the deflection of nematicons, reflection at interfaces, and the control of multiple-beam combinations, such as single-beam deflection, outward deflection of two beams, and inward deflection of two beams. In this way, the controllability of propagation forms is greatly enriched.

On this basis, the study of localized structures has further extended to broader director solitary structures. These structures can not only exist stably, but also be used to build switchable diffraction patterns and functional optical elements. This extends the study of solitons to the level of device applications. More broadly, patterned liquid-crystal director fields and topological defects can also steer optical beams and even mediate interactions between optical solitons and topological solitonic structures,

further expanding the concept of soliton control from beam trajectory manipulation to light-matter co-patterning.[16,48] Overall, these studies together outline a clear picture of spatial optical solitons in nematic liquid crystals. This field is developing along several connected directions, including the expansion of self-localized objects, enhanced control of propagation, functional localized structures, and programmable dynamics.

3.2. Spatial optical solitons in photorefractive crystals

Photorefractive crystals are another classical type of medium in the study of nonlocal spatial optical solitons. Different from liquid crystals, whose nonlocality is usually related to strong nonlocal response caused by molecular reorientation, photorefractive crystals more often involve the modulation of charge-carrier transport and the space-charge field inside the crystal by the optical field. Under nonuniform light illumination, electrons or holes in the impurity energy levels can be excited to the conduction band or the valence band. Then, under the action of an applied electric field, a photovoltaic field, and a concentration gradient, these carriers drift and diffuse. They are then retrapped by traps in dark regions. In this way, a space-charge field related to the optical intensity distribution is formed. This field further changes the refractive index of the crystal through the linear or quadratic electro-optic effect. In a typical case, it can be written as:

$$\Delta n = -\frac{1}{2} n^3 r_{\mathrm{eff}} E_{\mathrm{sc}} \tag{13}$$

where r_{eff} is the effective electro-optic coefficient, and E_{sc} is the space-charge field.[49] Since the carriers can move in the transverse direction, the refractive-index change at one point is determined not only by the local light intensity, but also by the nearby optical field, crystal anisotropy, and boundary conditions. Therefore, the photorefractive response has nonlocality, anisotropy, saturation, and a non-instantaneous nature. Because the space-charge field can be formed in different ways, and because the crystal properties and the spatial structure of the input beam can also be different, photorefractive self-localization shows many forms. On the one

hand, different charge-transport mechanisms can produce screening solitons, photovoltaic solitons, and composite solitons. They can also combine with artificial photonic lattices to form gap states and topological localized states. On the other hand, structured light beams with special amplitude, phase, or orbital angular momentum distributions can show energy redistribution, trajectory control, and multi-channel self-trapping behavior under photorefractive nonlinearity. These behaviors are different from those of ordinary Gaussian beams.

3.2.1. Formation types and control mechanisms of solitons in photorefractive crystals

According to the main source of the space-charge field, photorefractive solitons are usually divided into screening solitons, photovoltaic solitons, and composite solitons. Screening solitons are often formed in biased crystals such as SBN and $BaTiO_3$. After photogenerated carriers are redistributed, they produce nonuniform screening of the external electric field. As a result, a saturated refractive-index potential well related to the optical intensity is formed. When nonlinear self-focusing just balances diffraction, bright screening solitons can be generated. Under self-defocusing conditions or in a finite background optical field, dark solitons and gray solitons can be formed. The direction of the applied electric field, the dark irradiance, and waveguide confinement can strongly change their threshold power, transverse size, and stability.[50] Photovoltaic solitons mainly appear in crystals with the bulk photovoltaic effect, such as $LiNbO_3$ and $LiTaO_3$. Their space-charge field can be directly built by the photovoltaic current, so they do not necessarily depend on an applied bias. If the external field, the photovoltaic field, and even the pyroelectric field caused by temperature changes exist at the same time, screening–photovoltaic–pyroelectric composite solitons can be formed. The bright and dark branches of the solitons, as well as their self-bending direction, can be jointly controlled by different electric-field components.[51]

Besides the above classification, photorefractive solitons can also be divided, according to their field distribution and coherence properties, into scalar solitons, vector solitons, partially coherent solitons, and discrete or gap solitons. To provide a

clear overview of these different forms, Table 3 summarizes the main types of photorefractive solitons and their defining features. The slow time scale of the photorefractive response allows beams with different frequencies or mutually incoherent beams to couple through a jointly written refractive-index potential. However, the diffusion current can cause a displacement between the refractive-index potential and the optical-intensity peak, leading to soliton self-bending and time-dependent drift. Therefore, in recent years, the research focus has gradually shifted from simply realizing self-trapping to the active control of localization position, stability, and functional structures. In particular, optically induced photorefractive lattices have recently provided a flexible platform for nonlinear topological photonics, where topological edge states, corner states, and soliton-like localized states can be tuned by photorefractive nonlinearity.[52–54] Besides, Tari et al. used the charge-clamping effect on the electrode surface to suppress the transverse motion of photorefractive solitons, providing a new method for stable self-written waveguides.[55]

Table 3. Concise classification of photorefractive spatial solitons.

Soliton type	Main physical mechanism	Typical material	External bias	Main control parameter
Screening	Screening of an applied field	SBN, $BaTiO_3$	Usually yes	Bias field and dark irradiance
Photovoltaic	Bulk photovoltaic field	$LiNbO_3$, $LiTaO_3$	Not necessary	Photovoltaic current and illumination
Composite	Screening plus photovoltaic or pyroelectric fields	$LiNbO_3$-type crystals	Optional	Relative field strengths and temperature
Scalar / vector	One field or mutually coupled components	SBN and related crystals	System dependent	Polarization, coherence, and power ratio
Partially coherent	Slow response to time-averaged intensity	SBN	Often yes	Coherence length and background

Soliton type	Main physical mechanism	Typical material	External bias	Main control parameter
Discrete / gap	Localization in an induced photonic lattice	SBN lattice	Often yes	Lattice depth, detuning, and input position

3.2.2. Self-localized propagation of structured light beams in photorefractive crystals

The input beam of traditional photorefractive spatial solitons is usually a Gaussian beam, whose transverse intensity has only one main scale. Under

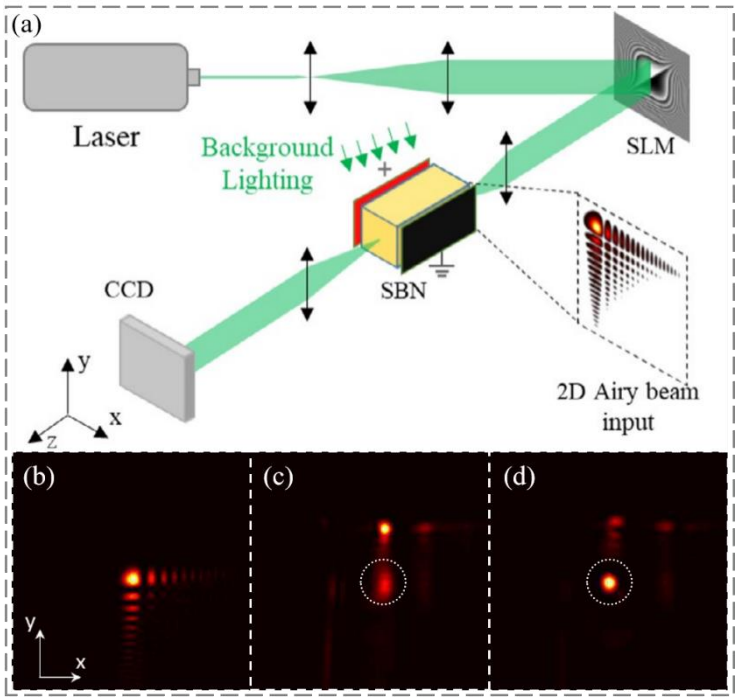


Fig. 6. Self-localized propagation of two-dimensional Airy beams in photorefractive SBN crystals. (a) Schematic diagram of the experimental setup; (b) transverse intensity distribution of the two-dimensional Airy beam at an input power of 2.8μW; (c) output intensity distribution under an applied electric field of 800V/cm, where a localized bright spot and residual Airy side-lobe structure coexist; (d) when the applied electric field is increased to 1200V/cm, the beam energy is further concentrated into the localized bright spot, showing clearer two-dimensional soliton-like features. The white dashed circles indicate the soliton formation positions. Reproduced with permission from Ref. [59].

self-focusing conditions, the Gaussian beam forms an approximately single-channel refractive-index potential. After the soliton condition is reached, its beam width and

peak position remain basically stable. In contrast, Airy beams, Bessel beams, and vortex beams have more complex amplitude and phase structures. Their self-localization does not simply appear as an unchanged overall beam width. Instead, it is often accompanied by side-lobe energy exchange, trajectory deflection, topological-charge evolution, and the formation of multi-channel refractive-index structures.

Finite-energy Airy beams have an asymmetric multi-side-lobe structure and transverse self-accelerating properties. Under weak photorefractive nonlinearity, their main lobe can be compressed while still keeping a curved trajectory. When the applied bias and the self-focusing strength increase, part of the energy in the main lobe and nearby side lobes can separate from the accelerating wave packet and form an “off-axis soliton” that propagates almost in a straight line. The remaining energy continues to propagate as an Airy-type wave packet.[56,57] This process is different from the overall contraction of a Gaussian beam into a central soliton. Instead, it shows selective energy redistribution among different spatial-frequency components inside a structured light beam. In addition, two counter-propagating Airy beams can also interact with each other under a large transverse mismatch. By adjusting the side-lobe size and the input displacement, the output position of the soliton can be shifted. Its effective interaction range is much larger than that of counter-propagating Gaussian beams.[58] Furthermore, the transition from a two-dimensional Airy beam to a two-dimensional soliton state has been experimentally observed in an SBN:61 crystal, as shown in Fig. 6. The researchers also showed that the soliton width and output position can be controlled only by tuning the external electric field, without changing the input beam shape,[59] as shown in Figs. 6(c) and 6(d).

A Bessel beam consists of a central bright spot and concentric rings. Its outer rings can continuously transfer energy to the central region, so it has nearly nondiffracting and self-healing properties. Flammini et al. observed the self-trapping of Bessel beams in photorefractive KLTN crystals. Because energy is continuously exchanged between the ring-shaped energy reservoir and the diffracting wave, the obtained nonlinear localized state shows periodic breathing along the propagation

direction, rather than the nearly steady propagation of a Gaussian soliton.[60] Beyond self-trapping, Bessel beams can also be used to optically induce long and complex waveguiding channels in photorefractive media, including multichannel and three-dimensional reconfigurable waveguide structures.[61–63]

Vortex beams carrying orbital angular momentum have a helical phase and a central phase singularity. In a uniform self-focusing medium, ring-shaped beams are easily affected by azimuthal modulation instability and split into several bright spots. Therefore, their stable self-localization is usually more difficult than that of Gaussian beams and Airy main lobes. The saturable nonlinearity, nonlocal diffusion, and externally induced photonic lattices in photorefractive crystals can provide extra confinement for vortex beams and make photorefractive platforms useful for studying nonlinear topological localization.[52,54] Recent work realized an "optical vortex ladder" based on synthetic spin pumping in a honeycomb photonic lattice in photorefractive SBN crystals. In this system, the topological charge of the beam can be converted step by step, and high-order vortex output with a Bessel-like transverse profile can be generated.[64] Although this result focuses more on the generation and control of topological charge than on vortex solitons in a uniform medium in the traditional sense, it shows that the combination of photorefractive nonlinearity and structured lattices can control the spatial localization, propagation trajectory, and orbital angular momentum of light beams at the same time. This provides a new way to realize stable high-order vortex states and reconfigurable OAM optical paths.

3.3. Spatial optical solitons in thermal nonlinear media

Thermal nonlinearity usually comes from the absorption of the propagating light by the medium. The absorbed optical energy is converted into heat, which increases the temperature in the illuminated region. The heat then spreads to the surrounding region through transverse thermal diffusion, and causes a refractive-index change through the thermo-optic effect. The refractive-index change can be written as:

$$\Delta n = \frac{\mathrm{d}n}{\mathrm{d}T} \times \Delta T \tag{14}$$

where $\mathrm{d}n/\mathrm{d}T$ is the thermo-optic coefficient of the medium, and ΔT is the temperature increase relative to the ambient temperature.[65] When the thermo-optic coefficient is positive, the refractive index in the heated region increases, and the medium usually shows thermally induced self-focusing. When the thermo-optic coefficient is negative, the medium mainly shows thermally induced self-defocusing.

Under the steady-state approximation, the transverse temperature field satisfies the thermal diffusion equation:

$$\kappa \nabla_{\perp}^{2} T = -\alpha I \tag{15}$$

where κ is the thermal conductivity of the medium, α is the optical absorption coefficient, I is the optical intensity, and $\nabla_{\perp}^{2}$ denotes the transverse Laplacian operator.[68,78] Since heat can diffuse from the beam center to a region much wider than the beam itself, the temperature and refractive-index change at one point are determined not only by the local optical intensity, but also by the whole intensity distribution, the sample size, and the boundary heat-dissipation conditions. Therefore, thermal nonlinearity usually has strong spatial nonlocality and a non-instantaneous temporal response.[66] In materials with a positive thermo-optic coefficient, such as lead glass, the heated central region can form a broad self-focusing refractive-index potential, providing conditions for the generation of spatial optical solitons. In media with a negative thermo-optic coefficient, thermally induced self-defocusing and the corresponding dark localized structures mainly appear.

Thermal nonlocality is distinct from the other two mechanisms. Its response is governed by an elliptic heat-diffusion equation and a boundary Green function, so the response range may approach the sample size and remains strongly boundary dependent. Liquid-crystal nonlocality instead follows elastic director reorientation and is anisotropic, whereas the photorefractive response follows carrier drift, diffusion, trapping, and a saturable space-charge field. Thermal media also respond more slowly and usually show stronger memory than the other platforms.

3.3.1. Thermal nonlocal response and basic types of spatial optical solitons

The formation of spatial optical solitons in thermal nonlinear media results from the balance between diffraction and thermally induced refractive-index modulation. Under self-focusing conditions, a local beam can produce a broad temperature distribution and a refractive-index potential well. When the focusing effect of this potential well just compensates for diffraction, fundamental bright solitons can be formed. In self-defocusing media and in the presence of a finite background optical field, dark solitons or gray solitons can be formed. According to different transverse field structures and polarization components, thermal nonlinear media can also support multipole solitons, vortex solitons, vector solitons, elliptical solitons, and periodically breathing soliton states.[67,68]

When the scale of the thermal response is much larger than the beam width, the medium enters the strongly nonlocal regime. Near the beam, the broad thermally induced refractive-index distribution can be approximately expanded into a parabolic potential. As a result, the beam propagation becomes similar to the eigenmodes of an

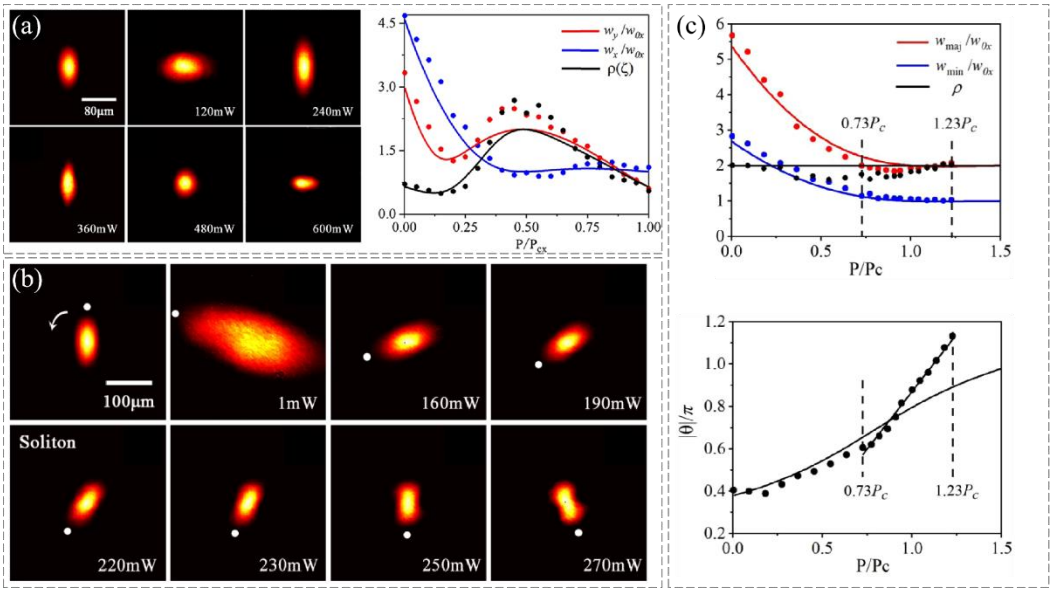


Fig. 7. Breathing propagation of elliptical beams and formation of rotating elliptical solitons in thermal nonlocal media. (a) Propagation results of elliptical beams without orbital angular momentum under different input powers, and the dependence of the semi-axis lengths and ellipticity of the output beam on the normalized input power. The solid lines are theoretical values, and the dotted data points are experimental results; (b) propagation results of elliptical beams carrying critical orbital angular momentum under different input powers. The white arrows indicate the rotation direction, and the white dots mark the long and short axes of the helical elliptical beam; (c) dependence of the semi-axis lengths, ellipticity, and rotation angle of the elliptical beam on the normalized input power. Reproduced with permission from Ref.

oscillator. This approximation is related to the Snyder-Mitchell model and can explain the self-trapping of fundamental modes, higher-order modes, and breathing states. However, in real thermal media, the curvature of the potential well, the central position, and the critical power for soliton formation are still determined by the sample size and the thermal boundary conditions.[69–71] Therefore, strong nonlocality does not mean that the soliton properties are completely independent of boundaries. Instead, the boundaries influence the localized beam indirectly through the whole temperature field.

In recent years, the research focus has gradually expanded from fundamental bright solitons to more complex spatial and polarization structures. Experiments have observed high-topological-charge vortex solitons and vector vortex solitons in lead glass with strong thermal nonlocality.[72,73] Cylindrical-vector vortex beams with radial or azimuthal polarization can also form stable solitons. Their ring-shaped intensity profile, polarization distribution, and topological charge can remain stable during propagation.[74] When the input power or the beam width deviates from the strict soliton condition, the beam often shows periodic breathing. The incoherent coupling between the Gaussian component and the vortex component can form vector vortex breathers, and can further transform into stable vector solitons within certain power-ratio ranges.[75,76] Elliptical beams can show beam-width oscillation and rotation in a nonlocal potential well,[77] as shown in Fig. 7. For elliptical beams without orbital angular momentum, thermally induced self-focusing can compress the beam. However, because diffraction and focusing in the two transverse directions cannot reach balance at the same time, the beam mainly shows breathing propagation, in which the semi-axis lengths and ellipticity change with power, as shown in Fig. 7(a). In comparison, when an elliptical beam carries suitable orbital angular momentum, the rotation effect can act together with thermally induced self-focusing and diffraction. In this case, the beam can keep an elliptical profile within a certain power range and form a rotating elliptical soliton, as shown in Fig. 7(b). Figure 7(c) shows that near the critical power, the two semi-axis lengths remain almost stable, the change in ellipticity is small, and the rotation angle increases with the input power.

This indicates that the shape and rotation properties of rotating elliptical solitons can be controlled by adjusting the spatial structure, orbital angular momentum, and power of the input beam.

3.3.2. Control of thermal solitons by long-range interactions and boundary conditions

An important feature of thermal nonlinearity is that its thermal response scale is much larger than the beam size. Each soliton not only forms a refractive-index potential near itself, but also produces a slowly varying temperature field in a wider region. Therefore, two solitons that do not overlap in the transverse plane can still interact with each other through the thermal field. Different from the short-range interaction in local media, which depends on the overlap of the beam tails, the interaction force in thermal nonlocal media can cross several beam widths. It is also effective for mutually incoherent or orthogonally polarized beams. In lead glass and other thermal nonlinear media, the interaction between two solitons can be described by an effective model similar to Newtonian gravity.[6,78] The soliton power acts like an effective "mass", and the broad thermally induced refractive-index distribution acts like a long-range potential field. As a result, two solitons can show mutual attraction, orbital motion, spiral propagation, or collision.[69]

This gravity-like description makes thermal media an important optical platform for studying long-range coupled wave packets. Under strongly nonlocal conditions, several solitons can share one overall refractive-index potential. Therefore, their motion is no longer a set of independent pairwise collisions, but may form rotating bound states, soliton clusters, and collective oscillations. Studies based on the Schrödinger–Poisson equations further show that the long-range potential field can not only confine coherent solitons, but may also make a random-wave background self-organize into large-scale incoherent localized structures.[78] In these structures, "hidden coherent solitons" that are difficult to identify directly from ordinary intensity images can be trapped.[79] At present, this result is mainly a prediction of the long-range nonlocal model, but its equation form is closely related to highly nonlocal thermal optical systems. This provides a theoretical direction for studying

self-organization and soliton generation of incoherent light in thermal media.

Boundary conditions are another key factor that determines the motion of thermal solitons. The temperature field must satisfy constant-temperature, thermally insulated, or convective heat-transfer conditions at the sample surface. Therefore, the thermally induced refractive-index distribution is essentially the combined result of

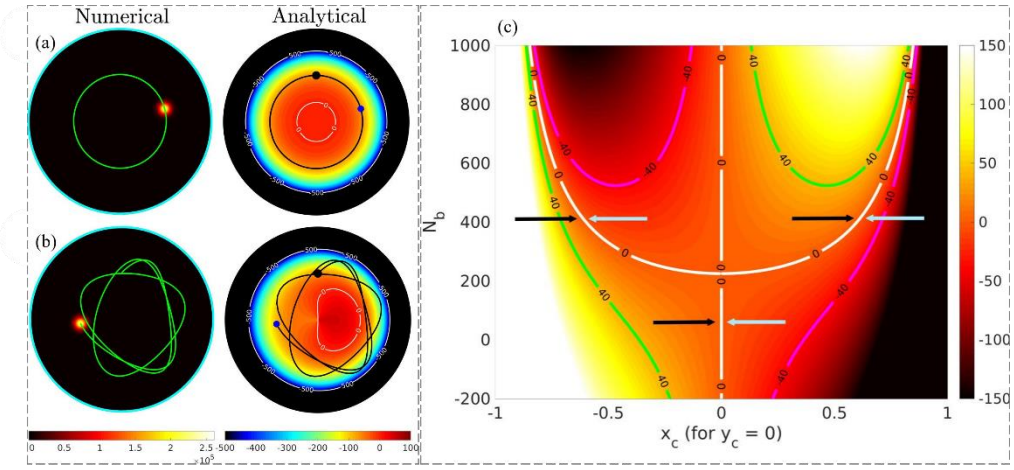


Fig. 8. Control of soliton trajectories and equilibrium positions by boundary conditions in finite cylindrical nonlocal media. (a) Under circularly symmetric constant-value boundary conditions, the soliton forms an almost circular trajectory. The left and right panels show the full numerical propagation result and the analytical model result, respectively; (b) after azimuthal modulation is added to the boundary condition, the effective force field loses circular symmetry, and the soliton equilibrium region and propagation trajectory are shifted and deformed; (c) as the constant-value boundary parameter increases, the central equilibrium point undergoes a supercritical pitchfork bifurcation, and two new stable equilibrium positions are produced. Reproduced with permission from Ref. [71].

the beam distribution and the boundary Green's function. Even when the beam is far from the boundary, the boundary can still exert an effective transverse force on the soliton by changing the whole temperature gradient. When the boundary condition exceeds a certain threshold, the central equilibrium position may undergo a supercritical pitchfork bifurcation, producing two new stable propagation positions,[71] as shown in Fig. 8(c). Under circularly symmetric constant-value boundary conditions, the soliton forms an almost circular orbit. When azimuthal modulation is added to the boundary, the circular symmetry is broken, and the propagation trajectory of the

soliton is clearly shifted and deformed, as shown in Figs. 8(a) and 8(b). The boundary can also change the stability of higher-order solitons. Reducing the transverse size of the sample usually strengthens the boundary confinement and makes the thermally induced potential well steeper. However, overly strong confinement may also cause mode reorganization in higher-order vector vortex states. By continuously tuning the effective transverse size of the sample, one can control the transition of higher-order solitons among stable propagation, periodic breathing, and azimuthal breakup.[70] Therefore, the stable state and position of a soliton are determined not only by the input conditions, but also by the sample size, thermal conductivity, and boundary conditions.

In addition, in an oscillatory strongly nonlocal response model related to boundary confinement, the interaction between two solitons can switch periodically among attraction, repulsion, and parallel propagation as the initial separation changes.[80] This result shows that by designing the boundary shape, segmented heat dissipation, or composite thermal structures, it may be possible to reshape the nonlocal response function. In this way, the direction of the soliton interaction force and the binding distance can be actively controlled. Overall, in thermal nonlocal media, boundaries are not only passive limits to soliton propagation. They are also important degrees of freedom for controlling the refractive-index potential, long-range interactions, and stable trajectories.[6,67,68]

Experimental realization has been demonstrated for several central states discussed above: fundamental and vortex nematicons in liquid crystals,[40,41] Airy-derived and Bessel self-trapped states in photorefractive crystals,[59,60] and vortex, vector-vortex, and rotating elliptic solitons in thermal media.[72–74,77] The main experimental tests are a nearly invariant transverse profile, a power-dependent localization threshold, and reproducible propagation over several diffraction lengths. Quantitative comparison with theory must also include absorption, response time, sample boundaries, walk-off, and input noise; these effects explain why some higher-order states remain numerical predictions or short-lived experimental states.

4. Long-range interactions, new localized wave structures, and controllable transport of nonlocal spatial optical solitons

Section 3 discussed three typical nonlocal nonlinear media, including liquid crystals, photorefractive crystals, and thermal nonlinear media. It focused on the formation mechanisms, propagation features, and material dependence of spatial optical solitons in these systems. Based on the study of typical media, it is necessary to further examine the long-range interactions, new localized wave structures, and controllable transport properties of nonlocal spatial optical solitons. The following chapter will discuss these issues, so as to reveal the deeper role of nonlocality in soliton dynamics and potential applications.

4.1. Long-range interactions of nonlocal solitons

Long-range interaction is a defining feature that differentiates nonlocal media from local media. In local Kerr media, the interaction strength between two spatial solitons decreases exponentially with their separation. Clear attraction or repulsion usually appears only when the solitons strongly overlap, normally within about two to three beam widths. The interaction also strongly depends on their relative phase: in-phase solitons attract each other, while out-of-phase solitons repel each other, usually with obvious energy exchange.[81]

The response mechanism in nonlocal media is quite different. The refractive-index change is determined by the convolution of the optical intensity distribution and the medium response function. Because of this convolution relation, the refractive-index modulation induced by a single soliton is not limited to the beam cross section, but extends outward over a characteristic length w_{m}, forming an extended refractive-index channel. When the inter-soliton separation d falls below the characteristic scale w_{m}, they can still interact through this channel even if their optical fields do not overlap. This constitutes the physical origin of long-range interactions.[14]

The experimental observation of this phenomenon can be traced back to 2006, when Rotschild et al. reported a milestone demonstration in thermal nonlinear lead

glass. They used a 488nm laser with a beam width of about 60μm and set the initial separation between two solitons to 600μm. At this distance, the field overlap between the two beams could be neglected. However, at the output, the soliton separation decreased to 514μm. In addition, the tangential velocity did not show a simple monotonic relation with the separation. These results indicated that the interaction did not originate from the remaining local overlap, but from the long-range modulational effect of the nonlocal refractive-index channel.[6] This result was consistent with earlier theoretical predictions in liquid crystals. Later experiments further confirmed that the degree of nonlocality can be continuously tuned by an external field, which provides a basis for controlling long-range interactions.[15]

Nematic liquid crystals are typical nonlocal platforms. Their response function is usually exponential, and the nonlocal length w_{m} is closely related to the pretilt angle. By applying a bias voltage to vary the pretilt angle of nematic liquid crystal molecules, researchers realized continuous control of the degree of nonlocality. They also observed the switching between attraction and repulsion for in-phase and out-of-phase soliton pairs. In this way, the relation between the degree of nonlocality and the sign or strength of interaction was directly established as an experimentally controllable parameter.[82]

Theoretically, the Snyder-Mitchell model in the strongly nonlocal limit provides a simple framework for understanding long-range interactions. When $w_{\mathrm{m}} \gg w_0$, where w_0 is the soliton beam width, and w_{m} is the nonlocal characteristic lengththe convolution can be expanded to second order, and the induced refractive index can be approximately written as a parabolic form:

$$\Delta n \approx \Delta n_0 - \frac{1}{2}\gamma\left(x^2 + y^2\right) \tag{16}$$

Here, $\gamma \propto R''(0)$ is the response curvature.[14] Under this approximation, two solitons can be regarded as particles moving in an effective potential:

$$V\left(x_1, x_2\right) \propto -P_1 P_2 \exp\left[-\left(x_1 - x_2\right)^2 \Big/ \left(2w_{\mathrm{m}}^2\right)\right] \tag{17}$$

The form of this potential shows that the interaction is attractive, and its range is determined by w_{m} rather than by w_0.[14,82,83] Solving this model in two-dimensional nonlocal media gives the analytical structure and binding conditions of soliton pairs, and clarifies the matching relation between separation and power.[84]

It should be noted that the shape of the response function can strongly modify the long-range tail. Studies of interactions under oscillatory response functions show that the long-range tail may alternate between positive and negative values. This can lead to short-range attraction and long-range repulsion, or even more complex oscillatory evolution paths.[80] When the long-range framework is extended to two-color systems, a temperature-dependent orientational nonlocal response can be introduced in nematic liquid crystals. In this case, the strength and sign of the interaction between two-color solitons can drift slowly with the thermal boundary conditions. This provides a new way to control long-range coupling by using a thermal field.[85]

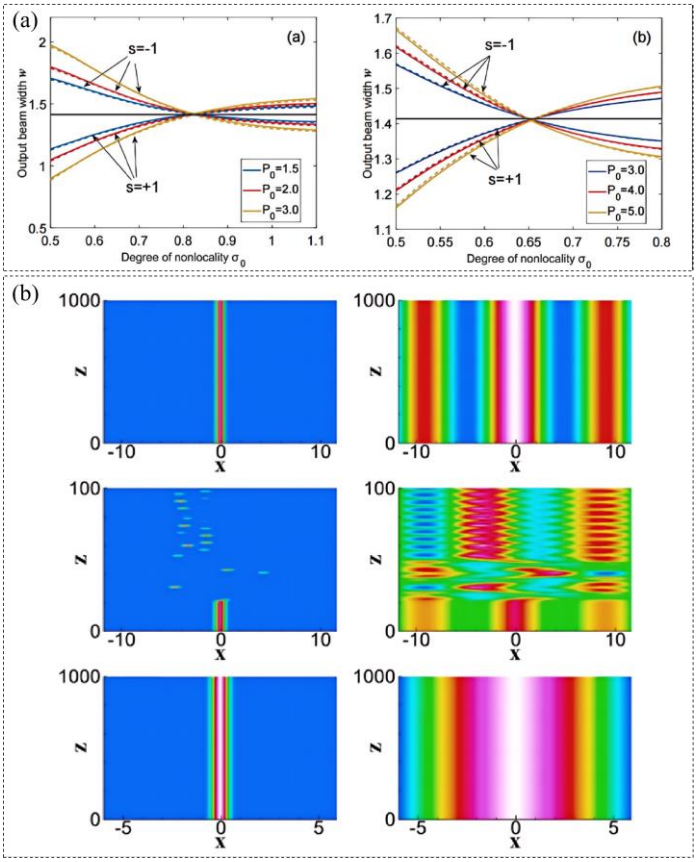


Fig. 9. Long-range interactions of nonlocal solitons. (a) Relation between beam focusing properties and the degree of nonlocality; (b) verification of soliton stability under boundary confinement. Reproduced with permission from Ref. [86,87].

When the response function loses positive definiteness and shows a sinusoidal

oscillatory form, the effective nonlinear coefficient of the medium can change sign. As a result, self-focusing and self-defocusing properties can be continuously controlled by the degree of nonlocality, as shown in Fig. 9(a). This is also accompanied by special modulation instability and new types of solitons. This theoretical framework has been successfully used to describe beam evolution in nematic liquid crystals with negative dielectric anisotropy.[86] On this basis, a rectangular Dirichlet boundary was further introduced, and the model was extended to a confined (1+2D) system. Nearly Gaussian-centered solitons and boundary-clipped oscillatory nonlinear refractive-index profiles were obtained, as shown in Fig. 9(b). This confirms that boundary confinement can effectively stabilize such oscillatory nonlocal solitons, which are usually unstable in unbounded space.[87]

Long-range interactions can also lead to spiral propagation and the formation of multi-soliton bound states. When two solitons enter the medium with a nonzero tangential velocity, the angular force component in the nonlocal potential bends their trajectories and forms spiral propagation paths.[67] In orientation-modulated nematic liquid crystals, the trajectories of counter-propagating solitons show direction sensitivity, suggesting the possibility of nonreciprocal interactions.[88] In addition, the existence of multi-soliton bound states is also a direct result of long-range interactions. Numerical simulations show that, by adding a local quintic nonlinear term to a weakly nonlocal medium, stable multi-peak bound states can be formed. Their separation is determined by the competition between the nonlocal length and the coefficient of the quintic term.[89] These studies show that long-range interactions not only break the distance limit of local media, but also produce rich dynamical behaviors through the response-function shape, external-field control, and many-body coupling.

4.2. New localized wave structures

The nonlocal response smoothes the optical intensity distribution through a spatial averaging effect. This can effectively suppress collapse and azimuthal instability, which are common in local media. Therefore, nonlocality makes it possible for complex localized wave structures to exist. In local self-focusing Kerr media, two-dimensional bright solitons collapse because the local intensity peak is too high. Vortex solitons carrying topological charge are also likely to suffer from azimuthal

instability and break into several lobes.[90,91] The introduction of nonlocality changes this situation. The extended refractive-index channel reduces the local intensity gradient and makes stable propagation of high-dimensional and higher-order structures possible.[92]

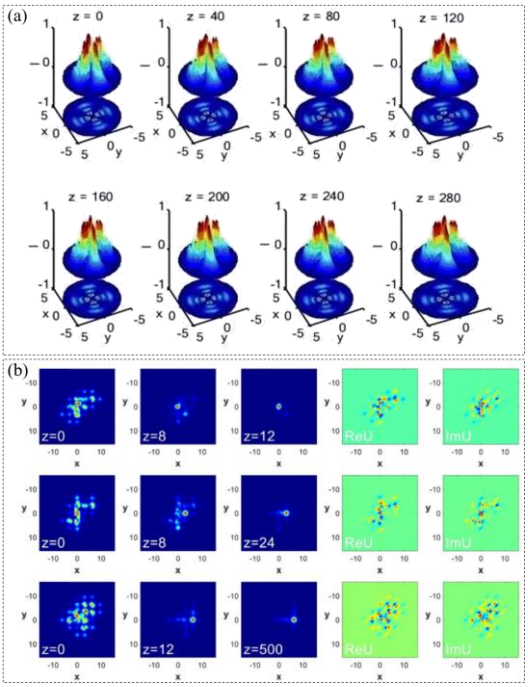


Fig. 10. Vortex solitons and multipole solitons. (a) Evolution of the intensity distribution of strongly nonlocal multipole solitons; (b) evolution of dipole solitons under different degrees of nonlocality. Reproduced with permission from Ref. [93,99].

Vortex solitons are typical objects for studying the stabilizing effect of nonlocality. In local models, vortex solitons with topological $m \geq 1$ have zero intensity at the phase singularity and high intensity around it. Therefore, they are very sensitive to azimuthal instability. In nonlocal media, the spatial averaging of the response function smooths the intensity distribution in the vicinity of the singularity and effectively suppresses the instability.[90,91] As shown in Fig. 10(a), Wang et al. obtained self-similar solutions of multipole solitons and vortex solitons in the strongly nonlocal nonlinear Schrödinger equation by using the variational method. They pointed out that nonlocal nonlinear response can remove the symmetry-breaking instability of vortices and provide a general potential-based explanation for the stable existence of vector vortex solitons (VVSs).[93,94] At the same time, by using linear stability analysis and the Petviashvili iteration method, researchers constructed the existence region of VVSs in nematic liquid crystals. They found that when the induced nonlocal potential provided by the lower-order component is deep enough,

the azimuthal instability of the higher-order vortex component can be completely suppressed. In particular, near the equal-beam-width point and within certain power-ratio ranges, VVSs can become fully linearly stable.[95] This conclusion can also be verified from the viewpoint of thermal nonlocal media. The observed adaptive evolution of higher-order vector vortices under thermally tunable transverse scales shows that increasing the degree of nonlocality deepens the effective potential well and extends the survival distance of vortex beams.[70]

Multipole solitons, including dipole, quadrupole, and multi-peak structures, are another class of complex structures that benefit from nonlocal stability. Analytical solutions of elliptic fundamental solitons, dipole solitons, and vortex solitons under anisotropic diffraction show that anisotropy not only changes the beam width, but also breaks axially symmetric solutions into preferred directions along the principal axes.[96,97] Furthermore, the variational method was used to construct approximate analytical solutions for vector multipole solitons and vortex soliton clusters in general nonlocal media. These studies revealed that the modulation depth and topological charge jointly determine the transverse mode structure, and also observed the rotation of multipole lobes during propagation.[93,94] When this study is extended to systems modified by external potentials, it is found that in nonlocal nonlinear media with superimposed Gaussian potential wells or barriers, the survival conditions of vortex dipoles, tripoles, and quadrupoles have clear thresholds. For example, the $m=2$ state can survive within a certain potential-well depth, while the $m=4$ state has a stronger tendency to split into lobes and requires stronger nonlocal confinement.[98] Recently, Chen et al. further extended the external potential to PT-symmetric periodic lattices. As shown in Fig. 10(b), they gave analytical solutions and evolution phase diagrams of two-dimensional vortex dipole solitons under a PT complex potential and nonlocality within a variational framework. They found that an intermediate degree of nonlocality can stabilize the fundamental-charge vortex dipole with $m=1$, while strong nonlocality instead induces the dipole to coalesce into a single soliton. For higher-order cases with $m=2,3$, even with the PT lattice, intermediate nonlocality still cannot make them fully stable, and their merging under strong nonlocality is not a fully steady state.[99] In addition, from the viewpoint of competing nonlocality and anisotropic diffraction, the bifurcation structures of elliptic solitons and dipole solitons have been obtained. These results show that anisotropy can make the

boundary of the stability region in parameter space show a “breaking-reconnection” behavior.[100]

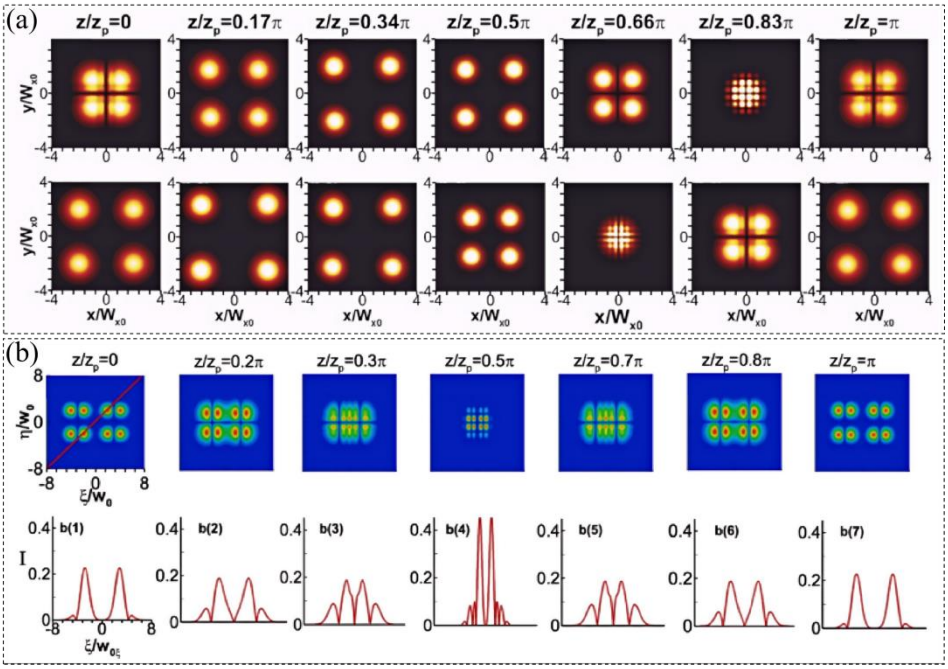


Fig. 11. Breathers and periodic solitons. (a) Periodic evolution of optical intensity patterns of breathers under different parameters; (b) periodic evolution of the intensity of two astigmatic hyperbolic-sine-Gaussian beams under out-of-phase conditions. Reproduced with permission from Ref. [12,101].

Breathers represent the periodic evolution of non-steady localized waves. When the input power P deviates from the critical power P_c in the Snyder-Mitchell model, the beam width no longer remains constant, but shows periodic oscillation. As shown in Fig. 11(a), Sun et al. classified the evolution of complex-variable sine-Gaussian cross-phase (CVSGCP) beams in strongly nonlocal media as “generalized higher-order breathers”. They showed that the sine terms and the cross-phase term can jointly control the intensity pattern, the periodic oscillation of the beam width, and the controllable rotation of the beam. Positive and negative parameters can also trigger a 90° pattern flip and reverse evolution.[12] Studies on Hermite-cos-Gauss (HcG) beams attribute the triggering mechanism of breathers to the curvature of the initial equal-phase surface, namely a nonzero wavefront curvature. When the wavefront is planar, the solution belongs to stable or unstable soliton branches. When the wavefront is curved, the beam becomes transversely asymmetric, and the beam width enters a periodic oscillation regime. The breathing period T_b, oscillation amplitude, and propagation distance have explicit dependence relations.[101] This result turns

breathers from a vague phenomenon into computable objects with clear triggering conditions and provides a basis for dynamic control in nonlocal media.

Periodic solitons usually refer to the periodic propagation of multi-peak structures. Their essence is the transverse periodic evolution of multi-soliton bound states. In convolution-type nonlocal systems, periodic solitons can be understood as periodic orbits in an effective potential. Two or more solitons propagate in pairs with fixed separations or phase differences, and their relative coordinates show periodic oscillation.[82]

As shown in Fig. 11(b), Bian et al. extended this idea to astigmatic hyperbolic-sine-Gaussian multi-beam clusters in strongly nonlocal media. They gave analytical solutions for the co-propagation of two and four AHSGBs. The results show that both in-phase and out-of-phase beams attract each other with a period of $2\pi z_{\mathrm{p}}$. A special phase arrangement of four beams can extend the period to $2\pi z_{\mathrm{p}}$ and a two-peak structure appears at the center in the in-phase case.[102] In addition, studies of the dynamics of Gaussian beams and super-Gaussian beams in weakly nonlocal media show that, under certain parameter conditions, these beams can evolve into periodic multi-peak structures. This further confirms the general existence of periodic solitons.[103]

Ince-Gaussian (IG), Hermite-Gaussian (HG), and Laguerre-Gaussian (LG) solitons form an analytical spectrum in the strongly nonlocal model. In the strongly nonlocal limit, the Snyder-Mitchell model is mathematically equivalent to the Schrödinger equation in a two-dimensional harmonic-oscillator potential. Its natural eigenfunction family is the LG family. The IG family is a continuous transition basis between the LG and HG families. The parameter $\varepsilon \in [0,1]$ can smoothly transform the solution from the HG type $(\varepsilon = 0)$ to the LG type $(\varepsilon = 1)$, while the solution remains approximately solvable during the whole transition.[70,104,105,106]

This analytical spectrum can also be extended to controllable mode conversion. In nonlocal media designed with anisotropic diffraction, one can artificially break or restore transverse isotropy. In this way, soliton modes can be continuously transformed between LG and HG modes, while the orbital angular momentum (OAM) evolves periodically. When the anisotropy is “twisted back” to isotropy at a certain propagation distance z, the target mode, such as a pure LG or pure HG mode, can be locked at a designed position.[105] As early as 2009, researchers gave the analytical

form of IG-type solutions in strongly nonlocal media, confirming the completeness of this spectrum.[107] In summary, nonlocality not only supports the existence of complex localized wave structures, but also provides a unified analytical framework for them.

4.3. Influence of nonlocality on soliton stability and critical power

The improvement of soliton stability is one of the core values of nonlocality. However, this improvement is not a simple monotonic process. It is strongly affected by boundary conditions, the shape of the response function, anisotropy, and competing nonlinearities. In the Snyder-Mitchell model, the critical power satisfies $P_c \propto 1/|\gamma|$, where γ is the response curvature. Therefore, a stronger degree of nonlocality usually leads to a smaller P_c.[89] However, in real systems, P_c and stability are not determined only by w_m.

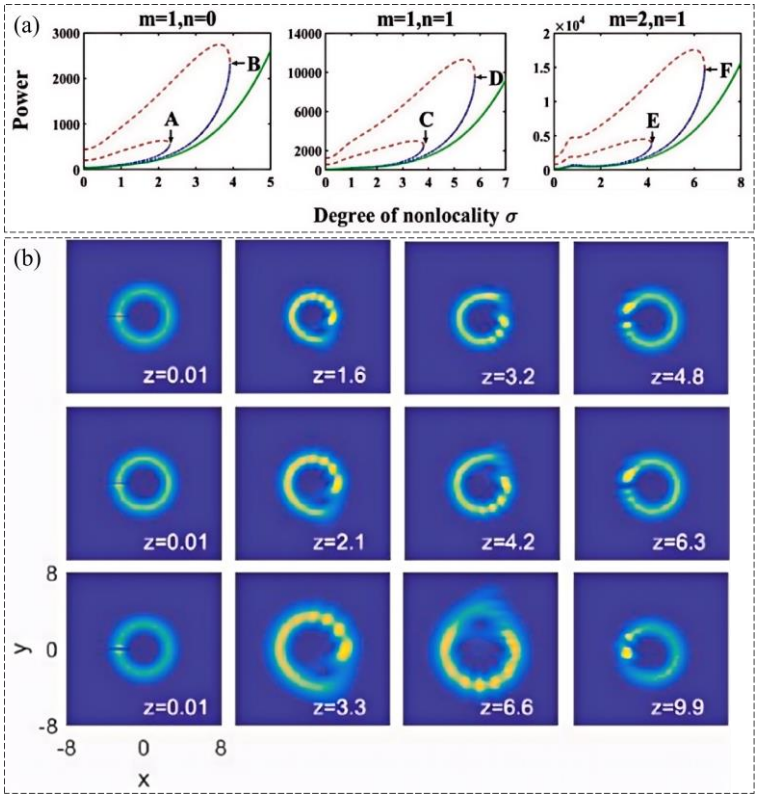


Fig. 12. (a) Power of HG soliton clusters as a function of the degree of nonlocality σ; (b) propagation of fractional vortex beams under different initial powers. Reproduced with permission from Ref. [108,115].

Competing nonlocal nonlinearities have become an important topic in recent years. In nematic liquid crystals, orientational nonlocality and thermal nonlocality can exist at the same time, but they have opposite signs. Orientational nonlocality is self-focusing, while thermal nonlocality is self-defocusing. Their competition can lead

to a nonmonotonic change of P_c. As shown in Fig. 12(a), Du et al. demonstrated this competing effect through experiments and simulations. When orientational nonlocality dominates, P_c decreases as the temperature increases. When thermal nonlocality becomes stronger, P_c increases instead, forming bistable branches.[108,109,110] Researchers further studied the shift of the modulation instability threshold in a competing cubic–quintic nonlocal model. They found that the addition of the quintic term can create a new turning point in the stable single-soliton branch and produce hysteresis in some parameter regions.[111,112] By using a variational framework, the existence condition of high-dimensional solitons can be written as an explicit inequality involving P/P_c and the nonlocal correction. This makes it possible to draw the boundary of "stability islands" clearly. The results show that the stability island becomes larger as the degree of nonlocality increases, but it is still limited by the finite boundary of the sample.[113]

The shape of the response function has a particularly strong influence on stability. When the response function contains high-frequency oscillatory components, the beam trajectory can change from regular propagation to chaotic motion. The threshold for chaos also decreases as the degree of nonlocality increases.[114] Oscillatory response models further show that the alternating positive and negative long-range tails can generate secondary minima in the effective soliton potential. As a result, solitons may be trapped in metastable states, which increases the risk of instability.[80] Anisotropic response changes the stability condition by introducing differences between the principal axes. Anisotropic diffraction models also show that the stability region of elliptic solitons moves toward higher power when the anisotropy changes, and that the stability condition is closely related to the ellipticity.[96,97] Competing nonlocal models also support this point, showing that anisotropy can distort the bifurcation boundary of multipole solitons.[97]

Boundary conditions and finite sample size are also important practical factors. When the sample size becomes comparable to w_m, the nonlocal response is truncated by the boundary. This reduces the effective w_m, increases P_c and makes the soliton more likely to bend toward the boundary.[92] Similar phenomena have also been found in self-defocusing Kerr-type nonlocal media, where boundary reflection can break the symmetric propagation of solitons.[90,91] In addition, the coupling between loss and

anisotropic diffraction can further complicate the stability picture.

As shown in Fig. 12(b), Pan et al. found that in nonlocal nonlinear media, the initial power P_0 is a key factor that determines whether a fractional-order vortex soliton can be formed. When P_0 is equal to the critical power P_c, the beam can maintain quasi-stable propagation and form a soliton. If $P_0 > P_c$, the nonlinear effect is stronger than diffraction, and the beam initially narrows. If $P_0 < P_c$, diffraction dominates, and the beam initially broadens. In addition, fractional-order vortex solitons with different topological charges have the same critical power, but their orbital angular momentum increases as the topological charge increases.[115] At the same time, under the fractional effect, the interaction force shows slower long-range decay, which may influence the stability of multi-soliton bound states.[116]

Recent studies have also focused on the difference between experimental measurements and theoretical predictions of the critical power. By systematically comparing hyperbolic soliton families in different nonlocal media, researchers found that the critical power predicted by the ideal Snyder-Mitchell model is usually lower than the experimental value. The difference mainly comes from the non-parabolic response of the medium and boundary effects.[117] These studies show that the stabilizing effect of nonlocality is a complex process controlled by many factors. It should be evaluated together with the response function, boundary conditions, and competing nonlinearities in specific media.

In summary, the response length sets the interaction range; the boundary condition shifts the effective potential and equilibrium position; the response-function shape determines whether the long-range tail is purely attractive or oscillatory; and competing nonlinearities change both the potential curvature and the critical power. Stability and critical power are therefore joint properties of the beam, response kernel, boundaries, and material nonlinearities, rather than functions of a single nonlocality parameter.

4.4. Controllable transport and application outlook

Because nonlocal solitons can act as self-written waveguides, they have unique advantages in beam transport and optical-field control. Different from traditional

optical waveguides, the refractive-index channel induced by a nonlocal soliton can be tuned in real time by external fields. This makes dynamic redirection of the beam path possible. The key point of controllable transport is that the trajectory, interaction, and transmission state of a soliton are jointly determined by the nonlocal response function and the induced potential. At the same time, the response function itself can be modulated by an applied electric field, temperature, boundary conditions, and other factors.[118,119]

Electric-field control is the most direct method. In nematic liquid crystals , the bias voltage changes the molecular pretilt angle and thus controls the degree of nonlocality. This further changes the depth and width of the effective soliton potential. Based on this mechanism, the switching of soliton interactions can be realized,[82] and the soliton trajectory can also be continuously deflected by dynamically tuning the bias voltage.[120] Temperature control mainly works by changing the diffusion length of the thermal nonlocal response. In addition, temperature-dependent two-color soliton interactions show that temperature changes can slowly shift the nonlocal length and thus modulate the strength of long-range coupling.[85] In general, Nematic liquid crystal solitons can already realize functions such as beam guiding, all-optical switching, and micromanipulation.[118,119]

Anisotropic diffraction design provides a new dimension for mode control. By introducing anisotropic diffraction, one can realize continuous conversion among IG, HG, and LG modes and periodic evolution of OAM. This provides new ideas for mode multiplexing and optical signal processing.[105] Studies of the anomalous propagation of Airy beams in nonlocal media show that their trajectories can be controlled by the degree of nonlocality and the initial wavefront. This can be used for beam deflection and particle manipulation.[121] Research on coherent interactions between Airy beams and solitons further shows that the trajectory distortion of a non-soliton beam in a nonlocal channel can be used to probe the nonlocal properties of the medium.[122]

Counter-propagation and orientation modulation provide possibilities for nonreciprocal transport. In orientation-modulated nematic liquid crystals, the trajectories of counter-propagating solitons show direction sensitivity, suggesting the potential to realize optical isolators.[88] Studies on controllable soliton transformation also show that soliton states can be controlled by adjusting the initial conditions and power ratio.[120] Together, these studies point to an important application prospect:

nonlocal solitons can act as reconfigurable optical channels and may play an important role in low-power all-optical routing, reconfigurable optical interconnects, and on-chip optical information processing. These functions are relevant to reconfigurable self-written waveguides, low-power all-optical routing, structured-light transport, and nonlinear topological photonics. For integrated platforms, the broad response may reduce sensitivity to small defects and provide remote coupling between channels, but device-scale use will require faster response, lower loss, stable boundaries, and reproducible material parameters.

However, it should also be recognized that most applications are still at the stage of basic research or proof of concept. In real systems, sample boundaries, response drift, and long-term stability have not been fully solved. For example, ion contamination in nematic liquid crystals can cause the degree of nonlocality to decay over time. Thermal diffusion in thermal media can limit the response speed. Photogenerated carrier drift in photorefractive crystals can introduce additional loss. Future research should make progress in the following directions. First, nonlocal media with higher stability and faster response should be developed. Second, integrated nonlocal soliton device structures should be designed. Third, hybrid integration of nonlocal solitons with other photonic systems, such as microcavities and waveguide arrays should be explored. Solving these problems will help promote nonlocal solitons from laboratory phenomena to practical applications.

5. Summary and outlook

This review has systematically summarized spatial optical solitons in nonlocal nonlinear media, including their theoretical basis, typical material systems, and recent research progress. The formation of spatial optical solitons comes from the dynamic balance between beam diffraction and nonlinear self-focusing. In nonlocal media, the refractive-index change at one point is determined not only by the optical intensity at that point, but also by the optical-field distribution in the surrounding region. Under strongly nonlocal conditions, the broad response can be approximately described as an effective parabolic potential near the beam. Based on this idea, the Snyder–Mitchell model provides a unified description for accessible solitons, fundamental modes, and

higher-order localized states. However, this model is still an ideal approximation. In real propagation, factors such as finite response width, boundary conditions, material anisotropy, and competing nonlinearities can also influence soliton behavior.

In typical media, liquid crystals rely on molecular reorientation, photorefractive crystals rely on charge-carrier transport and the space-charge field, and thermal media rely on absorption, thermal diffusion, and the thermo-optic effect to produce nonlocal refractive-index modulation. These different mechanisms can support many kinds of self-localized structures, including Gaussian, Bessel, Airy, vortex, vector, and multipole structures. They also shift the research focus from the formation and stable propagation of solitons to long-range interactions across several beam widths, the stabilization of complex localized wave structures, and the use of electric fields, temperature, boundaries, and structured light fields to realize trajectory deflection, binding, and switching. Recent studies on Airy beam self-localization, soliton interactions under competing nonlinearities, and higher-order multipole states further show this development trend.

Overall, nonlocality is not only a physical mechanism for stabilizing high-dimensional solitons, but also an important degree of freedom for building reconfigurable platforms for optical-field transport. Future studies may focus on developing unified models beyond the ideal parabolic potential, quantitatively including boundaries, non-instantaneous response, dissipation, and multi-field coupling. They may also strengthen the connection between theoretical predictions and repeatable experiments, and promote the combination of nonlocal solitons with programmable liquid crystals,[8] photonic lattices, and topological transport.[11] At the same time, nonlocal effects also show unique advantages in suppressing instabilities, improving the robustness of higher-order localized states, and realizing long-distance information transmission. Looking ahead, if problems such as response speed, loss, scale integration, and long-term stability can be further solved, nonlocal spatial optical solitons may play a more important role in all-optical routing, information processing, and new types of optical-field control.

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