

An improved CRE method and novel interaction solutions to a fifth-order nonlinear equation

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SPECIAL TOPIC — Nonlinear localized waves: Theory, methods and applications

An improved CRE method and novel interaction solutions to a fifth-order nonlinear equation

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The consistent Riccati expansion (CRE) method is powerful for deriving interaction solutions of nonlinear evolution equations. We improve the CRE method by extending the Riccati equation to the Riccati–Bernoulli equation. The improved CRE method solves nonlinear evolution equations by utilizing both the Riccati and Bernoulli equations. The effectiveness and convenience of the improved CRE method have been demonstrated by a new fifth-order equation. With this method, more types of interaction solutions can be constructed. These include not only soliton-cnoidal solutions but also exponential-elliptic and other interaction solutions. Moreover, the improved CRE method can be applied to construct interaction solutions of other nonlinear systems.

Keywords: consistent Riccati expansion method, Riccati equation, Bernoulli equation, interaction solutions

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1. Introduction

Nonlinear science is a significant development in natural sciences after quantum mechanics and relativity. Its rapid advancement has established nonlinear science as one of the frontier research topics across numerous disciplines. Nonlinear evolution equations (NEEs) can describe a wide range of natural phenomena, including nonlinear optics,[1–4] Bose–Einstein condensates,[5] ferrimagnetics,[6–8] and others.[9–11] Because they are essential for modeling complex natural phenomena, finding solutions to NEEs has become a central focus of nonlinear science. There are a number of direct approaches to constructing explicit solutions to NEEs, for example, the tanh algorithm,[12] the generalized tanh-function algorithm,[13] the Jacobi elliptic function expansion approach,[14] the consistent Riccati expansion (CRE) method,[15] Hirota's bilinear method,[16] the Darboux transformation,[17] and so on.[18–20] Among these, the CRE method is a very practical approach for generating interaction solutions between different waves, including soliton–cnoidal interaction solution, the resonant soliton solution, and interactions between soliton and triangular periodic waves.[15,21,22] The main idea of the CRE method is to express the solutions of an NEE as a polynomial in the solutions of the Riccati equation. Therefore, Bäcklund transform can be derived from the CRE method. Using the Bäcklund transform, different interaction solutions can be constructed. By applying the CRE method to many nonlinear systems, such as Bogoyavlenskii coupled KdV system,[23] Drinfel'd–Sokolov–Wilson equation,[24] $(3+1)$-dimensional P-type evolution equation,[25] and so on,[26,27] interesting results have been obtained. Moreover, the CRE method has been extended to fractional-order NEEs.[28,29]

In many real physical systems, such as shallow water waves, plasmas, and nonlinear optical fibers, waves often do not appear as isolated solitons but rather as solitons coexisting with a periodic background, for example, the propagation of a solitary wave on a swell background in the ocean.[30,31] Therefore, constructing interaction solutions that capture the coupling between solitons and background periodic waves is conducive to providing a better description and explanation of such practical phenomena, offering deeper insight into the underlying nonlinear dynamics. Consequently, searching for interaction solutions and developing new methods to derive more interactions are of great significance. In this paper, we will improve the CRE method by extending the Riccati equation to the Riccati–Bernoulli equation. The improved CRE method is more general, encompassing the original CRE method as a special case. The contribution of the improved CRE method to the fields of mathematical physics and nonlinear science is that it provides a systematic way to construct exact solutions that describe interactions of different waves. Using the improved CRE method, more novel types of interaction solutions will be directly constructed.

The improved CRE method will be illustrated by the following fifth-order model:

$$u_t + u_{xxt} + \mu u_x - u u_{xxx} - u u_x - 3u_x u_{xx} - \mu u_{xxxxx} = 0, \quad (1)$$

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where μ is a real parameter.[32] This equation can describe the water wave problem for long, small amplitude waves over the shallow horizontal bottom.[33] When $\mu = 0$, Eq. (1) reduces to the Gibson–Pickering (GP) model. Painlevé integrability, multiple soliton solutions, and trigonometric periodic solutions have been studied in Ref. [32]. A generalized Painlevé integrable fifth-order equation has been proposed and studied by Hirota's method.[33] Equation (1) is a new fifth-order nonlinear model and is different from the other fifth-order equations, such as the fifth-order KdV, Sawada–Kotera and Kaup–Kupershmidt equations.[34–37]

The remainder of this paper is structured as follows. The main steps of the improved CRE method are given in Section 2. Sections 3 and 4 demonstrate the application of the improved CRE method to Eq. (1) and construct interaction solutions for it. A summary of this paper is provided in Section 5.

2. A brief outline of the improved CRE method

In what follows, the improved CRE method is explained and its main steps are provided.

Step 1 Given a partial differential equation

$$P(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt},\ldots)=0. \tag{2}$$

The following ansatz is adopted for the solution of Eq. (2):

$$u=\sum_{j=0}^{J} a_j R^j(W), \tag{3}$$

where a_j $(j=0,1,2,\ldots,J)$ and W are differentiable functions with respect to x and t. The positive integer J is determined by the principle of homogeneous balance. $R(W)$ satisfies the following constant-coefficient Riccati–Bernoulli equation

$$R'=A+BR+MR^N, \tag{4}$$

where R' denotes the derivative of R with respect to W, and N is an integer with $N\geq 1$. Exact solutions of Eq. (4) have been studied in Refs. [10,11] and listed in Appendix A.

When $N=2$ and $M\neq 0$, Eq. (4) becomes the Riccati equation.

When $N\neq 0,1,2$, and $A=0, M\neq 0$, Eq. (4) becomes the Bernoulli equation.

Step 2 Substituting Eqs. (3) and (4) into Eq. (2), we obtain a system of overdetermined equations by collecting the coefficients of different $R^j(W)$. The unknown functions a_j and W will be determined by solving the system of equations. Since the overdetermined equations are usually difficult to solve, we introduce two hypotheses.

Case I $W=k_1x+k_2t+k_0$, where k_0, k_1 and k_2 are undetermined constants.

Case II $W=k_1x+k_2t+k_0+H(\eta)$, $\eta=n_1x+n_2t+n_0$, with n_0, n_1 and n_2 being constants. H' and H'' denote the first and second derivative of H with respect to η, and they satisfy the following Jacobian elliptic equation:

$$H''=\sqrt{c_0+c_1H'+c_2(H')^2+c_3(H')^3+c_4(H')^4}, \tag{5}$$

where c_0, c_1, c_2, c_3, and c_4 are constants. Solutions of Eq. (5) when c_0, c_1, c_2, c_3, and c_4 take different values have been summarized in Refs. [38,39].

Step 3 Substituting a_j, W, and $R(W)$ into Eq. (3).

3. Application of the improved CRE method to Eq. (1) when $N=2$

If we take $N=2$ into Eq. (4), then Eq. (4) becomes the Riccati equation and has the form

$$R'=A+BR+MR^2, \tag{6}$$

and the procedure in Section 2 becomes the CRE method.

Taking Eq. (3) with Eq. (6) into Eq. (1), we get $J=2$, so

$$u=a_0+a_1R(W)+a_2R^2(W), \tag{7}$$

where a_0, a_1, and a_2 are functions of x and t, $R(W)$ is a solution of Eq. (6).

Plugging Eqs. (7) and (6) into Eq. (1) and rewriting it in the form of a polynomial in terms of $R^j(W)$, we have

$$h_7R^7(W)+h_6R^6(W)+h_5R^5(W)+\cdots+h_1R(W)+h_0=0, \tag{8}$$

with h_0, h_1, …, h_7 being functions of a_0, a_1, a_2, W and their derivatives. Expressions of h_7, h_6, and h_5 are as follows:

$$\begin{aligned}
h_7 &= -720\mu a_2M^5W_x^5-60M^3W_x^3a_2^2,\\
h_6 &= -1200\mu M^4a_2W_x^3W_{xx}-2400\mu BM^4a_2W_x^5\\
&\quad -150BM^2W_x^3a_2^2-60M^3W_x^3a_1a_2-30M^2W_xW_{xx}a_2^2\\
&\quad -60M^2W_x^2a_{2x}a_2-600\mu M^4W_x^4a_{2x}-120\mu M^5a_1W_x^5,\\
h_5 &= -12MW_xa_{2x}^2-2MW_xa_2^2-12M^3W_x^3a_1^2-240\mu M^3W_x^3a_{2xx}\\
&\quad -12MW_xa_{2xx}a_2-12MW_{xx}a_{2x}a_2-36M^2W_x^2a_1a_{2x}\\
&\quad -124AM^2W_x^3a_2^2-122B^2MW_x^3a_2^2-24M^3a_0W_x^3a_2\\
&\quad +24M^3W_tW_x^2a_2-120M^4W_x^4a_{1x}\mu-36M^2W_x^2a_{1x}a_2\\
&\quad -360\mu a_2W_{xx}^2W_xM^3-240\mu a_2W_x^2W_{xxx}M^3\\
&\quad -720\mu a_{2x}W_x^2W_{xx}M^3-36W_xW_{xx}a_1a_2M^2\\
&\quad -54BMW_xW_{xx}a_2^2-240\mu a_1W_x^3W_{xx}M^4\\
&\quad -144BM^2W_x^3a_1a_2-108BMW_x^2a_{2x}a_2\\
&\quad -1680\mu a_{2x}W_x^4BM^3-1680\mu a_2W_x^5AM^4\\
&\quad -3000\mu a_2W_x^5B^2M^3-360\mu a_1W_x^5BM^4\\
&\quad -3360\mu a_2W_x^3W_{xx}BM^3-2MW_{xxx}a_2^2.
\end{aligned}$$

From h_7, h_6, and h_5, the coefficients a_0, a_1, and a_2 can be derived and they are as follows:

$$a_0 = \left(8\mu AMW_x^4+\mu B^2W_x^4+6\mu BW_x^2W_{xx}-3\mu W_{xx}^2\right.$$

$$+4\mu W_x W_{xxx} - \mu W_x^2 - W_x W_t\big)/(-W_x^2),$$
$$a_1 = -12\mu BMW_x^2 - 12\mu MW_{xx},\ \ a_2 = -12\mu M^2 W_x^2. \quad (9)$$

Substituting the above expressions of a_0, a_1 and a_2 into Eq. (8), we find that h_4, h_3, h_2, h_1, and h_0 can be reduced to the following two equations:

$$W_x W_t W_{xx} - 4\mu W_x W_{xx} W_{xxx} + 4\mu MAW_x^4 W_{xx} - \mu B^2 W_x^4 W_{xx}$$
$$-W_x^2 W_{xt} + \mu W_x^2 W_{xxxx} + 3\mu W_{xx}^3 = 0, \quad (10)$$
$$-W_x^2 W_{tt} + \mu W_x^2 W_{xxxt} - \mu B^2 W_x^4 W_{xt} + 3\mu W_{xx}^2 W_{xt} - \mu W_x W_{xt} W_{xxx}$$
$$-3\mu W_x W_{xx} W_{xxt} + W_x W_{xt} W_t + 4\mu AMW_x^4 W_{xt} = 0. \quad (11)$$

So, solving Eqs. (10) and (11) is essential for obtaining explicit solutions to Eq. (1). In the following, we consider the following cases.

3.1. W is a linear function

In this case, we take $W = k_1 x + k_2 t + k_0$, where k_0, k_1, and k_2 are constants, then Eqs. (10) and (11) are automatically satisfied. Under this circumstance,

$$a_0 = -8\mu AMk_1^2 - \mu B^2 k_1^2 + \mu + \frac{k_2}{k_1},$$
$$a_1 = -12\mu BMk_1^2, a_2 = -12\mu M^2 k_1^2,$$

and Eq. (7) becomes

$$u = -8\mu AMk_1^2 - \mu B^2 k_1^2 + \mu + \frac{k_2}{k_1}$$
$$-12\mu BMk_1^2 R(W) - 12\mu M^2 k_1^2 R^2(W).$$

For different A, B, and M, explicit solutions of Eq. (1) are as follows:

$$u_1 = -3k_1^2\mu(\pm\csc h(W) + \coth(W))^2 + 2k_1^2\mu + \gamma,\ \ (B=0,\ A=0.5,\ M=-0.5),$$
$$u_2 = -3k_1^2\mu(\tanh(W) \pm \mathrm{i}\,\mathrm{sech}(W))^2 + 2k_1^2\mu + \gamma,\ \ (B=0,\ A=0.5,\ M=-0.5),$$
$$u_3 = -\frac{3k_1^2\mu\tanh^2(W)}{(1\pm\mathrm{sech}(W))^2} + 2k_1^2\mu + \gamma,\ \ (B=0,\ A=0.5,\ M=-0.5),$$
$$u_4 = -\frac{3k_1^2\mu\coth^2(W)}{(1\pm\mathrm{i}\,\mathrm{csch}(W))^2} + 2k_1^2\mu + \gamma,\ \ (B=0,\ A=0.5,\ M=-0.5),$$
$$u_5 = -3k_1^2\mu(\tan(W) \pm \sec(W))^2 - 2k_1^2\mu + \gamma,\ \ (B=0,\ A=0.5,\ M=0.5),$$
$$u_6 = -3k_1^2\mu(\csc(W) - \cot(W))^2 - 2k_1^2\mu + \gamma,\ \ (B=0,\ A=0.5,\ M=0.5),$$
$$u_7 = -\frac{3k_1^2\mu\tan^2(W)}{(1\pm\sec(W))^2} - 2k_1^2\mu + \gamma,\ \ (B=0,\ A=0.5,\ M=0.5),$$
$$u_8 = -3k_1^2\mu(\cot(W) \pm \csc(W))^2 - 2k_1^2\mu + \gamma,\ \ (B=0,\ A=-0.5,\ M=-0.5),$$
$$u_9 = -3k_1^2\mu(\sec(W) - \tan(W))^2 - 2k_1^2\mu + \gamma,\ \ (B=0,\ A=-0.5,\ M=-0.5),$$
$$u_{10} = -\frac{3k_1^2\mu\cot^2(W)}{(1\pm\csc(W))^2} - 2k_1^2\mu + \gamma,\ \ (B=0,\ A=-0.5,\ M=-0.5),$$
$$u_{11} = -12k_1^2\mu\tanh^2(W) + 8k_1^2\mu + \gamma,\ \ (B=0,\ A=1,\ M=-1),$$
$$u_{12} = -12k_1^2\mu\coth^2(W) + 8k_1^2\mu + \gamma,\ \ (B=0,\ A=1,\ M=-1),$$
$$u_{13} = -12k_1^2\mu\tan^2(W) - 8k_1^2\mu + \gamma,\ \ (B=0,\ A=1,\ M=1),$$
$$u_{14} = -12k_1^2\mu\cot^2(W) - 8k_1^2\mu + \gamma,\ \ (B=0,\ A=-1,\ M=-1),$$
$$u_{15} = -\frac{48k_1^2\mu\tan^2(W)}{(1+\tan(W))^2} + \frac{48k_1^2\mu\tan(W)}{1+\tan(W)} - 20k_1^2\mu + \gamma,\ \ (B=-2,\ A=1,\ M=2),$$
$$u_{16} = -\frac{48k_1^2\mu\tan^2(W)}{(1-\tan(W))^2} - \frac{48k_1^2\mu\tan(W)}{1-\tan(W)} - 20k_1^2\mu + \gamma,\ \ (B=2,\ A=1,\ M=2),$$
$$u_{17} = -\frac{48k_1^2\mu\cot^2(W)}{(1+\cot(W))^2} + \frac{48k_1^2\mu\cot(W)}{1+\cot(W)} - 20k_1^2\mu + \gamma,\ \ (B=2,\ A=-1,\ M=-2),$$
$$u_{18} = -\frac{48k_1^2\mu\cot^2(W)}{(1-\cot(W))^2} - \frac{48k_1^2\mu\cot(W)}{1-\cot(W)} - 20k_1^2\mu + \gamma,\ \ (B=-2,\ A=-1,\ M=-2),$$
$$u_{19} = -\frac{12M^2k_1^2\mu}{(MW+A_0)^2} + \gamma,\ \ (B=0,\ A=0,\ M\neq 0),$$

where $\mathrm{i}^2 = -1$ and $\gamma = \mu + \frac{k_2}{k_1}$.

The above solutions include mixed soliton wave solutions u_1, u_2, u_3, u_4, kink wave solutions u_{11}, u_{12}, a rational wave solution u_{19}, and triangular periodic wave solutions. In particular, when $k_0 = 0$, $k_2 = -(8\mu k_1^3 + \mu k_1 - a_0 k_1)$, where a_0 is a constant, the solutions u_{11}, u_{12} here are the same as the kink solutions in Ref. [32], and the solutions u_{13}, u_{14} here are the same as the periodic solutions in Ref. [32].

3.2. W is not a linear function

In this case, we take

$$W = k_1 x + k_2 t + k_0 + H(\eta), \quad \eta = n_1 x + n_2 t + n_0, \tag{12}$$

with n_0, n_1, and n_2 being undetermined constants. Let $H' = \varphi$, where $\varphi = \varphi(\eta)$ is a differential function with respect to η, φ' denotes the first-order derivative, and it satisfies the following equation:

$$\varphi' = \sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}, \tag{13}$$

where c_0, c_1, c_2, c_3, and c_4 are constants. Equation (13) is the third kind Jacobian elliptic equation.

Substituting Eq. (12) with Eqs. (5) and (13) into Eqs. (10) and (11), we obtain

$$\begin{aligned} c_0 &= \left[\left(4AMk_1^3 - B^2k_1^3 + c_1n_1^3 - c_2k_1n_1^2 + c_3k_1^2n_1\right)k_1\right]/n_1^4, \\ c_4 &= -4AM + B^2, \\ k_2 &= (16\mu AMn_1k_1^3 - 4\mu B^2n_1k_1^3 + \mu n_1^4c_1 - 2\mu n_1^3c_2k_1 \\ &\quad + 3\mu n_1^2c_3k_1^2 - k_1n_2)/(-n_1), \end{aligned}$$

where A, B, M, k_0, k_1, n_0, n_1, n_2, c_1, c_2, and c_3 are free constants. If we take appropriate values of these free constants, interaction solutions of Eq. (1) can be derived.

3.2.1. A soliton–elliptic interaction solution

If we take $A = 1$, $B = 0$, $M = -1$, $k_1 = \frac{1}{2}n_1$, $c_1 = c_3 = 0$ and $c_2 = -m^2 - 1$, then $c_0 = m^2/4$, $c_4 = 4$, $k_2 = (1-m^2)n_1^3\mu + n_2/2$. Here, m $(0 < m \leq 1)$ is the modulus parameter. According to Appendix A and the results in Refs. [38,39], solutions of Eq. (6) and Eq. (13) are as follows:

$$R(W) = \tanh(W), \quad \varphi(\eta) = \frac{m}{2}\mathrm{sn}(\eta, m).$$

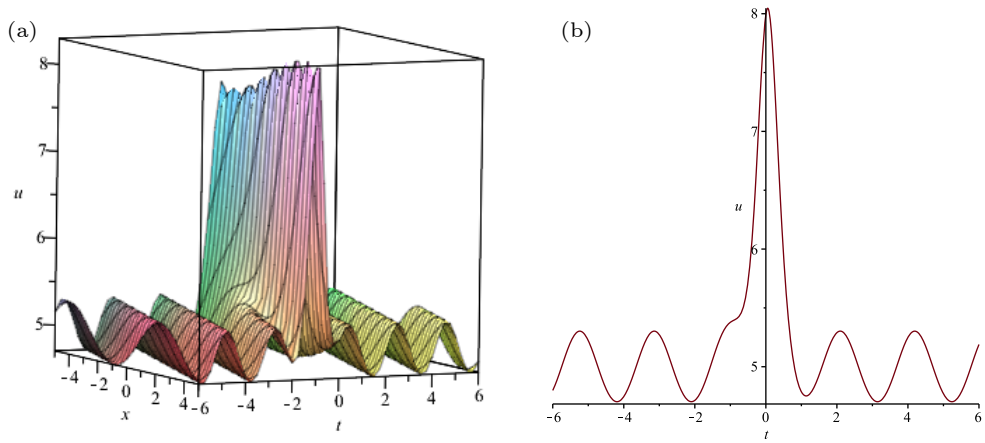


Fig. 1. The soliton–elliptic interaction solution (14) with $m = 0.05$: (a) a three-dimensional plot; (b) a two-dimensional plot with $x = 0$.

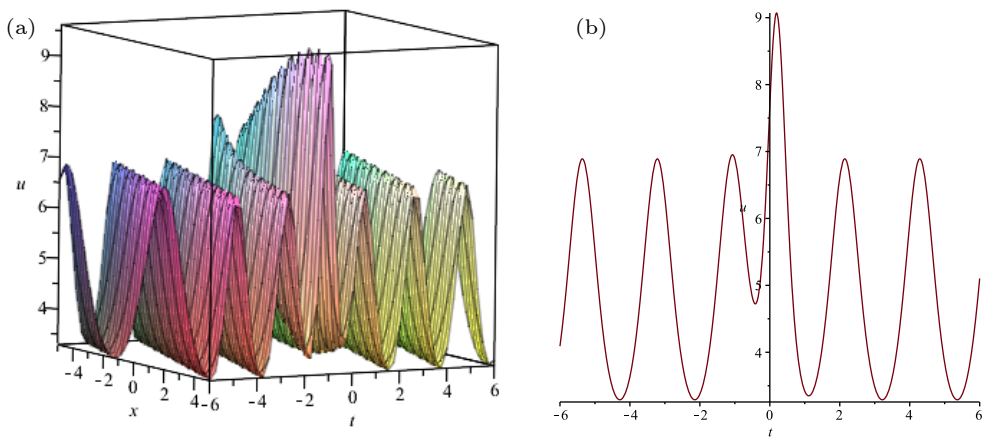


Fig. 2. The soliton–elliptic interaction solution (14) with $m = 0.3$: (a) a three-dimensional plot; (b) a two-dimensional plot with $x = 0$.

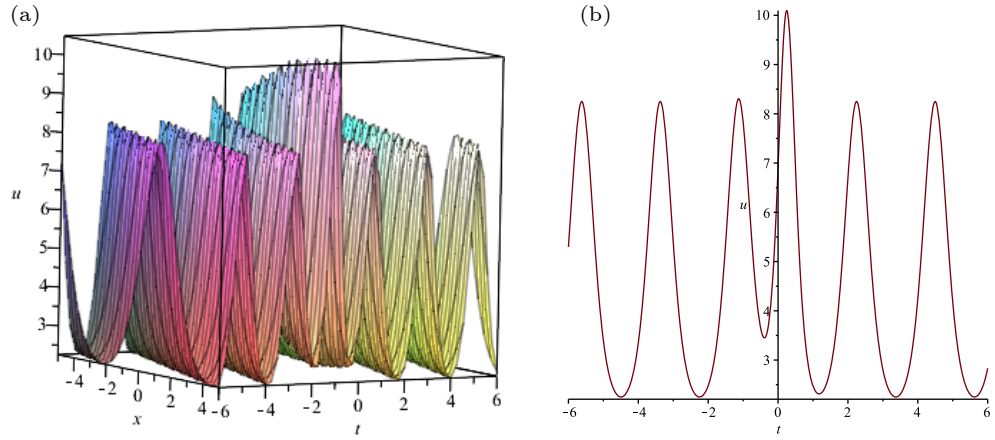


Fig. 3. The soliton–elliptic interaction solution (14) with $m = 0.5$: (a) a three-dimensional plot; (b) a two-dimensional plot with $x = 0$.

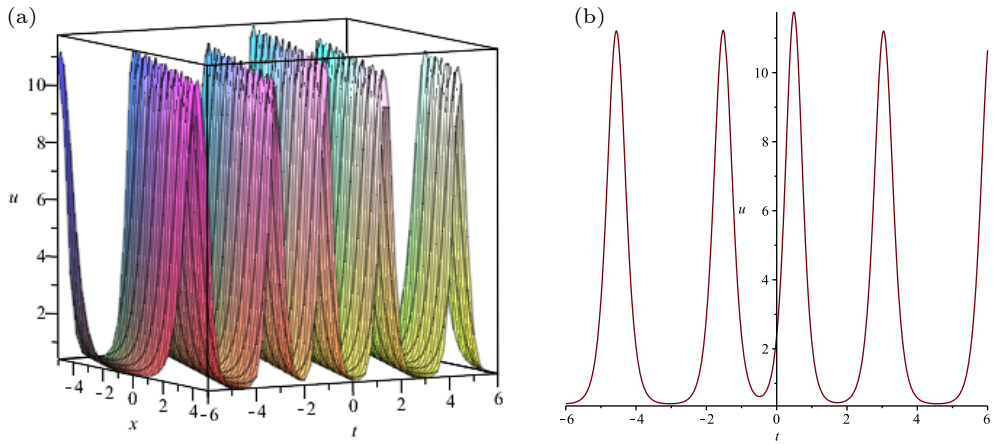


Fig. 4. The soliton–elliptic interaction solution (14) with $m = 0.9$: (a) a three-dimensional plot; (b) a two-dimensional plot with $x = 0$.

Then, $H(\eta) = \frac{1}{2}\ln(\mathrm{dn}(\eta,m) - m\mathrm{cn}(\eta,m))$, and a solution in which the soliton wave interacts with the elliptic wave is

$$\begin{aligned} u = \frac{1}{n_1}\Big\{ & -3n_1^3\mu\tanh^2(W)\,(m\mathrm{sn}(\eta,m)+1)^2 \\ & +6n_1^3 m\mu\,\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)\tanh(W) \\ & -3n_1^3m^2\mu\,\mathrm{sn}(\eta,m)^2 + 6n_1^3 m\mu\,\mathrm{sn}(\eta,m) \\ & +n_1^3m^2\mu + 4n_1^3\mu + n_1\mu + n_2\Big\}, \end{aligned} \tag{14}$$

where $W = -m^2\mu n_1^3 t + \mu n_1^3 t + \frac{1}{2}n_1 x + \frac{1}{2}n_2 t + \frac{1}{2}\ln(\mathrm{dn}(\eta,m) - m\mathrm{cn}(\eta,m)) + k_0$, $\eta = n_1 x + n_2 t + n_0$.

The above solution (14) is the soliton-elliptic interaction solution. Taking $\mu = 1$, $n_1 = 1$, $n_2 = 3$, $n_0 = k_0 = 0$, graphs of Eq. (14) can be obtained for different modulus parameter m.

It can be seen from the figures that as the modulus parameter m increases, the amplitude of the background periodic wave also increases. When $m < 0.5$, the solitary wave on the periodic background can be clearly observed. When $m \to 1$, the background periodic wave approaches a solitary wave.

3.2.2. Another soliton–elliptic interaction solution

If we take $A = 1$, $B = 0$, $M = -1$, $k_1 = \frac{1}{2}n_1$, $c_1 = c_3 = 0$, and $c_2 = -m^2 - 1$, then $c_0 = m^2/4$, $c_4 = 4$, $k_2 = (1-m^2)n_1^3\mu + n_2/2$. According to Appendix 1 and the results in Refs. [38,39], solutions of Eqs. (6) and (13) are as follows:

$$R(W) = \tanh(W), \quad \phi(\eta) = \frac{1}{2}\mathrm{ns}(\eta,m).$$

Then,

$$H(\eta) = \frac{1}{2}\ln\left(\frac{\mathrm{dn}(\eta,m) - \mathrm{cn}(\eta,m)}{\mathrm{sn}(\eta,m)}\right),$$

and another soliton–elliptic interaction solution of Eq. (1) is

$$\begin{aligned} u = n_1^{-1}\Big\{ & \mathrm{sn}(\eta,m)\cosh(W)\,(\mathrm{cn}(\eta,m) - \mathrm{dn}(\eta,m))\Big(m^2\mathrm{sn}(\eta,m)^2\mathrm{cn}(\eta,m) + \mathrm{dn}(\eta,m)^2\mathrm{cn}(\eta,m) \\ & -\mathrm{sn}(\eta,m)^2\mathrm{dn}(\eta,m) + \mathrm{sn}(\eta,m)\mathrm{cn}(\eta,m) - \mathrm{sn}(\eta,m)\mathrm{dn}(\eta,m) - \mathrm{cn}(\eta,m)^2\mathrm{dn}(\eta,m)\Big)\Big\}^{-2} \\ & \times\Big\{(\mathrm{sn}(\eta,m)+1)^2\Big[\Big(m^4\mathrm{sn}(\eta,m)^4 + 6m^2\mathrm{sn}(\eta,m)^4 + \mathrm{sn}(\eta,m)^4 + 4m^2\mathrm{sn}(\eta,m)^2\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m) \end{aligned}$$

$$
\begin{aligned}
&+\mathrm{sn}(\eta,m)^2\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)-8m^2\mathrm{sn}(\eta,m)^2-2\mathrm{sn}(\eta,m)^2-8\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)+8\Big)\\
&\times\left(n_1^3m^2\mu\,\mathrm{sn}(\eta,m)^2+n_1^3\mu\,\mathrm{sn}(\eta,m)^2+\mu n_1\mathrm{sn}(\eta,m)^2+n_2\mathrm{sn}(\eta,m)^2-6n_1^3\mu\right)\cosh^2(W)\\
&-6n_1^3\mu\sinh(W)\cosh(W)\Big(4m^4\mathrm{sn}(\eta,m)^6+4m^2\mathrm{sn}(\eta,m)^6+\mathrm{sn}(\eta,m)^4\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)(m^4+6m^2+1)\\
&-4(m^4+4m^2+1)\mathrm{sn}(\eta,m)^4-8m^2\mathrm{sn}(\eta,m)^2\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)+12m^2\mathrm{sn}(\eta,m)^2\\
&-8\mathrm{sn}(\eta,m)^2\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)+12\mathrm{sn}(\eta,m)^2+8\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)-8\Big)3n_1^3\mu\left(\mathrm{sn}(\eta,m)+1\right)^2\\
&\times\Big(m^4\mathrm{sn}(\eta,m)^4+6m^2\mathrm{sn}(\eta,m)^4+\mathrm{sn}(\eta,m)^4+4m^2\mathrm{sn}(\eta,m)^2\left(\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)-2\right)\\
&+4\mathrm{sn}(\eta,m)^2\left(\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)-2\right)-8\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)+8\Big)\Big]\Big\},
\end{aligned}
\tag{15}
$$

where

$$
W=-m^2\mu n_1^3t+\mu n_1^3t+\frac{1}{2}n_1x+\frac{1}{2}n_2t+\frac{1}{2}\ln\left(\frac{\mathrm{dn}(\eta,m)-\mathrm{cn}(\eta,m)}{\mathrm{sn}(\eta,m)}\right)+k_0,\quad \eta=n_1x+n_2t+n_0.
$$

3.2.3. A periodic–elliptic interaction solution

If we take $A=1$, $B=0$, $M=1$, $k_1=\frac{1}{2}mn_1$, $c_1=c_3=0$ and $c_2=2m^2-1$, then $c_0=-\frac{m^2}{4}(m^2-1)$, $c_4=-4$, $k_2=-mn_1^3\mu+\frac{1}{2}mn_2$. According to Appendix A and the results in Refs. [38,39], solutions of Eqs. (6) and (13) are as follows:

$$
R(W)=\tan(W),\quad \phi(\eta)=\frac{m}{2}\mathrm{cn}(\eta,m).
$$

Then, $H(\eta)=\frac{1}{2}\arctan\left(\frac{m\mathrm{sn}(\eta,m)}{\mathrm{dn}(\eta,m)}\right)$, and Eq. (1) admits a solution in which the trigonometric periodic wave interacts with the elliptic wave

$$
\begin{aligned}
u=&\frac{1}{n_1}\left(\mathrm{cn}(\eta,m)+1\right)^{-2}\Big\{3n_1^3m^2\mu\tan^2(W)\left(-\mathrm{sn}(\eta,m)^4+4\mathrm{sn}(\eta,m)^2\mathrm{cn}(\eta,m)+8\mathrm{sn}(\eta,m)^2-8\mathrm{cn}(\eta,m)-8\right)\\
&+6n_1^3m\mu\,\mathrm{sn}(\eta,m)\mathrm{dn}(\eta,m)\tan(W)\left(-\mathrm{sn}(\eta,m)^2+2\mathrm{cn}(\eta,m)+2\right)+3n_1^3m^2\mu\,\mathrm{sn}(\eta,m)^4+8n_1^3m^2\mu\,\mathrm{sn}(\eta,m)^2\\
&-8n_1^3\mu\,\mathrm{sn}(\eta,m)^2-n_1\mu\,\mathrm{sn}(\eta,m)^2-n_2\mathrm{sn}(\eta,m)^2-2\left(8n_1^3m^2\mu-n_1^3\mu-n_1\mu-n_2\right)\left(\mathrm{cn}(\eta,m)+1\right)\Big\},
\end{aligned}
\tag{16}
$$

where

$$
W=-m\mu n_1^3t+\frac{1}{2}mn_1x+\frac{1}{2}mn_2t+\frac{1}{2}\arctan\left(\frac{m\mathrm{sn}(\eta,m)}{\mathrm{dn}(\eta,m)}\right)+k_0,\quad \eta=n_1x+n_2t+n_0.
$$

4. Application of the improved CRE method to Eq. (1) when $N=3$

When $N=3$ and $A=0$, Eq. (4) becomes the following Bernoulli equation:

$$
R'=BR+MR^3. \tag{17}
$$

According to the procedure of the improved CRE method, taking Eq. (3) with Eq. (17) into Eq. (1), we get $J=4$ by the principle of homogeneous balance, so Eq. (3) becomes

$$
u=a_0+a_1R(W)+a_2R^2(W)+a_3R^3(W)+a_4R^4(W), \tag{18}
$$

where a_0, a_1, a_2, a_3, a_4, and W are functions of x and t.

Gathering the coefficients of different $R^j(W)$, we have

$$
\begin{aligned}
&-480M^3a_4W_x^3\left(48M^2\mu W_x^2+a_4\right)R^{14}(W)\\
&-693M^3a_3W_x^3\left(15M^2\mu W_x^2+a_4\right)R^{13}(W)+\cdots=0.
\end{aligned}
\tag{19}
$$

Here, the coefficients of $R^j(W)$ $(j=0,1,\ldots,14)$ are functions of a_i $(i=0,1,\ldots,4)$, W and their derivatives. Setting the coefficients of $R^{14}(W)$ and $R^{13}(W)$ to zero, we obtain

$$
a_4=-48M^2\mu W_x^2,\quad a_3=0. \tag{20}
$$

Substituting the expressions of a_4 and a_3 into Eq. (19), we have

$$
\begin{aligned}
&19200M^5\mu W_x^5\left(48MB\mu W_x^2+24M\mu W_{xx}+a_2\right)R^{12}(W)\\
&+14175M^5\mu W_x^5a_1R^{11}(W)+\cdots=0.
\end{aligned}
\tag{21}
$$

By setting the coefficients of $R^{12}(W)$ and $R^{11}(W)$ equal to zero, we obtain

$$
a_2=-48MB\mu W_x^2-24M\mu W_{xx},\quad a_1=0. \tag{22}
$$

Substituting the expressions of a_2 and a_1 into Eq. (21), we

have

$$9216\mu M^5W_x^5\Big(4\mu B^2W_x^2+12\mu BW_{xx}+a_0-\mu+\frac{4\mu W_{xxx}}{W_x}-\frac{3\mu W_{xx}^2}{W_x^2}-\frac{W_t}{W_x}\Big)R^{10}(W)+\cdots+h_1R(W)+h_0=0. \quad (23)$$

By setting the coefficient of $R^{10}(W)$ equal to zero, we obtain

$$a_0 = -4\mu B^2W_x^2-12\mu BW_{xx}+\mu-\frac{4\mu W_{xxx}}{W_x}+\frac{3\mu W_{xx}^2}{W_x^2}+\frac{W_t}{W_x}. \quad (24)$$

Substituting the above expression of a_0 into Eq. (23), we find that h_9, h_7, h_5, h_3, and h_1 are all zero, h_8, h_6, h_4, h_2, and h_0 can be reduced to the following two equations:

$$W_xW_tW_{xx}-4\mu W_xW_{xx}W_{xxx}-4\mu B^2W_x^4W_{xx}-W_x^2W_{xt}+\mu W_x^2W_{xxxx}+3\mu W_{xx}^3=0, \quad (25)$$

$$-W_x^2W_{tt}+\mu W_x^2W_{xxxt}-4\mu B^2W_x^4W_{xt}+3\mu W_{xx}^2W_{xt}-\mu W_xW_{xt}W_{xxx}-3\mu W_xW_{xx}W_{xxt}+W_xW_{xt}W_t=0. \quad (26)$$

So, solving Eqs. (25) and (26) is essential for obtaining explicit solutions to Eq. (1). In the following, we consider the following cases.

4.1. W is a linear function

In this case, we take $W=k_1x+k_2t+k_0$, where k_0, k_1, and k_2 are constants, then Eqs. (25) and (26) are automatically satisfied. Under this circumstance,

$$a_0=-4\mu B^2k_1^2+\mu+\frac{k_2}{k_1},\ \ a_1=0,$$
$$a_2=-48\mu BMk_1^2,\ \ a_3=0,\ \ a_4=-48\mu M^2k_1^2,$$

and Eq. (1) has the following solution:

$$u = -4\mu B^2k_1^2+\mu+\frac{k_2}{k_1}-48\mu BMk_1^2R^2(W)-48\mu M^2k_1^2R^4(W). \quad (27)$$

Analytical solutions of Eq. (1) for different choices of B and M are presented as follows.

4.1.1. An exponential function solution

When $B\neq M$, there is an exponential function solution of Eq. (1)

$$u = \frac{1}{-k_1\left(\mathrm{e}^{-2BW}K_1B-M\right)^2}\Big(4B^4K_1^2k_1^3\mu\,\mathrm{e}^{-4BW}+40B^3K_1Mk_1^3\mu\,\mathrm{e}^{-2BW}-B^2K_1^2k_1\mu\,\mathrm{e}^{-4BW}+4M^2k_1^3\mu B^2-B^2K_1^2k_2\,\mathrm{e}^{-4BW}+2BK_1Mk_1\mu\,\mathrm{e}^{-2BW}+2BK_1k_2M\,\mathrm{e}^{-2BW}-M^2k_1\mu-M^2k_2\Big), \quad (28)$$

where K_1 is a constant and $K_1\neq 0$.

4.1.2. A singular kink soliton solution

When $B=M$, Eq. (1) admits a singular kink soliton solution

$$\begin{aligned}u = &-\frac{1}{k_1[\tanh(BW)K_2-K_2+\tanh(BW)+1]^2}\\&\times\Big\{4B^2K_2^2k_1^3\mu\tanh^2(BW)-8B^2K_2^2k_1^3\mu\tanh(BW)\\&-40B^2K_2k_1^3\mu\tanh^2(BW)+4B^2k_1^3\mu\tanh^2(BW)\\&+8B^2k_1^3\mu\tanh(BW)-K_2^2k_1\mu\tanh^2(BW)\\&+2K_2^2k_1\mu\tanh(BW)-2K_2k_1\mu\tanh^2(BW)\\&+4B^2k_1^3\mu+4B^2K_2^2k_1^3\mu+40B^2K_2k_1^3\mu-k_1\mu\\&-k_2-k_2\tanh^2(BW)-2k_2\tanh(BW)\\&-k_1\mu\tanh^2(BW)+2K_2^2k_2\tanh(BW)\\&-K_2^2k_2\tanh^2(BW)-2K_2k_2\tanh^2(BW)\\&-2k_1\mu\tanh(BW)-K_2^2k_2+2K_2k_2\\&-K_2^2k_1\mu+2K_2k_1\mu\Big\},\end{aligned} \quad (29)$$

where K_2 is an arbitrary constant.

4.2. W is not a linear function

In this case, we take

$$W=k_1x+k_2t+k_0+H(\eta),\eta=n_1x+n_2t+n_0, \quad (30)$$

with n_0, n_1, and n_2 being undetermined constants. Let $\varphi=H'$, and φ satisfy the Jacobian elliptic equation (13).

Substituting Eqs. (30) and (13) into Eqs. (25) and (26), we obtain

$$c_0=-\frac{\left(4B^2k_1^3-c_1n_1^3+c_2k_1n_1^2-c_3k_1^2n_1\right)k_1}{n_1^4},$$
$$c_4=4B^2,$$
$$k_2=16B^2k_1^3\mu-\mu n_1^3c_1+2n_1^2c_2k_1\mu-3n_1c_3k_1^2\mu+\frac{k_1n_2}{n_1},$$

where B, M, k_0, k_1, n_0, n_1, n_2, c_1, c_2, and c_3 are free constants. If we take appropriate values of these free constants, we can obtain new types of interaction solutions of Eq. (1).

4.2.1. An exponential-elliptic interaction solution

If we take $B=m/2$, $n_1=k_1$, $c_1=c_3=0$, $c_2=-1-m^2$, then $c_0=1$, $c_4=m^2$, $k_2=2(m^2-1)\mu k_1^3+n_2$. According to Appendix A and the results in Refs. [38,39], exact solutions of Eqs. (17) and (13) are as follows:

$$R(W)=\frac{\sqrt{m}}{\sqrt{K_1m\,\mathrm{e}^{-mW}-2M}},\ \ \varphi(\eta)=\mathrm{sn}(\eta,m).$$

Then, $H(\eta)=\frac{1}{m}\ln\left(\mathrm{dn}(\eta,m)-m\mathrm{cn}(\eta,m)\right)$, and a new interaction solution between the exponential wave and the elliptic wave of Eq. (1) is

$$
\begin{aligned}
u = & \frac{1}{k_1}\Big\{m^2K_1^2\,\mathrm{e}^{-2m\xi}+4K_1Mm\left(m\mathrm{cn}(\eta,m)-\mathrm{dn}(\eta,m)\right)\mathrm{e}^{-m\xi}-4M^2\left(2m^2\mathrm{sn}^2(\eta,m)+2m\mathrm{cn}(\eta,m)\mathrm{dn}(\eta,m)-m^2-1\right)\Big\}^{-1}\\
&\times\Big\{\left[-K_1^2m^2\left(6k_1^3m^2\mu\mathrm{sn}^2(\eta,m)\right)+6k_1^3m\mu\mathrm{cn}(\eta,m)\mathrm{dn}(\eta,m)-k_1^3m^2\mu-k_1^3\mu-k_1\mu-n_2\right]\mathrm{e}^{-2m\xi}\\
&+4K_1Mm\left(12k_1^3m^2\mu\mathrm{sn}(\eta,m)+7k_1^3m^2\mu+k_1^3\mu+k_1\mu+n_2\right)\left(m\mathrm{cn}(\eta,m)-\mathrm{dn}(\eta,m)\right)\mathrm{e}^{-m\xi}\\
&+\left(2k_1^3m^2\mu+2k_1^3\mu-k_1\mu-n_2\right)\left(8M^2m^2\mathrm{sn}^2(\eta,m)+8M^2m\mathrm{cn}(\eta,m)\mathrm{dn}(\eta,m)\right)\\
&+4M^2\left(k_1^3m^4\mu-10k_1^3m^2\mu+k_1^3\mu+k_1m^2\mu+m^2n_2+k_1\mu+n_2\right)\Big\},
\end{aligned}
\tag{31}
$$

where $\xi=k_1x+(2m^2-2)\mu k_1^3t+n_2t+k_0$, $\eta=k_1x+n_2t+n_0$.

The new exponential-elliptic solution can model dissipative or unstable systems such as lossy optical fibers or collisional plasma.[40,41]

4.2.2. A singular-soliton–elliptic interaction solution

If we take $B=M=m/2$, $n_1=k_1$, $c_1=c_3=0$, $c_2=-1-m^2$, then $c_0=1$, $c_4=m^2$, $k_2=(2m^2-2)\mu k_1^3+n_2$. According to Appendix A and the results in Refs. [38,39], exact solutions of Eqs. (17) and (13) are as follows:

$$
R(W)=\sqrt{\frac{1-\tanh\left(\frac{-mW}{2}\right)}{(K_2-1)+(K_2+1)\tanh\left(\frac{-mW}{2}\right)}},
$$
$$
\varphi(\eta)=\mathrm{sn}(\eta,m).
$$

Then, $H(\eta)=(1/m)\ln\left(\mathrm{dn}(\eta,m)-m\mathrm{cn}(\eta,m)\right)$, and a new interaction solution between the soliton wave and the mixed elliptic waves of Eq. (1) is

$$
\begin{aligned}
u = & \frac{1}{\left((K_2+1)^2\tanh^2(\frac{m}{2}W)+\left(2-2K_2^2\right)\tanh(\frac{m}{2}W)+(K_2-1)^2\right)k_1}\Big\{\Big[-6k_1^3m^2\mu\left(K_2^2+1\right)\mathrm{sn}(\eta,m)^2+24K_2k_1^3m^2\mu\mathrm{sn}(\eta,m)\\
&-6k_1^3\mu m\left(K_2^2-1\right)\mathrm{cn}(\eta,m)\mathrm{dn}(\eta,m)+\mu\left(K_2^2m^2+14K_2m^2+K_2^2+m^2+2K_2+1\right)k_1^3+\mu\left(K_2+1\right)^2k_1+n_2\left(K_2+1\right)^2\Big]\\
&\times\tanh\left(\frac{m}{2}W\right)^2+\Big[12k_1^3m^2\mu\left(K_2^2-1\right)\mathrm{sn}(\eta,m)^2+12k_1^3\mu m\left(K_2^2+1\right)\mathrm{dn}(\eta,m)\mathrm{cn}(\eta,m)-2\left(\mu\left(m^2+1\right)k_1^3+\mu k_1+n_2\right)\\
&\times\left(K_2^2-1\right)\Big]\tanh(\frac{m}{2}W)-6k_1^3m^2\mu\left(K_2^2+1\right)\mathrm{sn}(\eta,m)^2-24K_2k_1^3m^2\mu\mathrm{sn}(\eta,m)-6k_1^3\mu m\left(K_2^2-1\right)\mathrm{cn}(\eta,m)\mathrm{dn}(\eta,m)\\
&+\mu\left(K_2^2m^2-14K_2m^2+K_2^2+m^2-2K_2+1\right)k_1^3+\mu\left(K_2-1\right)^2k_1+n_2\left(K_2-1\right)^2\Big\},
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
W &= k_1x+(2m^2-2)\mu k_1^3t+n_2t+k_0\\
&\quad+\frac{1}{m}\ln(\mathrm{dn}(\eta,m)-m\mathrm{cn}(\eta,m)),\\
\eta &= k_1x+n_2t+n_0.
\end{aligned}
$$

This solution represents the coexistence of three different types of elliptic waves with a singular kink soliton. It can model complex wave interactions in multicomponent systems, where solitary waves interact with different periodic backgrounds.

5. Conclusion

In this research, we propose the improved CRE method and apply it to the new Painlevé integrable fifth-order equation. Employing the Riccati equation and the Bernoulli equation, many new traveling wave solutions and interaction solutions have been derived. The traveling wave solutions include soliton solutions, exponential function solutions, trigonometric periodic solutions, and other types of solutions. Except for u_{11}, u_{12}, u_{13}, and u_{14}, all the other traveling wave solutions are different from those in Ref. [32]. The exponential wave solution (28) and the singular soliton wave solution (29) cannot be constructed by the CRE method. Interaction solutions include soliton–soliton, soliton–elliptic, and interaction solutions between tangent periodic waves and elliptic waves. Interactions between the solitary wave and the elliptic wave have been illustrated by three- and two-dimensional graphs. The exponential–elliptic interaction solution (31) and the singular–soliton–elliptic interaction solution (32) are new types of interaction solutions that have not been derived previously. It is worth mentioning that some exact solutions (e.g., (31) and (32)) have singularity, which is inherent to the construction of this class of solutions. For physical applications, we restrict the solutions to regions where they remain finite. The results underscore the significant role of the improved CRE method in deriving interaction solutions for NEEs. The improved CRE method can be applied to other NEEs arising in physics.

Appendix A

Table 1. Solutions of $\mathrm{d}\phi/\mathrm{d}\xi = A + B\phi + M\phi^N$.

$N = 2\ (\phi_1 - \phi_{12})$	
$B \neq 0, M = 0$	$\phi_1 = \dfrac{\exp(B\xi) - A}{B}$
$B = 0, A = 0.5, M = -0.5$	$\varphi_{2.1} = \coth(\xi) \pm \mathrm{csch}(\xi)$, $\varphi_{2.2} = \tanh(\xi) \pm \mathrm{isech}(\xi)$ $\varphi_{2.3} = \dfrac{\tanh(\xi)}{1 \pm \mathrm{sech}(\xi)}$, $\varphi_{2.4} = \dfrac{\coth(\xi)}{1 \pm \mathrm{icsch}(\xi)}$
$B = 0, A = 0.5, M = 0.5$	$\varphi_{3.1} = \tan(\xi) \pm \sec(\xi)$, $\varphi_{3.2} = \csc(\xi) - \cot(\xi)$ $\varphi_{3.3} = \dfrac{\tan(\xi)}{1 \pm \sec(\xi)}$
$B = 0, A = -0.5, M = -0.5$	$\varphi_{4.1} = \cot(\xi) \pm \csc(\xi)$, $\varphi_{4.2} = \sec(\xi) - \tan(\xi)$ $\varphi_{4.3} = \dfrac{\cot(\xi)}{1 \pm \csc(\xi)}$
$B = 0, A = 1, M = -1$	$\varphi_{5.1} = \tanh(\xi)$, $\varphi_{5.2} = \coth(\xi)$
$B = 0, A = 1, M = 1$	$\phi_6 = \tan(\xi)$
$B = 0, A = -1, M = -1$	$\phi_7 = \cot(\xi)$
$B = -2, A = 1, M = 2$	$\phi_8 = \dfrac{\tan(\xi)}{1 + \tan(\xi)}$
$B = 2, A = 1, M = 2$	$\phi_9 = \dfrac{\tan(\xi)}{1 - \tan(\xi)}$
$B = 2, A = -1, M = -2$	$\phi_{10} = \dfrac{\cot(\xi)}{1 + \cot(\xi)}$
$B = -2, A = -1, M = -2$	$\phi_{11} = \dfrac{\cot(\xi)}{1 - \cot(\xi)}$
$B = 0, A = 0, M \neq 0$	$\phi_{12} = \dfrac{-1}{M\xi + m_0}$
$N \neq 0, N \neq 1, N \neq 2\ (\phi_{13} - \phi_{14})$	
$B \neq M$	$\varphi_{13} = \left[\dfrac{-M}{B} + \dfrac{K_1}{\exp[B(N-1)\xi]}\right]^{\frac{1}{1-N}}$
$B = M$	$\varphi_{14} = \left[\dfrac{(K_2 - 1) + (K_2 + 1)\tanh\left(\frac{B(1-N)\xi}{2}\right)}{1 - \tanh\left(\frac{B(1-N)\xi}{2}\right)}\right]^{\frac{1}{1-N}}$

Here $i^2 = 1, K_1$ and K_2 are constants.

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