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SPECIAL TOPIC — A 40th Anniversary Celebration: Physics Research and Innovation at Ningbo University

Group invariant decompositions, elliptic and Painlevé solitons of the AKNS systems and NLS equations

Man Jia(贾曼)[1], Fa-Ren Wang(王法仁)[1,2], and S. Y. Lou(楼森岳)[1,2,†]

[1] *School of Physical Science and Technology, Ningbo University, Ningbo* 315211, *China*
[2] *Institute of Fundamental Physics and Quantum Technology, Ningbo University, Ningbo* 315211, *China*

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Decomposition algorithms are powerful methods for solving nonlinear mathematical physics problems. In this paper, we propose a novel decomposition method based on group-invariant space-time rather than the natural space-time. Taking the Ablowitz–Kaup–Newell–Segur (AKNS) system as a simple illustrative example, we apply the method to find interacting solutions among different types of nonlinear waves, including solitary waves, periodic cnoidal waves, Painlevé waves, rational/irrational algebraic waves, error-function/Kummer waves, and so on.

Keywords: group invariant symmetry decompositions, residual symmetries, Painlevé IV solitons, Ablowitz–Kaup–Newell–Segur (AKNS) systems, irrational algebraic solitons

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1. Introduction

The study of nonlinear partial differential equations (NLEs) has been a cornerstone of modern mathematical physics, providing the fundamental language for describing complex phenomena in fields ranging from fluid dynamics and plasma physics to nonlinear optics and quantum field theory. Among the vast landscape of NLEs, integrable systems occupy a privileged position. These are equations endowed with a rich mathematical structure that, in a precise sense, makes them "exactly solvable", despite their inherent nonlinearity. This integrability manifests through a constellation of remarkable features: the existence of an infinite number of conserved quantities/symmetries, a Lax pair formulation, the applicability of the inverse scattering transform (IST), and the presence of stable, particle-like solutions known as solitons.

The concept of symmetry is inextricably linked to integrability. In the context of differential equations, symmetries are transformations that map solutions to other solutions. The pioneering work of Sophus Lie laid the groundwork for the systematic analysis of continuous symmetries through Lie group theory, a framework comprehensively treated by Olver (1993).[1] For integrable NLEs, the symmetry structure is exceptionally rich. They often possess infinite-dimensional symmetry algebras, which are intimately connected to their infinite sets of conservation laws via Noether's theorem. A pivotal breakthrough was Peter Lax's discovery that many integrable equations arise as the compatibility condition of a linear system—the Lax pair.[2] This formulation not only provides a method for solving the initial value problem via the IST but also reveals a deep hidden symmetry: the equation can be viewed as an isospectral flow on a space of linear operators. The study of these symmetries, including master symmetries and recursion operators, and their connection to the IST framework, has been extensively developed in foundational works such as Ablowitz and Segur (1981).[3]

The symmetry integrability is related to several powerful integrable theories. Two of the most influential are the Hirota bilinear method and the Darboux transformations (DT) and/or B"acklund transformations (BT). The Hirota method, masterfully presented in Hirota (2004),[4] introduces a clever bilinear dependent variable transformation that converts a nonlinear equation into a quadratic form in terms of a τ-function. In this bilinear form, soliton solutions appear as elegant polynomials of exponentials, and the algebraic structure of the equation becomes transparent. By means of Hirota's bilinear form, the whole hierarchy of local and nonlocal symmetries can be written in some simple compact forms.[5] On the other hand, BT provide a non-linear superposition principle: they generate new solutions from known ones by solving a pair of related first-order equations. Iterative application of these transformations, often crystallized in DT for the associated linear spectral problem, allows for the systematic construction of multi-soliton, breather, rogue waves and even some cnoidal wave solutions, as detailed in Matveev and Salle (1991).[6] The nonlocal symmetries are just the infinitesimal forms of DT/BT.[7–10] By means of nonlocal symmetries, one can find finite Levi

†Corresponding author. E-mail: lousenyue@nbu.edu.cn

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transformations.[11–13] These methods decompose the problem of solving the nonlinear differential equation (PDE) into more manageable algebraic or linear steps.

Unraveling the intricate structure of integrable systems and constructing their solutions has motivated the development of powerful decomposition techniques that transcend linearization. Beyond the celebrated Hirota bilinear method and DT/BT approaches, which map nonlinearities into bilinear forms or provide nonlinear superposition principles, more profound algebraic-geometric decompositions have emerged. One influential approach is the nonlinearization method pioneered by Cao and colleagues,[14,15] where the Lax pair of an integrable hierarchy (e.g., the KdV hierarchy) is constrained by its own eigenfunctions. This procedure decomposes the infinite-dimensional integrable flow into finite-dimensional, completely integrable Hamiltonian systems (e.g., the Neumann or Bargmann systems), thereby linking soliton equations to classical integrable mechanics.

Closely related are symmetry constraints, which leverage the infinite-dimensional symmetry algebras of integrable systems. By imposing constraints that equate the symmetry (or potential) with a function of the eigenfunctions, the original $(1+1)$-dimensional PDE can be decomposed into two commuting finite-dimensional Hamiltonian flows or a set of $(0+1)$-dimensional ordinary differential equations. This method provides a systematic way to generate finite-dimensional integrable subsystems and particular solutions, as independently established by Cheng and Li[16] and Konopelchenko *et al.*[17] The nonlinearization approach[14] or the symmetry constraints method[16] is also called the variable separation approach.[18]

For broader applicability, including to some non-integrable models, the concept of formal variable separation was developed to decompose a multi-dimensional field into lower-dimensional components. While not guaranteeing integrability, it can facilitate the solution procedure for a wide class of equations, as demonstrated in early work by Lou and Chen.[19] More recently, the paradigm of integrable decomposition has been articulated, where a given nonlinear system is shown to be a consistent combination of two or more simpler, exactly solvable subsystems. This framework not only elucidates the system's structure but also facilitates the construction of new solutions through linear superposition of solutions to the decomposed parts, a principle exemplified in studies of the BKP hierarchy and its dispersionless limit by Hao and Lou.[20–22]

A remarkable and more stringent criterion for integrability is the Painlevé property, which demands that all movable singularities of solutions in the complex plane are poles (i.e., no branch points or essential singularities that depend on initial conditions). This property, originally studied for ordinary differential equations (ODEs) by Paul Painlevé and his school (classically summarized in Ince),[23] led to the classification of all second-order ODEs with this property. The result was the discovery of six new nonlinear ODEs whose general solutions define the Painlevé transcendents—nonlinear special functions that are now considered the ODE analogues of integrable PDEs. The connection to PDEs is forged through similarity reduction. When a symmetry (often scaling) is used to reduce an integrable PDE to an ODE, this resulting ODE is frequently one of the Painlevé equations or their higher-order/generalized versions, a connection extensively explored by Ablowitz and Clarkson.[24] This Painlevé test (or the Weiss–Tabor–Carnevale method for PDEs[25]) has become a powerful heuristic for detecting integrability, as discussed in the collective volume by Conte.[26] From the Painlevé test, one can find the so-called residual symmetries.[27] Applying the integrable decomposition method to $(1+1)$-dimensional cases, a novel method, the deformation algorithm, can be established to find higher-dimensional integrable systems.[28,29]

The traditional 'sech^2' profile solitons of the Korteweg-de Vries (KdV) equation exist on a constant background. Recent decades have witnessed the exploration of integrable systems on non-constant, spatially periodic backgrounds. This leads to the fascinating concept of elliptic solitons, where the soliton's profile and the background are expressed in terms of Jacobi elliptic functions (e.g., 'cn', 'sn', 'dn'). These solutions, which interpolate between periodic cnoidal waves and traditional solitons, reveal a deeper algebro-geometric structure connected to finite-gap integration on Riemann surfaces, as presented in the rigorous work of Gesztesy and Holden.[30]

Perhaps even more intriguing is the synthesis of the aforementioned ideas in the concept of Painlevé solitons or solutions with Painlevé asymptotics. These are solutions to integrable PDEs where, in certain regions of space-time, the profile is governed not by a hyperbolic function but by a Painlevé transcendent. A quintessential example arises in the study of asymptotic behavior. The second Painlevé equation (PII) famously describes the universal transition shape (the nonlinear modulation) near the edge of a rarefaction wave for the KdV equation in the small-dispersion limit[31] or in the semiclassical limit of the focusing nonlinear Schrödinger equation.[32] In this sense, the Painlevé functions act as "nonlinear special functions" that govern critical transitional patterns, bridging different asymptotic regions dominated by elliptic or solitonic behavior. These objects sit at the confluence of symmetry, reduction, and transcendentality, representing a frontier in the understanding of integrable systems.

In Section 2 of this paper, we outline the symmetry invariant decomposition method for general nonlinear systems. In Section 3, taking the celebrated AKNS system as a typical example, Lie point symmetries are derived in a standard manner, while nonlocal residual symmetries are obtained via

the truncated Painlevé expansion. To derive group-invariant solutions associated with these nonlocal symmetries, the localization of nonlocal symmetries is presented in Section 4. By combining both local and nonlocal symmetries, the corresponding symmetry invariant decompositions are established in Section 5. Section 6 focuses on plotting several special types of JacobiPQ solitons, where P and Q denote any two elements from the set C, D, N, S—such as JacobiSN, JacobiCN, and JacobiDN solitons, among others. In Section 7, Painlevé solitons—including rational/irrational algebraic solitons and Whittaker/cylindrical solitons—are investigated. The final section provides a brief summary and further discussions.

2. General aspect of the symmetry invariant decomposition method

For a general nonlinear autonomous $(1+1)$-dimensional PDE system

$$\begin{aligned} &F(\boldsymbol{u}) \equiv F(\boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{u}_x, \boldsymbol{u}_{xx}, \dots) = 0, \\ &\boldsymbol{u} = \{u_1, u_2, \dots, u_n\}, \end{aligned} \tag{1}$$

its Lie point symmetry takes the form

$$\boldsymbol{\sigma} = \boldsymbol{U} - X\,\boldsymbol{u}_x - T\,\boldsymbol{u}_t, \quad \boldsymbol{U} = \{U_1, U_2, \dots, U_n\}, \tag{2}$$

where $X \equiv X(x, t, \boldsymbol{u})$, $T \equiv T(x, t, \boldsymbol{u})$, and $U_i \equiv U_i(x, t, \boldsymbol{u})$ for $i = 1, 2, \dots, n$ are functions depending only on $\boldsymbol{u}$ and the independent variables $\{x, t\}$.

The group-invariant solutions of Eq. (1) associated with the symmetry Eq. (2) are obtained by solving the coupled system

$$F(\boldsymbol{u}) = 0 \ \text{ and } \ \boldsymbol{\sigma} = 0.$$

From the quasilinear Eq. (2) with $\boldsymbol{\sigma} = 0$, a group-invariant variable ξ can be introduced via

$$\mathrm{d}\xi = X^{-1}\,\mathrm{d}x - T^{-1}\,\mathrm{d}t, \tag{3}$$

up to an arbitrary function of ξ; equivalently, for simplicity, one may apply a transformation $\xi \to \zeta(\xi)$ with a suitable choice of ζ.

For instance, for the spacetime translation symmetry $X = x_0$, $T = t_0$ with arbitrary constants $\{x_0,\ t_0\}$, we obtain $\xi = t_0 x - x_0 t$. For the spacetime scaling invariance with $X = x$, $T = at$ and constant a, one may take $\xi = xt^{-1/a}$.

By means of the group-invariant variable ξ, we can make the spacetime transformation

$$\{x,\ t\} \longrightarrow \{\xi,\ t\}, \tag{4}$$

$$\boldsymbol{u}(x,\ t) \longrightarrow \boldsymbol{u}(\xi,\ t). \tag{5}$$

Under the transformation Eq. (5), the group-invariant condition $\boldsymbol{\sigma} = 0$ is changed to a simple t-dynamic system

$$\boldsymbol{u}_t = \frac{1}{T}(\boldsymbol{U} - X\,\boldsymbol{u}_x), \quad \boldsymbol{u} = \boldsymbol{u}(\xi,\ t). \tag{6}$$

By using the transformation Eq. (5) and the t-dynamical system Eq. (6), the original nonlinear PDE Eq. (1) becomes a ξ-dynamical system

$$F(\boldsymbol{u}) = 0 \to F'(x,\ t, \boldsymbol{u}, \boldsymbol{u}_\xi, \boldsymbol{u}_{\xi\xi}, \dots) = 0, \quad \boldsymbol{u} = \boldsymbol{u}(\xi,\ t). \tag{7}$$

If the original PDE system Eq. (1) is integrable, both the t-dynamical system Eq. (6) and the ξ-dynamical system must be consistent.

3. Lie point symmetries and residual symmetries of the AKNS system

For concreteness, we apply the symmetry invariant decomposition method to the celebrated AKNS system[33]

$$\begin{aligned} &p_t + \frac{1}{2}bip_{xx} - \mathrm{i}p^2 q = 0, \\ &q_t - \frac{1}{2}biq_{xx} + \mathrm{i}q^2 p = 0, \quad b^2 = 1, \ \ \mathrm{i}^2 = -1. \end{aligned} \tag{8}$$

If q is taken as the complex conjugate of p (i.e., $q = p^*$), the AKNS system reduces to the widely applicable nonlinear Schrödinger (NLS) equation

$$p_t + \frac{1}{2}bip_{xx} - \mathrm{i}|p|^2 p = 0, \quad \mathrm{i} = \sqrt{-1}, \ \ b^2 = 1. \tag{9}$$

Here, $b = 1$ corresponds to the normal-dispersion NLS equation, while $b = -1$ corresponds to the anomalous-dispersion case.

The NLS Eq. (9) describes the evolution of slowly varying wave envelopes in dispersive media and has found unparalleled applications across diverse fields, including nonlinear optics,[34,35] plasma physics,[36] and Bose-Einstein condensates.[37] Through the IST method, the NLS equation yields a rich solution structure. The most well-known are its fundamental solitons—localized, particle-like waves that retain their form and speed upon interaction. These solutions arise from the discrete spectrum of the associated scattering problem and are typically constructed on a vanishing (or constant) background. In many physically relevant and mathematically interesting scenarios, however, solitons propagate on non-trivial, spatially or temporally varying backgrounds. This gives rise to a broader class of solutions that intertwine solitonic features with the complex structure of the background field.[38–40]

A symmetry of the AKNS system, denoted by $\boldsymbol{\sigma} = \begin{pmatrix} \sigma^p \\ \sigma^q \end{pmatrix}$, is defined as a solution of the corresponding linearized equations

$$\begin{aligned} &\sigma^p_t + \frac{1}{2}bi\sigma^p_{xx} - \mathrm{i}p^2\sigma^q - 2\mathrm{i}pq\sigma^p = 0, \\ &\sigma^q_t - \frac{1}{2}bi\sigma^q_{xx} + \mathrm{i}q^2\sigma^p + 2\mathrm{i}qp\sigma^q = 0, \end{aligned} \tag{10}$$

which ensures that the original system Eq. (8) remains invariant under the infinitesimal transformation

$$p \to p+\varepsilon\sigma^p, \quad q \to q+\varepsilon\sigma^q,$$

with ε being an infinitesimal parameter.

The Lie point symmetry of the AKNS system can be expressed as

$$\sigma^p = P - Xp_x - Tp_t, \quad \sigma^q = Q - Xq_x - Tq_t, \tag{11}$$

where $P = P(x,\ t,\ p,\ q)$, $Q = Q(x,\ t,\ p,\ q)$, $X = X(x,\ t,\ p,\ q)$, and $T = T(x,\ t,\ p,\ q)$ are smooth functions of the indicated variables.

Substituting Eq. (11) into the symmetry condition Eq. (10), eliminating p_t and q_t via Eq. (8), and setting the coefficients of the differential polynomials in p and q to zero yields the determining equations for $P,\ Q,\ X$, and T

$$\begin{aligned}
&P_q = Q_p = X_p = X_q = T_x = T_p = T_q = P_{pp} = Q_{qq} \\
&\quad = T_t - 2X_x = 0, \\
&X_t - \tfrac{1}{2}\mathrm{i}bX_{xx} + \mathrm{i}bQ_{xq} = X_t + \tfrac{1}{2}\mathrm{i}bX_{xx} - \mathrm{i}bP_{xp} = 0, \\
&P_t - \mathrm{i}\left[p^2q(2X_x - P_p) + 2pqP + p^2Q - \tfrac{b}{2}P_{xx}\right] = 0, \\
&Q_t - \mathrm{i}\left[pq^2(Q_q - 2X_x) - q^2p - 2pqP + \tfrac{b}{2}Q_{xx}\right] = 0.
\end{aligned} \tag{12}$$

The general solution of Eq. (12) is given by

$$\begin{aligned}
&X = cx + at + x_0, \quad T = 2ct + t_0, \quad P = -\mathrm{i}abxp + (\phi_0 - c)p, \\
&Q = \mathrm{i}abxq - (\phi_0 + c)q,
\end{aligned} \tag{13}$$

where $c,\ a,\ x_0,\ t_0$, and ϕ_0 are arbitrary constants.

Consequently, the explicit form of the Lie point symmetries reads

$$\begin{aligned}
\sigma^p &= -\mathrm{i}abxp + (\phi_0 - c)p - (cx + at + x_0)p_x - (2ct + t_0)p_t, \\
\sigma^q &= \mathrm{i}abxq - (\phi_0 + c)q - (cx + at + x_0)q_x \\
&\quad -(2ct + t_0)q_t,
\end{aligned} \tag{14}$$

which constitutes a combination of space translation (the x_0 term), time translation (the t_0 term), spacetime scaling (the c term), Galilean transformation (the a term), and scaling of the fields p and q (the ϕ_0 term).

To obtain infinitely many generalized local symmetries, one may apply the known recursion operator of the AKNS system.[1,41] For an integrable system, besides the infinitely many generalized local symmetries, there also exist infinitely many nonlocal symmetries, which can be derived via inverse recursion operators,[41–44] truncated Painlevé analysis,[27] infinitesimal Darboux/Bäcklund transformations,[7] or the formal series symmetry approach.[45]

For the AKNS system Eq. (8), a special type of nonlocal symmetry can be obtained via the truncated Painlevé expansion

$$p = c_0\frac{\sigma^{p_1}}{f} + p_1, \quad q = c_0\frac{\sigma^{q_1}}{f} + q_1, \quad c_0^2 = b, \tag{15}$$

where

$$\sigma^{p_1} = f_x\,\mathrm{e}^{\mathrm{i}u}, \quad \sigma^{q_1} = f_x\,\mathrm{e}^{-\mathrm{i}u}, \tag{16}$$

$$p_1 = -c_0^{-1}\left(\mathrm{i}\frac{f_t}{f_x} + \frac{bf_{xx}}{2f_x} - \mathrm{i}b\lambda\right)\mathrm{e}^{\mathrm{i}u},$$

$$q_1 = c_0^{-1}\left(\mathrm{i}\frac{f_t}{f_x} - \frac{bf_{xx}}{2f_x} + \mathrm{i}b\lambda\right)\mathrm{e}^{-\mathrm{i}u}, \tag{17}$$

$$u_x = \frac{2bf_t}{f_x} + \lambda, \quad u_t = -\frac{bf_{xxx} + 8\lambda f_t}{2f_x} + \frac{3b}{4}\frac{f_{xx}^2 + 4f_t^2}{f_x^2}, \tag{18}$$

$$C_t = 3CC_x - \frac{1}{4}S_x + 2b\lambda C_x, \quad C \equiv \frac{f_t}{f_x}, \quad S \equiv \frac{f_{xxx}}{f_x} - \frac{3}{2}\frac{f_{xx}^2}{f_x^2}. \tag{19}$$

The Eq. (15) constitutes a special Bäcklund transformation relating the solutions $\{p,q\}$ and $\{p_1,q_1\}$. The symmetry

$$\sigma = \begin{pmatrix}\sigma^{p_1}\\ \sigma^{q_1}\end{pmatrix},$$

is precisely the residual symmetry associated with the solution $\{p_1,q_1\}$.

The singular manifold equation for f, namely the Schwarzian NLS Eq. (19), is invariant under the Möbius transformation

$$f \longrightarrow \frac{af+b}{cf+d}, \quad ad - bc \neq 0. \tag{20}$$

This Möbius invariance Eq. (20) corresponds to a particular Lie point symmetry of the Schwarzian NLS Eq. (19)

$$\sigma^f = a_2f^2 + a_1f + a_0, \tag{21}$$

where the term proportional to f^2 is directly linked to the nonlocal symmetry Eq. (16) for the fields p and q.

4. Localization of nonlocal symmetries of the AKNS system

In order to find the group-invariant solutions associated with the nonlocal symmetry Eq. (22) with $\{p_1,\ q_1\} \to \{p,\ q\}$, namely

$$\sigma^p = f_x\,\mathrm{e}^{\mathrm{i}u}, \quad \sigma^q = f_x\,\mathrm{e}^{-\mathrm{i}u}, \tag{22}$$

$$p = -c_0^{-1}\left(\mathrm{i}\frac{f_t}{f_x} + \frac{bf_{xx}}{2f_x} - \mathrm{i}b\lambda\right)\mathrm{e}^{\mathrm{i}u},$$

$$q = c_0^{-1}\left(\mathrm{i}\frac{f_t}{f_x} - \frac{bf_{xx}}{2f_x} + \mathrm{i}b\lambda\right)\mathrm{e}^{-\mathrm{i}u}, \tag{23}$$

the so-called localization procedure[9,10,39,44,46] should be employed.

From Eq. (22) we see that when the fields p and q undergo the infinitesimal transformation

$$(p,\ q) \longrightarrow (p,\ q) + \varepsilon(\sigma^p,\ \sigma^q),$$

the auxiliary quantities f, $g\equiv f_x$, u and λ must undergo corresponding transformations

$$(f,\ g,\ u,\ \lambda)\ \longrightarrow\ (f,\ g,\ u,\ \lambda)+\varepsilon(\sigma^f,\ \sigma^g,\ \sigma^u,\ \delta),$$

where the functions $(\sigma^f,\ \sigma^g,\ \sigma^u,\ \delta)$ are to be determined from the linearized equations of Eqs. (23) and (18), and the relation $f_x=g$

$$\begin{aligned}
\sigma^p &= -c_0^{-1}\left(\mathrm{i}\frac{\sigma^f_t}{f_x}-\mathrm{i}\frac{f_t\sigma^f f_x}{f_x^2}+\frac{b\sigma^f_{xx}}{2f_x}-\frac{bf_{xx}\sigma^f_x}{2f_x^2}-\mathrm{i}b\delta\right)\mathrm{e}^{\mathrm{i}u}\\
&\quad -c_0^{-1}\left(\mathrm{i}\frac{f_t}{f_x}+\frac{bf_{xx}}{2f_x}-\mathrm{i}b\lambda\right)\mathrm{i}\sigma^u\mathrm{e}^{\mathrm{i}u},\\
\sigma^q &= c_0^{-1}\left(\mathrm{i}\frac{\sigma^f_t}{f_x}-\mathrm{i}\frac{f_t\sigma^f_x}{f_x^2}-\frac{b\sigma^f_{xx}}{2f_x}+\frac{bf_{xx}\sigma^f_x}{2f_x^2}+\mathrm{i}b\delta\right)\mathrm{e}^{-\mathrm{i}u}\\
&\quad -c_0^{-1}\left(\mathrm{i}\frac{f_t}{f_x}-\frac{bf_{xx}}{2f_x}+\mathrm{i}b\lambda\right)\mathrm{i}\sigma^u\mathrm{e}^{-\mathrm{i}u},\\
\sigma^u_x &= \frac{2b\sigma^f_t}{f_x}-\frac{2bf_t\sigma^f_x}{f_x^2}+\delta,\\
\sigma^u_t &= -\frac{b\sigma^f_{xxx}+8\delta f_t+8\lambda\sigma^f_t}{2f_x}+\frac{bf_{xxx}+8\lambda f_t}{2f_x^2}\sigma^f_x\\
&\quad +\frac{3b}{4}\frac{f_{xx}^2+4f_t^2}{f_x^2}-\frac{3b}{2}\frac{f_{xx}^2+4f_t^2}{f_x^3}\sigma^f_x.
\end{aligned}\tag{24}$$

Using a procedure analogous to that employed in the previous section for the symmetry Eqs. (10) and (24), the Lie point symmetries of the prolonged AKNS system Eqs. (8), (18), and (23) can be found, which take the following form:

$$\begin{aligned}
\sigma^p &= -c_2\sqrt{b}g\,\mathrm{e}^{\mathrm{i}u}-\mathrm{i}abxp+(\phi_0-c)p\\
&\quad -(cx+at+x_0)p_x-(2ct+t_0)p_t,\\
\sigma^q &= -c_2\sqrt{b}g\,\mathrm{e}^{-\mathrm{i}u}+\mathrm{i}abxq-(\phi_0+c)q\\
&\quad -(cx+at+x_0)q_x-(2ct+t_0)q_t,\\
\sigma^f &= c_2f^2+c_1f+c_0-(cx+at+x_0)f_x\\
&\quad -(2ct+t_0)f_t,\\
\sigma^g &= 2c_2fg+(c_1-c)g-(cx+at+x_0)g_x\\
&\quad -(2ct+t_0)g_t,\\
\sigma^u &= -abx-\mathrm{i}\phi_0-(cx+at+x_0)u_x-(2ct+t_0)u_t,\\
\delta &= c\lambda-ab.
\end{aligned}\tag{25}$$

From Eq. (25) we observe that the nonlocal symmetry Eq. (22) of the original AKNS system Eq. (8) becomes a local symmetry for the extended AKNS system Eqs. (8), (18), and (23).

5. Group invariant decompositions of the AKNS system

According to the general symmetry decomposition theory summarized in Section 2, we can obtain two types of significant decompositions with the help of the group invariants $\xi=t_0x-t_0t$ and $\eta=xt^{-1/2}$.

5.1. Spacetime translation invariant decomposition

Removing the scaling and Galilean transformations in Eq. (25) by setting $c=a=0$, we apply the transformation

$$\begin{aligned}
&p=p(\xi,t),\ \ q=q(\xi,t),\ \ f=f(\xi,t),\\
&g=g(\xi,t),\ \ u=u(\xi,t),\ \ \xi\equiv x-x_0t,
\end{aligned}$$

where we have set $t_0=1$ without loss of generality.

Substituting Eq. (26) into the group-invariant conditions $\sigma^p=\sigma^q=\sigma^f=\sigma^g=\sigma^u=\delta=0$ yields a $(0+1)$-dimensional dynamical system

$$\begin{aligned}
u_t &= -\mathrm{i}\phi_0,\ \ f_t=c_2f^2+c_1f+c_0,\\
g_t &= (c_1+2c_2f)g,\ \ q_t=-\phi_0q-\sqrt{b}c_2g\,\mathrm{e}^{-\mathrm{i}u},\\
p_t &= \phi_0p-\sqrt{b}c_2g\,\mathrm{e}^{\mathrm{i}u}.
\end{aligned}\tag{26}$$

Combining transformation Eq. (26) with the t-dynamical system Eq. (26), the extended AKNS system described by Eqs. (8), (18) and (23) reduces to a ξ-dynamical system

$$\begin{aligned}
u_\xi &= \lambda-2bx_0+\frac{2b}{g}(c_2f^2+c_1f+c_0),\ \ f_\xi=g,\\
g_\xi &= -2ba_0qg\,\mathrm{e}^{\mathrm{i}u}+2\mathrm{i}(\lambda-bx_0)g+2b\mathrm{i}(c_2f^2+c_1f+c_0),\\
q_\xi &= b^{3/2}\left[\mathrm{i}(2c_2f+c_1-\phi_0)+\lambda x_0-\frac{b}{2}\lambda^2-pq\right]\mathrm{e}^{-\mathrm{i}u}\\
&\quad +\mathrm{i}(2bx_0-\lambda)q,\\
p_\xi &= -b^{3/2}\left[\mathrm{i}(2c_2f+c_1+\phi_0)-\lambda x_0+\frac{b}{2}\lambda^2+pq\right]\mathrm{e}^{\mathrm{i}u}\\
&\quad -\mathrm{i}(2bx_0-\lambda)p.
\end{aligned}\tag{27}$$

It is interesting to observe that the t-dynamical system Eq. (26) and the ξ-dynamical system Eq. (27) become consistent after eliminating one of the quantities $\{p,\ q,\ f,\ u,\ g\}$ via

$$\begin{aligned}
&\mathrm{i}a_0g\left(q\mathrm{e}^{\mathrm{i}u}-p\mathrm{e}^{-\mathrm{i}u}\right)+2(b\lambda-x_0)g\\
&+2\left(c_2f^2+c_1f+c_0\right)=0.
\end{aligned}\tag{28}$$

5.2. Spacetime scaling invariant decomposition

In this case, we may set $c=1$, $a=\lambda=0$, and $x_0=t_0=0$ without loss of generality, as these constants can be recovered via spacetime translations and Galilean transformations.

Applying the spacetime transformation $\{x,\ t\}\to\{xt^{-1/2},\ t\}$ with

$$\begin{aligned}
&p=p(\eta,\ t),\ \ q=q(\eta,\ t),\ \ f=f(\eta,\ t),\ \ g=g(\eta,\ t),\\
&u=u(\eta,\ t),\ \ \eta\equiv xt^{-1/2},
\end{aligned}\tag{29}$$

to the symmetry vanishing conditions $\sigma^p=\sigma^q=\sigma^f=\sigma^g=\sigma^u=\delta=0$, together with the extended AKNS system

Eqs. (8), (18), and (25), we obtain the following t-dynamical system:

$$\begin{aligned} 2tu_t &= -\mathrm{i}\phi_0,\ \ 2tf_t = c_2f^2+c_1f+c_0,\\ 2tg_t &= g(2c_2f+c_1-1),\\ 2tp_t &= (\phi_0-1)p-\sqrt{b}c_2g\,\mathrm{e}^{\mathrm{i}u},\\ 2tq_t &= -(\phi_0+1)q-\sqrt{b}c_2g\,\mathrm{e}^{-\mathrm{i}u}, \end{aligned} \tag{30}$$

and the corresponding η-dynamical system

$$\begin{aligned} u_\eta &= -b\eta+\frac{b}{\sqrt{t}g}(c_2f^2+c_1f+c_0),\ \ f_\eta=\sqrt{t}g,\\ g_\eta &= \frac{g}{b\sqrt{t}}(\mathrm{i}\sqrt{t}\eta-2\sqrt{b}tp\,\mathrm{e}^{-\mathrm{i}u})-\frac{\mathrm{i}}{b\sqrt{t}}(c_2f^2+c_1f+c_0),\\ p_\eta &= -\frac{\mathrm{i}b^{3/2}}{2\sqrt{t}}(2c_2f+c_1+\phi_0-1-2\mathrm{i}tpq)\,\mathrm{e}^{\mathrm{i}u}-\mathrm{i}b\eta p,\\ q_\eta &= -\frac{\mathrm{i}b^{3/2}}{2\sqrt{t}}(1+\phi_0-2c_2f-c_1-2\mathrm{i}tpq)\,\mathrm{e}^{-\mathrm{i}u}+\mathrm{i}b\eta q. \end{aligned} \tag{31}$$

It is straightforward to verify that the t-dynamical system Eq. (30) and the η-dynamical system Eq. (31) are consistent upon eliminating one of the quantities $\{p,q,f,u,g\}$ via

$$\left[a_0t(q\,\mathrm{e}^{\mathrm{i}u}-p\,\mathrm{e}^{-\mathrm{i}u})-\sqrt{t}\eta\right]g+c_2f^2+c_1f+c_0=0. \tag{32}$$

6. Elliptic solitons of the AKNS/NLS systems

In this section, we study the elliptic solutions of the AKNS/NLS systems, which arise from the t-dynamical system Eq. (26) and the ξ-dynamical system Eq. (27).

Denoting

$$\omega^2=c_1^2-4c_0c_2, \tag{33}$$

the general solution of the coupled dynamical systems Eqs. (26) and (27) can be readily expressed as follows:

$$\begin{aligned} f &= \frac{2c_0}{\omega^2-c_1^2}\left[c_1+\omega\tanh(\varphi)\right],\ \ \varphi\equiv\frac{1}{2}(F+\omega t),\\ F &= F(\xi)=F(x-x_0t),\\ g &= \frac{c_0\omega}{\omega^2-c_1^2}G\mathrm{sech}^2(\varphi),\ \ G(\xi)=G=F_\xi,\\ u &= U-\mathrm{i}\phi_0t,\ U=U(\xi),\ U_\xi=\lambda-2bx_0+2b\omega G^{-1},\\ p &= \sqrt{b}\left[\frac{1}{2}G\tanh(\varphi)-\mathrm{i}(\lambda-bx_0)-\frac{1}{2G}(G_\xi+2\mathrm{i}b\omega)\right]\\ &\quad\times\mathrm{e}^{\phi_0t+\mathrm{i}U},\\ q &= \frac{1}{\sqrt{b}}\left[\frac{b}{2}G\tanh(\varphi)+\mathrm{i}(b\lambda-x_0)-\frac{1}{2G}(bG_\xi-2\mathrm{i}\omega)\right]\\ &\quad\times\mathrm{e}^{-\phi_0t-\mathrm{i}U}, \end{aligned} \tag{34}$$

where the remaining undetermined function G is given by an elliptic integral, subject to the equation

$$\begin{aligned} G_\xi^2 &= G^4+C_1G^3+\left(12b\lambda x_0-6\lambda^2-4x_0^2-4\mathrm{i}b\phi_0\right)G^2\\ &\quad+8\omega(x_0-b\lambda)G-4\omega^2, \end{aligned} \tag{35}$$

with C_1 an arbitrary integration constant.

Owing to the invariance of the G-equation (Eq. (35)) under Möbius transformations (up to redefinitions of the constants),[47,48] its solutions can be expressed in terms of standard elliptic functions as

$$G=\frac{a_1J+a_0}{a_3J+a_2},\ \ a_1a_2\neq a_0a_3, \tag{36}$$

provided that suitable relations are imposed among the constants $\{a_0,a_1,a_2,a_3,C_1,\lambda,\phi_0,x_0,\omega\}$ and the parameters of the elliptic functions. In the representation Eq. (36), the function J can be chosen as the Weierstrass elliptic function $\mathcal{P}(d\xi,g_2,g_3)$ or as Jacobi elliptic functions $\mathrm{pq}(d\xi,m)$, where p and q are any pair from the set $\{\mathtt{s},\mathtt{c},\mathtt{d},\mathtt{n}\}$. Specific choices of J are discussed in the following subsections.

6.1. Square JacobiSN solitons of the AKNS/NLS equation

The first type of elliptic soliton is obtained by expressing G as

$$G=\frac{a_1\,\mathrm{sn}^2(d\xi,m)+a_0}{a_3\,\mathrm{sn}^2(d\xi,m)+a_2},\ \ a_1a_2\neq a_0a_3, \tag{37}$$

which corresponds to the choice $J=\mathrm{sn}^2(d\xi,m)$. Equivalently, one may take J to be the square of any Jacobi elliptic function $\mathrm{pq}(d\xi,m)$, where p and q are selected from the set $\{\mathtt{s},\mathtt{c},\mathtt{d},\mathtt{n}\}$.

Substituting Eq. (37) into Eq. (35) and setting the coefficients of the different powers of $\mathrm{sn}(d\xi,m)$ to zero, we have

$$\begin{aligned} m^2 &= -\frac{a_3}{a_2}-\frac{\Delta^2}{4a_3a_{23}a_2^2d^2},\ \ a_{23}\equiv a_2+a_3,\\ \Delta &\equiv a_0a_3-a_1a_2,\\ \omega^2 &= -\frac{a_0a_1a_{01}d^2}{a_2\Delta}-\frac{a_1a_{01}a_0^2}{4a_3a_{23}a_2^2},\ \ a_{01}\equiv a_0+a_1,\\ C_1 &= -\frac{4a_3a_{23}d^2}{\Delta}-\frac{a_1a_2}{a_3a_{23}}-\frac{2a_1+3a_0}{a_{23}}-\frac{2a_0a_3}{a_2a_{23}},\\ \lambda &= \frac{x_0}{b}+\frac{d^2(a_3a_0^2+a_2a_1^2+2a_0a_1a_{23})}{2b\omega a_2\Delta}\\ &\quad+\frac{a_0(a_0^2a_3+3a_0a_1a_2+2a_0a_1a_3+2a_2a_1^2)}{8b\omega a_3a_{23}a_2^2},\\ \phi_0 &= \mathrm{i}b\Bigg(\Lambda^2+\lambda\Lambda-\frac{\lambda^2}{2}-\frac{m^2d^2\Delta}{a_1a_3}+\frac{4ba_3\omega\Lambda}{a_1}\\ &\quad+\frac{3a_3^2\omega^2}{a_1^2}+\frac{a_1^2}{4a_3^2}\Bigg),\ \ \Lambda\equiv\lambda-bx_0, \end{aligned} \tag{38}$$

where the constants a_0, a_1, a_2, a_3, d, and x_0 remain free.

Corresponding to Eq. (37), the solution of the AKNS system takes the form

$$p=\frac{\mathrm{e}^{\phi_0t+\mathrm{i}U}}{\sqrt{b}J_1J_2}\left[\frac{1}{2}J_1^2\tanh(\varphi)-\mathrm{i}b\omega J_2^2+d\Delta\mathrm{SnCnDn}\right.$$

$$+\mathrm{i}b\Lambda(a_0a_2+\Delta_1\mathrm{Sn}^2+a_1a_3\mathrm{Sn}^4)\Big],$$

$$q = \frac{\sqrt{b}\,\mathrm{e}^{-\phi_0 t-\mathrm{i}U}}{J_1J_2}\Big[\frac{1}{2}bJ_1^2\tanh(\varphi)+\mathrm{i}b\omega J_2^2+d\Delta\,\mathrm{SnCnDn}$$

$$-\mathrm{i}\Lambda(a_0a_2+\Delta_1\mathrm{Sn}^2+a_1a_3\mathrm{Sn}^4)\Big], \tag{39}$$

where $\mathrm{Sn}\equiv\mathrm{sn}(d\xi,m)$, $\mathrm{Cn}\equiv\mathrm{cn}(d\xi,m)$, $\mathrm{Dn}\equiv\mathrm{dn}(d\xi,m)$, $J_1\equiv a_0+a_1\mathrm{Sn}^2$, $J_2\equiv a_2+a_3\mathrm{Sn}^2$, $\Delta_1=a_0a_3+a_1a_2$, $\varphi=\frac{1}{2}(F+\omega t)$, and

$$F = \frac{a_1\xi}{a_3}+\frac{\Delta}{da_2a_3}\,\pi\left(\mathrm{Sn},-\frac{a_3}{a_2},m\right),$$

$$U = \theta+\frac{A_1\xi}{a_1}-\frac{2b\omega\Delta}{da_0a_1}\,\pi\left(\mathrm{Sn},-\frac{a_1}{a_0},m\right),$$

$$A_1 = a_1\lambda+2ba_3\omega-2ba_1x_0, \tag{40}$$

where θ is an arbitrary constant and $\pi(z,\nu,m)$ denotes the standard incomplete elliptic integral of the third kind, defined by

$$\pi(z,\,\nu,\,m)=\int_0^z\frac{\mathrm{d}t}{(1-\nu t^2)\sqrt{(1-t^2)(1-m^2t^2)}}. \tag{41}$$

Figure 1 denotes the density plot of the strength $I\equiv|p|^2=pq$ for the elliptic soliton with the free parameters being fixed by $b=d=x_0=a_0=1$, $a_1=2$, $a_2=-5$, $a_3=3$.

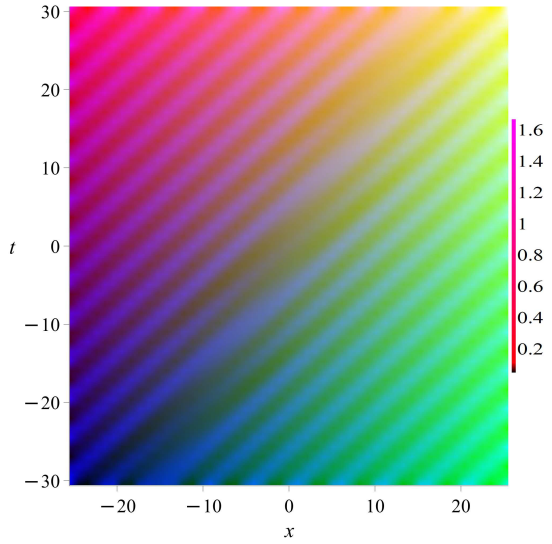

Fig. 1. Density plot of the elliptic soliton solution Eq. (40) for the quantity $I=|p|^2=pq$.

6.2. JacobiCN, cnoidal solitons of the AKNS/NLS system

The second type of elliptic soliton is obtained by expressing G as

$$G=\frac{a_1\,\mathrm{cn}(d\xi,m)+a_0}{a_3\,\mathrm{cn}(d\xi,m)+a_2}\equiv\frac{\mathrm{Cn}_1}{\mathrm{Cn}_2},\quad a_1a_2\neq a_0a_3. \tag{42}$$

In this case, the relations among all relevant parameters are

$$d^2 = \frac{\Delta^2}{\Omega_2^2},\quad m'^2\equiv1-m^2,\quad \Omega_2^2=(m^2a_2^2+m'^2a_3^2)(a_3^2-a_2^2),$$

$$\omega^2 = \frac{\Omega_1^2}{4\Omega_2^2},\quad \Omega_1^2=(m^2a_0^2+m'^2a_1^2)(a_0^2-a_1^2),$$

$$C_1 = \frac{2}{\Omega_2^2}\left[2\Delta m^2(a_2^2-a_3^2)+a_3(a_0a_2a_3+a_1a_2^2-2a_1a_3^2)\right],$$

$$\lambda = bx_0+\frac{2m^2\Delta(a_0^2-a_1^2)+a_1(a_0^2a_3+a_0a_1a_2-2a_1^2a_3)}{4b\omega\Omega_2^2},$$

$$\phi_0 = \frac{1}{2}\mathrm{i}bx_0^2$$

$$+\frac{\mathrm{i}\Delta^2\left[4m^4-2m^2(a_0^2-a_1^2)(a_0^2-3a_1^2)+a_1^2(a_0^2+2a_1^2)\right]}{8b\Omega_2^2\Omega_1^2}, \tag{43}$$

where the constants a_0, a_1, a_2, a_3, m, and x_0 remain free.

Corresponding to the G-function selection Eq. (42), the solution of the AKNS/NLS system can be expressed as

$$p = \frac{\mathrm{e}^{\phi_0 t+\mathrm{i}U}}{2\sqrt{b}\,\mathrm{Cn}_1\mathrm{Cn}_2}\Big[\sqrt{b}\,\mathrm{Cn}_1^2\tanh(\varphi)-d\sqrt{b}\,\Delta\,\mathrm{Sn}\,\mathrm{Dn}$$

$$-2\mathrm{i}b^{3/2}\omega\,\mathrm{Cn}_2^2+2\mathrm{i}\sqrt{b}(bx_0-\lambda)\mathrm{Cn}_1\mathrm{Cn}_2\Big],$$

$$q = \frac{\sqrt{b}\,\mathrm{e}^{-\phi_0 t-\mathrm{i}U}}{2\mathrm{Cn}_1\mathrm{Cn}_2}\Big[\sqrt{b}\,\mathrm{Cn}_1^2\tanh(\varphi)-d\sqrt{b}\,\Delta\,\mathrm{Sn}\,\mathrm{Dn}$$

$$+2\mathrm{i}b^{3/2}\omega\,\mathrm{Cn}_2^2-2\mathrm{i}b^{3/2}(bx_0-\lambda)\mathrm{Cn}_1\mathrm{Cn}_2\Big], \tag{44}$$

with

$$F = \frac{a_1}{a_3}\xi+\frac{a_2\Delta}{da_3(a_2^2-a_3^2)}\,\pi\left(\mathrm{Sn},\frac{a_3^2}{a_3^2-a_2^2},m\right)$$

$$+\frac{\Delta}{d\Omega_2}\,\mathrm{arctanh}\left[\frac{\Omega_2\,\mathrm{Sn}}{(a_3^2-a_2^2)\mathrm{Dn}}\right], \tag{45}$$

$$U = \frac{A_1}{a_1}\xi-\frac{2b\omega a_0\Delta}{da_1(a_0^2-a_1^2)}\,\pi\left(\mathrm{Sn},\frac{a_1^2}{a_1^2-a_0^2},m\right)$$

$$+\frac{2b\omega\Delta}{d\Omega_1}\arctan\left[\frac{\Omega_1\,\mathrm{Sn}}{(a_0^2-a_1^2)\mathrm{Dn}}\right]+\theta, \tag{46}$$

where all notations used in this solution, such as Sn, Cn, Dn, Π, θ, and A_1, are the same as those in the first type of solution Eqs. (37)–(41).

6.3. JacobiDN, dnoidal solitons of the AKNS/NLS system

For the third type of elliptic soliton, we take the solution of the G-Eq. (35) as

$$G=\frac{a_1\,\mathrm{dn}(d\xi,m)+a_0}{a_3\,\mathrm{dn}(d\xi,m)+a_2}\equiv\frac{\mathrm{Dn}_1}{\mathrm{Dn}_2},\quad a_1a_2\neq a_0a_3, \tag{47}$$

with the parameter relations

$$
\begin{aligned}
d^2 &= \frac{\Delta^2}{\omega_2^2},\ \ \omega_2^2=(a_2^2-m'^2a_3^2)(a_3^2-a_2^2),\ \ \omega^2=\frac{\omega_1^2}{4\omega_2^2},\ \ \omega_1^2=(a_0^2-m'^2a_1^2)(a_0^2-a_1^2),\\
C_1 &= \frac{2}{\omega_2^2}\left[2\Delta(a_2^2-a_3^2)+m^2a_3(a_0a_2a_3+a_1a_2^2-2a_1a_3^2)\right],\ \ \lambda=bx_0-\frac{2\Delta(a_0^2-a_1^2)+a_1m^2(a_0^2a_3+a_0a_1a_2-2a_1^2a_3)}{4b\omega\omega_2^2},\\
\phi_0 &= -\frac{1}{2}\mathrm{i}bx_0^2-\frac{\mathrm{i}\Delta^2\left[2m^2(a_0^2-a_1^2)(a_0^2-3a_1^2)-m^4a_1^2(a_0^2+2a_1^2)-4(a_0^2-a_1^2)^2\right]}{8b\omega_2^2\omega_1^2},
\end{aligned}
\tag{48}
$$

In this case, the solution of the AKNS/NLS system becomes

$$
\begin{aligned}
p &= \frac{\sqrt{b}\mathrm{e}^{\phi_0t+\mathrm{i}U}}{2\mathrm{Dn}_1\mathrm{Dn}_2}\Big[\mathrm{Dn}_1^2\tanh(\varphi)-dm^2\Delta\,\mathrm{Sn}\mathrm{Cn}\\
&\quad -2\mathrm{i}b\omega\,\mathrm{Dn}_2^2+2\mathrm{i}(bx_0-\lambda)\mathrm{Dn}_1\mathrm{Dn}_2\Big],\\
q &= \frac{\sqrt{b}\mathrm{e}^{-\phi_0t-\mathrm{i}U}}{2\mathrm{Dn}_1\mathrm{Dn}_2}\Big[\mathrm{Dn}_1^2\tanh(\varphi)-dm^2\Delta\,\mathrm{Sn}\mathrm{Cn}\\
&\quad +2\mathrm{i}b\omega\,\mathrm{Dn}_2^2-2\mathrm{i}b(bx_0-\lambda)\mathrm{Dn}_1\mathrm{Dn}_2\Big],
\end{aligned}
\tag{49}
$$

where

$$
\begin{aligned}
F &= \frac{a_1}{a_3}\xi+\frac{a_2\Delta}{da_3(a_2^2-a_3^2)}\pi\left(\mathrm{Sn},\frac{m^2a_3^2}{a_3^2-a_2^2},m\right)\\
&\quad -\frac{(B-ma_3)\Delta}{\sqrt{a_3^2-a_2^2}dBB_2}\arctanh\left(\frac{\sqrt{a_3^2-a_2^2}\mathrm{Sn}}{B_2(1+\mathrm{Cn})}\right)\\
&\quad -\frac{(B+ma_3)\Delta}{\sqrt{a_3^2-a_2^2}dBB_1}\arctanh\left(\frac{\sqrt{a_3^2-a_2^2}\mathrm{Sn}}{B_1(1+\mathrm{Cn})}\right),\\
U &= \frac{A_1}{a_3}\xi-\frac{2a_0b\omega\Delta}{da_1(a_0^2-a_1^2)}\pi\left(\mathrm{Sn},\frac{m^2a_1^2}{a_1^2-a_0^2},m\right)\\
&\quad +\frac{2b\omega(D-ma_1)\Delta}{\sqrt{a_0^2-a_1^2}dDD_2}\arctan\left(\frac{\sqrt{a_0^2-a_1^2}\mathrm{Sn}}{D_2(1+\mathrm{Cn})}\right)\\
&\quad +\frac{2b\omega(D+ma_1)\Delta}{\sqrt{a_0^2-a_1^2}dDD_1}\arctanh\left(\frac{\sqrt{a_0^2-a_1^2}\mathrm{Sn}}{D_1(1+\mathrm{Cn})}\right)+\theta,
\end{aligned}
\tag{50, 51}
$$

with

$$
\begin{aligned}
&B=\sqrt{a_2^2-m'^2a_3},\ \ B_1=\sqrt{a_2^2+a_3^2(2m^2-1)+2ma_3B},\\
&B_2=\sqrt{a_2^2+a_3^2(2m^2-1)-2ma_3B},\ \ D=\sqrt{a_0^2-m'^2a_1^2},\\
&D_1=\sqrt{a_0^2+a_1^2(2m^2-1)+2ma_1D},\\
&D_2=\sqrt{a_0^2+a_1^2(2m^2-1)-2ma_1D},
\end{aligned}
$$

while the other notations are the same as in the first and second types of solutions.

6.4. JacobiSC elliptic solitons of the AKNS/NLS system

For the fourth type of elliptic function, one can take the G-function of Eq. (35) in the form

$$
\begin{aligned}
&G=\frac{a_1\,\mathrm{sc}(d\xi,m)+a_0}{a_3\,\mathrm{sc}(d\xi,m)+a_2}\equiv\frac{\mathrm{Sc}_1}{\mathrm{Sc}_2},\\
&a_1a_2\neq a_0a_3,\ \mathrm{sc}(d\xi,m)\equiv\frac{\mathrm{sn}(d\xi,m)}{\mathrm{cn}(d\xi,m)},
\end{aligned}
\tag{52}
$$

with the parameter relations

$$
\begin{aligned}
d^2 &= \frac{\Delta^2}{o_2^2},\ \ o_2^2=(a_3^2+m'^2a_2^2)(a_3^2+a_2^2),\ \ \omega^2=\frac{o_1^2}{4o_2^2},\ \ o_1^2=(a_0^2(m^2-1)-a_1^2)(a_0^2+a_1^2),\\
C_1 &= \frac{2}{o_2^2}\left[m^2a_2(a_1a_2a_3+a_0a_3^2+2a_0a_2^2)-2(a_0^2+a_1^2)(a_2^2+a_3^2)\right],\\
\lambda &= bx_0-\frac{-2(a_0a_2+a_1a_3)(a_0^2+a_1^2)+a_0m^2(a_1^2a_2+a_0a_1a_3+2a_0^2a_2)}{4b\omega o_2^2},\\
\phi_0 &= -\frac{1}{2}\mathrm{i}bx_0^2-\frac{\mathrm{i}\Delta^2\left[m^4a_0^2(2a_0^2-a_1^2)-2m^2(a_0^2+a_1^2)(a_1^2+3a_0^2)+4(a_0^2+a_1^2)^2\right]}{8bo_2^2o_1^2},
\end{aligned}
\tag{53}
$$

In this case, we have

$$
\begin{aligned}
p &= \frac{\sqrt{b}\mathrm{e}^{\phi_0t+\mathrm{i}U}}{2\mathrm{Sc}_1\mathrm{Sc}_2}\left[\mathrm{Sc}_1^2\tanh(\varphi)+d\Delta\,\frac{\mathrm{Dn}}{\mathrm{Cn}^2}-2\mathrm{i}b\omega\,\mathrm{Sc}_2^2+2\mathrm{i}(bx_0-\lambda)\mathrm{Sc}_1\mathrm{Sc}_2\right],\\
q &= \frac{\sqrt{b}\mathrm{e}^{-\phi_0t-\mathrm{i}U}}{2\mathrm{Sc}_1\mathrm{Sc}_2}\left[\mathrm{Sc}_1^2\tanh(\varphi)+d\Delta\,\frac{\mathrm{Dn}}{\mathrm{Cn}^2}+2\mathrm{i}b\omega\,\mathrm{Sc}_2^2-2\mathrm{i}b(bx_0-\lambda)\mathrm{Sc}_1\mathrm{Sc}_2\right],
\end{aligned}
\tag{54}
$$

where the F and U functions are determined by

$$\begin{aligned} F &= \frac{(a_0a_2+a_1a_3)\xi}{a_2^2+a_3^2}+\frac{a_3\Delta}{da_2(a_2^2+a_3^2)}\pi\left(\mathrm{Sn},1+\frac{a_3^2}{a_2^2},m\right) \\ &\quad+\frac{\Delta}{do_2}\mathrm{arctanh}\left[\frac{(a_2^2+a_3^2)^2\mathrm{Dn}+o_2^2}{(a_2^2+a_3^2)(\mathrm{Dn}+1)o_2}\right], \qquad (55) \\ U &= \frac{A_2\xi}{a_0^2+a_1^2}+\frac{2ba_1\omega\Delta}{da_0(a_0^2+a_1^2)}\pi\left(\mathrm{Sn},1+\frac{a_1^2}{a_0^2},m\right) \\ &\quad-\frac{2b\omega\Delta}{do_1}\arctan\left[\frac{(a_0^2+a_1^2)^2\mathrm{Dn}-o_1^2}{(a_0^2+a_1^2)(\mathrm{Dn}+1)o_1}\right]+\theta, \qquad (56) \end{aligned}$$

with $A_2\equiv 2b\omega(a_0a_2+a_1a_3)+(a_0^2+a_1^2)(\lambda-2bx_0)$.

6.5. JacobiSD elliptic solitons of the AKNS/NLS system

The fifth type of elliptic G-function Eq. (35) can be taken as

$$\begin{aligned} G&=\frac{a_1\,\mathrm{sd}(d\xi,m)+a_0}{a_3\,\mathrm{sd}(d\xi,m)+a_2}\equiv\frac{\mathrm{Sd}_1}{\mathrm{Sd}_2}, \\ &a_1a_2\neq a_0a_3,\ \ \mathrm{sd}(d\xi,m)\equiv\frac{\mathrm{sn}(d\xi,m)}{\mathrm{dn}(d\xi,m)}, \qquad (57) \end{aligned}$$

where the following parameter conditions should be satisfied:

$$\begin{aligned} d^2 &= \frac{\Delta^2}{O_2^2},\ O_2^2=\alpha_2(a_3^2-m'^2a_2^2),\ \alpha_2=a_3^2+m^2a_2^2,\ \omega^2=\frac{O_1^2}{4O_2^2},\ O_1^2=\alpha_1(m'^2a_0^2-a_1^2),\ \alpha_1=m^2a_0^2+a_1^2, \\ C_1 &= -\frac{2}{O_2^2}\left[2a_0a_2^3m^4-2m^2a_2(a_0a_2^2-a_1a_2a_3-a_0a_3^2)-a_3(a_0a_2a_3+a_1a_2^2-2a_1a_3^2)\right], \\ \lambda &= bx_0+\frac{2a_0^3a_2m^4-a_1(a_0^2a_3+a_0a_1a_2-2a_1^2a_3)-2a_0m^2(a_0^2a_2-a_0a_1a_3-a_1^2a_2)}{4b\omega O_2^2}, \\ \phi_0 &= -\frac{1}{2}\mathrm{i}bx_0^2-\frac{\mathrm{i}\Delta^2\left[4m^6a_0^4-2a_0^2m^4(3a_0^2-4a_1^2)+m^2(2a_0^4-8a_0^2a_1^2+4a_1^4)-a_1^2(a_0^2+2a_1^2)^2\right]}{8bO_2^2O_1^2}, \qquad (58) \end{aligned}$$

In this case, fields p and q are determined by

$$\begin{aligned} p &= \frac{\sqrt{b}\mathrm{e}^{\phi_0t+\mathrm{i}U}}{2\,\mathrm{Sd}_1\mathrm{Sd}_2}\left[\mathrm{Sd}_1^2\tanh(\varphi)+d\Delta\frac{\mathrm{Cn}}{\mathrm{Dn}^2}-2\mathrm{i}b\omega\,\mathrm{Sd}_2^2\right. \\ &\quad\left.+2\mathrm{i}(bx_0-\lambda)\mathrm{Sd}_1\mathrm{Sd}_2\right], \\ q &= \frac{\sqrt{b}\mathrm{e}^{-\phi_0t-\mathrm{i}U}}{2\mathrm{Sd}_1\mathrm{Sd}_2}\left[\mathrm{Sd}_1^2\tanh(\varphi)+d\Delta\frac{\mathrm{Cn}}{\mathrm{Dn}^2}+2\mathrm{i}b\omega\,\mathrm{Sd}_2^2\right. \\ &\quad\left.-2\mathrm{i}(x_0-b\lambda)\mathrm{Sd}_1\mathrm{Sd}_2\right], \qquad (59) \end{aligned}$$

where the F and U functions are given by

$$\begin{aligned} F &= \frac{(a_0a_2m^2+a_1a_3)\xi}{\alpha_2}+\frac{a_3\Delta}{da_2\alpha_2}\pi\left(\mathrm{Sn},\frac{\alpha_2}{a_2^2},m\right) \\ &\quad-\frac{\Delta}{dO_2}\mathrm{arctanh}\left[\frac{\alpha_2^2\mathrm{Cn}+O_2^2}{\alpha_2(\mathrm{Cn}+1)O_2}\right], \qquad (60) \\ U &= \frac{A_3\xi}{\alpha_1}-\frac{2ba_1\omega\Delta}{da_0\alpha_1}\pi\left(\mathrm{Sn},\frac{\alpha_1}{a_0^2},m\right) \\ &\quad+\frac{2b\omega\Delta}{dO_1^2}\arctan\left[\frac{\alpha_1^2\mathrm{Cn}-O_1^2}{\alpha_1(\mathrm{Cn}+1)O_1}\right]+\theta, \qquad (61) \end{aligned}$$

with $A_3\equiv 2b\omega(a_0a_2m^2+a_1a_3)+\alpha_1(\lambda-2bx_0)$.

6.6. JacobiCD elliptic solitons of the AKNS/NLS system

The sixth type of elliptic soliton solution is related to

$$\begin{aligned} G&=\frac{a_1\,\mathrm{cd}(d\xi,m)+a_0}{a_3\,\mathrm{cd}(d\xi,m)+a_2}\equiv\frac{\mathrm{Cd}_1}{\mathrm{Cd}_2}, \\ &a_1a_2\neq a_0a_3,\ \ \mathrm{cd}(d\xi,m)\equiv\frac{\mathrm{cn}(d\xi,m)}{\mathrm{dn}(d\xi,m)}, \qquad (62) \end{aligned}$$

where the following parameter conditions should be satisfied:

$$\begin{aligned} d^2 &= \frac{\Delta^2}{\gamma_2^2},\ \gamma_2^2=\beta_2(a_2^2-a_3^2),\ \beta_2=m^2a_2^2-a3^2, \\ \omega^2 &= \frac{\gamma_1^2}{4\gamma_2^2},\ \gamma_1^2=\beta_1(a_1^2-a_0^2),\ \beta_1=m^2a_0^2-a_1^2, \\ C_1 &= \frac{2(2a_0a_2^2-a_0a_3^2-a_1a_2a_3)}{a_2(a_3^2-a_2^2)}-\frac{2a_3\Delta}{\beta_2a_2}, \\ x_0 &= b\lambda-\frac{1}{4}\frac{a_0m^2\mu-a_1\nu}{\omega\beta_2(a_2^2-a_3^2)}, \\ \mu &= 2a_0^2a_2-a_0a_1a_3-a_1^2a_2,\ \nu=a_0^2a_3+a_0a_1a_2-2a_1^2a_2, \\ \phi_0 &= -\frac{1}{2}\mathrm{i}bx_0^2+\frac{\mathrm{i}\lambda(a_0\mu m^2-a_1\nu)}{4\omega\gamma_2^2} \\ &\quad-\frac{\mathrm{i}b(\mu_0a_1^3+\mu_1m^2+\mu_2a_0^3m^4)}{4\gamma_1^2\gamma_1^2}, \qquad (63) \end{aligned}$$

where $\mu_0=2a_0^3a_2a_3-3a_0^2a_1a_3^2-a_1^3a_2^2+2a_1^3a_3^2$, $\mu_1=6a0^4a_1^2a_3^2-a_0^6a_3^2-3a_0^4a_1^2a_2^2-4a_0^3a_1^3a_2a_3+6a_0^2a_1^4a_2^2-3a_0^2a_1^4a_3^2-a_1^6a_2^2$ and $\mu_2=2a_0^3a_2^2-a_0^3a_3^2-3a_0a_1^2a_2^2+2a_1^3a_2a_3$.

In this case, the quantities p and q are fixed as

$$\begin{aligned} p &= \frac{\sqrt{b}\mathrm{e}^{\phi_0t+\mathrm{i}U}}{2\,\mathrm{Cd}_1\mathrm{Cd}_2}\left[\mathrm{Cd}_1^2\tanh(\varphi)-dm'^2\Delta\frac{\mathrm{Sn}}{\mathrm{Dn}^2}\right. \\ &\quad\left.-2\mathrm{i}b\omega\,\mathrm{Cd}_2^2+2\mathrm{i}(bx_0-\lambda)\mathrm{Cd}_1\mathrm{Cd}_2\right], \\ q &= \frac{\sqrt{b}\mathrm{e}^{-\phi_0t-\mathrm{i}U}}{2\mathrm{Cd}_1\mathrm{Cd}_2}\left[\mathrm{Cd}_1^2\tanh(\varphi)-dm'^2\Delta\frac{\mathrm{Sn}}{\mathrm{Dn}^2}+2\mathrm{i}b\omega\,\mathrm{Cd}_2^2\right. \\ &\quad\left.-2\mathrm{i}(bx_0-\lambda)\mathrm{Cd}_1\mathrm{Cd}_2\right], \qquad (64) \end{aligned}$$

where the F and U functions are given by

$$F = \frac{(a_0a_2m^2-a_1a_3)\xi}{\beta_2} - \frac{a_2a_3m'^2\Delta}{da_2\gamma_2^2}\pi\left(\mathrm{Sn},\frac{\beta_2^2}{\gamma_2^2},m\right) - \frac{\Delta}{d\gamma_2}\mathrm{arctanh}\left(\frac{\beta_2\mathrm{Sn}}{\gamma_2}\right), \tag{65}$$

$$U = \frac{A_4\xi}{\beta_1} - \frac{2a_0a_1bm'^2\Delta}{d\gamma_1^2}\pi\left(\mathrm{Sn},-\frac{\beta_1^2}{\gamma_1^2},m\right) + \frac{2b\omega\Delta}{d\gamma_1}\arctan\left(\frac{\beta_1\mathrm{Sn}}{\gamma_1}\right)+\theta, \tag{66}$$

with $A_4 \equiv 2b\omega(a_0a_2m^2-a_1a_3)+\beta_1(\lambda-2bx_0)$.

6.7. Weierstrass elliptic solitons of the AKNS system

The first six types of elliptic solitons presented in Subsections 6.1–6.6 serve not only as solutions to the AKNS system Eq. (8) but also to the NLS Eq. (9), as the complex conjugate condition $q=p^*$ can be readily fulfilled by appropriately choosing the free parameters. In this subsection and the next, we propose two examples that are solutions to the AKNS system but not to the NLS equation, as the complex conjugate condition cannot be satisfied in these cases.

For the seventh example of the elliptic soliton solution, we take the G-function of Eq. (35) in the form

$$G = \frac{a_1\mathcal{P}(d\xi,g_2,g_3)+a_0}{a_3\mathcal{P}(d\xi,g_2,g_3)+a_2} \equiv \frac{W_1}{W_2},\quad a_1a_2\neq a_0a_3, \tag{67}$$

where $\mathcal{P}(z,g_2,g_3)\equiv$ is the standard Weierstrass elliptic function, which is a solution of

$$\mathcal{P}_z^2 = 4\mathcal{P}^3-g_2\mathcal{P}-g_3. \tag{68}$$

In this case, all the parameters satisfy the following conditions:

$$\begin{aligned}
d^2 &= \frac{\Delta^2}{\rho_2^2},\quad \rho_2^2 = a_3(g_2a_2a_3^2-g_3a_2^3-4a_2^3),\\
\omega^2 &= \frac{\rho_1^2}{4\rho_2^2},\quad \rho_1^2 = a_1(4a_0^3+g_3a_3^3-g_2a_0a_1^2),\\
C_1 &= \frac{4g_3a_1a_3^3+4a_2^2(3a_0a_3+a_1a_2)-g_2a_1^2(3a_0a_3+a_1a_2)}{\rho_2^2},\\
\lambda &= bx_0 + \frac{g_2a_1^2(3a_0a_3+a_1a_2)-4g_3a_1^3a_3-4a_0^2(a_0a_3+3a_1a_2)}{8b\omega\rho_2^2},\\
\phi_0 &= -\frac{1}{2}\mathrm{i}bx_0^2 + \frac{3\mathrm{i}\Delta^2(g_2^2a_1^4+8g_2a_0^2a_1^2-32g_3a_0a_1^3+16a_0^4)}{32\rho_1^2\rho_2^2}.
\end{aligned} \tag{69}$$

Correspondingly, the solutions of the AKNS system read

$$\begin{aligned}
p &= \frac{\sqrt{b}\mathrm{e}^{\phi_0t+\mathrm{i}U}}{2W_1W_2}\Big[W_1^2\tanh(\varphi)+d\Delta\mathcal{P}'-2\mathrm{i}b\omega W_2^2 +2\mathrm{i}(bx_0-\lambda)W_1W_2\Big],\\
q &= \frac{\sqrt{b}\mathrm{e}^{-\phi_0t-\mathrm{i}U}}{2W_1W_2}\Big[W_1^2\tanh(\varphi)+d\Delta\mathcal{P}'+2\mathrm{i}b\omega W_2^2 -2\mathrm{i}(bx_0-\lambda)W_1W_2\Big],
\end{aligned} \tag{70}$$

where $\mathcal{P}'$ is the derivative of the Weierstrass function $\mathcal{P}(z,g_2,g_3)$, while the F and U functions are given by

$$F = \frac{a_1\xi}{a_3}+\frac{\Delta}{d\rho_2}\left\{2d\zeta(Z,\ g_2,\ g_3)\xi +\ln\left[\frac{\sigma(d\xi-Z,\ g_2,\ g_3)}{\sigma(d\xi+Z,\ g_2,\ g_3)}\right]\right\}, \tag{71}$$

$$U = \frac{A_1\xi}{a_1}+\frac{2\mathrm{i}b\omega\Delta}{d\rho_1}\left\{2d\zeta(z,\ g_2,\ g_3)\xi +\ln\left[\frac{\sigma(d\xi-z,\ g_2,\ g_3)}{\sigma(d\xi+z,\ g_2,\ g_3)}\right]\right\}+\theta, \tag{72}$$

where $\sigma(y,g_2,g_3)$ and $\zeta(f,g_2,g_3)$ are the Weierstrass σ function and Weierstrass ζ function, respectively, $A_1\equiv a_1(\lambda-2bx_0)+2ba_3\omega$, while the constants Z and z are determined by $a_3\mathcal{P}(Z,\ g_2,\ g_3)+a_2=0$ and $a_1\mathcal{P}(z,\ g_2,\ g_3)+a_0=0$, respectively.

Based on the expressions in Eq. (70), the complex conjugate condition $q=p^*$ requires that both F and U be real. However, from Eqs. (71) and (72) it is evident that no choice of parameters can satisfy this reality condition. Consequently, the solution given by Eq. (70) together with Eqs. (69), (71), and (72) is a solution of the AKNS system only, and not of the NLS equation.

6.8. JacobiSN elliptic solitons of the AKNS system

If we select the solution of the G- Eq. (35) in the form

$$G = \frac{a_1\mathrm{JacobiSN}(d\xi,m)+a_0}{a_3\mathrm{JacobiSN}(d\xi,m)+a_2} \equiv \frac{\mathrm{Sn}_1}{\mathrm{Sn}_2},\quad a_1a_2\neq a_0a_3, \tag{73}$$

then we obtain the eighth type of elliptic soliton solution,

$$\begin{aligned}
p &= \frac{\sqrt{b}\mathrm{e}^{\phi_0t+\mathrm{i}U}}{2\mathrm{Sn}_1\mathrm{Sn}_2}\Big[\mathrm{Sn}_1^2\tanh(\varphi)+d\Delta\mathrm{CnDn}-2\mathrm{i}b\omega\,\mathrm{Sn}_2^2 +2\mathrm{i}(bx_0-\lambda)\mathrm{Sn}_1\mathrm{Sn}_2\Big],\\
q &= \frac{\sqrt{b}\mathrm{e}^{-\phi_0t-\mathrm{i}U}}{2\mathrm{Sn}_1\mathrm{Sn}_2}\Big[\mathrm{Sn}_1^2\tanh(\varphi)+d\Delta\mathrm{CnDn}+2\mathrm{i}b\omega\,\mathrm{Sn}_2^2 -2\mathrm{i}(bx_0-\lambda)\mathrm{Sn}_1\mathrm{Sn}_2\Big],
\end{aligned} \tag{74}$$

where

$$\begin{aligned}
F &= \frac{a_1\xi}{a_3}+\frac{\Delta}{da_2a_3}\pi\left(\mathrm{Sn},\frac{a_3^2}{a_2^2},m\right) - \frac{\Delta}{dr_2}\mathrm{arctanh}\left(\frac{B_+\mathrm{Sn}^2+2r_2a_3^2\mathrm{CnDn}+C_-}{B_-\mathrm{Sn}^2+2r_2a_3^2\mathrm{CnDn}+C_+}\right),\\
B_\pm &= m^2a_3^2(2a_2^2-a_3^2)\pm a_3^2(a_2^2-a_3^2),
\end{aligned}$$

$$C_\pm = 2a_3^4-(1+m^2)a_2^2a_3^2\mp a_2^4, \tag{75}$$

and $A_1=(\lambda-2bx_0)a_1+2b\omega a_3$,

$$U = \frac{A_1\xi}{a_1}-\frac{2b\omega\Delta}{da_0a_1}\pi\left(\mathrm{Sn},\frac{a_1^2}{a_0^2},m\right) -\frac{2\mathrm{i}b\omega\Delta}{dr_1}\operatorname{arctanh}\left(\frac{B'_+\mathrm{Sn}^2+2\mathrm{i}r_2a_1^2\mathrm{CnDn}+C'_-}{B'_-\mathrm{Sn}^2+2\mathrm{i}r_1a_1^2\mathrm{CnDn}+C'_+}\right)+\theta,$$

$$B'_\pm = (2m^2\pm1)a_0^2a_1^2-(1+m^2)a_1^4,$$

$$C'_\pm = 2a_1^4-(1+m^2)a_0^2a_1^2\pm a_0^4. \tag{76}$$

In this case, the parameters $\{d,\ \omega,\ C_1,\ \lambda,\ \phi_0,\ r_1,\ r_2\}$ are related to the free parameters $\{a_1,\ a_2,\ a_3,\ a_4,\ m,\ x_0\}$ by

$$\begin{aligned}
d^2 &= \frac{\Delta^2}{r_2^2},\ r_2^2=(a_2^2-a_3^2)(m^2a_2^2-a_3^2),\ \omega^2=\frac{r_1^2}{4r_2^2},\ r_1^2=(a_1^2-a_0^2)(m^2a_0^2-a_1^2),\\
C_1 &= \frac{2}{mr_2^2}\left[a_3(a_0a_2a_3+a_1a_2^2-2a_1a_3^2)-m^2a_2(2a_0a_2^2-a_0a_3^2-a_1a_2a_3)\right],\\
\lambda &= bx_0+\frac{m^2a_0(2a_0^2a_2-a_0a_1a_3-a_1^2a_2)-a_1(a_0^2a_3+a_0a_1a_2-2a_1^2a_3)}{4b\omega r_2^2},\\
\phi_0 &= -\frac{1}{2}\mathrm{i}bx_0^2+\frac{\mathrm{i}\Delta^2(m^4a_0^2(2a_0^2+a_1^2))+m^2(2a_0^4-10a_0^2a_1^2+2a_1^4)+a_1^2(a_0^2+2a_1^2)}{8br_1^2r_2^2}.
\end{aligned} \tag{77}$$

Based on the expressions in Eqs. (76) and (77), it is clear that no choice of parameters can make U real; therefore, the complex conjugate condition $q=p^*$ cannot be satisfied. Consequently, the solution given by Eq. (74) together with Eqs. (77), (75), and (76) is a solution of the AKNS system only, and not of the NLS equation.

7. Painlevé solitons of the AKNS systems

7.1. General Painlevé IV soliton of the AKNS system

Solving the spacetime scaling invariant decomposition system Eqs. (30) and (31), it is not difficult to find that the extended AKNS system Eqs. (8), (18), and (23) possesses the solution

$$\begin{aligned}
f &= A\tanh(\omega\phi),\ \ \phi\equiv\ln(t)+F,\ F=F(\eta),\ \eta=\frac{x}{\sqrt{t}},\\
u &= -\frac{1}{2}\mathrm{i}\phi_0\ln(t)+U,\ \ U=U(\eta),\\
g &= Gt^{-1/2}\mathrm{sech}^2(\omega\phi),\ \ G=G(\eta),\\
p &= \frac{\sqrt{b}}{A}t^{\frac{1}{2}(\phi_0-1)}\left[G\,\mathrm{e}^{\mathrm{i}\,U}\,\tanh(\omega\,\phi)+P\right],\ \ P=P(\eta),\\
q &= \frac{\sqrt{b}}{A}t^{-\frac{1}{2}(\phi_0+1)}\left[G\,\mathrm{e}^{-\mathrm{i}\,U}\,\tanh(\omega\,\phi)+Q\right],\ \ Q=Q(\eta),
\end{aligned} \tag{78}$$

while the group invariant functions F, U, G, P and Q are determined by

$$\begin{aligned}
&G=A\omega F_\eta,\\
&P=\frac{A}{2}\left(\mathrm{i}b\eta-F_{\eta\eta}F_\eta^{-1}-2\mathrm{i}bF_\eta^{-1}\right)\mathrm{e}^{\mathrm{i}U},\\
&Q=\frac{A}{2}\left(2\mathrm{i}bF_\eta^{-1}-\mathrm{i}b\eta-F_{\eta\eta}F_\eta^{-1}\right)\mathrm{e}^{-\mathrm{i}U},\\
&U_\eta=2bF_\eta^{-1}-b\eta,
\end{aligned}$$

$$\begin{aligned}
&F_\eta F_{\eta\eta\eta}-2\omega^2F_\eta^4-\frac{1}{2}(2\mathrm{i}b\phi_0+\eta^2)F_\eta^2\\
&-\frac{3}{2}F_{\eta\eta}^2+4\eta F_\eta-6=0.
\end{aligned} \tag{79}$$

From Eqs. (79) and (78), we observe that once the quantity F is determined from the F-equation (the last equation in Eq. (79)), the solution of the extended AKNS system is completely specified.

The presence of hyperbolic functions in the solution expression Eq. (78) indicates the existence of a soliton propagating on a background governed by F. Moreover, the function F is intimately connected to the Painlevé IV equation

$$w_{zz}=\frac{1}{2}\frac{w_z^2}{w}+\frac{3}{2}w^3+4zw^2+2(z^2-\alpha)w+\frac{\beta}{w}, \tag{80}$$

via the relations

$$\begin{aligned}
&F_\eta=-\frac{4\mu}{w(\mu\eta)}=-\frac{4\mu}{w(z)}=-\frac{4\mu}{w},\\
&\mu^4=-\frac{1}{4},\ \ \phi_0=-2\mathrm{i}b\mu^2\alpha,\ \ \omega^2=-\frac{\beta}{32}.
\end{aligned} \tag{81}$$

Accordingly, we refer to the solution Eq. (78) as a *Painlevé soliton* — that is, a soliton embedded in a background dictated by the Painlevé IV equation.

Although the expressions Eq. (78) together with Eqs. (79) and (81) constitute a family of solutions to the AKNS system for any given special solution of the Painlevé IV Eq. (80), it remains highly nontrivial to extract physically relevant solutions of the NLS equation from Eq. (80) due to the complex nature of its argument $z=\mu\eta=\frac{1}{2}(\delta_1+\delta_2\mathrm{i})\frac{x}{\sqrt{t}}$, with $\delta_1^2=1$, $\delta_2^2=1$. The difficulty is further compounded by the reality condition $q=p^*$, which forces F to be real.

To construct real solutions for F, it is more convenient to work directly with the original F-equation, or equivalently,

with the following bilinear equation

$$\begin{aligned}&2V_\eta V_{\eta\eta\eta}+16\omega\eta VV_\eta-3V_{\eta\eta}^2\\&-(\eta^2+2\mathrm{i}b\phi_0)V_\eta^2-48\omega^2V^2=0,\end{aligned}\tag{82}$$

under the transformation

$$F=\frac{1}{2\omega}\ln(V),\tag{83}$$

for complex ϕ_0.

Since the Painlevé IV Eq. (81) and the V-Eq. (82) are both derived from the same underlying F-equation, we refer to Eq. (82) as the variant Painlevé IV (VPIV) equation.

7.2. Polynomials of the variant Painlevé IV Eq. (82)

It is well known that for Painlevé equations, certain families of rational solutions exist, which are associated with specific types of polynomials under appropriate choices of parameters. For example, rational solutions of the Painlevé II equation are governed by the so-called Yablonskii–Vorob'ev polynomials,[49,50] while those of the Painlevé IV equation can be expressed in terms of the Okamoto polynomials.[51]

In this subsection, we present several families of rational solutions to the VPIV Eq. (82) by introducing new classes of polynomials.

7.2.1. Polynomial solutions of the variant Painlevé IV Eq. (82)

Formally, the variant Painlevé IV equation can be solved by Laurent/power series solutions in the form

$$V=\sum_{k=0}^{\infty}a_k\eta^{k-\theta},\tag{84}$$

with the recursion relation

$$\begin{aligned}&\sum_{j=0}^{k}a_j\big\{48\omega^2a_{k-j-4}+\big[(j-\theta)\big(a_{k-j-4}(k-j-4-16\omega-\theta)\\&-a_{k-j}(k-j-\theta)(k-j-1-\theta)\\&+2\mathrm{i}\,ba_{k-j-2}(k-j-2-\theta)\phi_0\big)\big]\big\},\ k=0,\ 1,\ \ldots,\infty,\end{aligned}\tag{85}$$

with $a_j=0,\ \forall\ j<0$.

Substituting $k=1$ into Eq. (85), it is found that θ may be taken as $\theta=0,\ 1$ and -1. It is interesting that for some specially selected parameters ω and ϕ_0, the series will be truncated. Here are some special cases:

Case 1. Complex variant Hermite polynomials If we fix the parameters ω and ϕ_0 as

$$\omega=\frac{n}{4},\ \ \phi_0=-(n-1)b,\ \ n=0,\ 1,\ 2,\ \ldots,\ \infty,\tag{86}$$

then the solution Eq. (84) is truncated to the polynomials (which we call complex variant Hermite polynomials) for $\theta=0,\ 1$ with the linear recursion relation

$$V_{k+1}=\eta V_k+\mathrm{i}\,kV_{k-1},\ V_0=1,\ V_1=\eta,\tag{87}$$

while the next four polynomials read

$$\begin{aligned}V_2&=\eta^2+\mathrm{i},\\V_3&=\eta^3+3\mathrm{i}\eta,\\V_4&=\eta^4+6\mathrm{i}\eta^2-3,\\V_5&=\eta^5+10\mathrm{i}\eta^3-15\,\eta.\end{aligned}\tag{88}$$

Case 2. Special real polynomials and real truncated Laurent series In addition to the complex variant Hermite polynomial solutions Eq. (87), there are several other interesting special cases of truncated Laurent series solutions. Here we only list some real ones

$$\begin{aligned}V_1&=\eta,\ \ \omega=\frac{1}{4},\ \ \phi_0=0,\\V_a&=\eta,\ \ \omega=\frac{1}{12},\ \ \phi_0=0,\\V_b&=\eta^5+45\eta,\ \ \omega=\frac{5}{12},\ \ \phi_0=0,\\V_c&=\frac{1}{\eta},\ \ \omega=-\frac{1}{4},\ \ \phi_0=0,\\V_d&=\frac{1}{\eta},\ \ \omega=-\frac{1}{12},\ \ \phi_0=0,\\V_e&=\eta^3-\frac{3}{\eta},\ \ \omega=\frac{3}{4},\ \ \phi_0=0,\\V_f&=\eta^7+126\eta^3-\frac{567}{\eta},\ \ \omega=\frac{7}{12},\ \ \phi_0=0.\end{aligned}\tag{89}$$

In addition to the polynomial solutions, there are infinitely many other types of exact solutions of the VPIV Eq. (82) for some special choices of the parameters ω and ϕ_0.

7.3. Rational soliton of the AKNS system

Applying the polynomial solutions listed in the last subsection to Eq. (78) together with Eq. (79), we immediately obtain the corresponding solutions of the AKNS/NLS systems. For the complex polynomial solutions, namely, the complex variant Hermite polynomials given in Eq. (88), we obtain solutions of the AKNS system, but not of the NLS equation, since the complex conjugation condition $q=p^*$ cannot be satisfied.

The first three nontrivial rational soliton solutions of the AKNS system read

$$\begin{aligned}p&=\frac{c_0}{2}\frac{\sqrt{b}\,\mathrm{e}^{\mathrm{i}\,c_1}[\mathrm{i}\,(b+3)\,x^2+(b-1)(\mathrm{i}-t)]}{x^{1+b}[\mathrm{i}\,(1+x^2)-t]},\\q&=-\frac{1}{2}\frac{\sqrt{b}\,\mathrm{e}^{-\mathrm{i}\,c_1}[\mathrm{i}\,(b-3)\,x^2+(b+1)(\mathrm{i}-t)]}{c_0x^{1-b}[\mathrm{i}\,(1+x^2)-t]},\\I&\equiv|p|^2+|q|^2=\frac{(x^2-1)^2+t^2+4}{C[(1+x^2)^2+t^2]},\ \ C\equiv\frac{c_0^2}{1+c_0^4},\end{aligned}\tag{90}$$

$$\begin{aligned}p&=-\frac{c_0\sqrt{b}\,\mathrm{e}^{\mathrm{i}\,c_1}[(b+2)\,x^4+3\,\mathrm{i}\,(b+1)\,tx^2+(b-1)x-3t^2]}{(\mathrm{i}\,x^2-t)^{1+b}[\mathrm{i}\,(1+x^3)-3xt]},\\q&=\frac{\sqrt{b}\,\mathrm{e}^{-\mathrm{i}\,c_1}[(b-2)\,x^4+3\,\mathrm{i}\,(b-1)tx^2+(b+1)x+3t^2]}{c_0(\mathrm{i}\,x^2-t)^{1-b}[\mathrm{i}\,(1+x^3)-3tx]},\\I&\equiv|p|^2+|q|^2=\frac{(x^4-3t^2-2x)^2+9}{C[(1+x^3)^2+9t^2x^2]},\end{aligned}\tag{91}$$

and

$$p=-\frac{c_0\sqrt{b}\,\mathrm{e}^{\mathrm{i}\,c_1}[3(1-b)(\mathrm{i}\,x^2+3t^3-t)+\mathrm{i}\,(3b+5)(9t^2-x^4)x^2+3(7b+9)\,tx^4]}{2(\mathrm{i}\,x^3-3xt)^{1+b}[x^4+6\mathrm{i}x^2t-3t^2+1]},$$
$$q=\frac{\sqrt{b}\,\mathrm{e}^{-\mathrm{i}\,c_1}[3(1+b)(t-\mathrm{i}\,x^2-3t^3)+\mathrm{i}\,(3b-5)(9t^2-x^4)x^2+3(7b-9)\,tx^4]}{2C(\mathrm{i}\,x^3-3xt)^{1-b}[x^4+6\mathrm{i}x^2t-3t^2+1]},$$
$$I\equiv|p|^2+|q|^2=\frac{x^4(2x^4-9t^2-6)^2+36t^2(3t^2-1)^2+459t^4x^4+36t^2x^4+64}{4C[x^4(x^4+30t^2+2)+(3t^2-1)^2]},\tag{92}$$

with two arbitrary real constants c_1 and c_0. The rational solutions Eqs. (90), (91), and (92) are related to the complex polynomial solutions V_2, V_3, and V_4 given in Eq. (88).

Remark 1 From the solutions Eqs. (90), (91), and (92), we see that these solutions hold only for the AKNS system but not for the NLS equation, since $q\neq p^*$. Consequently, the quantity pq is complex, i.e., its imaginary part is nonzero.

Remark 2 The quantities $I_{b=1,V=V_k}\equiv(|p|^2+|q|^2)\big|_{b=1,V=V_k}$ and $I_{b=-1,V=V_k}$ are exactly the same for all $k\geq 2$, where V_k is defined by Eq. (87). In other words, the intensity I of the rational solutions is the same for the AKNS systems regardless of whether they exhibit normal dispersion and/or anomalous dispersion.

7.4. Rational soliton of the AKNS/NLS system

To obtain rational soliton solutions for both the AKNS system and the NLS equation from the VPIV Eq. (82), we need to employ the real polynomial (and/or rational) solutions.

Substituting V_1 or V_c into the solution Eq. (78) together with Eq. (79), we obtain a time-independent rational solution

$$p=\frac{c_0\sqrt{b}\,\mathrm{e}^{\mathrm{i}c_1}}{x+1},\quad q=\frac{\sqrt{b}\,\mathrm{e}^{-\mathrm{i}c_1}}{c_0(x+1)},\tag{93}$$

where c_1 and c_0 are two arbitrary constants.

It should be emphasized that the stationary rational solution Eq. (93) is a solution of the AKNS system for both $b=1$ and $b=-1$. However, Eq. (93) is a solution of the NLS equation only for $b=1$, but not for the $b=-1$ case.

Applying the real polynomial solution V_e given in Eq. (89) to Eq. (78) together with Eqs. (79) and (83), we obtain

$$p=\frac{c_0\sqrt{b}\,\mathrm{e}^{\mathrm{i}c_1}[2\mathrm{i}btx(3t^2-x^4-x)-2x^7-10t^2x^3+x^4-t^2]}{(3t^2-x^4-x)(x^4+t^2)^{1-b}(\mathrm{i}x^2+t)^{2b}},$$
$$q=-\frac{\sqrt{b}\,\mathrm{e}^{-\mathrm{i}c_1}[2\mathrm{i}btx(3t^2-x^4-x)+2x^7+10t^2x^3-x^4+t^2]}{c_0(3t^2-x^4-x)(x^4+t^2)^{1-b}(\mathrm{i}x^2-t)^{2b}},\tag{94}$$

where c_1 and c_0 are two real constants.

It can be readily verified that solution Eq. (94) is valid only for the AKNS system but not for the NLS equation, since $q\neq p^*$ for any real c_1 and c_0.

For the AKNS system, the intensity

$$I=|p|^2+|q|^2=\frac{[(2x^3+1)^2+36t^2x^2](c_0^4+1)}{c_0^2(x^4-3t^2+x)^2},\tag{95}$$

remains the same regardless of whether the system exhibits normal dispersion and/or anomalous dispersion.

7.5. Irrational algebraic solitons of the AKNS/NLS system

Although rational solutions to a wide range of nonlinear systems, including the AKNS systems, NLS equations, and KP equations, have been extensively investigated by mathematical physicists, the study of irrational solutions remains largely untouched. It is interesting that some types of real polynomial (and/or rational) solutions of the VPIV Eq. (82) will lead to irrational algebraic solitons for the AKNS/NLS systems.

For the polynomial solutions V_a (or V_d), V_b, and V_f given in Eq. (89), the corresponding solutions of the AKNS system can be written as

$$p=\frac{c_0\sqrt{b}[3t+\mathrm{i}\,bx(x+t^{1/3})]\,\mathrm{e}^{\mathrm{i}\left(c_1-\frac{bx^2}{3t}\right)}}{3t(x+t^{1/3})},\quad q=\frac{\sqrt{b}[3t-\mathrm{i}\,bx(x+t^{1/3})]\,\mathrm{e}^{-\mathrm{i}\left(c_1-\frac{bx^2}{3t}\right)}}{3c_0t(x+t^{1/3})},$$
$$I=|p|^2+|q|^2=\frac{9t^{8/3}+x^2t^{4/3}+x^4t^{2/3}+2tx^3}{9Ct^{8/3}(x+t^{1/3})^2},\quad C\equiv\frac{c_0^2}{1+c_0^4},\tag{96}$$
$$p=-\frac{c_0\sqrt{b}[\mathrm{i}\,bxA(9t^2-x^4)+6x^3t^{8/3}-9tx^8-1215t^5]\,\mathrm{e}^{\mathrm{i}\left(c_1-\frac{bx^2}{3t}\right)}}{3tA(x^4+9t^2)^{1-b}(3t-\mathrm{i}\,x^2)^{2b}},\ A\equiv x^5+t^{5/3}+45xt^2,$$
$$q=\frac{\sqrt{b}[\mathrm{i}\,bxA(9t^2-x^4)-6x^3t^{8/3}+9tx^8+1215t^5]\,\mathrm{e}^{\mathrm{i}\left(c_1-\frac{bx^2}{3t}\right)}}{3c_0tA(x^4+9t^2)^{1-b}(3t+\mathrm{i}\,x^2)^{2b}},$$

$$I=|p|^2+|q|^2=\frac{x^2(9t^2-x^4)^2A^2-36x^3t^{11/3}(3x^8+405t^4-x^3t^{5/3})+81t^2(x^8+135t^4)^2}{9Ct^2(x^4+9t^2)^2A^2}, \tag{97}$$

$$p=\frac{c_0\sqrt{b}(9Nt^{10/3}+12tx^3M+\mathrm{i}\,bxAD)\,\mathrm{e}^{\mathrm{i}\left(c_1-\frac{bx^2}{3t}\right)}}{3tDB^{1-b}(x^4+9t^2+6\,\mathrm{i}\,tx^2)^{2b}},\quad q=\frac{c_0\sqrt{b}(9Nt^{10/3}+12tx^3M-\mathrm{i}\,bxAD)\,\mathrm{e}^{-\mathrm{i}\left(c_1-\frac{bx^2}{3t}\right)}}{3tDB^{1-b}(x^4+9t^2-6\,\mathrm{i}\,tx^2)^{2b}},$$

$$I=|p|^2+|q|^2=\frac{x^2A^2D^2+144t^2x^6M^2+81N^2t^{20/3}N^2+216NMx^3t^{13/3}}{9Ct^2B^2D^2}, \tag{98}$$

where c_0 and c_1 are arbitrary real constants, and the other notations are defined by $A=x^8+18t^2x^4+405t^4$, $B=x^8+54t^2x^4+81t^4$, $D=t^{7/3}x+x^8+126x^4t^2-567t^4$, $M=x^{12}+81t^2x^8+4131t^4x^4+25515t^6$, and $N=27t^4-x^8-18t^2x^4$.

Figure 2 displays the intensity profiles of the irrational algebraic solitons, where $I=|p|^2+|q|^2$, corresponding to solutions Eqs. (97) and (98), respectively, with the parameters chosen as $c_0=b=1$.

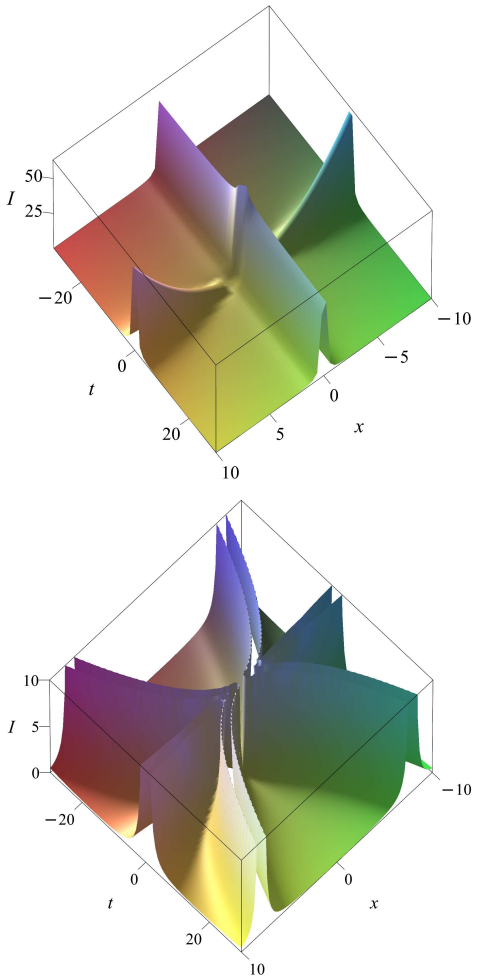

Fig. 2. Intensity distributions of the irrational algebraic solitons for $I=|p|^2+|q|^2$: (a) corresponding to solution Eq. (97) and (b) corresponding to solution Eq. (98), with parameters $c_0=b=1$.

As in the case of the rational algebraic soliton solutions with real V, the irrational algebraic solitons listed in this subsection are solutions of the AKNS system for both $b=1$ and $b=-1$. However, they are solutions of the NLS equation only for $b=1$.

Similar to the rational case, the intensity $I=|p|^2+|q|^2$ of the irrational algebraic solitons is the same for the AKNS system regardless of normal or anomalous dispersion.

7.6. Kummer solitons of the AKNS system

In addition to the polynomial/rational solutions, there are many other types of solutions. If we make the transformation

$$V=K\left(\frac{\mathrm{i}}{2}\eta^2\right)\equiv K(z)\equiv K, \tag{99}$$

and fix the parameters ω and ϕ_0 as

$$\omega=\frac{1}{4}(1-2\mu),\quad \phi_0=2b\mu, \tag{100}$$

the V Eq. (83) becomes an extended Kummer equation

$$\begin{aligned}&2zK_{zz}=(2z-1)K_z+(2\mu-1)K-2ZH_z,\\&[2zZ_z+(Z-2z-1)Z]K_z-2(2\mu-1)ZK=0.\end{aligned} \tag{101}$$

Especially, taking $Z=0$, the extended nonlinear Kummer Eq. (101) becomes a special linear variant Kummer equation

$$2zK_{zz}=(2z-1)K_z+(2\mu-1)K, \tag{102}$$

which possesses the general solution

$$V=K=\sqrt{z}\left[c_1K_M\left(\mu,\frac{3}{2},z\right)+c_2K_U\left(\mu,\frac{3}{2},z\right)\right], \tag{103}$$

where the Kummer-M function $K_M(\mu,\nu,z)$ and Kummer-U function $K_U(\mu,\nu,z)$ are two independent solutions of the Kummer equation

$$zK_{zz}+(\nu-z)K_z-\mu K=0. \tag{104}$$

Taking $c_1=0$ and $\mu=-n/2$, the solutions Eq. (103) are reduced back to the complex variant Hermite polynomial solutions Eq. (87).

By fixing some different parameters μ, c_1 and c_2 in the Kummer-soliton solutions Eq. (78) together with Eqs. (79), (82) and (103), one can find rich structures of Kummer solitons for the AKNS system (but not the NLS equation). Here we just list two more results for the fields p and q

$$p=\frac{(1+b)\mathrm{i}\,c_0\sqrt{t}\,\mathrm{e}^{\mathrm{i}\,bc_1}}{2(E_g+\sqrt{t})}-\frac{\mathrm{i}\,(1-b)c_0\,\mathrm{e}^{-\mathrm{i}\,bc_1}E_g(\mathrm{i}\,x^2+t+\sqrt{t}E_g)}{2t^2(E_g+\sqrt{t})},\quad E_g\equiv\exp\left(\frac{\mathrm{i}\,x^2}{2t}\right),$$
$$q=\frac{(b-1)\sqrt{t}\,\mathrm{e}^{\mathrm{i}\,bc_1}}{2c_0(E_g+\sqrt{t})}-\frac{(1+b)\,\mathrm{e}^{-\mathrm{i}\,bc_1}E_g(\mathrm{i}\,x^2+t+\sqrt{t}E_g)}{2c_0t^2(E_g+\sqrt{t})},\tag{105}$$

$$p=\frac{(b-1)c_0x\sqrt{-b}\,\mathrm{e}^{\mathrm{i}[c_1-bx^2/(2t)]}}{2t^{1-b/2}}+\frac{(1-\mathrm{i})dc_0\sqrt{b}t^{b/2}\,\mathrm{e}^{\mathrm{i}[c_1-(b-1)x^2/(2t)]}}{\sqrt{\pi t}\left[d\,\mathrm{erf}\left(\frac{(1-\mathrm{i})x}{2\sqrt{(t)}}\right)+1\right]},$$
$$q=-\frac{(1+b)x\sqrt{-b}\,\mathrm{e}^{\mathrm{i}[c_1-bx^2/(2t)]}}{2c_0t^{1+b/2}}+\frac{(1-\mathrm{i})d\sqrt{b}t^{-b/2}\,\mathrm{e}^{\mathrm{i}[(b+1)x^2/(2t)-c_1]}}{c_0\sqrt{\pi t}\left[d\,\mathrm{erf}\left(\frac{(1-\mathrm{i})x}{2\sqrt{(t)}}\right)+1\right]},\tag{106}$$

where d, c_1 and c_0 are three real constants and $\mathrm{erf}(z)$ is the standard error function defined by

$$\mathrm{erf}(z)=\frac{2\int_0^z\mathrm{e}^{-t^2}\,\mathrm{d}t}{\sqrt{\pi}}.$$

Figure 3 is a plot of the intensity profiles of the special Kummer solitons, where $I=|p|^2+|q|^2$, corresponding to the solution Eq. (105) with the parameters fixed as $c_0=b=d=1$.

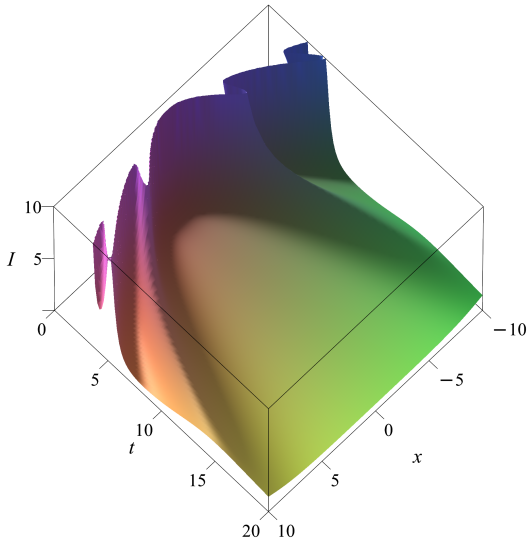

Fig. 3. Intensity plot of the special Kummer soliton for $I=|p|^2+|q|^2$ corresponding to the solution Eq. (105) with parameters $c_0=b=d=1$.

8. Summary and discussion

In this paper, we have proposed and systematically developed a novel symmetry decomposition method for solving nonlinear evolution equations, based on the concept of group-invariant spacetime rather than the usual natural spacetime. Taking the Ablowitz–Kaup–Newell–Segur (AKNS) system as a concrete and important example, we have demonstrated how this method can be used to construct a remarkable diversity of exact solutions, ranging from classical solitary waves to highly nontrivial elliptic, rational, irrational, and Painlevé-type solitons. The key idea is to enforce the vanishing of the symmetry generator not in the original coordinates, but in a transformed frame where the group-invariant variable becomes the new spatial coordinate. This procedure systematically decomposes the original $(1+1)$-dimensional PDE into a pair of consistent lower-dimensional dynamical systems: a t-dynamical system and a ξ-dynamical system (or an η-dynamical system for scaling invariance). The consistency condition between these two subsystems provides a powerful integrability constraint, which in turn leads to the explicit construction of the solutions.

A central technical achievement of this work is the successful localization of the nonlocal residual symmetries of the AKNS system, which originate from its Painlevé property and are intimately related to the Möbius invariance of the singular manifold equation. By extending the original system to include the auxiliary fields f, g, and u, we have shown that these nonlocal symmetries become local in the prolonged system. This localization is crucial because it allows the full power of standard Lie group theory to be applied. Consequently, the group-invariant decomposition yields closed-form expressions for a vast array of solutions. The general solution is expressed in terms of a single function $G(\xi)$ that satisfies a fourth-order elliptic equation, whose general solution is given by the ratio of two linear combinations of Jacobi elliptic functions, the Weierstrass $\mathcal{P}$ function, or their degenerations. This result elegantly unifies the description of periodic cnoidal waves and localized solitons within a single framework.

The richness of the obtained solutions is truly remarkable. By choosing different representations for the function G, we have constructed at least eight distinct families of elliptic soliton solutions, including those expressed via JacobiSN, JacobiCN, JacobiDN, JacobiSC, JacobiSD, JacobiCD, the Weierstrass $\mathcal{P}$ function, and JacobiSN again (but with a different parameterization). Some of these families, particularly those involving the Weierstrass $\mathcal{P}$ function and the second JacobiSN representation, are genuine solutions of the full AKNS system but do not satisfy the complex conjugate reduction to the NLS equation, thus highlighting the greater generality of the AKNS system. For those families that do satisfy the NLS reduction, they represent new classes of elliptic solitons propagating on nontrivial periodic backgrounds, generalizing the well-known

solitons on a constant background.

Beyond the elliptic realm, the spacetime scaling invariant decomposition has led to the discovery of a new class of solutions that we term Painlevé solitons. In this case, the reduced equations are shown to be intimately connected to the fourth Painlevé equation (PIV). By establishing a connection between the F-equation and a bilinear variant of PIV (the VPIV equation), we have been able to generate an infinite hierarchy of rational, irrational, and special function solutions. In particular:

(i) We have introduced new classes of polynomials, such as the complex variant Hermite polynomials, which generate rational soliton solutions for the AKNS system. These solutions are distinct from the classical rational solutions of the NLS equation.

(ii) We have constructed genuine rational and irrational algebraic solitons that satisfy both the AKNS system and the NLS equation (for the $b = 1$ case). The irrational solitons, expressed in terms of fractional powers and algebraic combinations of x and t, are particularly novel and have been largely unexplored in the literature.

(iii) By connecting the VPIV equation to the Kummer equation, we have derived solutions in terms of the Kummer functions K_M and K_U, leading to what we call Kummer solitons. These include solutions expressed through elementary exponentials and the error function, the latter representing a type of diffusive soliton with a front-like structure.

The results presented in this paper open up several interesting avenues for future research. First, the symmetry decomposition method should be applicable to a wide range of other integrable and possibly non-integrable systems. Extending this approach to higher-dimensional systems, such as the Kadomtsev–Petviashvili (KP) equation or the Davey-Stewartson system, is a natural next step. The deformation conjecture, which suggests that lower-dimensional integrable systems can be deformed into higher-dimensional ones using conservation laws, may provide a guiding principle for such extensions. Second, the physical interpretation of the new solutions, particularly the irrational and Kummer solitons, deserves further investigation. For the NLS equation, these solutions may describe novel nonlinear wave patterns in optical fibers or deep-water waves, where the background itself is not constant but varies algebraically or in a way governed by Painlevé transcendents. The error-function soliton, for instance, might be relevant to the description of dispersive shock waves or the merging of solitons with a diffusive background. Third, the connection between the group-invariant decomposition and the IST on non-constant backgrounds is a deep theoretical question. The elliptic solutions presented here likely correspond to finite-gap potentials, and it would be interesting to clarify their spectral characterization. Finally, the discovery that the intensity $I = |p|^2 + |q|^2$ for many of the rational and irrational solutions is identical for both the normal ($b = 1$) and anomalous ($b = -1$) dispersion regimes of the AKNS system suggests the existence of a hidden symmetry or an underlying reality condition that transcends the specific dispersion sign. This observation merits further analytical exploration.

In conclusion, the group-invariant spacetime decomposition method provides a powerful, systematic, and unified approach for constructing exact solutions to nonlinear PDEs. Its application to the AKNS system has yielded a treasure trove of new solutions, including elliptic, rational, irrational, and Painlevé solitons, thereby significantly enriching our understanding of this fundamental integrable system and opening new perspectives for the study of nonlinear wave phenomena.

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