**SPECIAL TOPIC — Nonlinear localized waves: Theory, methods and applications**

# Checking out the $(3+1)$-dimensional Hirota–Wazwaz–El Sherif–El Tantawy bilinear equation modeling oceanic nonlinear wave phenomena

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Ocean wave dynamics has emerged as a research focus in *Chin. Phys. B*. This study investigates the $(3+1)$-dimensional Hirota–Wazwaz–El Sherif–El Tantawy bilinear equation, a mathematical framework for modeling certain nonlinear wave phenomena in oceans and seas. We combine the equivalent form to that equation, the non-characteristic movable singular manifold, and symbolic computation to deal with the equation. Our findings include an auto-Bäcklund transformation, four families of solitonic solutions, and two families of rational solutions for the equation. All our results hinge on the coefficients in the equation, under the constraints that these coefficients are respectively non-zero. Various choices of these coefficients might aid in exploring certain nonlinear wave phenomena in oceans and seas through our auto-Bäcklund transformation, solitons, and rational solutions.

**Keywords:** nonlinear wave phenomena in oceans and seas, $(3+1)$-dimensional Hirota–Wazwaz–El Sherif–El Tantawy bilinear equation, equivalent form, singular manifold, symbolic computation, Bäcklund transformation, solitons, rational solutions

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## 1. Introduction

Ocean wave dynamics/physics has emerged as a research focus in *Chin. Phys. B*[1–3] and other leading journals, with recent relevant contributions including certain boundary effects on the modal shape in a deep ocean via non-integer-order parabolic cylinder functions,[1] topographic oceanic Rossby solitary waves via physics-informed neural networks,[2] tubular actuators constructed with dielectric elastomer materials for ocean exploration,[3] triad resonance for internal waves via rogue waves and breathers, aiming at energy transfer processes in the oceans,[4] performance assessment for ocean engineering applications via boundary element methods,[5] an open-source solver for deterministic nonlinear ocean wave propagation,[6] the impact of ocean currents on surface waves generated by tropical cyclones,[7] as well as similarity reductions and Bäcklund transformation for a $(2+1)$-dimensional generalized variable-coefficient Date–Jimbo–Kashiwara–Miwa equation with marine-engineering and ocean-dynamics applications.[8]

To begin our story, Ref. [9] has confirmed the Lax integrability and presented certain superpositions of solitons, breathers, and lump waves for a non-Painlevé-integrable extension of the Boiti–Leon–Manna–Pempinelli equation, which is

$$\mathcal{U}_{xyt}+\alpha_1\left[\mathcal{U}_{xxxxy}+3\left(\mathcal{U}_x\mathcal{U}_y\right)_{xx}\right]+\alpha_2\mathcal{U}_{xyy}+\alpha_3\mathcal{U}_{xxy}+\alpha_4\mathcal{U}_{xxz}=0, \tag{1}$$

where the subscripts denote the partial derivatives, $\mathcal{U}(x,y,z,t)$ is a real differentiable function of the scaled spatial variables $x$, $y$, $z$, and temporal variable $t$, while $\alpha_1\ldots\alpha_4$ are all real constants satisfying[10]

$$\alpha_1\neq 0,\ \alpha_2\neq 0,\ \alpha_3\neq 0,\ \text{and}\ \alpha_4\neq 0. \tag{2}$$

A potential-function relation, i.e.,

$$\mathcal{U}_x(x,y,z,t)=\Upsilon(x,y,z,t),$$

has helped people link Eq. (1) with Eq. (3) as follows.[10]

Originally proposed in Ref. [10] and "applied to delve into the study of nonlinear wave phenomena in oceans and seas",[11] the $(3+1)$-dimensional Hirota–Wazwaz–El Sherif–El Tantawy bilinear equation, i.e.,

$$\Upsilon_{yt}+\alpha_1\left(\Upsilon_{xxxy}+6\Upsilon_y\Upsilon_x+3\Upsilon\Upsilon_{xy}+3\Upsilon_{xx}\int\Upsilon_y\,\mathrm{d}x\right)+\alpha_2\Upsilon_{yy}+\alpha_3\Upsilon_{xy}+\alpha_4\Upsilon_{xz}=0, \tag{3}$$

will be the subject that the present study seeks to explore.

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Reference [11] has discussed how Eq. (3) can model certain nonlinear wave phenomena in oceans, and presented certain lump-multi-stripe, lump-multi-soliton, and interaction solutions for Eq. (3). Reference [10] has shown that Eq. (3) does not satisfy the Painlevé integrability test, but that the integrability characteristics of Eq. (3) can be justified in the Lax sense, and presented certain multi-soliton, lump, and rational solutions for Eq. (3).

By the way, all different from Eq. (3), there have existed some other equations named the "$(3+1)$-dimensional Hirota bilinear equation"s, as seen in Refs. [12–21] and references therein. More relevant nonlinear contributions can also be seen, e.g., in Refs. [22–55].

To our knowledge, there has been no Bäcklund-transformation work on Eq. (3) as yet. In Section 2 of this paper, we will present an equivalent form to Eq. (3). In Section 3, combining the equivalent form, singular manifold, and symbolic computation,[56–59] we will obtain an auto-Bäcklund transformation for Eq. (3). In Section 4, we will construct four families of solitons and two families of rational solutions for Eq. (3). Section 5 will present our conclusions.

## 2. Equivalent form to Eq. (3)

With the introduction of

$$\Gamma(x,y,z,t)=\int \Upsilon_y(x,y,z,t)\,\mathrm{d}x, \tag{4}$$

Eq. (3) can be equivalently rewritten as

$$\begin{aligned}&\Upsilon_{yt}+\alpha_1\left(\Upsilon_{xxxy}+6\Upsilon_y\Upsilon_x+3\Gamma\Upsilon_{xy}+3\Upsilon_{xx}\Gamma\right)\\&+\alpha_2\Upsilon_{yy}+\alpha_3\Upsilon_{xy}+\alpha_4\Upsilon_{xz}=0,\end{aligned} \tag{5a}$$

$$\Gamma_x=\Upsilon_y. \tag{5b}$$

## 3. Auto-Bäcklund transformation for Eq. (3)

Through the application of symbolic computation, we utilize the Painlevé expansion, which is a form of the generalized Laurent series centered on a non-characteristic movable singular manifold characterized by the analytic function $\lambda(x,y,z,t)=0$,[8,60] whereby

$$\Upsilon(x,y,z,t)=\lambda^{-\Omega}(x,y,z,t)\sum_{\omega=0}^{\infty}\Upsilon_\omega(x,y,z,t)\lambda^{\omega}(x,y,z,t), \tag{6a}$$

$$\Gamma(x,y,z,t)=\lambda^{-\Xi}(x,y,z,t)\sum_{\xi=0}^{\infty}\Gamma_\xi(x,y,z,t)\lambda^{\xi}(x,y,z,t), \tag{6b}$$

with $\Upsilon_\omega(x,y,z,t)$'s and $\Gamma_\xi(x,y,z,t)$'s as the analytic functions of $x$, $y$, $z$, and $t$, $\Omega$ and $\Xi$ as the positive integers, $\Upsilon_0(x,y,z,t)\neq 0$, $\Gamma_0(x,y,z,t)\neq 0$, $\lambda_x(x,y,z,t)\neq 0$ and $\lambda_y(x,y,z,t)\neq 0$.

Matching the exponents of $\lambda(x,y,z,t)$ at the lowest orders results in $\Omega=\Xi=2$.

The truncated Painlevé expansion, denoted by

$$\Upsilon(x,y,z,t)=\frac{\Upsilon_0(x,y,z,t)}{\lambda(x,y,z,t)^2}+\frac{\Upsilon_1(x,y,z,t)}{\lambda(x,y,z,t)}+\Upsilon_2(x,y,z,t), \tag{7a}$$

$$\Gamma(x,y,z,t)=\frac{\Gamma_0(x,y,z,t)}{\lambda(x,y,z,t)^2}+\frac{\Gamma_1(x,y,z,t)}{\lambda(x,y,z,t)}+\Gamma_2(x,y,z,t), \tag{7b}$$

may consequently be substituted into Eq. (5), where the coefficients corresponding to the identical powers of $\lambda(x,y,z,t)$ are set to zero, respectively. As a result, the Painlevé–Bäcklund equations can be obtained symbolically as follows:

$$(\lambda^{-6},\lambda^{-3}):\quad \Gamma_0=-2\lambda_y\lambda_x, \tag{8}$$

$$\Upsilon_0=-2\lambda_x^2, \tag{9}$$

$$(\lambda^{-5},\lambda^{-2}):\quad \Gamma_1=2\lambda_{xy}, \tag{10}$$

$$\Upsilon_1=2\lambda_{xx}, \tag{11}$$

$$(\lambda^{-4},-):\quad 3\alpha_1\lambda_x\lambda_y\Upsilon_2+3\alpha_1\lambda_x^2\Gamma_2+\alpha_1\lambda_{xxx}\lambda_y-3\alpha_1\lambda_{xy}\lambda_{xx}+3\alpha_1\lambda_x\lambda_{xxy}+\alpha_2\lambda_y^2+\alpha_3\lambda_x\lambda_y+\alpha_4\lambda_z\lambda_x+\lambda_t\lambda_y=0, \tag{12}$$

$$\begin{aligned}(\lambda^{-3},-):\quad &6\alpha_1\lambda_x^3\Upsilon_{2,y}+6\alpha_1\lambda_y\lambda_x^2\Upsilon_{2,x}+9\alpha_1\lambda_{xy}\lambda_x^2\Upsilon_2+9\alpha_1\lambda_y\lambda_{xx}\lambda_x\Upsilon_2+18\alpha_1\lambda_{xx}\lambda_x^2\Gamma_2+7\alpha_1\lambda_{xxxy}\lambda_x^2+3\alpha_1\lambda_{xx}\lambda_{xxy}\lambda_x\\&-4\alpha_1\lambda_{xy}\lambda_{xxx}\lambda_x+2\alpha_1\lambda_y\lambda_{xxxx}\lambda_x-3\alpha_1\lambda_{xy}\lambda_{xx}^2+\alpha_1\lambda_y\lambda_{xx}\lambda_{xxx}+\alpha_2\lambda_{yy}\lambda_x^2+4\alpha_2\lambda_y\lambda_{xy}\lambda_x+\alpha_2\lambda_y^2\lambda_{xx}\\&+3\alpha_3\lambda_{xy}\lambda_x^2+3\alpha_3\lambda_y\lambda_{xx}\lambda_x+3\alpha_4\lambda_{xz}\lambda_x^2+3\alpha_4\lambda_z\lambda_{xx}\lambda_x+\lambda_{yt}\lambda_x^2+2\lambda_y\lambda_{xt}\lambda_x+2\lambda_t\lambda_{xy}\lambda_x+\lambda_t\lambda_y\lambda_{xx}=0,\end{aligned} \tag{13}$$

$$\begin{aligned}(\lambda^{-2},-):\quad &3\alpha_1\lambda_x^2\Upsilon_{2,xy}+12\alpha_1\lambda_{xy}\lambda_x\Upsilon_{2,x}+18\alpha_1\lambda_{xx}\lambda_x\Upsilon_{2,y}+3\alpha_1\lambda_y\lambda_x\Upsilon_{2,xx}+6\alpha_1\lambda_y\lambda_{xx}\Upsilon_{2,x}+9\alpha_1\lambda_{xxy}\lambda_x\Upsilon_2+9\alpha_1\lambda_{xy}\lambda_{xx}\Upsilon_2\\&+3\alpha_1\lambda_y\lambda_{xxx}\Upsilon_2+12\alpha_1\lambda_{xxx}\lambda_x\Gamma_2+9\alpha_1\lambda_{xx}^2\Gamma_2+5\alpha_1\lambda_{xxxxy}\lambda_x-2\alpha_1\lambda_{xxy}\lambda_{xxx}+4\alpha_1\lambda_{xx}\lambda_{xxxy}-\alpha_1\lambda_{xy}\lambda_{xxxx}\\&+\alpha_1\lambda_y\lambda_{xxxxx}+2\alpha_2\lambda_{xyy}\lambda_x+2\alpha_2\lambda_{xy}^2+\alpha_2\lambda_{yy}\lambda_{xx}+2\alpha_2\lambda_y\lambda_{xxy}+3\alpha_3\lambda_{xxy}\lambda_x+3\alpha_3\lambda_{xy}\lambda_{xx}+\alpha_3\lambda_y\lambda_{xxx}\\&+3\alpha_4\lambda_{xxz}\lambda_x+3\alpha_4\lambda_{xz}\lambda_{xx}+\alpha_4\lambda_z\lambda_{xxx}+2\lambda_{xyt}\lambda_x+2\lambda_{xt}\lambda_{xy}+\lambda_{yt}\lambda_{xx}+\lambda_y\lambda_{xxt}+\lambda_t\lambda_{xxy}=0,\end{aligned} \tag{14}$$

$$\begin{aligned}(\lambda^{-1},-):\quad &3\alpha_1\lambda_{xx}\Upsilon_{2,xy}+3\alpha_1\lambda_{xy}\Upsilon_{2,xx}+6\alpha_1\lambda_{xxy}\Upsilon_{2,x}+6\alpha_1\lambda_{xxx}\Upsilon_{2,y}+3\alpha_1\lambda_{xxxy}\Upsilon_2+3\alpha_1\lambda_{xxxx}\Gamma_2+\alpha_1\lambda_{xxxxy}\\&+\alpha_2\lambda_{xxyy}+\alpha_3\lambda_{xxxy}+\alpha_4\lambda_{xxxz}+\lambda_{xxyt}=0,\end{aligned} \tag{15}$$

$$(-,\lambda^{-1}):\quad \text{(satisfied automatically)}$$

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$$(\lambda^0,\lambda^0):\quad \Upsilon_{2,yt}+\alpha_1\left(\Upsilon_{2,xxxy}+6\Upsilon_{2,y}\Upsilon_{2,x}+3\Upsilon_2\Upsilon_{2,xy}+3\Upsilon_{2,xx}\Gamma_2\right)+\alpha_2\Upsilon_{2,yy}+\alpha_3\Upsilon_{2,xy}+\alpha_4\Upsilon_{2,xz}=0, \tag{16}$$

$$\Gamma_{2,x}=\Upsilon_{2,y}. \tag{17}$$

For Eq. (3) (or equivalently, for Eq. (5)), Eqs. (7)–(17), taken collectively, constitute an auto-Bäcklund transformation, with $\Upsilon_2(x,y,z,t)$ and $\Gamma_2(x,y,z,t)$ serving as the seed solutions.[8,60] This assertion hinges on the fact that Eqs. (7)–(17) are interconnected in a self-consistent manner in the sense that they can be explicitly solved to determine the functions $\lambda(x,y,z,t)$, $\Upsilon_0(x,y,z,t)$, $\Gamma_0(x,y,z,t)$, $\Upsilon_1(x,y,z,t)$, $\Gamma_1(x,y,z,t)$, $\Upsilon_2(x,y,z,t)$ and $\Gamma_2(x,y,z,t)$, as elaborated subsequently.

Auto-Bäcklund transformation Eqs. (7)–(17), involving the variables $\Upsilon(x,y,z,t)$ and $\Gamma(x,y,z,t)$, hinges on the coefficients $\alpha_1\ldots\alpha_4$, under coefficient constraints Eq. (2).

Various choices of $\alpha_1\ldots\alpha_4$ might aid in exploring certain oceanic nonlinear wave phenomena through auto-Bäcklund transformation Eqs. (7)–(17).

## 4. Families of the solitonic and rational solutions for Eq. (3)

To launch the investigation, we will conduct symbolic computation on a number of solitonic instances with explicit solvability, under the assumptions that

$$\lambda(x,y,z,t)=\mathrm{e}^{a_1(z,t)x+a_2(z,t)y+a_3(z,t)}+1, \tag{18}$$

$$\Upsilon_2(x,y,z,t)=b_1(z,t)x+b_2(z,t)y+b_3(z,t), \tag{19}$$

$$\Gamma_2(x,y,z,t)=d_1(z,t)x+d_2(z,t)y+d_3(z,t), \tag{20}$$

where $a_1(z,t)$, $a_2(z,t)$, $a_3(z,t)$, $b_1(z,t)$, $b_2(z,t)$, $b_3(z,t)$, $d_1(z,t)$, $d_2(z,t)$, and $d_3(z,t)$ are real differentiable functions of $z$ and $t$, with $a_1(z,t)\neq 0$ and $a_2(z,t)\neq 0$ since $\lambda_x\neq 0$ and $\lambda_y\neq 0$.

Inserting Assumptions Eqs. (18)–(20) into auto-Bäcklund transformation Eqs. (7)–(17) and carrying out symbolic computation, we obtain

$$(-,\lambda^0):\quad b_2(z,t)=d_1(z,t), \tag{21}$$

$$(\lambda^{-4}x^1,-):\quad b_1(z,t)=-\frac{\alpha_4 a_{1,z}(z,t)}{3\alpha_1 a_2(z,t)}-\frac{a_{1,t}(z,t)}{3\alpha_1 a_1(z,t)}-\frac{a_1(z,t)d_1(z,t)}{a_2(z,t)}, \tag{22}$$

$$(\lambda^{-4}y^1,-):\quad d_1(z,t)=-\frac{a_{2,t}(z,t)}{3\alpha_1 a_1(z,t)}-\frac{a_1(z,t)d_2(z,t)}{a_2(z,t)}-\frac{\alpha_4 a_{2,z}(z,t)}{3\alpha_1 a_2(z,t)}, \tag{23}$$

$$(\lambda^{-4},-):\quad d_3(z,t)=-\frac{a_2(z,t)a_{3,t}(z,t)}{3\alpha_1 a_1(z,t)^2}-\frac{\alpha_2 a_2(z,t)^2}{3\alpha_1 a_1(z,t)^2}-\frac{\alpha_3 a_2(z,t)}{3\alpha_1 a_1(z,t)}-\frac{a_2(z,t)b_3(z,t)}{a_1(z,t)}-\frac{a_1(z,t)a_2(z,t)}{3}-\frac{\alpha_4 a_{3,z}(z,t)}{3\alpha_1 a_1(z,t)}, \tag{24}$$

$$(\lambda^{-3},-):\quad a_2(z,t)=a_4-\alpha_4\int a_{1,z}(z,t)\,\mathrm{d}t\neq 0, \tag{25}$$

$$(\lambda^0,-):\quad d_2(z,t)=0\ \text{(assumed)}, \tag{26}$$

$$a_1(z,t)=a_5(t)=\text{real non-zero differentiable function of } t, \tag{27}$$

based on which $a_4$ turns out to be a real non-zero constant.

We now assemble the first family of solitonic solutions for Eq. (3) (or equivalently, for Eq. (5)), i.e.,

$$\Upsilon^{(\mathrm{I})}(x,y,z,t)=-\frac{a_5(t)^2}{2}\tanh^2\left[\frac{a_5(t)x+a_4y+a_3(z,t)}{2}\right]-\frac{a_5'(t)x}{3\alpha_1 a_5(t)}+b_3(z,t)+\frac{a_5(t)^2}{2}, \tag{28a}$$

$$\Gamma^{(\mathrm{I})}(x,y,z,t)=-\frac{a_4a_5(t)}{2}\tanh^2\left[\frac{a_5(t)x+a_4y+a_3(z,t)}{2}\right]-\frac{a_4a_{3,t}(z,t)}{3\alpha_1 a_5(t)^2}-\frac{\alpha_4 a_{3,z}(z,t)}{3\alpha_1 a_5(t)}-\frac{a_4b_3(z,t)}{a_5(t)}-\frac{a_4^2\alpha_2}{3\alpha_1 a_5(t)^2}-\frac{a_4\alpha_3}{3\alpha_1 a_5(t)}+\frac{a_4a_5(t)}{6}, \tag{28b}$$

where the "$'$" sign represents differentiation with respect to $t$.

Again, symbolically and explicitly, for solitonic solutions under the assumptions that*

$$\lambda(x,y,z,t)=\mathrm{e}^{f_1(t)x+f_2(t)y+f_3(t)z+f_4(t)}+1, \tag{29}$$

$$\Upsilon_2(x,y,z,t)=g_1(t)x+g_2(t)y+g_3(t)z+g_4(t), \tag{30}$$

$$\Gamma_2(x,y,z,t)=h_1(t)x+h_2(t)y+h_3(t)z+h_4(t), \tag{31}$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$, $g_1(t)$, $g_2(t)$, $g_3(t)$, $g_4(t)$, $h_1(t)$, $h_2(t)$, $h_3(t)$, and $h_4(t)$ are real differentiable functions of $t$, with $f_1(t)\neq 0$ and $f_2(t)\neq 0$ since $\lambda_x\neq 0$ and $\lambda_y\neq 0$, we can substitute them into auto-Bäcklund transformation Eqs. (7)–(17) and obtain

$$(-,\lambda^0):\quad g_2(t)=h_1(t), \tag{32}$$

$$(\lambda^{-4}x^1,-):\quad g_1(t)=-\frac{f_1'(t)}{3\alpha_1 f_1(t)}-\frac{f_1(t)h_1(t)}{f_2(t)}, \tag{33}$$

$$(\lambda^{-4}y^1,-):\quad h_1(t)=-\frac{f_2'(t)}{3\alpha_1 f_1(t)}-\frac{f_1(t)h_2(t)}{f_2(t)}, \tag{34}$$

$$(\lambda^{-4}z^1,-):\quad h_3(t)=-\frac{f_2(t)f_3'(t)}{3\alpha_1 f_1(t)^2}-\frac{f_2(t)g_3(t)}{f_1(t)}, \tag{35}$$

$$(\lambda^{-4},-):\quad h_4(t)=-\frac{\alpha_2 f_2(t)^2}{3\alpha_1 f_1(t)^2}-\frac{\alpha_3 f_2(t)}{3\alpha_1 f_1(t)}$$

*The set of Assumptions Eqs. (29)–(31) can be considered as a special case of Assumptions Eqs. (18)–(20), but we hereby prefer to take one more look at Assumptions Eqs. (29)–(31).

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$$-\frac{\alpha_4 f_3(t)}{3\alpha_1 f_1(t)}-\frac{f_2(t)f_4'(t)}{3\alpha_1 f_1(t)^2}$$
$$-\frac{f_2(t)g_4(t)}{f_1(t)}-\frac{f_1(t)f_2(t)}{3}, \tag{36}$$

$$(\lambda^{-3},-): \quad f_2(t)=f_5=\text{real non-zero constant}, \tag{37}$$

$$(\lambda^{0},-): \quad h_2(t)=h_5=\text{real non-zero constant}, \tag{38}$$

$$f_1(t)=\pm\frac{f_5}{\sqrt{2\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}}, \tag{39}$$

where $f_6$ is another real constant, and $\sqrt{-f_5^2 f_6-6\alpha_1 h_5 t}\neq 0$. Due to the "$\pm$" sign, we now assemble the second and third families of solitonic solutions for Eq. (3) (or equivalently, for Eq. (5)), i.e.,

$$\begin{aligned}\Upsilon^{(\mathrm{II})\&(\mathrm{III})}(x,y,z,t) =& \frac{f_5^2}{4\left(f_5^2 f_6+6\alpha_1 h_5 t\right)}\tanh^2\left[\frac{\pm\frac{f_5 x}{\sqrt{2\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}}+f_5 y+f_3(t)z+f_4(t)}{2}\right]\\ &+\frac{h_5 x}{2\left(f_5^2 f_6+6\alpha_1 h_5 t\right)}\mp\frac{h_5 y}{\sqrt{2\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}}+g_3(t)z+\frac{f_5^2}{4\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}+g_4(t),\end{aligned} \tag{40a}$$

$$\begin{aligned}\Gamma^{(\mathrm{II})\&(\mathrm{III})}(x,y,z,t) =& \mp\frac{f_5^2}{\sqrt{8\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}}\tanh^2\left[\frac{\pm\frac{f_5 x}{\sqrt{2\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}}+f_5 y+f_3(t)z+f_4(t)}{2}\right]\\ &\mp\frac{h_5 x}{\sqrt{2\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}}+h_5 y\pm\frac{\sqrt{2}f_5^2 f_6 g_3(t)z}{\sqrt{-f_5^2 f_6-6\alpha_1 h_5 t}}+\frac{4h_5 t f_3'(t)z}{f_5}\pm\frac{6\sqrt{2}\alpha_1 h_5 t g_3(t)z}{\sqrt{-f_5^2 f_6-6\alpha_1 h_5 t}}\\ &+\frac{2f_5 f_6 f_3'(t)z}{3\alpha_1}\mp\frac{\alpha_3\sqrt{2\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}}{3\alpha_1}\mp\frac{\alpha_4 f_3(t)\sqrt{2\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}}{3\alpha_1 f_5}\\ &\mp g_4(t)\sqrt{2\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}\pm\frac{f_5^2}{6\sqrt{2\left(-f_5^2 f_6-6\alpha_1 h_5 t\right)}}+\frac{2f_5 f_6 f_4'(t)}{3\alpha_1}\\ &+4\alpha_2 h_5 t+\frac{4h_5 t f_4'(t)}{f_5}+\frac{2\alpha_2 f_5^2 f_6}{3\alpha_1}.\end{aligned} \tag{40b}$$

Similarly, solitonically, symbolically and explicitly, under the assumptions that

$$\lambda(x,y,z,t)=\mathrm{e}^{m_1x^2+m_2x+m_3y+m_4z+m_5(t)}+1, \tag{41}$$
$$\Upsilon_2(x,y,z,t)=n_1x^2+n_2x+n_3y+n_4z+n_5(t), \tag{42}$$
$$\Gamma_2(x,y,z,t)=s_1x^2+s_2x+s_3y+s_4z+s_5(t), \tag{43}$$

where $m_1$, $m_2$, $m_3$, $m_4$, $n_1$, $n_2$, $n_3$, $n_4$, $s_1$, $s_2$, $s_3$, and $s_4$ are real constants, $m_5(t)$, $n_5(t)$, and $s_5(t)$ are real differentiable functions of $t$, with $m_3\neq 0$ since $\lambda_y\neq 0$, we can go through auto-Bäcklund transformation Eqs. (7)–(17) and obtain

$$(-,\lambda^0x^1): \quad s_1=0, \tag{44}$$
$$(-,\lambda^0): \quad s_2=n_3, \tag{45}$$
$$(\lambda^0x^1,-): \quad n_1\neq 0 \ \text{ but } \ n_3=0, \tag{46}$$
$$(\lambda^0y^1,-): \quad s_3=0, \tag{47}$$
$$(\lambda^0z^1,-): \quad s_4=0, \tag{48}$$
$$(\lambda^0,-): \quad s_5(t)=0, \tag{49}$$
$$(\lambda^{-4}z^1,-): \quad n_4=0, \tag{50}$$
$$(\lambda^{-4}x^3,-): \quad n_1=-\frac{4m_1^2}{3}\neq 0, \tag{51}$$
$$(\lambda^{-4}x^2,-): \quad n_2=-\frac{4m_1m_2}{3}, \tag{52}$$
$$(\lambda^{-4}x^1,-): \quad n_5(t)=-\frac{\alpha_3}{3\alpha_1}-\frac{\alpha_4m_4}{3\alpha_1m_3}-2m_1-\frac{m_2^2}{3}, \tag{53}$$
$$(\lambda^{-4},-): \quad m_5(t)=m_6-\alpha_2m_3t, \tag{54}$$

with $m_6$ as another real constant.

We now derive the fourth family of solitonic solutions for Eq. (3) (or equivalently, for Eq. (5)), i.e.,

$$\begin{aligned}\Upsilon^{(\mathrm{IV})}(x,y,z,t) =& -\frac{(2m_1x+m_2)^2}{2}\\ &\times\tanh^2\left(\frac{m_1x^2+m_2x+m_3y+m_4z-\alpha_2m_3t+m_6}{2}\right)\\ &+2m_1\tanh\left(\frac{m_1x^2+m_2x+m_3y+m_4z-\alpha_2m_3t+m_6}{2}\right)\\ &+\frac{2m_1^2x^2}{3}+\frac{2m_1m_2x}{3}+\frac{m_2^2}{6}-\frac{\alpha_3}{3\alpha_1}-\frac{\alpha_4m_4}{3\alpha_1m_3},\end{aligned} \tag{55a}$$

$$\begin{aligned}\Gamma^{(\mathrm{IV})}(x,y,z,t) =& -\frac{m_3(2m_1x+m_2)}{2}\\ &\times\tanh^2\left(\frac{m_1x^2+m_2x+m_3y+m_4z-\alpha_2m_3t+m_6}{2}\right)\end{aligned}$$

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$$+m_1m_3x+\frac{m_2m_3}{2}, \tag{55b}$$

with $2m_1x+m_2\neq 0$ since $\lambda_x\neq 0$. We also prefer to have $m_1\neq 0$.

Let us now symbolically compute some explicitly solvable rational examples, firstly under the assumptions that

$$\lambda(x,y,z,t)=k_1(z,t)x+k_2(z,t)y+k_3(z,t)\ , \tag{56}$$
$$\Upsilon_2(x,y,z,t)=l_1(z,t)x+l_2(z,t)y+l_3(z,t)\ , \tag{57}$$
$$\Gamma_2(x,y,z,t)=p_1(z,t)x+p_2(z,t)y+p_3(z,t)\ , \tag{58}$$

where $k_1(z,t)$, $k_2(z,t)$, $k_3(z,t)$, $l_1(z,t)$, $l_2(z,t)$, $l_3(z,t)$, $p_1(z,t)$, $p_2(z,t)$ and $p_3(z,t)$ are real differentiable functions of $z$ and $t$, with $k_1(z,t)\neq 0$ and $k_2(z,t)\neq 0$ since $\lambda_x\neq 0$ and $\lambda_y\neq 0$.

Substituting assumptions (Eqs. (56)–(58)) into auto-Bäcklund transformation (Eqs. (7)–(17)) gives

$$(-,\lambda^0):\quad l_2(z,t)=p_1(z,t), \tag{59}$$
$$(\lambda^{-4}x^1,-):\quad p_1(z,t)=-\frac{\alpha_4k_{1,z}(z,t)}{3\alpha_1k_1(z,t)}-\frac{k_{1,t}(z,t)k_2(z,t)}{3\alpha_1k_1(z,t)^2}-\frac{k_2(z,t)l_1(z,t)}{k_1(z,t)}, \tag{60}$$
$$(\lambda^{-4}y^1,-):\quad p_2(z,t)=\frac{\alpha_4k_{1,z}(z,t)k_2(z,t)}{3\alpha_1k_1(z,t)^2}-\frac{\alpha_4k_{2,z}(z,t)}{3\alpha_1k_1(z,t)}+\frac{k_{1,t}(z,t)k_2(z,t)^2}{3\alpha_1k_1(z,t)^3}-\frac{k_{2,t}(z,t)k_2(z,t)}{3\alpha_1k_1(z,t)^2}+\frac{k_2(z,t)^2l_1(z,t)}{k_1(z,t)^2}, \tag{61}$$
$$(\lambda^{-4},-):\quad p_3(z,t)=-\frac{\alpha_2k_2(z,t)^2}{3\alpha_1k_1(z,t)^2}-\frac{\alpha_3k_2(z,t)}{3\alpha_1k_1(z,t)}-\frac{\alpha_4k_{3,z}(z,t)}{3\alpha_1k_1(z,t)}-\frac{k_2(z,t)k_{3,t}(z,t)}{3\alpha_1k_1(z,t)^2}-\frac{k_2(z,t)l_3(z,t)}{k_1(z,t)}, \tag{62}$$
$$(\lambda^{-3},-):\quad k_2(z,t)=k_4(z)-\alpha_4\int k_{1,z}(z,t)\,\mathrm{d}t\neq 0, \tag{63}$$
$$(\lambda^0,-):\quad l_1(z,t)=0\ \text{(assumed)}, \tag{64}$$
$$k_1(z,t)=k_5(z)=\text{real non-zero differentiable function of } z, \tag{65}$$

with $k_4(z)$ as another real differentiable function of $z$, and with $k_4(z)-\alpha_4tk_5'(z)\neq 0$. Thus, we now derive the first family of rational solutions for for Eq. (3) (or equivalently, for Eq. (5)), i.e.

$$\Upsilon^{(\mathrm{V})}(x,y,z,t)=-\frac{2k_5(z)^2}{\left\{k_5(z)x+\left[k_4(z)-\alpha_4tk_5'(z)\right]y+k_3(z,t)\right\}^2}-\frac{\alpha_4k_5'(z)y}{3\alpha_1k_5(z)}+l_3(z,t)\ , \tag{66a}$$

$$\Gamma^{(\mathrm{V})}(x,y,z,t)=-\frac{2k_5(z)\left[k_4(z)-\alpha_4tk_5'(z)\right]}{\left\{k_5(z)x+\left[k_4(z)-\alpha_4tk_5'(z)\right]y+k_3(z,t)\right\}^2}-\frac{\alpha_4k_5'(z)x}{3\alpha_1k_5(z)}+\frac{\alpha_4^2tk_5''(z)y}{3\alpha_1k_5(z)}-\frac{2\alpha_4^2tk_5'(z)^2y}{3\alpha_1k_5(z)^2}-\frac{\alpha_4k_4'(z)y}{3\alpha_1k_5(z)}+\frac{2\alpha_4k_4(z)k_5'(z)y}{3\alpha_1k_5(z)^2}+\frac{\alpha_4tk_5'(z)k_{3,t}(z,t)}{3\alpha_1k_5(z)^2}-\frac{\alpha_4k_{3,z}(z,t)}{3\alpha_1k_5(z)}-\frac{k_4(z)k_{3,t}(z,t)}{3\alpha_1k_5(z)^2}+\frac{\alpha_4tk_5'(z)l_3(z,t)}{k_5(z)}-\frac{k_4(z)l_3(z,t)}{k_5(z)}-\frac{\alpha_2\alpha_4^2t^2k_5'(z)^2}{3\alpha_1k_5(z)^2}+\frac{2\alpha_2\alpha_4tk_4(z)k_5'(z)}{3\alpha_1k_5(z)^2}+\frac{\alpha_3\alpha_4tk_5'(z)}{3\alpha_1k_5(z)}-\frac{\alpha_2k_4(z)^2}{3\alpha_1k_5(z)^2}-\frac{\alpha_3k_4(z)}{3\alpha_1k_5(z)}. \tag{66b}$$

Closing up the work, more rational examples rely on the assumptions that

$$\lambda(x,y,z,t)=c_1x^2+c_2x+c_3y+c_4z+c_5(t), \tag{67}$$
$$\Upsilon_2(x,y,z,t)=j_1x^2+j_2x+j_3y+j_4z+j_5(t), \tag{68}$$
$$\Gamma_2(x,y,z,t)=q_1x^2+q_2x+q_3y+q_4z+q_5(t), \tag{69}$$

where $c_1$, $c_2$, $c_3$, $c_4$, $j_1$, $j_2$, $j_3$, $j_4$, $q_1$, $q_2$, $q_3$ and $q_4$ are real constants, $c_5(t)$, $j_5(t)$ and $q_5(t)$ are real differentiable functions of $t$, with $c_3\neq 0$ since $\lambda_y\neq 0$.

Substituting Assumptions Eqs. (67)–(69) into auto-Bäcklund transformation Eqs. (7)–(17) gives

$$(;-;\lambda^0x^1):\quad q_1=0, \tag{70}$$
$$(-,\lambda^0):\quad q_2=j_3, \tag{71}$$
$$(\lambda^0x^1,-):\quad j_1=0\ \text{with}\ c_1\neq 0, \tag{72}$$
$$(\lambda^{-4}x^3,-):\quad j_3=0, \tag{73}$$
$$(\lambda^{-4}y^1,-):\quad q_3=0, \tag{74}$$
$$(\lambda^{-4}x^2z^1,-):\quad q_4=0, \tag{75}$$
$$(\lambda^{-4}x^2,-):\quad q_5(t)=-\frac{c_3j_2}{2c_1}, \tag{76}$$
$$(\lambda^{-4}x^1z^1,-):\quad j_4=0, \tag{77}$$
$$(\lambda^{-4}x^1,-):\quad j_5(t)=-\frac{\alpha_3}{3\alpha_1}-\frac{\alpha_4c_4}{3\alpha_1c_3}+\frac{c_2j_2}{2c_1}, \tag{78}$$
$$(\lambda^{-2},-):\quad j_2=0, \tag{79}$$
$$(\lambda^{-3},-):\quad c_5(t)=c_6-\alpha_2c_3t, \tag{80}$$

with $c_6$ as another real constant.

Thus, we now assemble the second family of rational solutions for Eq. (3) (or equivalently, for Eq. (5)), i.e.,

$$\Upsilon^{(\mathrm{VI})}(x,y,z,t)=-\frac{2(2c_1x+c_2)^2}{(c_1x^2+c_2x+c_3y+c_4z-\alpha_2c_3t+c_6)^2}+\frac{4c_1}{c_1x^2+c_2x+c_3y+c_4z-\alpha_2c_3t+c_6}-\frac{\alpha_3}{3\alpha_1}-\frac{\alpha_4c_4}{3\alpha_1c_3}, \tag{81a}$$

$$\Gamma^{(\mathrm{VI})}(x,y,z,t)=-\frac{2c_3(2c_1x+c_2)}{(c_1x^2+c_2x+c_3y+c_4z-\alpha_2c_3t+c_6)^2}. \tag{81b}$$

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Solitonic solutions Eqs. (28), (40), and (55), together with rational solutions Eqs. (66) and (81), involving the variables $\Upsilon(x,y,z,t)$ and $\Gamma(x,y,z,t)$, hinge on the coefficients $\alpha_1 \ldots \alpha_4$, under coefficient constraints Eq. (2).

## 5. Conclusion

This study has investigated the $(3+1)$-dimensional Hirota–Wazwaz–El Sherif–El Tantawy bilinear equation, a mathematical framework for modeling certain nonlinear wave phenomena in oceans and seas, i.e., Eq. (3) (or equivalently, Eq. (5)). This study has combined the equivalent form to Eq. (3), the non-characteristic movable singular manifold, and symbolic computation to deal with Eq. (3). Our findings have included auto-Bäcklund transformation (Eqs. (7)–(17)), solitonic solutions (Eqs. (28), (40), and (55)), together with rational solutions (Eqs. (66) and (81) for Eq. (3)). All our results have hinged on the coefficients $\alpha_1 \ldots \alpha_4$ in Eq. (3), under coefficient constraints (Eq. (2)). Various choices of $\alpha_1 \ldots \alpha_4$ might aid in exploring certain nonlinear wave phenomena in oceans and seas through our auto-Bäcklund transformation, solitons, and rational solutions.

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## References

[1] Zhan J K, Piao S C and Gong L J 2026 *Chin. Phys. B* **35** 024303
[2] Liu W X, Dao L Y and Zhang R G 2025 *Chin. Phys. B* **34** 070504
[3] Zhang C G 2025 *Chin. Phys. B* **34** 074501
[4] Yin H M, Pan Q and Chow K W 2024 *Phys. Rev. E* **109** 024204
[5] Zabala I, Kelly T E, Henriques J C, Pena-Sanchez Y, Penalba M and Blanco J M 2026 *Ocean Eng.* **343** 123168
[6] Ducrozet G, Charlou M and Bonnefoy F 2026 *Comput. Phys. Commun.* **323** 110098
[7] Papandreou A, Ginis I and Hara T 2026 *J. Phys. Oceanogr.* **56** 355
[8] Gao X Y, Liu J G and Wang G W 2026 *China Ocean Eng.* **40** 695
[9] Asadi E, Hosseini K and Madadi M 2024 *Phys. Scr.* **99** 125242
[10] Wazwaz A M, El-Sherif L and El-Tantawy S 2023 *Int. J. Numer. Methods Heat Fluid Flow* **33** 1837
[11] Liu N 2025 *Int. J. Numer. Methods Heat Fluid Flow* **35** 4366
[12] Hosseini K, Samavat M, Mirzazadeh M, Ma W X and Hammouch Z 2020 *Regul. Chaotic Dyn.* **25** 383
[13] Ma W B and Bilige S 2024 *Mod. Phys. Lett. B* **38** 2450240
[14] Aldwoah K, Alabdi I, Alsulami A, Touati A, Egami R and Hassan T S 2025 *J. Nonlinear Math. Phys.* **32** 62
[15] Younas U, Ismael H F, Sulaiman T A, Murad M A, Shah N A and Sharifpur M 2023 *Results Phys.* **54** 107124
[16] Ismael H F, Nabi H R, Sulaiman T A, Shah N A, Eldin S M and Bulut H 2023 *Results Phys.* **53** 106978
[17] Wang B J, Ma Z M and Xiong S H 2023 *Nonlinear Dyn.* **111** 9477
[18] Li W T, Jiao A L, Liu W and Guo Z Y 2023 *Math. Biosci. Eng.* **20** 19360
[19] Ay N G and Yasar E 2022 *J. Taibah Univ. Sci.* **16** 1287
[20] Tian Y and Liu J G 2021 *Nonlinear Dyn.* **104** 1507
[21] Madadi M and Inc M 2025 *Qual. Theory Dyn. Syst.* **24** 138
[22] Zhang H W, Zhang R F and Liu Q R 2025 *Chin. Phys. Lett.* **42** 100002
[23] Li Y, Yao R X and Lou S Y 2026 *Chin. Phys. B* **35** 040201
[24] Ramady A, Ahmed H M, Ismail K A and Rabie W B 2026 *J. Appl. Anal. Comput.* **16** 2264
[25] Iftikhar M, Tariq H, Wajid H A and Akram G 2026 *J. Taibah Univ. Sci.* **20** 2687767
[26] Ceesay B, Ahmed N, Akgul A, Karaca Y and Baber M Z 2026 *Math. Comput. Model. Dyn. Syst.* **32** 2667030
[27] Liu H D, Tian B, Gao X T and Shan H W 2026 *Eur. Phys. J. Plus* **141** 655
[28] Mares-Rincon D R, Macias S, Macias-Diaz J E and Guerrero-Diaz-de-Leon J A 2026 *J. Comput. Appl. Math.* **488** 117821
[29] Riano O and Roudenko S 2026 *Nonlinear Anal. Real World Appl.* **92** 104641
[30] Erfanifar R and Hajarian M 2026 *Appl. Math. Comput.* **529** 130154
[31] Gao X Y 2026 *Commun. Theor. Phys.* in press
[32] Caceres-Pablo I, Rosado A, Hernandez-Martin L, Corredera P and Ania-Castanon J D 2026 *Opt. Commun.* **617** 132200
[33] Mohapatra S, Kevrekidis P G, Yang S and Chandramouli S 2026 *Phys. D* **495** 135279
[34] Maurya D, Singh A and Jiwari R 2026 *Math. Comput. Simulat.* **248** 384
[35] Gao X Y 2027 *Int. J. Mod. Phys. C* in press
[36] Arnous A H 2026 *Commun. Nonlinear Sci. Numer. Simulat.* **161** 110018
[37] Kalashnikov V L, Sorokin E, Rudenkov A and Sorokina I T 2026 *Chaos Solitons Fract.* **210** 118609
[38] Kudryashov N A 2026 *Phys. Lett. A* **585** 131644
[39] Gao X Y 2026 *Phys. Lett. A* **591** 131868
[40] Shan H W, Tian B and Liu H D 2026 *Chaos Solitons Fract.* **204** 117744
[41] Yang Z P, Zhong W P and Belic M 2024 *Chin. Phys. Lett.* **41** 044201
[42] Yin H M and Chow K W 2025 *Phys. Rev. E* **112** 064202
[43] Yin H M and Chow K W 2025 *Appl. Math. Lett.* **171** 109656
[44] Chen Y Q, Wu Z T, Xiao P Y, Xiao W D and Liu W J 2024 *Opt. Lett.* **49** 2437
[45] Xing X W, Liu Y X, Han J F, Liu W J and Wei Z Y 2023 *ACS Photonics* **10** 2264
[46] Liu W J, Xiong X L, Liu M L, Xing X W, Chen H L, Ye H, Han J F and Wei Z Y 2022 *Appl. Phys. Lett.* **120** 053108
[47] Yuan W X and Guo R 2024 *Phys. Rev. E* **110** 065307
[48] Huang Y H and Guo R 2024 *Chaos* **34** 103103
[49] Liu Y H, Guo R and Zhang J W 2025 *Physica D* **472** 134528
[50] Wang Y, Lü X and Ma W X 2025 *Nonlinear Dyn.* **113** 21725
[51] Chen Y, Lü X, Jia X J and Tian H 2025 *Expert Syst. Appl.* **291** 128279
[52] Guo M H, Lü X and Jin Y X 2025 *Physica D* **481** 134689
[53] Liu F Y and Zhou Q 2025 *New J. Phys.* **27** 023019
[54] Ding C C, Zhou Q and Malomed B A 2025 *Phys. Rev. E* **111** 044203
[55] Liu H D, Tian B, Chen Y Q, Cheng C D and Gao X T 2025 *Nonlinear Dyn.* **113** 3655
[56] Gao X Y and Lee C H 2026 *Qual. Theory Dyn. Syst.* **25** 110
[57] Liu F Y and Zhou Q 2025 *Physica D* **472** 134540
[58] Wang G W, Tan Z X, Gao X Y and Liu J G 2026 *Appl. Math. Lett.* **172** 109720
[59] Shan H W, Tian B, Gao X T, Feng C H and Liu H D 2025 *Chaos Solitons Fractals* **201** 117199
[60] Gao X Y 2026 *Chaos Solitons Fractals* **208** 118301

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