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SPECIAL TOPIC — Nonlinear localized waves: Theory, methods and applications

Multiple solitons, soliton molecules, D'Alembert waves, and symmetry analysis for a nonlocal Nizhnik–Novikov–Veselov equation

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A nonlocal form of the Nizhnik–Novikov–Veselov (NNV) equation, which is a $(2+1)$-dimensional extension of the Korteweg–de Vries (KdV) equation, is introduced by employing the consistent correlated bang (CCB) method. By applying the symmetry and anti-symmetry separation method, the nonlocal NNV equation is transformed into a local equation, enabling the derivation of exact solutions from the local counterpart under specific parity and time-reversal constraints. Multiple soliton solutions are constructed via Hirota's bilinear method, and soliton molecules are generated using velocity resonance mechanisms. In addition, a variety of special solutions, including bright and dark–bright solitons, lump waves, and D'Alembert-type waves, are obtained. The Lie symmetry approach is further employed to derive symmetry groups and symmetry reduction solutions.

Keywords: nonlocal Nizhnik–Novikov–Veselov equation, multiple soliton, soliton molecule, D'Alembert waves, symmetry reduction solution

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1. Introduction

The study of nonlocal integrable systems has attracted considerable interest since Ablowitz and Musslimani introduced the nonlocal nonlinear Schrödinger (NNLS) equation in 2013,[1]

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) \pm q(x,t)q^*(-x,t)q(x,t), \tag{1}$$

where $q^*(-x,t)$ denotes space reversal of the complex conjugate of $q(x,t)$. This equation was shown to be integrable, possessing a Lax pair and infinite conservation laws. Since then, various nonlocal extensions of classical integrable systems have been developed and investigated using methods such as inverse scattering transformation,[2,3] Riemann–Hilbert techniques,[4] Hirota's bilinear method,[5] and Darboux transformations.[6] These studies have revealed that nonlocal integrable systems exhibit counterintuitive behaviors compared with their local counterparts. For instance, unlike their local counterparts, some nonlocal systems only admit even-number soliton solutions with odd-number solitons prohibited.[7,8]

The Nizhnik–Novikov–Veselov (NNV) equation is a natural $(2+1)$-dimensional generalization of the Korteweg–de Vries (KdV) equation,[9–11] which takes the form

$$\begin{aligned} &u_t + cu_x + du_y + au_{xxx} + bu_{yyy} + 3a(uv)_x + 3b(uw)_y = 0,\\ &u_x - v_y = 0,\\ &u_y - w_x = 0, \end{aligned} \tag{2a}$$

with a, b, c, d being coupling constants. The NNV equation (2) is integrable in the sense of possessing a Lax pair, infinite conservation laws, and a bilinear representation via Hirota's method.[12] It has been extensively studied due to its wide range of applications in fluid dynamics, plasma physics, nonlinear optics, and nonlinear acoustics, as well as its rich solution structures including solitons, lump waves, dromions, complexiton solutions, and D'Alembert waves.[13–16] For the generalized NNV equation, in the framework of symmetry analysis, the classical Lie point symmetry groups of both the standard and asymmetric NNV equations have been derived, and the full symmetry group, including discrete transformations, has been obtained.[17–19] However, nonlocal versions of the NNV equation remain largely unexplored.

Recently, by incorporating shifted parity $\hat{P}_{\mathrm{s}}$, delayed time reversal $\hat{P}_{\mathrm{d}}$, and charge conjugation operations $\hat{C}$, Lou proposed the consistent correlated bang (CCB) method as a systematic method to generate nonlocal nonlinear systems from local ones while preserving integrability.[20,21] This method has been successfully applied to several systems, including the coupled KdV equation,[22] the KP equation,[8] the Sawada–Kotera equation,[23] and the $(2+1)$-dimensional Ito equation,[24] leading to novel solutions such as multiple solitons, molecules, lump waves, etc. Among the various methods to solve nonlocal integrable systems, the symmetry and anti-symmetry separation method has been proven to be very efficient. It decomposes nonlocal equations into coupled lo-

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cal ones via a suitable discrete symmetry group, thereby enabling the application of all established methods for local systems.[25,26]

In this work, we extend the CCB method to the NNV equation to construct its nonlocal counterpart with shifted space and delayed time reversal properties. To this end, we substitute

$$u=\frac{A+B}{2},\quad v=\frac{C+E}{2},\quad w=\frac{F+G}{2},$$

with nonlocal variables satisfying

$$B = \hat{P}_{\rm s}^{x}\hat{P}_{\rm s}^{y}\hat{T}_{\rm d}A(x,y,t)=A(-x+x_0,-y+y_0,-t+t_0), \quad (3)$$

$$E = \hat{P}_{\rm s}^{x}\hat{P}_{\rm s}^{y}\hat{T}_{\rm d}C(x,y,t)=C(-x+x_0,-y+y_0,-t+t_0), \quad (4)$$

$$G = \hat{P}_{\rm s}^{x}\hat{P}_{\rm s}^{y}\hat{T}_{\rm d}F(x,y,t)=F(-x+x_0,-y+y_0,-t+t_0), \quad (5)$$

into Eq. (2), and a nonlocal Nizhnik–Novikov–Veselov (nNNV) equation is generated as

$$\begin{aligned}&\frac{\partial A}{\partial t}+a\frac{\partial^3 A}{\partial x^3}+\left(c+\frac{9}{4}aC+\frac{3}{4}aE\right)\frac{\partial A}{\partial x}\\&+\left(d+\frac{9}{4}bF+\frac{3}{4}bG\right)\frac{\partial A}{\partial y}+\frac{3}{4}a(C-E)\frac{\partial B}{\partial x}\\&+\frac{3}{4}b(F-G)\frac{\partial B}{\partial y}+\frac{3}{4}a(3A+B)\frac{\partial C}{\partial x}\\&+\frac{3}{4}a(A-B)\frac{\partial E}{\partial x}+\frac{3}{4}b(3A+B)\frac{\partial F}{\partial y}\\&+\frac{3}{4}b(A-B)\frac{\partial G}{\partial y}+b\frac{\partial^3 A}{\partial y^3}=0, \quad (6a)\end{aligned}$$

$$\frac{\partial A}{\partial x}-\frac{\partial C}{\partial y}=0, \quad (6b)$$

$$\frac{\partial A}{\partial y}-\frac{\partial F}{\partial x}=0. \quad (6c)$$

The paper is organized as follows. In Section 2, using the symmetry and anti-symmetry separation method, we transform the nNNV equation (6) into a localized system. This allows us to derive exact solutions from known solutions of the local NNV equation (2) under appropriate parity and time-reversal conditions. Multiple soliton solutions are obtained via Hirota's bilinear method and analyzed. In Section 3, we generate soliton molecules through applying velocity resonance mechanisms among different solitons. Lump and D'Alembert waves of various types are also given. In Section 4, the classical Lie group method is carried out on the nNNV equation to present Lie symmetry analysis and give symmetry reduction solutions from which the D'Alembert waves obtained in Section 3 can be recovered. A summary and discussion are given in Section 5.

2. Multi-soliton solutions

To investigate exact solutions of the nNNV equation, an effective way is to transform it into a local system. To this end, we separate the dependent variables in Eq. (6) in the form

$$\begin{aligned}&A=u+u_1,\ B=u-u_1,\ C=v+v_1,\ E=v-v_1,\\&F=w+w_1,\ G=w-w_1,\end{aligned} \quad (7)$$

where the local functions satisfy the following parity condition under the operator $\hat{g}=\hat{P}_{\rm s}^{x}\hat{P}_{\rm s}^{y}\hat{T}_{\rm d}$:

$$\hat{g}\{u,v,w,u_1,v_1,w_1\}=\{u,v,w,-u_1,-v_1,-w_1\}. \quad (8)$$

After substituting Eq. (7) into Eq. (6), a local nonlinear system is obtained as

$$\begin{aligned}u_t=&-a(3u_xv+3uv_x+u_{xxx})\\&-b(3u_yw+3uw_y+u_{yyy})-cu_x-du_y,\end{aligned} \quad (9a)$$

$$u_y=v_x, \quad (9b)$$

$$u_y=w_x, \quad (9c)$$

$$\begin{aligned}u_{1,t}=&-a(3u_xv_1+3vu_{1,x}+3uv_{1,x}+3v_xu_1+u_{1,xxx})\\&-b(3u_yw_1+3uw_{1,y}+3wu_{1,y}\\&+3w_yu_1+u_{1,yyy})-cu_{1,x}-du_{1,y},\end{aligned} \quad (9d)$$

$$u_{1,y}=v_{1,x}, \quad (9e)$$

$$u_{1,y}=w_{1,x}, \quad (9f)$$

where Eqs. (9a)–(9c) are the NNV equation and Eqs. (9d)–(9f) are their linearized forms.

It is known that the NNV equation can be bilinearized by the following transformation:[13]

$$u=2(\ln f)_{xy},\quad v=2(\ln f)_{xx},\quad w=2(\ln f)_{yy}, \quad (10)$$

into the form

$$\begin{aligned}&(f\partial_x-2f_x)D_y(2aD_x^3+2cD_x+D_t)f\cdot f\\&+(f\partial_y-2f_y)D_x(2bD_y^3+2dD_y+D_t)f\cdot f=0,\end{aligned} \quad (11)$$

where Hirota's bilinear operator D_{x_i} is defined as

$$D_{x_i}^n f\cdot g=\left(\partial_{x_i}-\partial_{x_i'}\right)^n f(x_i)g(x_i')\Big|_{x_i'=x_i},$$

with operators $\{D_y,D_t\}$ being defined similarly. Equation (11) can be further turned into the bilinear form

$$D_y(2aD_x^3+2cD_x+D_t+D_z)f\cdot f=0, \quad (12a)$$

$$D_x(2bD_y^3+2dD_y+D_t-D_z)f\cdot f=0, \quad (12b)$$

with z being an auxiliary parameter.

Based on the bilinear form of Eq. (12), the multiple soliton solutions of the NNV equation can be obtained by fixing f as

$$f=\sum_{\{v\}}K_{\{v\}}\cosh\left(\frac{1}{2}\sum_{i=1}^{N}v_i\xi_i\right), \quad (13)$$

where

$$\xi_i = k_i(x-\frac{x_0}{2})+l_i(y-\frac{y_0}{2})+q_i(z-\frac{z_0}{2})+r_i(t-\frac{t_0}{2})+\xi_{i0},$$
$$K_{\{\nu\}} = \prod_{i<j} a_{ij}, \tag{14a}$$
$$a_{ij} \equiv \sqrt{(l_i-\nu_i\nu_j l_j)(k_i-\nu_i\nu_j k_j)},$$
$$r_i = -ak_i^3 - bl_i^3 - ck_i - dl_i,\ \ q_i = ak_i^3 - bl_i^3 + ck_i - dl_i, \tag{14b}$$

with k_i, l_i, ξ_{i0} being arbitrary constants, and the summations of $\{\nu\}=\{\nu_1,\nu_2,\ldots,\nu_N\}$ are done for all non-dual permutations of $\nu_i=1$ or $\nu_i=-1$, $(i=1,2,\ldots,N)$. The auxiliary parameter z in Eqs. (13) and (14) can be absorbed into the arbitrary constants ξ_{i0}.

Substituting Eq. (13) into Eq. (10) yields the N-soliton solutions of the NNV equation as

$$u = 2\left[\ln\sum_{\nu} K_\nu \cosh\left(\frac{1}{2}\sum_{j=1}^{N}\nu_j\xi_j\right)\right]_{xy} \equiv 2(\ln F_N)_{xy},$$
$$v = 2(\ln F_N)_{xx},\ w = 2(\ln F_N)_{yy}, \tag{15}$$

where the parameter z is absorbed into the arbitrary constants ξ_{i0}, which are then set to zero so that the symmetry condition of Eq. (8) holds.

The NNV equation is a $(2+1)$-dimensional extension of the KdV equation, which has infinitely many symmetries that satisfy the symmetry Eqs. (9d)–(9f). Among these symmetries, we select a simple one

$$u_1 = c_1u_y + c_2u_x,\ \ v_1 = c_1v_y + c_2v_x,\ \ w_1 = c_1w_y + c_2w_x, \tag{16}$$

with arbitrary constants c_1, c_2, to construct exact solutions of the nNNV equation by using Eq. (7). It is obvious that when u, v, w is even under the operator $\hat{g}$, the symmetry in Eq. (16) is odd, so we obtain the N-soliton solutions of the nNNV equation by substituting Eq. (16) with Eq. (15) into Eq. (7)

$$A = u + c_1u_y + c_2u_x,\ \ C = v + c_1v_y + c_2v_x,$$
$$F = w + c_1w_y + c_2w_x. \tag{17}$$

For some lower-order N-soliton $(N=2,3)$ solutions of the nNNV equation, the explicit expressions can be given by Eq. (17) with Eq. (15), where

$$\begin{aligned} F_2 \ =\ & \delta_{12,-}\cosh\left[\frac{1}{2}((k_1+k_2)x+(l_1+l_2)y\right. \\ & \left.+(q_1+q_2)z+(r_1+r_2)t)\right] \\ & +\delta_{12,+}\cosh\left[\frac{1}{2}((k_1-k_2)x+(l_1-l_2)y\right. \\ & \left.+(q_1-q_2)z+(r_1-r_2)t)\right], \end{aligned} \tag{18}$$

$$\begin{aligned} F_3 \ =\ & \delta_{12,-}\delta_{13,-}\delta_{23,-}\cosh\left[\frac{1}{2}((k_1+k_2+k_3)x\right. \\ & \left.+(l_1+l_2+l_3)y+(q_1+q_2+q_3)z+(r_1+r_2+r_3)t)\right] \\ & +\delta_{12,+}\delta_{13,+}\delta_{23,-}\cosh\left[\frac{1}{2}((k_1-k_2-k_3)x\right. \\ & \left.+(l_1-l_2-l_3)y+(q_1-q_2-q_3)z+(r_1-r_2-r_3)t)\right] \\ & +\delta_{12,+}\delta_{13,-}\delta_{23,+}\cosh\left[\frac{1}{2}((k_1-k_2+k_3)x\right. \\ & \left.+(l_1-l_2+l_3)y+(q_1-q_2+q_3)z+(r_1-r_2+r_3)t)\right] \\ & +\delta_{12,-}\delta_{13,+}\delta_{23,+}\cosh\left[\frac{1}{2}((k_1+k_2-k_3)x\right. \\ & \left.+(l_1+l_2-l_3)y+(q_1+q_2-q_3)z\right. \\ & \left.+(r_1+r_2-r_3)t)\right], \end{aligned} \tag{19}$$

with

$$\delta_{ij,\pm} = \sqrt{(k_i\pm k_j)(l_i\pm l_j)}.$$

The N-soliton $(N\geq 4)$ solutions can be written out in a similar way.

Figures 1(a)–1(c) give three-dimensional depictions of the N-soliton collision interaction dynamics of the nNNV equation for $N=2,3,4$, respectively. It can be concluded from the pictures that collisions among different solitons are elastic, i.e., the amplitude for every soliton remains unchanged during interaction but with the phase changed. The parameters are all properly chosen so that they lead to real solutions of the nNNV equation. The parameters are fixed as

$$k_1=2,\ \ k_2=3/2,\ \ c_1=c_2=a=b=l_1=1,\ \ c=-2,$$
$$d=-1/2,\ l_2=1/2,\ x_0=y_0=z_0=t_0=\eta_{01}=\eta_{02}=0, \tag{20}$$

for Fig. 1(a);

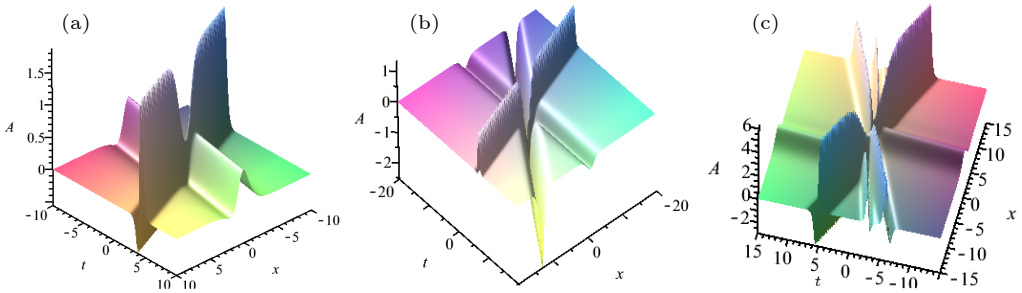


Fig. 1. Three-dimensional plots of the N-soliton solutions of the nNNV Eq. (6) at $y=z=0$: (a) two-soliton, (b) three-soliton, and (c) four-soliton.

$$k_1=2,\ k_2=3/2,\ k_3=5/2,\ l_1=1,\ l_2=-1/2,$$
$$l_3=c=-2,\ a=b=1,\ c_1=c_2=d=1/2,$$
$$x_0=y_0=z_0=t_0=\eta_{01}=\eta_{02}=\eta_{03}=0, \tag{21}$$

for Fig. 1(b); and

$$c_1=c_2=1, k_1=-1.9, k_2=0.9, k_3=1.7, k_4=1.8,$$
$$l_4=a=d=2, l_3=1.5, l_2=1.3, l_1=-2.7, b=c=-1,$$
$$x_0=y_0=z_0=t_0=\eta_{01}=\eta_{02}=\eta_{03}=\eta_{04}=0, \tag{22}$$

for Fig. 1(b).

3. Soliton molecules, lump waves, and D'Alembert waves

To obtain soliton molecules from multiple soliton solutions, one effective way is to force the velocities of all the solitons to be equal so that the usual elastic scattering behavior breaks down, and they lock together to form a molecule: this mechanism is called the velocity resonance mechanism.[27] Using this mechanism, we force the velocities of any two of the N-solitons of the nNNV equation to be equal by

$$\frac{k_i}{k_j}=\frac{l_i}{l_j},\ \frac{l_i}{l_j}=\frac{ak_i^3+bl_i^3+ck_i+dl_i}{ak_j^3+bl_j^3+ck_j+dl_j},$$
$$k_i\neq\pm k_j,\quad l_i\neq\pm l_j,\quad (i,j=1,\ldots,N). \tag{23}$$

Solving the two equations of (23) for l_i and l_j directly, we have

$$l_i=-\sqrt[3]{\frac{a}{b}}k_i,\quad l_j=-\sqrt[3]{\frac{a}{b}}k_j, \tag{24}$$

so the i-th variable in Eq. (15) becomes

$$\xi_i=k_i\left[\left(x-\frac{x_0}{2}\right)-\sqrt[3]{\frac{a}{b}}\left(y-\frac{y_0}{2}\right)+\left(d\sqrt[3]{\frac{a}{b}}-c\right)\left(t-\frac{t_0}{2}\right)+\eta_{0i}\right]. \tag{25}$$

To obtain a three-soliton molecule, we substitute Eq. (15) with Eq. (25) into Eq. (17) with $N=3$, and arbitrary N-soliton molecule solutions can be obtained similarly. Figures 2(a) and 2(b) give density plots of the three-soliton molecule and four-soliton molecule at $y=z=0$, respectively, with parameters being fixed by

$$a=b=c_1=c_2=k_2=1,\ k_1=1.1,\ k_3=0.9,\ c=2,$$
$$d=1/2,\ x_0=y_0=z_0=t_0=\eta_{01}=\eta_{02}=\eta_{03}=0, \tag{26}$$

for Fig. 2(a) and

$$a=b=c_1=c_2=k_2=1,\ k_1=1.1,\ k_3=0.9,\ k_4=0.8,$$
$$c=2,\ d=1/2,$$
$$x_0=y_0=z_0=t_0=\eta_{01}=\eta_{02}=\eta_{03}=\eta_{04}=0, \tag{27}$$

for Fig. 2(b). Figure 2(c) gives a density plot of interaction behaviors between a three-soliton molecule and a soliton at $y=z=0$ with parameters being fixed by

$$a=b=c_1=c_2=k_2=1,\ k_1=1.1,\ k_3=0.9,$$
$$k_4=-0.35,\ l_4=0.6,\ c=2,\ d=1/2,$$
$$x_0=y_0=z_0=t_0=\eta_{01}=\eta_{02}=\eta_{03}=\eta_{04}=0. \tag{28}$$

It is obvious that the interaction between soliton and soliton molecule is elastic, keeping amplitudes unchanged but having a phase shift after interaction.

It is found that the NNV Eq. (2) has D'Alembert wave solutions;[13] we use this type of solution that will satisfy the symmetry condition of (8) to generate D'Alembert waves for the nNNV Eq. (6) through Eq. (17). Below are some examples.

Compound three-soliton By substituting the three-soliton molecule solution of the NNV equation

$$u=\frac{6}{(e_1-\tau^2)^2}\exp\left[-\frac{2}{(e_1-\tau^2)^2}\right],$$
$$v=\frac{1}{\sqrt[3]{(\frac{a}{b})^2}}w=-\frac{1}{\sqrt[3]{\frac{a}{b}}}u, \tag{29}$$

with

$$\tau=x-\sqrt[3]{\frac{a}{b}}y-\left(c-d\sqrt[3]{\frac{a}{b}}\right)t \tag{30}$$

into Eq. (17), we obtain a compound three-soliton solution of the nNNV equation. Figure 3(a) displays the structure of the soliton solution A at $t=0$ and $c_1=c_2=-1$, $a=b=c=d=e_1=1$.

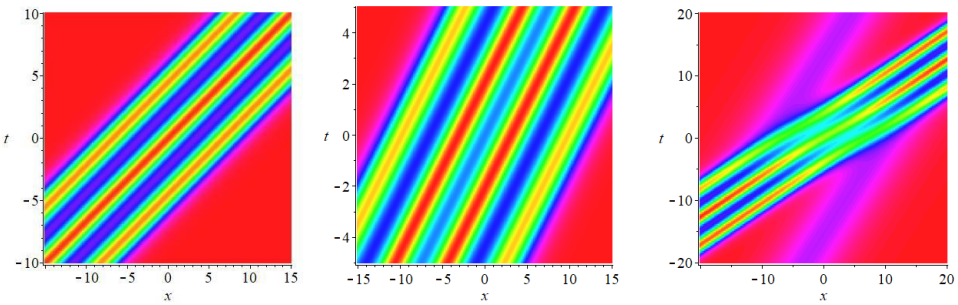


Fig. 2. The density plots of the soliton molecules of the nNNV Eq. (6) at $y=z=0$: (a) three-soliton molecule, (b) four-soliton molecule, and (c) three-soliton molecule interacting with a soliton.

Stripe soliton on periodic wave background Substituting a periodic kink solution of the NNV equation

$$u = \tanh\{\tau[1-0.5\cos(0.1\tau)]\}[1-0.1\cos(2\tau)],$$
$$v = \frac{1}{\sqrt[3]{(\frac{a}{b})^2}}w = -\frac{1}{\sqrt[3]{\frac{a}{b}}}u, \tag{31}$$

into Eq. (17), we obtain a stripe soliton on periodic wave background of the nNNV equation, which is depicted in Fig. 3(b) for A at $t=0$ and $a=b=c=d=1$, $c_1=c_2=-1$.

Bright-dark soliton 1 By substituting the hyperbolic function solution of the NNV equation as

$$u = \frac{d_1\tau^2\operatorname{sech}^2\tau}{d_2\tanh^2\tau},$$
$$v = \frac{1}{\sqrt[3]{(\frac{a}{b})^2}}w = -\frac{4d_1\exp(2\tau)\tau^2}{d_2\sqrt[3]{\frac{a}{b}}[\exp(2\tau)-1]^2}, \tag{32}$$

with arbitrary constants d_1, d_2 into Eq. (17), we obtain a bright–dark soliton solution of the nNNV equation, which is depicted for A in Fig. 3(c) at $y=0$ and $c_1=c_2=b=c=d_1=d_2=1$, $a=-1$.

Double bright–dark soliton Similarly, by substituting the following hyperbolic function solution into Eq. (17)

$$u = d_1\tau^2\operatorname{sech}^2\tau,$$
$$v = \frac{1}{\sqrt[3]{(\frac{a}{b})^2}}w = -\frac{4d_1\exp(2\tau)\tau^2}{\sqrt[3]{\frac{a}{b}}(\exp(2\tau)+1)^2}, \tag{33}$$

with an arbitrary constant d_1, we obtain a double bright–dark soliton solution of the nNNV equation, which is depicted in Fig. 3(d) for A at $y=0$ with the same parameters as in Fig. 3(c).

Bright soliton By substituting the following hyperbolic function solution

$$u = \frac{d_1\tau^2(d_3+\cosh\tau)}{d_2\sinh^2\tau},$$
$$v = \frac{1}{\sqrt[3]{(\frac{a}{b})^2}}w$$
$$= -\frac{2d_1\exp(\tau)\tau^2[2d_3\exp(\tau)+\exp(2\tau)+1]}{d_2\sqrt[3]{\frac{a}{b}}[\exp(2\tau)-1]^2}, \tag{34}$$

with arbitrary constants d_1, d_2, d_3 into Eq. (17), we get a bright soliton solution of the nNNV equation, which is depicted in Fig. 3(d) for A at $y=0$ and $c_1=c_2=b=c=d_1=d_2=d_3=1$, $a=-1$.

Hybrid bright–dark soliton By substituting the following solution of the NNV equation:

$$u = \frac{d_1\tau^2(d_3+\cosh\tau)}{d_2\cosh^2\tau},$$
$$v = \frac{1}{\sqrt[3]{(\frac{a}{b})^2}}w$$
$$= -\frac{2d_1\exp(\tau)\tau^2[2d_3\exp(\tau)+\exp(2\tau)+1]}{d_2\sqrt[3]{\frac{a}{b}}[\exp(2\tau)+1]^2}, \tag{35}$$

with arbitrary constants d_1, d_2, d_3 into Eq. (17), we obtain a hybrid bright–dark soliton solution of the nNNV equation, which is depicted for A in Fig. 3(f) at $y=0$ with the same parameters being fixed as for Fig. 3(e).

Bright-dark soliton 2 By choosing a simple kink soliton solution of the NNV equation as

$$u = -\sqrt[3]{\frac{a}{b}}v = -\frac{1}{\sqrt[3]{\frac{a}{b}}}w = d_1\tanh^2\tau, \tag{36}$$

with d_1 being an arbitrary constant and substituting it into Eq. (17), we obtain another bright–dark soliton solution of the nNNV equation. Figure 3(g) depicts the solution for A at $y=0$ and $c_1=c_2=b=c=d_1=1$, $a=-1$.

Periodic wave solution The NNV equation admits a Jacobi elliptic function solution

$$u = d_1 + \frac{d_2\operatorname{sn}(\tau,m_1)}{[m_1\operatorname{sn}^2(\tau,m_1)+1]^2},$$
$$v = \frac{1}{\sqrt[3]{(\frac{a}{b})^2}}w = -\frac{d_2\operatorname{sn}(\tau,m_1)}{\sqrt[3]{\frac{a}{b}}[m_1\operatorname{sn}^2(\tau,m_1)+1]^2}, \tag{37}$$

where d_1, d_2 are arbitrary constants and $0\le m_1\le 1$ is the modulus, which can be used to generate a periodic wave solution of the nNNV equation by using Eq. (17). The modulus m controls the periodic shape. When $m\to 1$, it degenerates into a soliton, and when $m\to 0$, it approaches a sine wave. This type of periodic wave solution can be applied to describe nonlinear lattices or repetitive waves. Figure 3(h) is the plot of the periodic wave structure of A at $y=0$ and $b=c=d=d_1=d_2=1$, $a=-1$, $m_1=0.6$, $c_1=c_2=1$.

Unlike the above special solutions that were derived from known D'Alembert wave solutions of the NNV equation, by fixing f in Eq. (10) as different functions, we can also obtain some interesting special solutions of the nNNV equation through Eq. (17). Below are some examples.[14]

W-shape soliton By fixing f in Eq. (10)

$$f = a_1\cosh\left[\sqrt[3]{\frac{a}{b}}y+\left(c-d\sqrt[3]{\frac{a}{b}}\right)t-x\right]$$
$$+a_2\cos\left[\sqrt[3]{\frac{a}{b}}y+\left(c-d\sqrt[3]{\frac{a}{b}}\right)t-x\right], \tag{38}$$

with a_1, a_2 being arbitrary real constants, we get a complexiton solution of the NNV equation that leads to a W-shape soliton solution of the nNNV equation from Eq. (17), which is three-dimensional plotted for A in Fig. 3(i) at $t=3$ with $c_1=c_2=a=b=c=d=a_1=a_2=1$.

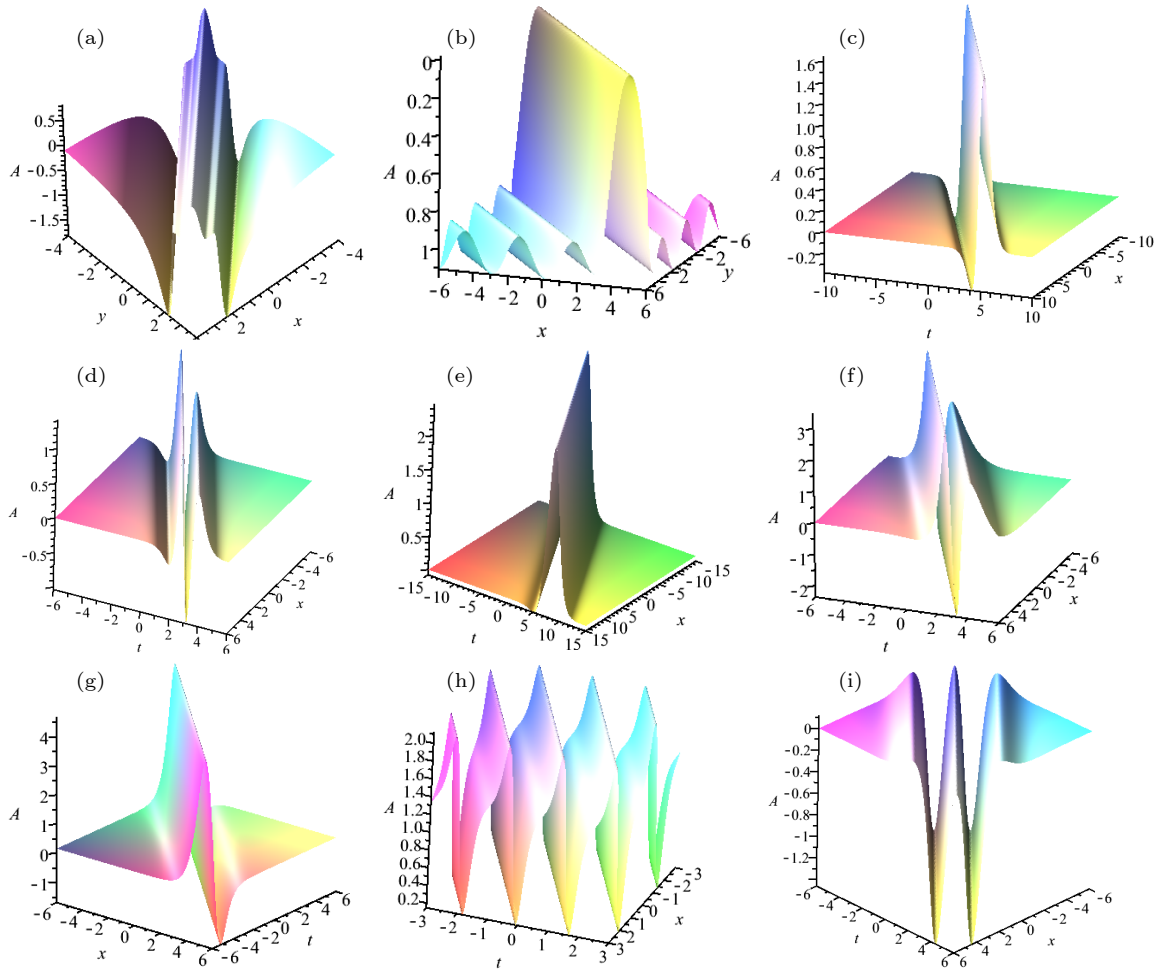


Fig. 3. Three-dimensional plots of 9 special solutions of the nNNV Eq. (6): (a) three-soliton molecule at $t = 0$, (b) bright soliton on periodic wave background at $t = 0$, (c) bright–dark soliton 1 at $y = 0$, (d) double bright–dark soliton at $y = 0$, (e) bright soliton at $y = 0$, (f) hybrid dark–bright soliton at $y = 0$, (g) bright–dark soliton 2 at $y = 0$, (h) elliptic periodic wave solution at $y = 0$, and (i) W-shape soliton at $y = 0$.

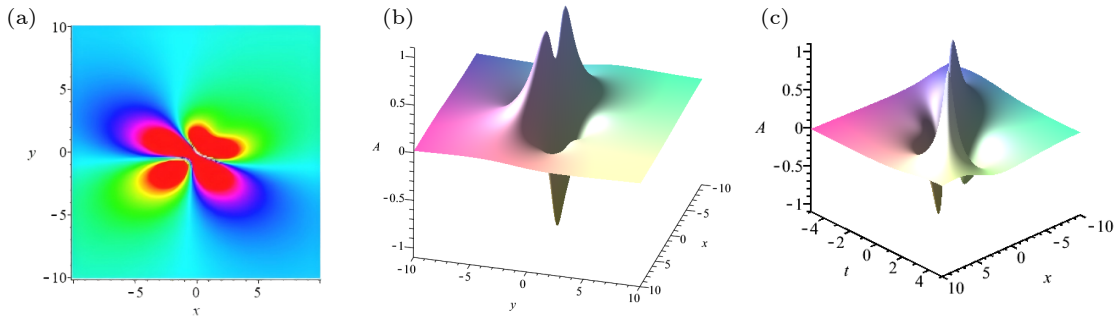


Fig. 4. Lump solutions of the nNNV system (6) with parameters being fixed by Eq. (40): (a) density plot of the lump at $t = 0$, (b) three-dimensional plot of the lump at $t = 0$, and (c) three-dimensional plot of the lump at $y = 0$.

Lump Lump solutions of the nNNV Eq. (6) can also be obtained by taking f in the form

$$f = b_1 + e_1\left[x - \frac{1}{2}x_0 - c\left(t - \frac{1}{2}t_0\right)\right]^2 - d_1\left[y - y_0 - d\left(t - \frac{1}{2}t_0\right)\right]^2, \tag{39}$$

with $b_1, d_1, e_1, x_0, y_0, t_0$ being arbitrary constants. Figures 4(a) and 4(b) show the lump structure by density distribution and the three-dimensional plot at $t = 0$, respectively, with parameters being fixed by

$$c_1 = c_2 = e_1 = a = b = 1,\ d_1 = -2,\ b_1 = 3,$$
$$x_0 = y_0 = z_0 = t_0 = 0. \tag{40}$$

Figure 4(c) shows a three-dimensional structure of the lump solution at $y = 0$ with parameters also being given by Eq. (40). It is remarked that the lump solutions of the nNNV equation obtained here are not derived from known solutions of D'Alembert waves and complexiton solutions in Refs. [13,14], which shows that a great variety of interesting solutions of the nNNV equation can be generated through Eq. (17).

4. Symmetry reduction solutions

In this section, we use the classical Lie symmetry method to investigate the nNNV Eq. (6) to give its symmetry reduction solutions; to this end, we first assume its general Lie point symmetry of the system has the form

$$\begin{aligned} V \;=\; & X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+T\frac{\partial}{\partial t}+\Gamma_1\frac{\partial}{\partial A}+\Lambda_1\frac{\partial}{\partial B} \\ & +\Gamma_2\frac{\partial}{\partial C}+\Lambda_2\frac{\partial}{\partial E}+\Gamma_3\frac{\partial}{\partial F}+\Lambda_3\frac{\partial}{\partial G}, \end{aligned} \tag{41}$$

where $X, Y, T, \Gamma_1, \Gamma_2, \Gamma_3, \Lambda_1, \Lambda_2, \Lambda_3$ are functions of $x, y, t, A, B, C, E, F, G$ to be determined, which can also be written in function forms as

$$\sigma_A = XA_x + YA_y + TA_t - \Gamma_1, \tag{42a}$$
$$\sigma_B = XB_x + YB_y + TB_t - \Lambda_1, \tag{42b}$$
$$\sigma_C = XC_x + YC_y + TC_t - \Gamma_2, \tag{42c}$$
$$\sigma_E = XE_x + YE_y + TE_t - \Lambda_2, \tag{42d}$$
$$\sigma_F = XF_x + YF_y + TF_t - \Gamma_3, \tag{42e}$$
$$\sigma_G = XG_x + YG_y + TG_t - \Lambda_3. \tag{42f}$$

The symmetry equations of the nNNV equation take the form

$$\begin{aligned} & \sigma_{A,t}+\frac{3}{4}a(\sigma_E+3\sigma_C)A_x+\frac{3}{4}b(\sigma_G+3\sigma_F)A_y \\ & -\frac{3}{4}a(\sigma_E-\sigma_C)B_x-\frac{3}{4}b(\sigma_G-\sigma_F)B_y+\frac{3}{4}a(\sigma_B+3\sigma_A)C_x \\ & +\frac{3}{4}a(\sigma_A-\sigma_B)E_x+\frac{3}{4}b(\sigma_B+3\sigma_A)F_y+\frac{3}{4}b(\sigma_A-\sigma_B)G_y \\ & +a\sigma_{A,xxx}+b\sigma_{A,yyy}+(c+\frac{9}{4}aC+\frac{3}{4}aE)\sigma_{A,x} \\ & +(d+\frac{9}{4}bF+\frac{3}{4}bG)\sigma_{A,y}-\frac{3}{4}a(E-C)\sigma_{B,x} \\ & +\frac{3}{4}b(F-G)\sigma_{B,y}+\frac{3}{4}a(3A+B)\sigma_{C,x}-\frac{3}{4}a(B-A)\sigma_{E,x} \\ & +\frac{3}{4}b(3A+B)\sigma_{F,y}-\frac{3}{4}b(B-A)\sigma_{G,y}=0, \end{aligned} \tag{43a}$$
$$\sigma_{A,x}-\sigma_{C,y}=0, \tag{43b}$$
$$\sigma_{A,y}-\sigma_{F,x}=0, \tag{43c}$$

with the following additional nonlocal conditions:

$$\begin{aligned} \sigma_A &= \sigma_B(-x+x_0,-y+y_0,-t+t_0), \\ \sigma_C &= \sigma_E(-x+x_0,-y+y_0,-t+t_0), \\ \sigma_F &= \sigma_G(-x+x_0,-y+y_0,-t+t_0). \end{aligned} \tag{44}$$

First we substitute Eq. (42) into Eqs. (43a)–(43c) with $A_t, B_t, C_y, E_y, F_x, G_x$ being eliminated by Eq. (6). Next we vanish all different derivatives of A, B, C, E, F, and G, then an overdetermined system of $X, Y, T, \Gamma_1, \Gamma_2, \Gamma_3, \Lambda_1, \Lambda_2, \Lambda_3$ is obtained. Solved with the aid of software, e.g., maple, and considering the nonlocal condition of Eq. (44), we get the symmetry group of the nNNV equation

$$X=\frac{1}{3}\dot{g}_1x+g_2,\ Y=\frac{1}{3}\dot{g}_1y+g_3,\ T=g_1, \tag{45a}$$
$$\Gamma_1=-\frac{2}{3}\dot{g}_1A+e_1(B-A),\ \Lambda_1=-\frac{2}{3}\dot{g}_1B-e_1(B-A), \tag{45b}$$
$$\Gamma_2=\frac{1}{9a}\ddot{g}_1x-\frac{2(3aC+c)}{9a}\dot{g}_1+\frac{\dot{g}_2}{3a}+e_1(E-C), \tag{45c}$$
$$\Lambda_2=\frac{1}{9a}\ddot{g}_1x-\frac{2(3aE+c)}{9a}\dot{g}_1+\frac{\dot{g}_2}{3a}-e_1(E-C), \tag{45d}$$
$$\Gamma_3=\frac{1}{9b}\ddot{g}_1y-\frac{2(3bF+d)}{9b}\dot{g}_1+\frac{\dot{g}_3}{3b}-e_1(F-G), \tag{45e}$$
$$\Lambda_3=\frac{1}{9b}\ddot{g}_1y-\frac{2(3bG+d)}{9b}\dot{g}_1+\frac{\dot{g}_3}{3b}+e_1(F-G), \tag{45f}$$

where e_1 is an arbitrary constant and $g_i \equiv g_i(t)$, $(i=1,\ldots,3)$ are arbitrary functions that satisfy the following property:

$$\hat{T}_{\rm d}\{g_1,g_2,g_3\}=\{-g_1,-g_2,-g_3\}. \tag{46}$$

So we have the explicit expression of the symmetry by substituting Eq. (45) into Eq. (42)

$$\begin{aligned} \sigma_A \;=\; & g_1A_t+(\frac{1}{3}\dot{g}_1x+g_2)A_x+(\frac{1}{3}\dot{g}_1y+g_3)A_y \\ & +\frac{2}{3}\dot{g}_1A-e_1(B-A), \end{aligned} \tag{47a}$$
$$\begin{aligned} \sigma_B \;=\; & g_1B_t+(\frac{1}{3}\dot{g}_1x+g_2)B_x+(\frac{1}{3}\dot{g}_1y+g_3)B_y \\ & +\frac{2}{3}\dot{g}_1B+e_1(B-A), \end{aligned} \tag{47b}$$
$$\begin{aligned} \sigma_C \;=\; & g_1C_t+(\frac{1}{3}\dot{g}_1x+g_2)C_x+(\frac{1}{3}\dot{g}_1y+g_3)C_y \\ & -\frac{1}{9a}\ddot{g}_1x+\frac{2(3aC+c)}{9a}\dot{g}_1-\frac{\dot{g}_2}{3a}-e_1(E-C), \end{aligned} \tag{47c}$$
$$\begin{aligned} \sigma_E \;=\; & g_1E_t+(\frac{1}{3}\dot{g}_1x+g_2)E_x+(\frac{1}{3}\dot{g}_1y+g_3)E_y \\ & -\frac{1}{9a}\ddot{g}_1x+\frac{2(3aE+c)}{9a}\dot{g}_1-\frac{\dot{g}_2}{3a}+e_1(E-C), \end{aligned} \tag{47d}$$
$$\begin{aligned} \sigma_F \;=\; & g_1F_t+(\frac{1}{3}\dot{g}_1x+g_2)F_x+(\frac{1}{3}\dot{g}_1y+g_3)F_y \\ & -\frac{1}{9b}\ddot{g}_1y+\frac{2(3bF+d)}{9b}\dot{g}_1-\frac{\dot{g}_3}{3b}+e_1(F-G), \end{aligned} \tag{47e}$$
$$\begin{aligned} \sigma_G \;=\; & g_1G_t+(\frac{1}{3}\dot{g}_1x+g_2)G_x+(\frac{1}{3}\dot{g}_1y+g_3)G_y \\ & -\frac{1}{9b}\ddot{g}_1y+\frac{2(3bG+d)}{9b}\dot{g}_1-\frac{\dot{g}_3}{3b}-e_1(F-G). \end{aligned} \tag{47f}$$

To obtain group invariant solutions of the nNNV equation, we set $\sigma_A=\sigma_B=\sigma_C=\sigma_E=\sigma_F=\sigma_G=0$ in Eq. (47), or equivalently, solve the characteristic equation

$$\begin{aligned} \frac{dx}{\frac{1}{3}\dot{g}_1x+g_2} &= \frac{dy}{\frac{1}{3}\dot{g}_1y+g_3}=\frac{dt}{g_1}=\frac{dA}{-\frac{2}{3}\dot{g}_1A+e_1(B-A)} \\ &= \frac{dB}{-\frac{2}{3}\dot{g}_1B-e_1(B-A)} \\ &= \frac{dC}{\frac{1}{9a}\ddot{g}_1x-\frac{2(3aC+c)}{9a}\dot{g}_1+\frac{\dot{g}_2}{3a}+e_1(E-C)} \end{aligned}$$

$$= \frac{dE}{\frac{1}{9a}\ddot{g}_1 x - \frac{2(3aE+c)}{9a}\dot{g}_1 + \frac{\dot{g}_2}{3a} - e_1(E-C)}$$
$$= \frac{dF}{\frac{1}{9b}\ddot{g}_1 y - \frac{2(3bF+d)}{9b}\dot{g}_1 + \frac{\dot{g}_3}{3b} - e_1(F-G)}$$
$$= \frac{dG}{\frac{1}{9b}\ddot{g}_1 y - \frac{2(3bG+d)}{9b}\dot{g}_1 + \frac{\dot{g}_3}{3b} + e_1(F-G)}. \quad (48)$$

Without loss of generality, we assume $g_1 = d_1 t^3$ with d_1 being an arbitrary constant and $g_2 = g_3 = e_1 = 0$, we get

$$A = \frac{M_1}{t^2}, B = \frac{N_1}{t^2}, C = -\frac{c-\xi}{3a} + \frac{M_2}{t^2},$$
$$E = -\frac{c-\xi}{3a} + \frac{N_2}{t^2}, F = -\frac{d-\eta}{3b} + \frac{M_3}{t^2},$$
$$G = -\frac{d-\eta}{3b} + \frac{N_3}{t^2}, \quad (49a)$$

where M_1, N_1, M_2, N_2, M_3, N_3 are group invariant functions of two group invariant variables

$$\xi = \frac{x}{t}, \eta = \frac{y}{t}. \quad (50)$$

By substituting Eq. (49) into the nNNV Eq. (6) and considering the nonlocal condition (3), we get the corresponding symmetry reduction equations

$$3aN_{2,\xi}N_1 + 3bN_1N_{3,\eta} + 3bN_{1,\eta}N_3 + 3aN_2N_{1,\xi}$$
$$+aN_{1,\xi\xi\xi} + bN_{1,\eta\eta\eta} = 0, \quad (51)$$
$$N_{1,\xi} - N_{2,\eta} = 0, \quad (52)$$
$$N_{1,\eta} - N_{3,\xi} = 0, \quad (53)$$

with

$$M_1 = N_1, M_2 = N_2, M_3 = N_3. \quad (54)$$

Solving the solutions of Eqs. (51)–(53) and substituting them into Eq. (49), we acquire symmetry reduction solutions of the nNNV equation. As an illustration, we first consider travelling wave solutions of Eqs. (51)–(53) as

$$N_1 = d_0 + d_1 H(\tau), N_2 = \frac{d_1 H(\tau)}{c_1},$$
$$N_3 = d_1 c_1 H(\tau), \tau = \xi + c_1 \eta, \quad (55)$$

where $H(\tau)$ is an arbitrary function of τ with $c_1 = -\sqrt[3]{a/b}$ and d_0, d_1 being arbitrary constants. By substituting Eq. (55) with Eq. (50) into Eq. (49), we can get nontrivial solutions of the nNNV Eq. (6). Below are some examples.

Eagle-shape soliton By fixing

$$N_1 = \frac{c_1}{d_1}N_2 = \frac{1}{c_1 d_1}N_3 = \frac{d_1\tau^2(d_3+\cosh(\tau))}{d_2\cosh^2(\tau)}, \quad (56)$$

we get an eagle-shaped soliton solution of the nNNV Eq. (6) as A being depicted in Fig. 5(a) at $t = 1$ with $a = b = d_0 = d_1 = d_2 = d_3 = 1$, $c_1 = -1$.

Elliptic wave By fixing

$$N_1 = \frac{c_1}{d_1}N_2 = \frac{1}{c_1 d_1}N_3 = d_1 + \frac{d_2\mathrm{sn}^2(\tau, m_1)}{(m_1\mathrm{sn}^2(\tau, m_1)+1)^2}, \quad (57)$$

where $\mathrm{sn}(\tau, m_1)$ is a Jacobi elliptic function with modulus m_1, we get an elliptic wave solution of the nNNV Eq. (6) as A being depicted in Fig. 5(b) at $t = 2$ with $a = b = d_0 = d_1 = d_2 = 1$, $m_1 = 0.6$.

It is interesting that the solutions we obtained in Section 3 can also be recovered by fixing appropriate functions of $H(\tau)$ in Eq. (55), e.g., the elliptic wave solution obtained using Eq. (57) is the same as the solution generated by Eq. (37). Another example is by fixing

$$N_1 = \frac{c_1}{d_1}N_2 = \frac{1}{c_1 d_1}N_3 = \frac{6}{c_1 - \tau^2}\exp\left[-\frac{2}{(c_1-\tau^2)^2}\right], \quad (58)$$

we get the solution of the nNNV equation, which is the same as the D'Alembert wave solution generated by Eq. (29) in Section 3.

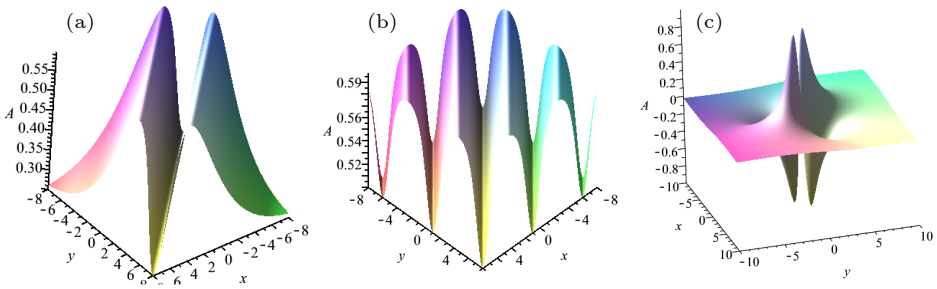


Fig. 5. Three-dimensional plots of three special solutions of the nNNV Eq. (6): (a) eagle-shaped soliton at $t = 1$, (b) elliptic wave at $t = 2$, and (c) lump solution at $t = 1$.

On the other hand, we can get a lump solution of the nNNV Eq. (6) by giving solutions of Eqs. (51)–(53) as

$$N_1 = -\frac{8e_1e_2\xi\eta}{(e_1\xi^2+e_2\eta^2+e_3)^2},$$
$$N_2 = -\frac{4(e_1\xi^2-e_2\eta^2-e_3)e_1}{(e_1\xi^2+e_2\eta^2+e_3)^2},$$
$$N_3 = \frac{4(e_1\xi^2-e_2\eta^2+e_3)e_2}{(e_1\xi^2+e_2\eta^2+e_3)^2}, \quad (59)$$

with arbitrary constants e_1, e_2, e_3. By substituting Eq. (59) with Eq. (50) into Eq. (49), we get a lump solution of the nNNV equation, which is depicted in Fig. 5(c) at $t = 1$ with $e_1 = e_2 = e_3 = 1$.

5. Summary and discussion

In summary, we have systematically investigated a nonlocal form of the Nizhnik–Novikov–Veselov (NNV) equation with shifted space and delayed time reversal properties, which is constructed by employing the consistent correlated bang (CCB) method. By applying the symmetry and anti-symmetry separation method, the nonlocal NNV system is successfully transformed into a localized system, enabling the derivation of exact solutions from the local NNV equation under specific parity and time-reversal constraints. Based on Hirota's bilinear method, multiple soliton solutions are explicitly constructed, and their elastic interaction properties are illustrated through graphical analysis. Furthermore, by utilizing the velocity resonance mechanism, we have generated soliton molecules-bound states of multiple solitons, which represent a novel class of coherent structures in nonlocal integrable systems. In addition to soliton molecules, a variety of special solutions have been obtained, including lumps, W-shaped solitons, and various types of D'Alembert-type waves, demonstrating the rich solution diversity of the nonlocal NNV equation.

A classical Lie symmetry analysis is further performed on the nonlocal NNV system, yielding its symmetry group and symmetry reduction solutions. Notably, many of the special solutions obtained in Section 3 can be recovered from these symmetry reductions, confirming the consistency and completeness of the Lie symmetry analysis method.

Compared with conventional methods for solving nonlocal systems such as the inverse scattering transform, Darboux transformations, and Riemann–Hilbert techniques, the symmetry and anti-symmetry separation method employed here provides a simpler and more efficient pathway to construct multiple solitons, soliton molecules, and various localized wave solutions from known solutions of the corresponding local system. In particular, the explicit multiple soliton solutions derived from Hirota's bilinear form offer a direct and convenient framework for generating soliton molecules via velocity resonance, a feature that is not easily accessible through other conventional methods. In addition, the D'Alembert-type wave solutions obtained in this work have not been reported in the existing literature on nonlocal systems obtained via the CCB method.

In recent years, topics such as long-range instability, long-time asymptotics,[28,29] and the experimental observation of soliton dynamics in local nonlinear systems have attracted considerable attention.[30,31] Extending these studies to nonlocal systems remains an important direction for future research. We anticipate that the results presented in this work will stimulate further theoretical and experimental investigations into the properties of nonlocal integrable systems.

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