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Hierarchy of average correlation, quantum steering and Bell nonlocality in two-qubit systems

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Nonclassicality is seen as an important physical resource and offers quantum advantages in modern quantum information processing. Motivated by recent work on the interplay between Bell nonlocality and nonclassicality quantified by average correlation [*Phys. Rev. Res.* **5** 023063 (2023)], in this paper we aim to establish the intrinsic relationship between the average correlation and the violation of the three-setting linear steering inequality for arbitrary two-qubit states. The results suggest that average correlation is closely related to the violation of the three-setting linear steering inequality, like its relationship with Bell inequality violation. Moreover, the dynamical behaviors of average correlation and steering are carefully analyzed under the influence of local unital and nonunital noisy channels. Particularly, for any two-qubit states, the hierarchy of average correlation-steering-Bell nonlocality is demonstrated by using the singular value decomposition theorem.

Keywords: quantum steering, Bell nonlocality, nonclassicality

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1. Introduction

Nonclassicality, which is recognized as a central concept in quantum theory providing invaluable physical resource for quantum information processing, has been widely investigated.[1–4] One of the most fascinating research topics is to characterize and quantify the nonclassicality of a quantum system. Consequently, various measures of nonclassicality, such as quantum discord, measurement-induced nonlocality, and so on, have been proposed in the past few years.[5–9] Traditionally, Bell inequality[10] often serves as a sufficient but not necessary condition for nonlocality (*e.g.*, the Clauser–Horne–Shimony–Holt inequality[11]): $|\langle\mathfrak{B}_{\rm CHSH}\rangle_\rho| \leq 2$, where $\mathfrak{B}_{\rm CHSH} = \boldsymbol{a}\cdot\boldsymbol{\sigma}\otimes(\boldsymbol{b}+\boldsymbol{b}')\cdot\boldsymbol{\sigma} + \boldsymbol{a}'\cdot\boldsymbol{\sigma}\otimes(\boldsymbol{b}-\boldsymbol{b}')\cdot\boldsymbol{\sigma}$ depends on four unit measurement directions $(\boldsymbol{a},\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}')$, $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ is the vector of Pauli matrices. This inequality holds true for all local hidden variable theories, but can be violated in quantum mechanics.[7] To better understand the nature of nonlocality, the concept of steering (created by Schrödinger in 1935[12]) was renewed and formalized by Wiseman *et al.* in 2007.[13] They showed that steerability was a different class of nonlocality intermediate between entanglement and Bell nonlocality.

At present, Bell nonlocality, steerability, and entanglement are regarded as three special aspects of nonclassicality, especially they are all used to capture some nonclassical aspects of quantum states, and provide quantum advantages for quantum computing and quantum information tasks compared to classical algorithms, such as quantum key distribution,[14,15] entanglement-based quantum cryptography,[16] boson sampling,[17] and so on. Nevertheless, there are also subtle differences among them: Bell nonlocality is more related to outcomes of experimental measurement statistics,[18] and it can actually be viewed as a specific nonclassicality. Particularly, Bell nonlocality defies any local hidden variable theories and is device-independent. Entanglement has its root in superposition of quantum states and captures superposition properties that cannot be represented by a probabilistic mixture of product states.[19] While steerability lying between Bell nonlocality and entanglement is also a characteristic of nonclassicality but exhibits an inherent asymmetry.[20]

More recently, Tschaffon *et al.*[21,22] have proposed a necessary and sufficient condition for any two-qubit state ρ with nonclassicality in terms of average correlation $\Sigma(\rho)$, which is defined as the average absolute value of the two-qubit correlation function over all measurement directions. Specifically, a necessary condition for a two-qubit state ρ with nonclassicality is required $\Sigma(\rho) \geq 1/4$, whereas $\Sigma(\rho) \geq 1/2\sqrt{2}$ is a sufficient condition for nonclassicality. However, it is worth emphasizing here that this concept of average correlation is very different from the above-mentioned quantum nonlocality and quantum discord,[5] which quantifies all nonclassical correlations of a quantum system. For example, the violation of Bell inequality is strongly dependent on the measurement

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directions and the underlying quantum states, while quantum discord involved in the computation process suffers from an extremum over all measurements. In contrast, average correlation independent of measurements has the operational and calculable advantages that do not require the control of measurement directions and the optimization procedure.

Beyond the characterization and quantification of nonclassicality, another interesting research topic is to address the interplay among different quantifiers. To begin with, an interesting finding in Ref. [23] is that violating the Bell inequality (referred to as nonlocality) implies some amount of entanglement quantified by concurrence C. Subsequently, an inequality relation between Bell inequality violation and concurrence for a two-qubit state with a fixed concurrence is given by the relation:[24] $2\sqrt{2}C \leq \mathfrak{B}_{\text{CHSH}} \leq 2\sqrt{1+C^2}$. Analogous to the relation between Bell nonlocality and concurrence, the more general relations between Bell nonlocality and several entanglement witnesses such as the negativity and the relative entropy of entanglement have also been explored.[25–28] In Refs. [29,30], the relationship between the violation of Bell inequality and geometric measure of quantum discord D_{G} for Bell diagonal states has been derived by the inequality: $4\sqrt{D_{\text{G}}} \leq \mathfrak{B}_{\text{CHSH}} \leq 2\sqrt{1+2D_{\text{G}}}$. Interestingly, nonlocality-nonclassicality interplay was generalized to the coherence case where the quantitative relation between the Bell nonlocality and the first-order coherence was established by an equality for the pure-state,[31] but by an inequality for the mixed-state.[32] Very recently, the connection of the maximal violation of Bell's inequality to average correlation has been studied.[21] The result shows that the violation of Bell's inequality $|\langle\mathfrak{B}_{\text{CHSH}}\rangle_\rho| > 2$ implies the average correlation, but the presence of average correlation does not necessarily violate this inequality.

Inspired by the above relation of average correlation to Bell nonlocality, we wonder whether a similar relationship, or what kind of relationship, exists between average correlation and another nonlocality, quantified by quantum steering. This is an interesting question since quantum steering is also recognized as a kind of nonlocal correlation, lying between Bell nonlocality and entanglement. Particularly, the relations of quantum steering to both entanglement and coherence have attracted considerable attention.[33–35] In this paper, we address this question by considering average correlation and the maximum violation of the three-setting linear steering inequality for any two-qubit state, and derive both upper and lower bounds between average correlation and the maximum violation of the three-setting linear steering. To gain further insight into their connections, we demonstrate the dynamical behaviors of these three quantifiers (average correlation, steering, and Bell nonlocality) in the presence of noisy channels, including local unital and nonunital channels. Our results show that the average correlation is closely related to the three-setting linear steering inequality violation, similar to its relationship with Bell inequality violation. In particular, we explicitly prove that the hierarchy of average correlation-steering-Bell nonlocality holds for any two-qubit state (not just special classes) using the singular value decomposition theorem.

This manuscript is organized as follows. In Section 2, we give a brief introduction to average correlation and quantum steering. For arbitrary two-qubit states, the link of the average correlation to the three-setting linear steering inequality violation is established in Section 3. In Section 4, we compare the dynamical behaviors of average correlation with those of quantum steering under decoherence. Section 5 gives the conclusion and discussion.

2. Preliminaries

Before revealing the hierarchy of average correlation, quantum steering, and Bell nonlocality in two-qubit systems, we first recall their definitions and present some recent results.

2.1. Average correlation

Recently, an alternative nonclassical quantifier named as average correlation is introduced by Tschaffon *et al.*,[21] it is defined as the average absolute value of the two-qubit correlation function

$$\Sigma(\rho) = \frac{1}{16\pi^2}\int \mathrm{d}\Omega_{\boldsymbol{a}} \int \mathrm{d}\Omega_{\boldsymbol{b}}\, |E(\boldsymbol{a},\boldsymbol{b})|\,, \tag{1}$$

where $E(\boldsymbol{a},\boldsymbol{b}) = \mathrm{Tr}[\rho\, \boldsymbol{a}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{b}\cdot\boldsymbol{\sigma}]$ denotes the correlation function measured by two observers averaging over all measurement directions $\boldsymbol{a}$ and $\boldsymbol{b} \in R^3$. $\mathrm{d}\Omega_{\boldsymbol{a}} = \sin\theta_a \mathrm{d}\theta_a \mathrm{d}\phi_a$ (and similarly for $\mathrm{d}\Omega_{\boldsymbol{b}}$) is the solid angle element. To evaluate this quantity, one can rewrite the correlation function as

$$E(\boldsymbol{a},\boldsymbol{b}) = \boldsymbol{a}^{\mathrm{T}} T \boldsymbol{b} \tag{2}$$

for any arbitrary ρ. Here the correlation matrix $T = \mathrm{Tr}[\rho\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}]$, there always exists a singular decomposition:[36] $UTV^{\mathrm{T}} = \mathrm{diag}(\alpha,\beta,\gamma)$ with nonnegative singular value $0 \leq \gamma \leq \beta \leq \alpha \leq 1$.

It turns out that the average correlation is recognized as a good indicator for nonclassicality ranging from a minimum of $\Sigma(\rho) = 1/4$ to a maximum of $\Sigma(\rho) = 1/2$.[21] Specifically, $\Sigma(\rho) \geq 1/4$ represents a necessary condition for the nonclassicality of an unknown two-qubit state, whereas $\Sigma(\rho) \geq 1/2\sqrt{2}$ is a sufficient condition.

2.2. Quantum steering

Quantum steering[12] was first introduced by Schrödinger in 1935, but later developed by Wiseman *et al.*[13] in 2007. It captures the ability of an observer on one side by the local measurements to remotely steer the state on the other side. In quantum information theory, quantum steering is recognized

as a kind of nonlocal correlation lying between Bell nonlocality and entanglement. Steerable states can be tested by the violation of steering inequality and are shown to have many potential applications in quantum information tasks.[37–40]

A linear steering inequality based on a finite sum of bilinear expectation values was introduced by Cavalcanti *et al.*[41] to diagnose the steerability of a quantum state. For a given two-qubit state ρ, the steering inequality can be expressed as

$$F_n(\rho,\mu)=\frac{1}{\sqrt{n}}\left|\sum_{i=1}^{n}\langle A_i\otimes B_i\rangle\right|\leq 1, \tag{3}$$

where $A_i=\boldsymbol{a}_i\cdot\boldsymbol{\sigma}$ and $B_i=\boldsymbol{b}_i\cdot\boldsymbol{\sigma}$ are Hermitian operators acting on qubits A and B, respectively. Both $\boldsymbol{a}_i$ and $\boldsymbol{b}_i\in R^3$ are two unit vectors, and $\mu=\{\boldsymbol{a}_i,\boldsymbol{b}_i\}$ is the set of measurement directions. Violation of this inequality implies that the state ρ is steerable, and the maximum steering inequality violation often serves as a steering measure[41]

$$\mathbb{S}_n(\rho)=\max\left\{0,\frac{s_n-1}{\sqrt{n}-1}\right\}. \tag{4}$$

Due to the normalized factor $\sqrt{n}-1$, $\mathbb{S}_n(\rho)$ ranges from 0 to 1. For any two-qubit states in the Hilbert–Schmidt space, the degree of steerability s_n is

$$s_n=\begin{cases}\sqrt{\alpha^2+\beta^2}, & \text{for } n=2,\\ \sqrt{\alpha^2+\beta^2+\gamma^2}, & \text{for } n=3.\end{cases} \tag{5}$$

Here $n=2,3$ refers to the two- or three-setting measurements per site. Obviously, $s_n\leq 1$ implies that the state obeys the n-setting linear steering inequality, while $s_n>1$ implies that the state violates this inequality and becomes steerable.

3. Relationship between average correlation and quantum steering

By comparing the notions of average correlation with quantum steering defined in Eqs. (1) and (4), both of which are put forward to characterize some kind of nonclassical correlations from different perspectives, this allows us to establish the possible relation between them. In order to do so, we define the expectation value $\mathbb{E}[X]$ with respect to X for all the measurement directions as follows:

$$\begin{aligned}\mathbb{E}[X]&=\frac{1}{4\pi}\int \mathrm{d}\Omega X\\&=\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}X\sin\theta\,\mathrm{d}\theta,\end{aligned} \tag{6}$$

where $\mathrm{d}\Omega=\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ is the solid angle element. The factor $1/(4\pi)$ ensures normalization: $\mathbb{E}[1]=1$.

As a result, the average correlation can be expressed as

$$\begin{aligned}\Sigma(\rho)&=\mathbb{E}_{\boldsymbol{a},\boldsymbol{b}}[|\boldsymbol{a}^{\mathrm{T}}T\boldsymbol{b}|]\\&=\mathbb{E}_{\boldsymbol{b}}[\mathbb{E}_{\boldsymbol{a}}[|\boldsymbol{a}^{\mathrm{T}}T\boldsymbol{b}|]]\\&=\frac{1}{2}\mathbb{E}_{\boldsymbol{b}}[||T\boldsymbol{b}||],\end{aligned} \tag{7}$$

where $||T\boldsymbol{b}||=\sqrt{\boldsymbol{b}^{\mathrm{T}}T^{\mathrm{T}}T\boldsymbol{b}}=\sqrt{\alpha^2b_1^2+\beta^2b_2^2+\gamma^2b_3^2}$, with $\boldsymbol{b}=(b_1,b_2,b_3)=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ is the unit vector.

Now, we prove that the upper boundary of average correlation is bounded by the three-setting linear quantum steering s_3

$$\begin{aligned}\Sigma(\rho)&=\frac{1}{2}\mathbb{E}_{\boldsymbol{b}}[||T\boldsymbol{b}||]\\&\leq\frac{1}{2}\sqrt{\mathbb{E}_{\boldsymbol{b}}[||T\boldsymbol{b}||^2]}\\&=\frac{1}{2}\sqrt{\mathbb{E}_{\boldsymbol{b}}[\boldsymbol{b}^{\mathrm{T}}T^{\mathrm{T}}T\boldsymbol{b}]}\\&=\frac{1}{2}\sqrt{\mathrm{Tr}(T^{\mathrm{T}}T\mathbb{E}_{\boldsymbol{b}}[\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}])}\\&=\frac{1}{2}\sqrt{\mathrm{Tr}\left(T^{\mathrm{T}}T\cdot\frac{1}{3}I\right)}\\&=\frac{1}{2}\sqrt{\frac{1}{3}(\alpha^2+\beta^2+\gamma^2)}\\&=\frac{s_3}{2\sqrt{3}},\end{aligned} \tag{8}$$

here the three-setting linear quantum steering $s_3=\sqrt{\alpha^2+\beta^2+\gamma^2}$. In the proof, the Cauchy–Schwarz inequality was used in Eq. (8) where $\mathbb{E}_{\boldsymbol{b}}[\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}]$ represents a matrix whose elements $\mathbb{E}_{\boldsymbol{b}}[b_ib_j]=(1/3)\delta_{ij}$. Note that $\mathbb{E}_{\boldsymbol{b}}[b_1^2+b_2^2+b_3^2]=\mathbb{E}_{\boldsymbol{b}}[b_1^2]+\mathbb{E}_{\boldsymbol{b}}[b_2^2]+\mathbb{E}_{\boldsymbol{b}}[b_3^2]\equiv 1$.

On the other hand, the lower boundary of the average correlation is also bounded by the three-setting linear quantum steering s_3

$$\Sigma(\rho)\geq\begin{cases}\dfrac{s_3}{4}, & \text{if } s_3<1,\\ \dfrac{1}{4}\displaystyle\int_0^{\pi/2}\sqrt{1-(2-s_3^2)\sin^2\phi}\,\mathrm{d}\phi, & \text{if } s_3\geq 1,\end{cases} \tag{9}$$

which is proved in Appendix A.

Obviously, the maximal violation of the three-setting linear steering inequality $s_3(\rho)\geq 1$ (equivalent to $\mathbb{S}_3(\rho)\geq 0$) implies quantum steering, which in turn implies nonclassicality quantified by $\Sigma(\rho)\geq 1/4$, while a value higher than $\Sigma(\rho)=1/2\sqrt{3}$ guarantees that the state is nonclassical. By contrast, the results shown in Ref. [21] where the maximum violation of the Bell inequality, *i.e.*,

$$\mathcal{B}_{\mathrm{CHSH}}(\rho)=\max\left\{0,\frac{\sqrt{\alpha^2+\beta^2}-1}{\sqrt{2}-1}\right\},$$

namely, $\mathcal{B}_{\mathrm{CHSH}}(\rho)=\mathbb{S}_2(\rho)$ implies the average correlation $\Sigma(\rho)>1/4$, but the presence of average correlation $\Sigma(\rho)>1/4$ does not necessarily violate this inequality. Besides, due to the fact that s_3 is always larger than s_2 for any ρ, this means that the presence of s_2 to witness steering guarantees the presence of $s_3(\rho)$ for steering. These results allow us to establish

the hierarchy of $\mathcal{B}_{\rm CHSH}(\rho)=\mathbb{S}_2(\rho)\subseteq\mathbb{S}_3(\rho)\subseteq\Sigma(\rho)$, meaning that the existence of Bell nonlocality implies steering, which in turn implies nonclassicality, while the converse is not true, as shown in Fig. 1.

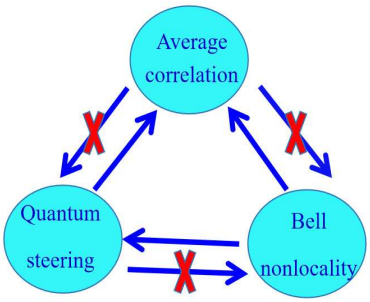


Fig. 1. Hierarchy of average correlation, quantum steering and Bell nonlocality.

4. Average correlation *versus* quantum steering under noisy channels

To further gain insights into the relationship between average correlation and quantum steering, in this section we would like to compare their performances under the influence of local unital and nonunital noisy channels. Usually, the evolved state of a two-qubit system under local noises can be described as a completely positive trace preserving map, $\varepsilon[\rho]=\sum_{ij}(E_i\otimes E_j)\rho(E_i\otimes E_j)^\dagger$ with the local Kraus operators satisfying $\sum_k E_k^\dagger E_k=1$. For simplicity, we restrict ourselves to the case where the initial states are prepared in the maximally mixed marginal states: $\rho(0)=(1/4)\left(I\otimes I+\sum_{i=1}^3 c_i\sigma_i\otimes\sigma_i\right)$, and show how environmental noises influence the average correlation versus quantum steering.

4.1. Unital noise

We first consider a type of local unital noisy channels[42] which satisfy the unital condition

$$\varepsilon\left(\frac{1}{2}I\right)=\sum_\mu E_\mu\left(\frac{1}{2}I\right)E_\mu^\dagger=\frac{1}{2}I.$$

For example, bit-flip, bit-phase-flip, and phase-flip channels belong to this category of noises. The corresponding Kraus operators are denoted by $E_0=\sqrt{1-p/2}I$ and $E_1=\sqrt{p/2}\sigma_i$, where σ_i are the three Pauli matrices, and $p=1-{\rm e}^{-\Gamma t}$ represents decoherence probabilities, where Γ is the decay rate of the channel. More specifically, σ_i $(i=1,2,3)$ correspond to bit-flip, bit-phase-flip, and phase-flip channels, respectively. Assume that the subsystem is subjected to the same quantum channel; the evolved state of such a system is determined by

$$\varepsilon[\rho]=\frac{1}{4}\left(I\otimes I+\sum_{i=1}^3 c_i'\sigma_i\otimes\sigma_i\right)\tag{10}$$

with $c_i'=c_i$ and $c_j'=c_j(1-p)^2$ for $i\neq j$. Here $i=1,2,3$ represents the system suffering from bit-flip, bit-phase-flip, and phase-flip channels, respectively.

On the other hand, both the phase damping channel and the depolarizing channel are also referred to as unital noisy channels. Their Kraus operators are respectively $E_0=\mathrm{diag}(1,\sqrt{1-p})$ and $E_1=\mathrm{diag}(0,\sqrt{p})$ for the phase damping channel, as well as $E_0=\sqrt{1-3p/4}I$, $E_1=\sqrt{p/4}\sigma_1$, $E_2=\sqrt{p/4}\sigma_2$, and $E_3=\sqrt{p/4}\sigma_3$ for the depolarizing channel. The corresponding evolved states under these unital channels are determined by

$$c_{1,2}'=c_{1,2}(1-p),\tag{11a}$$

$$c_3'=c_3\tag{11b}$$

for the phase damping channel and

$$c_{1,2,3}'=c_{1,2,3}(1-p)^2\tag{12}$$

for the depolarizing channel, respectively.

Using Eqs. (4) and (5), we immediately obtain the degree of steering under these unital noisy channels

$$s_2(\varepsilon[\rho])=\sqrt{\sum_{i=1}^3|c_i'|^2-\gamma^2},\tag{13}$$

$$s_3(\varepsilon[\rho])=\sqrt{\sum_{i=1}^3|c_i'|^2},\tag{14}$$

where $\gamma=\min\{|c_i'|,|c_j'|\}$. At the same time, the average correlation under unital noisy channels is strictly determined by Eq. (1), where $\alpha=\max\{|c_i'|,|c_j'|\}$. Obviously, both $\mathbb{S}_n(\xi[\rho])\leq\mathbb{S}_n(\rho)$ and $\Sigma(\xi[\rho])\leq\Sigma(\rho)$ are strictly satisfied due to the fact that $|c_i'|\leq|c_i|$ $(i=1,2,3)$ always holds. This indicates that the unital noises never increase the amount of quantum steering and average correlation.

The behavior of average correlation and quantum steering against Γt under unital noisy channels is depicted in Fig. 2(a), where we can clearly see that the amount of quantum steering and average correlation decreases monotonically under the above local unital channels. Moreover, the average correlation $\Sigma(\varepsilon[\rho])$ is more robust against decoherence than both $\mathbb{S}_2(\varepsilon[\rho])$ and $\mathbb{S}_3(\varepsilon[\rho])$, and they obey the hierarchy of robustness

$$\mathbb{S}_2(\varepsilon[\rho])\subseteq\mathbb{S}_3(\varepsilon[\rho])\subseteq\Sigma(\varepsilon[\rho]).\tag{15}$$

To support our statement for the hierarchy of robustness under arbitrary lossy channels, we resort to the initial states $\boldsymbol{c}=(c_1=c_2=c_3=|c|)$, and let t_{s_2}, t_{s_3}, and t_Σ be the sudden-death time or the shortest time where the corresponding $\mathbb{S}_2(\varepsilon[\rho])$, $\mathbb{S}_3(\varepsilon[\rho])$, and $\Sigma(\varepsilon[\rho])$ disappear, respectively,

$$t_{s_2}=-\frac{\ln\left[\dfrac{1-|c|^2}{|c|^2}\right]}{4\Gamma},\tag{16a}$$

$$t_{s_3}=-\frac{\ln\left[\dfrac{1-|c|^2}{2|c|^2}\right]}{4\Gamma},\tag{16b}$$

$$t_\Sigma = -\frac{\ln A}{4\Gamma}. \tag{16c}$$

Here A satisfies the equation

$$\operatorname{arcsinh}\left[\sqrt{\frac{1-A}{A}}\right] = \frac{1-|c|}{|c|}\frac{\sqrt{1-A}}{A}.$$

As expected, for any fixed c, $t_{s_2} \le t_{s_3} \le t_\Sigma$ is strictly satisfied. This implies $\mathbb{S}_2(\varepsilon[\rho])$ vanishes before $\mathbb{S}_3(\varepsilon[\rho])$, while $\mathbb{S}_3(\varepsilon[\rho])$ vanishes before $\Sigma(\varepsilon[\rho])$. These analytical results give strong support to the hierarchy claimed by Eq. (15). However, the question naturally arises whether this phenomenon still occurs under the influence of nonunital channels or more general two-qubit states.

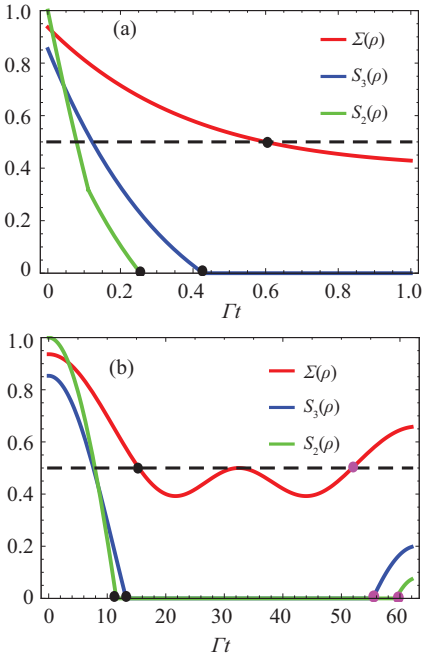


Fig. 2. Normalized average correlation $2\Sigma(\rho)$ and two or three-setting linear steering $S_2(\rho)$ ($S_3(\rho)$) as a function of Γt (a) under the local unital noise with initial state $(c_1, c_2, c_3) = (0.8, 1, 1)$, and (b) under local nonunital noise ($\Gamma = 0.005\kappa$) with initial state $(c_1, c_2, c_3) = (0.8, 1, 1)$. The decay order follows $t_{s_2} \le t_{s_3} \le t_\Sigma$ (black dot markers), while the revival order follows the reverse hierarchy $t_{s_2} \ge t_{s_3} \ge t_\Sigma$ (red dot markers).

4.2. Nonunital noise

The generalized amplitude damping channel is usually regarded as a nonunital noisy channel[43] which is defined as

$$\begin{aligned}\varepsilon\left(\frac{1}{2}I\right) &= \sum_\mu E_\mu \left(\frac{1}{2}I\right) E_\mu^\dagger \\ &= (1-p/2)|0\rangle\langle 0| + p/2|1\rangle\langle 1| \neq \frac{1}{2}I.\end{aligned}$$

The corresponding Kraus operators are denoted by $E_0 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|$ and $E_1 = \sqrt{p}|0\rangle\langle 1|$. Here

$$p = 1 - \mathrm{e}^{-\Gamma t}\left[\cos\left(\frac{Dt}{2}\right) + \frac{\Gamma}{D}\sin\left(\frac{Dt}{2}\right)\right]^2$$

with $D = \sqrt{2\kappa\Gamma - \Gamma^2}$, where Γ is the decay rate of the noisy channel and κ is the coupling strength between the system and noisy channel. In particular, for the case where $\Gamma < 2\kappa$, the system is located in the strong-coupling regime, while for the case where $\Gamma > 2\kappa$, the system lies in the weak-coupling regime. Proceeding in a similar way to the nonunital case, the evolved state under this nonunital channel is expressed as

$$\varepsilon[\rho] = \frac{1}{4}\left(I\otimes I + p(\sigma_3\otimes I + I\otimes\sigma_3) + \sum_{i=1}^{3} c_i'\sigma_i\otimes\sigma_i\right) \tag{17}$$

with $c_{1,2}' = c_{1,2}(1-p)$ and $c_3' = c_3(1-p)^2 + p^2$ leading to the dynamics of average correlation and quantum steering no longer monotonic. Figure 2(b) demonstrates the dynamics of average correlation and quantum steering against Γt under nonunital noisy channels. It can be seen that the decay and revival of average correlations and quantum steering occur. More interestingly, they follow a particular order: The order of decay satisfies $t_{s_2} \le t_{s_3} \le t_\Sigma$ which is the same hierarchy as in the case of unital channels, while the revival order follows in the reverse hierarchy $t_{s_2} \ge t_{s_3} \ge t_\Sigma$.

It is worth pointing out that the above hierarchy is only demonstrated for a special class of two-qubit states. One may also concern whether the obtained hierarchy holds for general two-qubit states ρ_{AB}. The answer is yes. Because, on one hand, the quantifiers $\mathcal{B}_{\mathrm{CHSH}}(\rho_{AB})$, $\mathbb{S}_3(\rho_{AB})$ and $\Sigma(\rho_{AB})$ are only dependent on the correlation matrix $T = \mathrm{Tr}[\rho_{AB}\sigma\otimes\sigma]$, on the other hand, the state ρ_{AB} is locally unitary equivalent to ρ_{AB}, that is, $\rho_{AB} = (U\otimes V)\rho_{AB}(U^\dagger\otimes V^\dagger)$, whose correlation matrix $T' = \mathrm{diag}\{c_1, c_2, c_3\}$ is a diagonal matrix. According to the singular value decomposition theorem in linear algebra, the matrix $T = \mathrm{Tr}[\rho_{AB}\sigma\otimes\sigma]$ can always be written as $U\,\mathrm{diag}\{\alpha,\beta,\gamma\}V^\dagger$[44] with $0 \le \gamma \le \beta \le \alpha \le 1$. Without loss of generality we assume that $c_1 \ge c_2 \ge c_3$, then the singular values $\alpha = c_1$, $\beta = c_2$, and $\gamma = c_3$. In this case, $\mathcal{B}_{\mathrm{CHSH}}(\rho_{AB})$, $\mathbb{S}_3(\rho_{AB})$, and $\Sigma(\rho_{AB})$ are invariant under local unitary transformations. This result presents strong evidence that the hierarchy holds for any two-qubit state.

5. Discussion and conclusion

Before concluding, there are still some open theoretical problems for further consideration. Specifically, the established hierarchical relationship among average correlation, quantum steering, and Bell nonlocality is currently confined to the bipartite qubit systems. Whether such a hierarchy holds in higher-dimensional bipartite systems (*e.g.*, qubit-qutrit or two-qutrit states) remains to be thoroughly explored. Indeed, the quantity $\Sigma(\rho)$ in Eq. (1) is specifically defined using the correlation function $E(\boldsymbol{a},\boldsymbol{b}) = \mathrm{Tr}[\rho\,\boldsymbol{a}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{b}\cdot\boldsymbol{\sigma}]$, which relies on the Pauli matrices. For higher-dimensional systems, one would need a suitable generalization of the correlation function, *e.g.*, using Gell–Mann matrices or other operator bases. Moreover, the singular value decomposition argument used in the two-qubit case does not directly apply to higher dimensions,

which have a more complicated algebraic structure. Therefore, extending the result to higher-dimensional bipartite systems would likely require numerical approaches or additional assumptions.

In addition, their applications in quantum information tasks deserve studying. As we know, two-qubit mixed states violating the Bell nonlocality are useful for teleportation.[45] However, the result in Ref. [46] indicates that there exist some states which violate the three-setting linear steering inequality but obey Bell inequality and are still useful for teleportation. This naturally inspires us to explore whether there exist some states ρ whose average correlation $\Sigma(\rho) > 1/4$ and obey the three-setting linear steering inequality $s_3(\rho) \leq 1$ are useful for teleportation.

In summary, we have established the connection between the average correlation and the violation of the three-setting linear steering inequality for arbitrary two-qubit states. In particular, we have set up the hierarchy of average correlation-steering-Bell nonlocality for two-qubit states. The results show that the average correlation is closely related to the maximum violation of the three-setting linear steering inequality, like its relationship with the maximum violation of the Bell inequality. To gain further insight into their connections, we have also explored the dynamical behaviors of these two quantifiers under the influence of local unital and nonunital noisy channels. The results suggest that unital noisy channels never increase nonclassicality (quantified by average correlation) and nonlocality (quantified by the two- or three-setting linear steering inequality), while nonunital noisy channels can induce revival of average correlations and quantum steering. Particularly, for any two-qubit states subject to decoherence, the aforementioned hierarchy $\mathcal{B}_{\mathrm{CHSH}}(\rho) = \mathbb{S}_2(\rho) \subseteq \mathbb{S}_3(\rho) \subseteq \Sigma(\rho)$ is also valid, as it follows from the hierarchy that the latter resource implies the former. In short, we hope our results in this paper would help us to better understand the relation between nonlocality and nonclassicality of the resource states.

Appendix A

In this appendix, we prove the lower boundary of average correlation with respect to quantum steering for arbitrary two-qubit states.

$$\begin{aligned}\Sigma(\rho) &= \frac{1}{2}\mathbb{E}_{\boldsymbol{b}}[||T\boldsymbol{b}||] \\ &= \frac{1}{2}\mathbb{E}_{\boldsymbol{b}}[\sqrt{\alpha^2 b_1^2+\beta^2 b_2^2+\gamma^2 b_3^2}] \\ &= \frac{1}{8\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi} f(\alpha,\beta,\gamma)\sin\theta\,\mathrm{d}\theta, \end{aligned} \tag{A1}$$

where

$$f(\alpha,\beta,\gamma) = \sqrt{\alpha^2 b_1^2+\beta^2 b_2^2+\gamma^2 b_3^2}, \tag{A2}$$

we first show that $f(\alpha,\beta,\gamma)$ increases monotonically in γ since

$$\begin{aligned}\frac{\partial f}{\partial\gamma} &= \frac{1}{2}(\alpha^2 b_1^2+\beta^2 b_2^2+\gamma^2 b_3^2)^{-1/2}\frac{\partial}{\partial\gamma}(\gamma^2 b_3^2) \\ &= \frac{\gamma b_3^2}{\sqrt{\alpha^2 b_1^2+\beta^2 b_2^2+\gamma^2 b_3^2}} \\ &\geq 0, \end{aligned} \tag{A3}$$

which implies that $\Sigma(\rho)$ is also strictly monotonically increasing in γ. Consequently, the average correlation $\Sigma(\rho)$ becomes minimal for $\gamma = 0$. In this case, $f(\alpha,\beta,\gamma)$ reduces to $f(\alpha,\beta,0)$, and $s_3 = \sqrt{\alpha^2+\beta^2}$.

On the other hand,

$$\begin{aligned}\frac{\partial f}{\partial\alpha} &= \frac{1}{2}(\alpha^2 b_1^2+\beta^2 b_2^2)^{-1/2}\frac{\partial}{\partial\alpha}[\alpha^2 b_1^2+(s_3^2-\alpha^2)b_2^2] \\ &= \frac{\alpha(b_1^2-b_2^2)}{\sqrt{\alpha^2 b_1^2+(s_3^2-\alpha^2)b_2^2}}. \end{aligned} \tag{A4}$$

This result indicates that $\Sigma(\rho)$ has the minimal value located at either $\alpha = 0$ if $\partial f/\partial\alpha \geq 0$, or $\alpha \leq \min(1, s_3)$ if $\partial f/\partial\alpha < 0$. However, for $\alpha = 0$, which means $\beta = \gamma = 0$ and $s_3 = 0$, hence we only consider the case of $\alpha \leq \min(1, s_3)$.

Case 1: $s_3 \geq 1$

Since $\alpha \leq 1$, the constraint $\alpha^2+\beta^2 = s_3^2 \geq 1$ forces $\alpha = 1$ and $\beta = \sqrt{s_3^2-1}$ (with $\beta \leq 1$ automatically satisfied). Thus, the minimal $\Sigma(\rho)$ is obtained by substituting $\alpha = 1, \beta = \sqrt{s_3^2-1}, \gamma = 0$ into the integral

$$\begin{aligned}\Sigma(\rho) &= \frac{1}{8\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi} f(\alpha,\beta,0)\sin\theta\,\mathrm{d}\theta, \\ &= \frac{1}{16}\int_0^{2\pi}\mathrm{d}\phi\sqrt{\cos^2\phi+(s_3^2-1)\sin^2\phi}, \\ &= \frac{1}{4}\int_0^{\pi/2}\sqrt{1-(2-s_3^2)\sin^2\phi}\,\mathrm{d}\phi. \end{aligned} \tag{A5}$$

Case 2: $0 < s_3 < 1$

The maximal value α is s_3 (because $s_3 = \sqrt{\alpha^2+\beta^2}$ leads to the maximal value $\alpha = s_3$ and $\beta = 0$). In this case $\Sigma(\rho)$ has a minimal value

$$\begin{aligned}\Sigma(\rho) &= \frac{1}{8\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi} f(\alpha,0,0)\sin\theta\,\mathrm{d}\theta, \\ &= \frac{1}{16}\int_0^{2\pi}\mathrm{d}\phi|\cos\phi|\alpha, \\ &= \frac{\alpha}{4}. \end{aligned} \tag{A6}$$

Therefore, the lower bound of the average correlation $\Sigma(\rho)$ for a fixed s_3

$$\Sigma_{\min}(\rho) = \begin{cases} \dfrac{s_3}{4}, & \text{if } s_3 < 1, \\ \dfrac{1}{4}\displaystyle\int_0^{\pi/2}\sqrt{1-(2-s_3^2)\sin^2\phi}\,\mathrm{d}\phi, & \text{if } s_3 \geq 1. \end{cases} \tag{A7}$$

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