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SPECIAL TOPIC — Nonlinear localized waves: Theory, methods and applications

Propagation of kinks embedded in nonlocal periodic waves in the inhomogeneous fiber with higher-order effect

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When employing ultrashort pulses to realize the transmission of higher-bit-rate solitons, the solitons become shorter. Under these circumstances, the higher-order nonlinear Schrödinger equation needs to be utilized to describe the situation. In this article, we first verify the conditions for the generation of nonlocal symmetry in the higher-order variable-coefficient nonlinear Schrödinger equation, which contains third-order dispersion and time-delay correlation in the cubic term, via the Painlevé truncation expansion. After localizing the nonlocal symmetry into a local one, the Lie point symmetry reduction is introduced to construct the interaction solutions between kinks and nonlocal waves. Finally, the propagation of kinks on nonlocal waves, as well as the influence of higher-order dispersion and time-delay correlation on kinks in the higher-order nonlinear Schrödinger equation, are analyzed.

Keywords: higher-order nonlinear Schrödinger equation, nonlocal symmetry, symmetry reduction, kink-periodic wave interaction

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1. Introduction

In the past few decades, research advancements in optical solitons have demonstrated that, to achieve large channel capacity and high-speed transmission, it is essential to transmit solitary waves at a high bit rate by means of ultrashort pulses. However, in such optical soliton transmission systems, as optical solitons shorten, higher-order effects, including third-order dispersion, stimulated Raman scattering, and self-steepening, become prominent. Considering these higher-order effects, Hasegawa *et al.*[1,2] put forward the higher-order nonlinear Schrödinger equation. In recent years, numerous scholars have analyzed the impact of higher-order terms on solitons from various perspectives based on the higher-order nonlinear Schrödinger equation,[3–10] such as Painlevé analysis,[11] inverse scattering transform,[12] Hirota direct method,[13,14] Darboux–Bäcklund transform,[15,16] conservation laws,[17,18] and numerical methods.[19] They have derived a variety of exact solutions, such as optical shock solutions, bright N-soliton solutions, higher-order rogue wave solutions, and higher-order periodic wave solutions for specific parameter values.[20–23] Nevertheless, it is readily evident from the existing research results that almost all the existing solutions are transmitted against the background of a plane wave.

In the present publication, to gain a comprehensive understanding of the propagation of kinks against the backdrop of more complex nonlocal waves, the variable-coefficient higher-order nonlinear Schrödinger equation is taken into consideration. This equation represents a refinement of the nonlinear Schrödinger equation, introducing terms that account for third-order dispersion and external potential in inhomogeneous optical fibers. In dimensionless form, it is expressed as follows:[24,25]

$$\begin{aligned} &v_x+\alpha_1 v_t+\alpha_2 v+\mathrm{i}\alpha_3 v_{tt}+\alpha_4 v_{ttt}\\ &+\mathrm{i}\alpha_5|v|^2 v-\alpha_6|v|^2 v_t=0, \end{aligned} \tag{1}$$

where v denotes the slowly varying envelope of the electric field, and the subscripts x and t denote the spatial and temporal partial derivatives in retarded-time coordinates. In Eq. (1), α_j $(j=1\text{–}6)$ are all real-valued functions with respect to x. α_3 and α_5 are the functions related to group velocity dispersion and self-phase modulation, respectively, and the two terms with coefficients α_4 and α_6 are respectively responsible for third-order dispersion and time-delay correlation in the cubic term. For picosecond light pulses, α_1, α_2, α_4 and α_6 can be set to zero, and Eq. (1) simplifies to the standard nonlinear Schrödinger (NLS) equation.[26,27]

The paper is organized as follows. In Section 1, the nonlocal residual symmetry of the higher-order NLS equation is explicitly constructed using the Painlevé truncation expansion. Considering the nonlocality of the residual symmetry, it is transformed into a local Lie point symmetry by means of the localization method. In Section 2, the interactions between solitons and nonlocal waves, such as Jacobi elliptic function

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periodic waves, parabolic periodic waves, and trigonometric function periodic waves of the higher-order NLS equation, are obtained through symmetry reduction, and the propagation of kinks embedded in nonlocal periodic waves is analyzed. Finally, some discussions and conclusions are provided in Section 3.

2. Nonlocal symmetry and its localization

In this section, to construct the nonlocal residual symmetry for the variable-coefficient higher-order nonlinear Schrödinger equation (1), the complex conjugation function $u = v^*$ is introduced. Subsequently, Eq. (1) is reformulated as the following system of equations:

$$\begin{aligned} &v_x + \alpha_1 v_t + \alpha_2 v + \mathrm{i}\alpha_3 v_{tt} + \alpha_4 v_{ttt} + \mathrm{i}\alpha_5 v^2 u - \alpha_6 v_t v u = 0, \\ &u_x + \alpha_1 u_t + \alpha_2 u - \mathrm{i}\alpha_3 u_{tt} + \alpha_4 u_{ttt} - \mathrm{i}\alpha_5 u^2 v - \alpha_6 u_t u v = 0. \end{aligned} \tag{2}$$

Under the dependent variable transformation

$$u = \mathrm{e}^{\mathrm{i}\phi} p, \quad v = \mathrm{e}^{-\mathrm{i}\phi} q, \tag{3}$$

with $\phi \equiv \phi(x,t)$ being a phase function and complex function $p = q^*$, Eq. (2) is equivalent to

$$\begin{aligned} &q_x - \mathrm{i}q\phi_x + \alpha_1(q_t - \mathrm{i}q\phi_t) + \alpha_2 q \\ &+\alpha_3(\mathrm{i}q_{tt} + q\phi_{tt} + 2q_t\phi_t - \mathrm{i}q\phi_t^2) \\ &+\alpha_4[q_{ttt} - \mathrm{i}q\phi_{ttt} - 3(q\phi_t + \mathrm{i}q_t)\phi_{tt} - 3\mathrm{i}q_{tt}\phi_t + \mathrm{i}q\phi_t^3 - 3q_t\phi_t^2] \\ &+\mathrm{i}\alpha_5 pq^2 - \alpha_6(pqq_t - \mathrm{i}pq^2\phi_t) = 0, \\ &p_x + \mathrm{i}p\phi_x + \alpha_1(p_t + \mathrm{i}p\phi_t) + \alpha_2 p \\ &+\alpha_3(-\mathrm{i}p_{tt} + p\phi_{tt} + 2p_t\phi_t + \mathrm{i}p\phi_t^2) \\ &+\alpha_4[p_{ttt} + \mathrm{i}p\phi_{ttt} - 3(p\phi_t - \mathrm{i}p_t)\phi_{tt} + 3\mathrm{i}p_{tt}\phi_t - \mathrm{i}p\phi_t^3 - 3p_t\phi_t^2] \\ &-\mathrm{i}\alpha_5 p^2 q - \alpha_6(pqp_t + \mathrm{i}p^2 q\phi_t) = 0. \end{aligned} \tag{4}$$

Within the framework of the truncated Painlevé expansion,[28,29] in order to seek the nonlocal residual symmetry for Eq. (4), we consider introducing the following expansions around a real singularity manifold $f \equiv f(x,t)$, which is assumed to be an analytic function of the independent variables x and t

$$p = p_0 + \frac{p_1}{f}, \quad q = q_0 + \frac{q_1}{f}, \tag{5}$$

where $q_0 = p_0^*$ and $q_1 = p_1^*$. Substituting Eq. (5) into Eq. (4) and separating the coefficients of different powers of f yield

$$f^{-4}: \quad 6\alpha_4 f_t^2 - \alpha_6 p_1 q_1 = 0, \tag{6}$$

$$\begin{aligned} f^{-3}: \quad &2\alpha_3 q_1 f_t^2 - 6\alpha_4(\phi_t q_1 f_t^2 + \mathrm{i}q_1 f_t f_{tt} + \mathrm{i}q_{1t} f_t^2) + \alpha_5 q_1^2 p_1 - \mathrm{i}\alpha_6(q_0 q_1 f_t p_1 + q_1^2 p_0 f_t + \mathrm{i}\phi_t q_1^2 p_1 - q_1 q_{1t} p_1) = 0, \\ &2\alpha_3 p_1 f_t^2 - 6\alpha_4(\phi_t p_1 f_t^2 - \mathrm{i}p_1 f_t f_{tt} - \mathrm{i}p_{1t} f_t^2) + \alpha_5 p_1^2 q_1 + \mathrm{i}\alpha_6(p_0 p_1 f_t q_1 + p_1^2 q_0 f_t - \mathrm{i}\phi_t p_1^2 q_1 - p_1 p_{1t} q_1) = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} f^{-2}: \quad &\alpha_1 q_1 f_t + \alpha_3(\mathrm{i}q_1 f_{tt} + 2if_t q_{1t} + 2q_1 f_t\phi_t) - \mathrm{i}\alpha_4(-3\mathrm{i}\phi_t^2 q_1 f_t + 3\phi_{tt} q_1 f_t + 3\phi_t q_1 f_{tt} + 6\phi_t q_{1t} f_t + \mathrm{i}q_1 f_{ttt} \\ &+3\mathrm{i}q_{1tt} f_t + 3\mathrm{i}q_{1t} f_{tt}) - \mathrm{i}\alpha_5(p_0 q_1 + 2p_1 q_0) q_1 - \mathrm{i}\alpha_6(-\mathrm{i}q_0 q_1 p_0 f_t + 2\phi_t q_0 q_1 p_1 + \phi_t q_1^2 p_0 + \mathrm{i}q_0 q_{1t} p_1 \\ &+\mathrm{i}q_1 p_0 q_{1t} + \mathrm{i}q_1 q_{0t} p_1) + q_1 f_x = 0, \\ &\alpha_1 p_1 f_t + \alpha_3(-\mathrm{i}p_1 f_{tt} - 2if_t p_{1t} + 2p_1 f_t\phi_t) + \mathrm{i}\alpha_4(3\mathrm{i}\phi_t^2 p_1 f_t + 3\phi_{tt} p_1 f_t + 3\phi_t p_1 f_{tt} + 6\phi_t p_{1t} f_t - \mathrm{i}p_1 f_{ttt} \\ &-3\mathrm{i}p_{1tt} f_t - 3\mathrm{i}p_{1t} f_{tt}) + \mathrm{i}\alpha_5(q_0 p_1 + 2q_1 p_0) p_1 + \mathrm{i}\alpha_6(\mathrm{i}p_0 p_1 q_0 f_t + 2\phi_t p_0 p_1 q_1 + \phi_t p_1^2 q_0 - \mathrm{i}p_0 p_{1t} q_1 \\ &-\mathrm{i}p_1 q_0 p_{1t} - \mathrm{i}p_1 p_{0t} q_1) + p_1 f_x = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} f^{-1}: \quad &\mathrm{i}\alpha_1(\mathrm{i}q_{1t} + \phi_t q_1) - \alpha_2 q_1 + \alpha_3(-\phi_{tt} q_1 + \mathrm{i}\phi_t 2q_1 - 2\phi_t q_{1t} - \mathrm{i}q_{1tt}) + \mathrm{i}\alpha_4(-3\mathrm{i}\phi_{tt}\phi_t q_1 - 3\mathrm{i}\phi_t^2 q_{1t} - \phi_t^3 q_1 \\ &+\mathrm{i}q_{1ttt} + \phi_{ttt} q_1 + 3\phi_{tt} q_{1t} + (3\phi_t q_{1tt} - \mathrm{i})\alpha_5 q_0(2p_0 q_1 + p_1 q_0) - \mathrm{i}\alpha_6(\mathrm{i}q_0 p_0 q_{1t} + \mathrm{i}q_0 q_{0t} p_1 + \mathrm{i}q_1 p_0 q_{0t} \\ &+\phi_t q_0^2 p_1 + 2\phi_t q_0 q_1 p_0) + \mathrm{i}\phi_x q_1 - q_{1x} = 0, \\ &-\mathrm{i}\alpha_1(-\mathrm{i}p_{1t} + \phi_t p_1) - \alpha_2 p_1 + \alpha_3(-\phi_{tt} p_1 - \mathrm{i}\phi_t 2p_1 - 2\phi_t p_{1t} + \mathrm{i}p_{1tt}) - \mathrm{i}\alpha_4(3\mathrm{i}\phi_{tt}\phi_t p_1 + 3\mathrm{i}\phi_t^2 p_{1t} - \phi_t^3 p_1 \\ &-\mathrm{i}p_{1ttt} + \phi_{ttt} p_1 + 3\phi_{tt} p_{1t} + (3\phi_t p_{1tt} + \mathrm{i})\alpha_5 p_0(2q_0 p_1 + q_1 p_0) + \mathrm{i}\alpha_6(-\mathrm{i}p_0 q_0 p_{1t} - \mathrm{i}p_0 p_{0t} q_1 - \mathrm{i}p_1 q_0 p_{0t} \\ &+\phi_t p_0^2 q_1 + 2\phi_t p_1 q_0) - \mathrm{i}\phi_x p_1 - p_{1x} = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} f^0: \quad &q_{0x} - \mathrm{i}q_0\phi_x + \alpha_1(q_{0t} - \mathrm{i}q_0\phi_t) + \alpha_2 q_0 + \alpha_3(\mathrm{i}q_{0tt} + q_0\phi_{tt} + 2q_{0t}\phi_t - \mathrm{i}q_0\phi_t^2) + \alpha_4[q_{0ttt} - \mathrm{i}q_0\phi_{ttt} \\ &-3(q_0\phi_t + \mathrm{i}q_{0t})\phi_{tt} - 3\mathrm{i}q_{0tt}\phi_t + \mathrm{i}q_0\phi_t^3 - 3q_{0t}\phi_t^2] + \mathrm{i}\alpha_5 p_0 q_0^2 - \alpha_6(p_0 q_0 q_{0t} - \mathrm{i}p_0 q_0^2\phi_t) = 0, \\ &p_{0x} + \mathrm{i}p_0\phi_x + \alpha_1(p_{0t} + \mathrm{i}p_0\phi_t) + \alpha_2 p_0 + \alpha_3(-\mathrm{i}p_{0tt} + p_0\phi_{tt} + 2p_{0t}\phi_t + \mathrm{i}p_0\phi_t^2) + \alpha_4[p_{0ttt} + \mathrm{i}p_0\phi_{ttt} \\ &-3(p_0\phi_t - \mathrm{i}p_{0t})\phi_{tt} + 3\mathrm{i}p_{0tt}\phi_t - \mathrm{i}p_0\phi_t^3 - 3p_{0t}\phi_t^2] - \mathrm{i}\alpha_5 q_0 p_0^2 - \alpha_6(q_0 p_0 p_{0t} + \mathrm{i}q_0 p_0^2\phi_t) = 0. \end{aligned} \tag{10}$$

By simply solving Eqs. (6)–(10), one obtains

$$p_1 = q_1 = \frac{\sqrt{6\alpha_4}}{\sqrt{\alpha_6}} f_t, \tag{11}$$

$$q_0 = p_0^* = -\frac{\sqrt{6\alpha_4}}{2\sqrt{\alpha_6}} \frac{f_{tt}}{f_t} + \frac{\mathrm{i}\sqrt{6\alpha_4}}{2\sqrt{\alpha_6}} \phi_t + \frac{\mathrm{i}\sqrt{6\alpha_4}\alpha_5}{2\sqrt{\alpha_6}\alpha_6}, \tag{12}$$

$$\phi_{ttt} = -2\phi_t^3 - \frac{6\alpha_5}{\alpha_6}\phi_t^2 - \frac{3\alpha_5^2}{\alpha_6^2}\phi_t - \frac{\phi_x}{\alpha_4} + \frac{3\alpha_5^3}{2\alpha_6^3} + \left(\frac{3\phi_t}{\alpha_4} + \frac{3\alpha_5}{\alpha_4\alpha_6}\right)\frac{f_x}{f_t}, \tag{13}$$

$$f_{ttt} = \frac{3f_{tt}^2}{2f_t} + \left(\frac{3\phi_t^2}{2} + \frac{3\alpha_5\phi_t}{\alpha_6}\right) f_t - \frac{f_x}{\alpha_4}, \tag{14}$$

where the coefficients of Eq. (1) are required to meet the constraint conditions

$$\alpha_1 = -\frac{3\alpha_4 c^2}{2}, \quad \alpha_2 = \frac{\alpha_4\alpha_{6x} - \alpha_6\alpha_{4x}}{2\alpha_4\alpha_6}, \quad \alpha_3 = -3\alpha_4 c, \quad \alpha_5 = c\alpha_6, \tag{15}$$

with c being an arbitrary constant. Upon comparing Eq. (4) with Eq. (10), it can be readily verified that p_0 and q_0 are precisely the solutions of the original higher-order nonlinear Schrödinger equation subject to the constraints Eq. (15). Meanwhile, p_1 and q_1, as expressed by Eq. (11), are exactly the residual symmetries of the same equation, which can be directly proved by substituting p_1 and q_1 into the symmetry equations of Eq. (4).

In general, owing to the nonlocality of the symmetries Eq. (11), it is not feasible to utilize them for reducing the dimensions of the system Eq. (4) and constructing group-invariant solutions. Therefore, we consider localizing the nonlocal symmetries Eq. (11) into local Lie point symmetries of the extended higher-order nonlinear Schrödinger equations. The enlarged higher-order nonlinear Schrödinger equations consist of Eqs. (4) and (12) with p_0 and q_0 being rewritten by p and q, Eqs. (13) and (14), and the potential equation

$$f_t = f_1. \tag{16}$$

To ensure that the enlarged higher-order nonlinear Schrödinger system remains invariant under the transformation

$$\{x,t,p,q,f,\phi,f_1,\alpha_6,\alpha_7\} \to \{x+\varepsilon X,\ t+\varepsilon T,\ p+\varepsilon P,\ q+\varepsilon Q,\ f+\varepsilon F,\ \phi+\varepsilon\Phi,\ f_1+\varepsilon F_1,\ \alpha_6+\varepsilon A_6,\ \alpha_7+\varepsilon A_7\}, \tag{17}$$

where for the sake of simplicity, α_4 is replaced by $\alpha_7^2 = \alpha_4\alpha_6$ with α_7 being an arbitrary function of x, now we assume that the Lie point symmetry of the enlarged higher-order nonlinear Schrödinger system possesses the expressions[30,31]

$$\sigma^\phi = X\phi_x + T\phi_t - \Phi,\ \sigma^p = Xp_x + Tp_t - P,\ \sigma^q = Xq_x + Tq_t - Q,\ \sigma^f = Xf_x + Tf_t - H,\ \sigma^{f_1} = Xf_{1x} + Tf_{1t} - F_1,\ \sigma^{\alpha_7} = X\alpha_{7x} - A_7,\ \sigma^{\alpha_6} = X\alpha_{6x} - A_6, \tag{18}$$

with X, T, Φ, P, Q, H, F_1 being the functions of $x,t,p,q,f,\phi,f_1,\alpha_6,\alpha_7$, and A_6 and A_7 with respect to x,α_6,α_7. By substituting the symmetries Eq. (18) into the linearized system of the extended higher-order nonlinear Schrödinger equations, eliminating p_x, q_x, f_{1t}, f_x, f_t, ϕ_x, ϕ_t, f_{1x} in accordance with Eqs. (4), (12)–(14), (16), and their compatibility conditions, and collecting the coefficients of p, q, f, ϕ, f_1, α_6, α_7 and their derivatives, we obtain the solutions for the infinitesimals X, T, P, Q, H, F_1, Φ, A_6 and A_7 through a direct calculation using Maple

$$X = g_1,\ T = c_5,\ P = Q = g_2 p + \frac{\sqrt{6}c_1\alpha_7 f_1}{2\alpha_6},\ H = -\frac{1}{2}c_1 f^2 + c_2 f + c_3,\ F_1 = -c_1 f f_1 + c_2 f_1,\ \Phi = c_4,\ A_6 = -\alpha_6 g_{1x} - 2\alpha_6 g_2,\ A_7 = -\alpha_7 g_{1x} - 2\alpha_7 g_2, \tag{19}$$

where g_1 and g_2 are arbitrary functions of x, and c_j $(j = 1\text{–}5)$ are five free constants. Substituting the infinitesimals Eq. (19) into Eq. (18) results in the symmetry of the enlarged higher-order nonlinear Schrödinger equations

$$\begin{aligned}
\sigma^\phi &= g_1\phi_x + c_5\phi_t - c_4,\\
\sigma^p &= g_1 p_x + c_5 p_t - \left(g_2 p + \frac{\sqrt{6}c_1\alpha_7 f_1}{2\alpha_6}\right),\\
\sigma^q &= g_1 q_x + c_5 q_t - \left(g_2 p + \frac{\sqrt{6}c_1\alpha_7 f_1}{2\alpha_6}\right),\\
\sigma^f &= g_1 f_x + c_5 f_t + \left(\frac{1}{2}c_1 f^2 - c_2 f - c_3\right),\\
\sigma^{f_1} &= g_1 f_{1x} + c_5 f_{1t} + (c_1 f f_1 - c_2 f_1),\\
\sigma^{\alpha_7} &= g_1\alpha_{7x} + (\alpha_7 g_{1x} + 2\alpha_7 g_2),\\
\sigma^{\alpha_6} &= g_1\alpha_{6x} + (\alpha_6 g_{1x} + 2\alpha_6 g_2).
\end{aligned} \tag{20}$$

From Eq. (20), it is not difficult to find that the c_1-dependent generators $f_1\frac{\partial}{\partial p}$ and $f_1\frac{\partial}{\partial q}$ of the relevant Lie symmetry groups of Eq. (4) precisely correspond to the infinitesimal transformations associated with the nonlocal residual symmetries Eq. (11). Presently, the corresponding group-invariant solutions for the enlarged higher-order nonlinear Schrödinger system can be obtained through the symmetry constraint conditions $\sigma^p = \sigma^q = \sigma^f = \sigma^{f_1} = \sigma^\phi = \sigma^{\alpha_6} = \sigma^{\alpha_7} = 0$ presented in Eq. (20).

3. Symmetry reduction for constructing kink-nonlocal wave interaction solutions

In this section, subsequent to solving Eq. (20), the group-invariant solutions for the higher-order nonlinear Schrödinger system will be explored. Without loss of generality, the arbitrary functions g_1 and g_2 are rewritten as $g_1 = \frac{1}{g_{3x}}$ and $g_2 = \frac{g_{4x}}{g_4 g_{3x}}$, where g_3 and g_4 are two other arbitrary functions of x. By solving the symmetry Eq. (20) with $\sigma^p = \sigma^q = \sigma^f = \sigma^{f_1} = \sigma^{\phi} = \sigma^{\alpha_6} = \sigma^{\alpha_7} = 0$, the similarity reduction solutions are as follows:

$$p = g_4 P_1 - \frac{\sqrt{6g_{3x}}c_1c_7\Psi}{\sqrt{\alpha_6(2c_1c_3+c_2^2)}} \times \tanh\left[\frac{\sqrt{2c_1c_3+c_2^2}}{2}(F+g_3)\right], \tag{21}$$

$$q = g_4 Q_1 - \frac{\sqrt{6g_{3x}}c_1c_7\Psi}{\sqrt{\alpha_6(2c_1c_3+c_2^2)}} \times \tanh\left[\frac{\sqrt{2c_1c_3+c_2^2}}{2}(F+g_3)\right], \tag{22}$$

$$f = \frac{c_2}{c_1} + \frac{\sqrt{2c_1c_3+c_2^2}}{c_1} \times \tanh\left[\frac{\sqrt{2c_1c_3+c_2^2}}{2}(F+g_3)\right], \tag{23}$$

$$f_1 = -\Psi \operatorname{sech}^2\left[\frac{\sqrt{2c_1c_3+c_2^2}}{2}(F+g_3)\right], \tag{24}$$

$$\phi = c_4 g_3 + \Theta, \quad g_3 = \int \frac{\alpha_7^2}{\alpha_6 c_7^2}dx + c_8, \quad g_4 = \frac{c_6\alpha_7}{\alpha_6 c_7}, \tag{25}$$

where the similarity variable is $\xi = -c_5 g_3 + t$, c_6, c_7, and c_8 are three integral constants, and the corresponding group-invariant functions Ψ, F, P_1, and Q_1 associated with the similarity variable ξ satisfy the reduction equations

$$P_1 = \frac{1}{\sqrt{2}F_\xi}[-\sqrt{3}c_7F_{\xi\xi} + \omega],$$
$$Q_1 = -\frac{1}{\sqrt{2}F_\xi}[\sqrt{3}c_7F_{\xi\xi} + \omega], \tag{26}$$
$$\Psi = -\frac{2c_1c_3+c_2^2}{2c_1}F_\xi, \quad \Theta_\xi = -c + \frac{\mathrm{i}\sqrt{3}\omega}{3c_7F_\xi}, \tag{27}$$
$$-3c_7^2(2c_1c_3+c_2^2)\omega^2F_\xi^4 + [6c_7^2\omega\omega_{\xi\xi} + (-9c^2c_7^2+6c_5)\omega^2]F_\xi^2 - 12(c_7^2\omega\omega_\xi F_{\xi\xi} + \omega^2)F_\xi + 3c_7^2\omega^2F_{\xi\xi}^2 - \omega^4 = 0,$$
$$\omega^2 = -2c_7^2F_\xi F_{\xi\xi\xi} + 3c_7^2F_{\xi\xi}^2 + c_7^2(2c_1c_3+c_2^2)F_\xi^4 + (2c_5 - 3c^2c_7^2)F_\xi^2 - 2F_\xi. \tag{28}$$

Specifically, by performing the transformation

$$F_\xi = R(\xi), \tag{29}$$

one can verify that Eq. (28) simplifies to the reduction equation

$$R_\xi^2 = a_0 + a_1R + a_2R^2 + a_3R^3 + a_4R^4, \tag{30}$$

where the constants satisfy any one of the following four cases:

$$a_0 = 0,\ a_1 = \frac{1}{c_7^2},\ a_2 = 3c^2 - \frac{2c_5}{c_7^2},\ a_4 = 2c_1c_3 + c_2^2, \tag{31}$$

$$a_1 = -\frac{2}{c_7^2},\ a_2 = -\frac{3c^2}{2} + \frac{c_5}{c_7^2},\ a_3 = 0,\ a_4 = \frac{c_1c_3}{2} + \frac{c_2^2}{4}, \tag{32}$$

$$a_1 = -\frac{2}{c_7^2},\ a_2 = 3c^2 - \frac{2c_5}{c_7^2} + \frac{3}{a_0c_7^4},$$
$$a_3 = -\frac{6c^2}{a_0c_7^2} + \frac{4c_5}{a_0c_7^4} - \frac{4}{a_0^2c_7^6},\ a_4 = 2c_1c_3 + c_2^2, \tag{33}$$

$$a_0 = \frac{(-3c^2c_7^2 + c_7^2a_2 + 2c_5)(3c^2c_7^2 + 2c_7^2a_2 - 2c_5)}{9c_7^4(2c_1c_3+c_2^2)},$$
$$a_1 = -\frac{2}{c_7^2},\ a_3 = -\frac{6(2c_1c_3+c_2^2)}{-3c^2c_7^2 + a_2c_7^2 + 2c_5},$$
$$a_4 = 2c_1c_3 + c_2^2. \tag{34}$$

Equation (30) is known as an elliptic equation, and its solutions have been widely discussed and applied.[32] If we consider Eq. (30) with coefficients in the form of Eq. (31), then

$$R_\xi^2 = \frac{1}{c_7^2}R + (3c^2 - \frac{2c_5}{c_7^2})R^2 + a_3R^3 + (2c_1c_3 + c_2^2)R^4. \tag{35}$$

By referring to Ref. [32], it can be concluded that Eq. (35) has a solution

$$R = b_0 + b_1S_n, \tag{36}$$

where $S_n \equiv S_n\ (b\xi, m)$ is the Jacobian elliptic sine function with modulus m, and

$$b_0 = \frac{\sqrt{2}}{c_7\sqrt{a_3(1-m^2)}},\ b_1 = \frac{\sqrt{2}m}{c_7\sqrt{a_3(1-m^2)}},$$
$$b = \frac{a_3^{1/4}}{2^{3/4}c_7^{1/2}(1-m^2)^{1/4}},\ c_2 = \frac{a_3^{3/4}c_7^{1/2}(1-m^2)^{1/4}}{2^{5/4}},$$
$$c_3 = 0,\ c_5 = \frac{3c^2c_7^2}{2} + \frac{\sqrt{2a_3}c_7(m^2-5)}{8(1-m^2)^{1/2}}. \tag{37}$$

Substituting Eq. (36) with Eq. (37) into Eq. (29) and integrating once with respect to ξ, the solution of the reduced Eq. (29) can be obtained as

$$F = b_0\xi + \frac{b_1}{bm}\ln(D_n - mC_n), \tag{38}$$

with $C_n \equiv C_n(b\xi, m)$ and $D_n \equiv D_n(b\xi, m)$ representing the Jacobian elliptic cosine function and the Jacobian elliptic function of the third kind, respectively. Continuing by inserting

solution Eq. (38) with Eq. (37) into the reduction Eq. (26), we obtain

$$\Psi = -\frac{c_2^2}{2c_1}(b_0+b_1S_n),\ P_1 = -\frac{\sqrt{3}bb_1c_7C_nD_n-\omega}{\sqrt{2}(b_0+b_1S_n)},$$
$$Q_1 = -\frac{\sqrt{3}bb_1c_7C_nD_n+\omega}{\sqrt{2}(b_0+b_1S_n)},$$
$$\omega^2 = -2b^2b_1c_7^2S_n(b_0+b_1S_n)(2m^2S_n^2-m^2-1)$$
$$+3b_1^2b^2c_7^2C_n^2D_n^2+c_2^2c_7^2(b_0+b_1S_n)^4$$
$$+(2c_5-3c^2c_7^2)(b_0+b_1S_n)^2-2(b_0+b_1S_n). \quad (39)$$

After replacing F, Ψ, P_1, and Q_1 in Eqs. (21) and (22) with Eq. (39), one has

$$p = \frac{\sqrt{6g_{3x}}c_2c_7(b_0+b_1S_n)}{2\sqrt{\alpha_6}} \times \tanh\left[\frac{c_2}{2}(g_3+b_0\xi+\frac{b_1}{bm}\ln(D_n-mC_n))\right] - \frac{\sqrt{g_{3x}}\left(\sqrt{3}bb_1c_7C_nD_n-\omega\right)}{\sqrt{2\alpha_6}(b_0+b_1S_n)},$$

$$q = \frac{\sqrt{6g_{3x}}c_2c_7(b_0+b_1S_n)}{2\sqrt{\alpha_6}} \times \tanh\left[\frac{c_2}{2}(g_3+b_0\xi+\frac{b_1}{bm}\ln(D_n-mC_n))\right] - \frac{\sqrt{g_{3x}}\left(\sqrt{3}bb_1c_7C_nD_n+\omega\right)}{\sqrt{2\alpha_6}(b_0+b_1S_n)}$$
$$\phi = (cc_5+c_4)g_3-ct+c_9, \quad (40)$$

where c_9 is another integral constant. Until now, by substituting solution Eq. (40) back into solution Eq. (3), we obtain the kink and nonlocal periodic wave interaction solution for the original higher-order nonlinear Schrödinger equation (1). Of course, when we set the coefficients a_j $(j=0$–$4)$ in Eq. (30) to other values as those in Eqs. (32)–(34), we can construct many other different solutions for Eq. (1). For example, when the coefficients of Eq. (30) match those of Eqs. (32)–(34), according to the properties of elliptic functions, this equation has solutions in the form of the Weierstrass elliptic function. Then, Eq. (1) possesses an interaction solution between a kink and a Weierstrass elliptic wave.

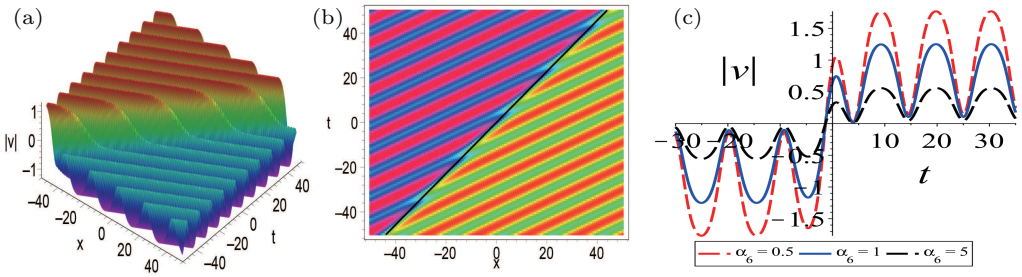


Fig. 1. Propagation of a kink embedded in a Jacobian elliptic periodic wave. Parameters are fixed as $m=0.5$, $a_3=1$, $g_3=x$, and $c=1$. (a) Three-dimensional visualization of the kink solution propagating on a Jacobian elliptic cnoidal wave background, with $\alpha_6=1$. (b) Corresponding density plot of (a), where the solid line shows the propagation trajectory of the kink. (c) Time evolution diagrams of the kink at $x=0$. As α_6 increases, the amplitudes of both the background periodic wave and the kink decrease ($\alpha_6=0.5$, 1, and 5 are plotted with red dashed, blue solid, and black dashed lines, respectively).

The following paragraphs analyze the evolution of kinks on various nonlocal periodic waves with different shapes. When all the coefficients associated with Eq. (1) are constants, the solution presented in Eq. (3) with Eq. (40) gives rise to an intriguing interaction between a kink and a Jacobian elliptic periodic wave. This particular solution is visually illustrated in Fig. 1(a), where the specific parameters are fixed as $m=0.5$, $a_3=c=\alpha_6=1$, and $g_3=x$. Under this parameter selection, the remaining coefficients of Eq. (1) take the values $\alpha_1=-1.5$, $\alpha_2=0$, $\alpha_3=-3$, and $\alpha_4=\alpha_5=1$. In Fig. 1(a), the kink is attached to a Jacobian elliptic periodic wave and propagates along the straight line $5.4x-4.7t=0$. During the propagation process, it collides with each wave peak of the periodic wave without changing its amplitude or speed, and no phase shift occurs, as can be observed from Fig. 1(b). If we alter the value of the coefficient α_6 associated with the time-delay correlation in the cubic term by either increasing or decreasing it, the amplitudes of the periodic wave and the kink exhibit an inverse relationship with α_6. Specifically, when α_6 increases, the amplitude of the periodic wave diminishes, and the amplitude of the kink correspondingly decreases. Conversely, when α_6 decreases, the amplitudes of both the periodic wave and the kink increase accordingly. This phenomenon is clearly depicted in Fig. 1(c). It is worth noting that any variation in the value of α_6 directly influences the value of α_5, which is associated with self-phase modulation. In essence, this implies that as the coefficient associated with the nonlinear term in Eq. (1) increases, the kink propagates along a periodic wave with a smaller amplitude. Conversely, when the coefficient associated with the nonlinear term decreases, the kink travels on a periodic wave with a larger amplitude. This dynamic interplay highlights the intricate connection between the coefficients and their impact on the wave structures within the equation.

It is unsurprising that when $m\to 0$ in solution Eq. (40), the kink-cnoidal wave interaction solution reduces to the kink

solution, whose interaction behavior is displayed in Fig. 2(a), with the wave parameters chosen as $m = 0.001$, $a_3 = c = \alpha_6 = 1$, and $g_3 = x$. However, when m approaches 1, the amplitudes of the kink and the periodic wave tend to become equal, as shown in Fig. 2(b).

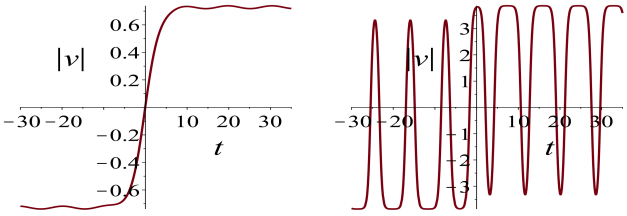


Fig. 2. Two-dimensional time evolution diagrams of the interaction between a kink and a Jacobian elliptic periodic wave at $x = 0$. The parameters are selected as $\alpha_6 = a_3 = c = 1$ and $g_3 = x$, with the modulus of the Jacobian elliptic function set to (a) $m = 0.001$ and (b) $m = 0.99$.

Furthermore, when the parameters of solution Eq. (40) are specifically selected as $m = 0.5$, $a_3 = c = 1$, $g_3 = 0.5x^2$, and $\alpha_6 = 2x$, such that the coefficients of Eq. (1) are linearly related to the variable x, i.e., $\alpha_1 = -1.5x$, $\alpha_2 = 0$, $\alpha_3 = -3x$, $\alpha_4 = x$, and $\alpha_5 = 2x$, the kink appears to propagate smoothly on a parabolic periodic wave. This particular propagation behavior can be clearly observed in Fig. 3(a). On the other hand, when the parameters g_3 and α_6 are chosen in the form of trigonometric functions, specifically $g_3 = 15\sin(x)$ and $\alpha_6 = \cos(x)$, the kink is embedded within a periodic wave modulated by these trigonometric functions. This interesting phenomenon, where the kink interacts with the trigonometric-function-modulated periodic wave, can be easily observed in Fig. 3(b). Besides this, there is another situation. When the periodic wave is modulated not by trigonometric functions but by hyperbolic functions, the solution Eq. (3) combined with Eq. (40) manifests as a kink that propagates along a periodic wave modulated by hyperbolic functions, as shown in Fig. 3(c). Here, we set $g_3 = \tanh(x)$ and $\alpha_6 = 0.5\mathrm{sech}(x)$, which results in the coefficients of Eq. (1) being $\alpha_1 = -1.5 + 1.5\tanh(x)$, $\alpha_2 = 0.5\tanh(x)$, $\alpha_3 = -3 + 3\tanh^2(x)$, $\alpha_4 = 1 - \tanh^2(x)$, and $\alpha_5 = 0.5\mathrm{sech}(x)$. By selecting more diverse types of parameters, solution Eq. (3) with Eq. (40) can exhibit a greater variety of interaction behaviors between kinks and nonlocal periodic waves.

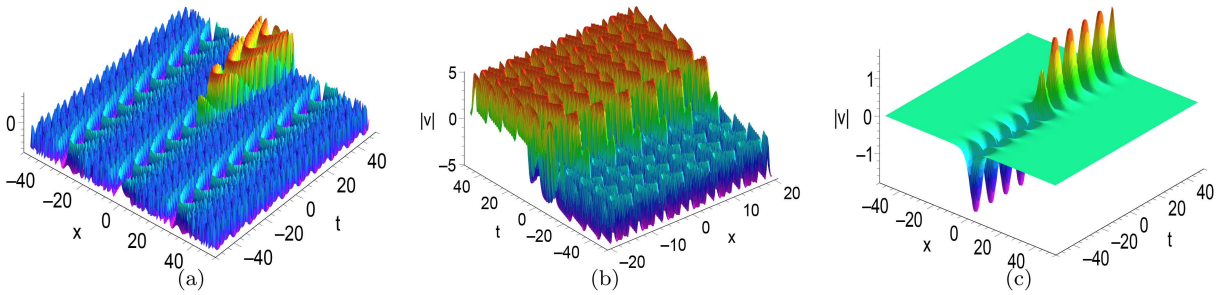


Fig. 3. Three-dimensional evolutions of kink solutions embedded in diverse nonlocal wave backgrounds, obtained from solution (3) in conjunction with (40). (a) A kink propagates on a parabolic periodic wave, where $g_3 = 0.5x^2$ and $\alpha_6 = 2x$. (b) A kink is embedded in a periodic wave modulated by trigonometric functions, with $g_3 = 15\sin(x)$ and $\alpha_6 = \cos(x)$. (c) A kink is embedded in a periodic wave modulated by a hyperbolic function, with $g_3 = \tanh(x)$ and $\alpha_6 = 0.5\mathrm{sech}(x)$. The other parameters are set as $m = 0.5$ and $a_3 = c = 1$.

4. Discussion and conclusion

Owing to their advantages in long-distance transmission, high bit rates, and rapid switching, subpicosecond or femtosecond optical solitons have emerged as an increasingly active research topic in recent years. As the duration of optical solitons decreases, various higher-order effects and nonlinear correction terms become increasingly significant. In this paper, the propagation of optical solitons with higher-order effects and external potentials in inhomogeneous optical fibers is investigated. We introduce the variable-coefficient higher-order nonlinear Schrödinger equation (1). To investigate the propagation of solitons embedded in nonlocal waves, we first identify the nonlocal symmetry of the higher-order nonlinear Schrödinger equation within the framework of the Painlevé truncation expansion method. By localizing the nonlocal symmetry into a local Lie point symmetry, the similarity reduction solutions associated with the nonlocal symmetry are obtained. Due to the introduction of the nonlocal symmetry, the similarity reduction solutions incorporate an additional kink. That is to say, when the part related to the nonlocal symmetry is removed, i.e., $c_1 = 0$, the similarity reduction solution Eq. (3) with Eq. (40) reduces to the commonly known symmetry reduction. Based on the similarity reduction solutions, we subsequently analyze the propagation of kinks on different nonlocal periodic waves by selecting distinct wave parameters related to higher-order dispersion and nonlinear correction terms. For instance, when the free variable coefficient α_6, representing the time-delay correlation of the cubic term, is set as a constant, the group-invariant solution Eq. (3) along with Eq. (40) characterizes the interaction between a kink and a Jacobian elliptic periodic wave. If the nonlinear effect is enhanced, the amplitudes of both the periodic wave and the kink decrease correspondingly, while the shapes and propagation speeds of the periodic wave and the kink remain unchanged. When α_6

is chosen as a function proportional to x, the similarity reduction solution describes the propagation of a kink embedded in a parabolic periodic wave, as depicted in Fig. 3(a). Additionally, when the coefficient of the nonlinear correction term is taken to be a trigonometric or hyperbolic function, the similarity reduction solution describes the interaction between kinks and periodic waves modulated by trigonometric or hyperbolic functions, respectively (see Figs. 3(b) and 3(c)). Therefore, according to our research findings, when different forms of higher-order dispersion and nonlinear effects are specified, the optical solitons propagate on different forms of periodic waves, and the shape of the background periodic waves is controllable.

Furthermore, it is worthy of further investigation whether the symmetry reduction method related to nonlocal symmetry employed in this paper can be applied to more higher-order nonlinear Schrödinger equations and utilized to uncover more substantial integrable properties, thus revealing more realistic physical phenomena.

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