SPECIAL TOPIC — A 40th Anniversary Celebration: Physics Research and Innovation at Ningbo University

Higher-order *b*-family systems: Abundant pseudo-peakons, peakons, and a conjecture

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We investigate the fifth-, seventh-, and ninth-order *b*-family systems and construct their pseudo-peakon and peakon solutions. The fifth-order system admits only solutions (both pseudo-peakons and peakons) that are independent of the system parameter b, whereas the seventh- and ninth-order systems support pseudo-peakons that are independent of b and peakons that may be either independent of or explicitly dependent on b. Pseudo-peakon solutions contain arbitrary constants whose number increases with system order, which yields a diverse range of profiles (M-shaped, W-shaped, double-peakon, and soliton-like) and permits parameter-driven transformations from lower- to higher-order pseudo-peakons. Conversely, under appropriate constraints, it is possible to recover pseudo-peakon solutions from peakons. The absence of b-dependent solutions in the fifth-order case highlights a qualitative distinction between lower- and higher-order members of the family and emphasizes the role of higher-order terms in enabling parameter-dependent coherent structures. Motivated by these results, we conjecture that a general $(2n+1)$-order *b*-family system admits a third-order pseudo-peakon of the proposed form; this conjecture has been verified for the fifth-, seventh-, ninth-, and eleventh-order cases. Together, these findings clarify how the parameter b and increased differential order shape the existence and diversity of coherent solutions in the *b*-family hierarchy.

Keywords: higher-order systems, *b*-independent pseudo-peakons, *b*-independent/dependent peakons, transition from lower- to higher order

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1. Introduction

The *b*-family system

$$m_t + um_x + bmu_x = 0,\ m = u - u_{xx}, \tag{1}$$

with real parameter b, was originally introduced in Refs. [1–3] as a model for shallow water wave dynamics.[4] Two values of b are of particular importance because they yield integrable equations. For $b = 2$, the system reduces to the Camassa–Holm (CH) equation

$$m_t + um_x + 2mu_x = 0,\ \ m = u - u_{xx}, \tag{2}$$

which was first derived via asymptotic expansions of the Hamiltonian for inviscid, incompressible Euler equations in the shallow water regime.[5] The CH equation is completely integrable, possessing a bi-Hamiltonian structure, a Lax pair, an infinite number of conservation laws in involution,[6] and peaked solitary wave solutions (peakons) that model shallow water waves with discontinuous derivatives at their crests.[7] The CH equation has also been extended to a two-component version,[8,9] a nonlocal version,[10,11] a $(2+1)$-dimensional version,[12] and a rotational-type version.[13] In addition, a super CH equation has been established in Ref. [14]. For $b = 3$, the system yields the Degasperis–Procesi (DP) equation

$$m_t + um_x + 3mu_x = 0,\ \ m = u - u_{xx}, \tag{3}$$

which also arises as a shallow water model and was shown to be asymptotically integrable to third order.[15] Its complete integrability was later established via a Lax pair and a bi-Hamiltonian structure associated with a third-order spectral problem,[16] and multi-peakon solutions and N-soliton solutions have since been constructed.[17] New results have been found about the DP equation; for example, the higher-dimensional DP equation is established in Ref. [18], and the 3-component DP equation is related to the first negative flow in a modified Yajima-Oikawa hierarchy.[18]

It is now well known that every member of the *b*-family admits *b*-independent peakon solutions of the form

$$u = c\,\mathrm{e}^{|x-ct|}, \tag{4}$$

where c is an arbitrary constant. The solution (Eq. (4)) is referred to as a single peakon because its spatial derivatives are not classically defined along the characteristic line $x = ct$. The peakon travels with a speed equal to its peak amplitude and is nonanalytic, exhibiting a discontinuity in the first derivative at its crest.

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More recently, a special b-family system was proposed in Ref. [19]

$$m_t + um_x + bmu_x = 0,\ m = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u,$$
$$\alpha \neq \beta,\ \alpha\beta \neq 0, \tag{5}$$

where α and β are real constants. Since the definition of m involves fourth-order spatial differentiation, the resulting system for u contains derivatives up to fifth order; accordingly, the model is classified as a fifth-order partial differential system. Besides admitting the peakon solution, the b-family model (5) was found to possess a pseudo-peakon

$$u = \frac{c}{\alpha-\beta}\left(\alpha e^{-\frac{|x-ct|}{\alpha}} - \beta e^{-\frac{|x-ct|}{\beta}}\right), \tag{6}$$

which is continuous together with its first and second derivatives at $x = ct$ and therefore is referred to as a third-order pseudo-peakon.

The fact that the solutions of system (Eq. (5)) are independent of the parameter b leads to several natural and meaningful questions concerning the structure and generality of its solutions: (I) Are there peakon or pseudo-peakon solutions that depend explicitly on b? (II) Beyond the currently known solutions, does the system admit other, possibly higher-order pseudo-peakon solutions? (III) In the degenerate limit when $\alpha = \beta$, what is the structure and regularity of the traveling-wave solutions? (IV) Can these phenomena be extended systematically to higher-order generalizations of the b-family while preserving peakon or pseudo-peakon solutions?

In fact, higher-order systems are fundamental to accurately understanding and modeling the complexity of the physical world, engineering applications, and advanced phenomena that cannot be captured by simple, first-order models. Such systems exhibit a range of distinctive and essential properties that lower-order descriptions often fail to capture. For example, higher-order structures like hypergraphs and simplicial complexes have been advocated as effective frameworks for representing multi-body interactions, which are not merely aggregates of pairwise links but reflect genuine group dynamics. This methodological shift is necessary to explain new, emergent physical phenomena.[20] Similarly, in photonics, a higher-order quantum spin Hall effect has been demonstrated in two-dimensional photonic crystals, where nontrivial higher-order topology and pseudospin-pseudospin coupling lead to directional localization of photons at corners — a behavior that extends the conventional quantum spin Hall effect into a higher-order regime.[21] Furthermore, in nonlinear optics and fluid dynamics, models such as the higher-order Chen-Lee-Liu equation describe the propagation of optical pulses in media with high-order dispersion and quintic nonlinearity, offering insights into nonlinear wave interactions and complex oceanic phenomena across both shallow and deep-water contexts.[22] Collectively, these examples underscore how higher-order systems provide indispensable tools for advancing scientific knowledge, enabling more accurate representations of reality where complexity, collective behavior, and multi-scale interactions prevail.

Building upon these considerations and the practical relevance of higher-order systems in modeling complex phenomena, we propose a novel generalized $(2n+1)$th-order b-family system

$$m_t + um_x + bmu_x = 0, \qquad m = \prod_{i=1}^{n}(1-\alpha_i^2\partial_x^2)u, \tag{7}$$

with $\alpha_i > 0$ $(i = 1,2,\ldots,n)$ and carry out a systematic investigation of peakon and pseudo-peakon solutions. Our main findings are the following. The generalized system admits b-independent pseudo-peakon solutions of various orders as well as both b-independent and b-dependent peakon branches; under specific parameter choices, a b-dependent peakon may degenerate into pseudo-peakons. Based on the observed structure, we conjecture that the generalized b-family Eq. (7) admits a third-order, b-independent pseudo-peakon of the form

$$\begin{aligned} u = {} & \frac{c}{\alpha_1-\alpha_2}\left(\alpha_1 e^{-\frac{|x-ct|}{\alpha_1}} - \alpha_2 e^{-\frac{|x-ct|}{\alpha_2}}\right) \\ & + \sum_{i=3}^{n} a_i\left[\alpha_1(\alpha_2-\alpha_i)e^{-\frac{|x-ct|}{\alpha_1}} + \alpha_2(\alpha_i-\alpha_1)e^{-\frac{|x-ct|}{\alpha_2}}\right. \\ & \left. + \alpha_i(\alpha_1-\alpha_2)e^{-\frac{|x-ct|}{\alpha_i}}\right], \end{aligned} \tag{8}$$

with a_i $(i = 3,4,\ldots,n)$ being free constants. Here we adopt the convention that an n-th-order pseudo-peakon is a traveling solution whose spatial derivatives are continuous up to order $n-1$, while the n-th derivative has a jump discontinuity.

These results clarify how parameter choices (including b and α_i) control waveform morphology and regularity in higher-order b-family models. They also provide a systematic framework for constructing explicit pseudo-peakon and peakon solutions and for studying limiting (degenerate) cases.

The paper is structured as follows. Section 2 investigates a fifth-order b-family system, establishing the existence of both pseudo-peakon and peakon solutions that are independent of the parameter b. Section 3 extends the analysis to a general seventh-order system. It is shown that this system supports not only b-independent third- and fifth-order pseudo-peakons but also both b-dependent and b-independent peakon solutions. Section 4 considers a ninth-order system, demonstrating the existence of b-independent pseudo-peakons of third, fifth, and seventh order, alongside both b-dependent and b-independent peakons. Section 5 examines a special degenerate case of the ninth-order system, revealing further constraints and solution behaviors under specific parametric conditions. Section 6 is a short summary.

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2. A general fifth order b-family system

The general fifth-order b-family system can be represented as follows:

$$m_t + um_x + bmu_x = 0, \quad m = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u, \quad (9)$$

where b, $\alpha > 0$ and $\beta > 0$ are real constants. To derive soliton solutions for the fifth-order b-family system described in Eq. (9), we investigate the possibility of a single weak peakon and/or pseudo-peakon solution characterized by the expression

$$u = c\left(a_1 \mathrm{e}^{-\frac{|X|}{\alpha}} + a_2 \mathrm{e}^{-\frac{|X|}{\beta}}\right), \quad X = x - ct. \quad (10)$$

By employing the ansatz Eq. (10), we obtain the following results:

$$\begin{aligned}
u_x &= \frac{c}{\alpha\beta}\mathrm{sgn}(X)\left(a_1\beta\,\mathrm{e}^{-\frac{|X|}{\alpha}} + a_2\alpha\,\mathrm{e}^{-\frac{|X|}{\beta}}\right),\\
m &= \frac{8c\delta(X)}{\alpha\beta}\left\{a_1\beta\left[\frac{3}{2}\alpha\beta^2\delta(X) + \left(\frac{\beta^2}{4}-\frac{1}{4}\right)\alpha^2 - \frac{5\beta^2}{4} - \alpha\beta^2\mathrm{sgn}(X)\right]\mathrm{e}^{-\frac{|X|}{\alpha}}\right.\\
&\quad \left. + a_2\alpha\left[\frac{3}{2}\beta\alpha^2\delta(X) + \left(\frac{\beta^2}{4}+\frac{5}{4}\right)\alpha^2 - \beta\alpha^2\mathrm{sgn}(X)\right]\mathrm{e}^{-\frac{|X|}{\beta}}\right\},\\
m_t &= \frac{8ca_1}{\alpha^2}\left\{\left[\frac{3}{2}c\alpha\beta^2\mathrm{sgn}(X) - 2\alpha^2\beta^2\right]\delta(X)^2 + \left[3c\alpha^2\beta^2\delta'(X) + \frac{c}{4}\left(\alpha^2\beta^2 - \alpha^2 + 5\beta^2\right)\mathrm{sgn}(X) - c\alpha\beta^2\right]\delta(X)\right.\\
&\quad \left. + \left[\frac{c\alpha}{4}\left(\alpha^2\beta^2 - \alpha^2 + 5\beta^2\right) - c\alpha^2\beta^2\mathrm{sgn}(X)\right]\delta'(X)\right\}\mathrm{e}^{-\frac{|X|}{\alpha}} + \frac{8ca_2}{\beta^2}\left\{\left[\frac{3}{2}c\alpha^2\beta\mathrm{sgn}(\xi) - 2\alpha^2\beta^2\right]\delta(X)^2\right.\\
&\quad + \left[3c\alpha^2\beta^2\delta'(X) + \frac{c}{4}\left(\alpha^2\beta^2 + 5\alpha^2 - \beta^2\right)\mathrm{sgn}(X) - c\alpha^2\beta\right]\delta(X)\\
&\quad \left. + \left[\frac{c\alpha}{4}\left(\alpha^2\beta^2 + 5\alpha^2 + \beta^2\right) - c\alpha^2\beta^2\mathrm{sgn}(X)\right]\delta'(X)\right\}\mathrm{e}^{-\frac{|X|}{\beta}}.
\end{aligned} \quad (11)$$

Here, $\mathrm{sgn}(X)$ denotes the signum function of X, $\delta(X)$ corresponds to the Dirac δ function of X, and $\delta'(X)$ refers to the first derivative of $\delta(X)$ with respect to X.

Substituting the proposed solution ansatz Eq. (10) into the system expressed in Eq. (9) and imposing the condition that the resultant be a zero distribution yields two distinct potential solutions

$$a_1 = \frac{\alpha}{\alpha-\beta}, \quad a_2 = -\frac{\beta}{\alpha-\beta}, \quad (12)$$

$$a_1 = \frac{\alpha^2}{\alpha^2-\beta^2}, \quad a_2 = -\frac{\beta^2}{\alpha^2-\beta^2}. \quad (13)$$

Case 1. A b-independent third-order pseudo-peakon

The first solution Eq. (12) leads to a third-order pseudo-peakon solution for Eq. (9) given by

$$u = \frac{c}{\alpha-\beta}\left(\alpha\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}} - \beta\,\mathrm{e}^{-\frac{|x-ct|}{\beta}}\right). \quad (14)$$

It is straightforward to verify that the solution in Eq. (14) is continuous in its first and second spatial derivatives, while its third spatial derivative exhibits a discontinuity. Here the continuity means $u_x(X=0^+) = u_x(X=0^-)$ and $u_{xx}(X=0^+) = u_{xx}(X=0^-)$, while the discontinuity refers to $u_{xxx}(X=0^+) \neq u_{xxx}(X=0^-)$. This result has been previously reported in Ref. [19].

Figure 1 illustrates the pseudo-peakon solution represented in Eq. (14), with parameters selected as $c=1$, $\alpha=0.8$, and $\beta=1$ at $t=0$ in panel (a), alongside the first-order derivative of u with respect to x in panel (b) and the second-order derivative of u with respect to x in panel (c).

In the special case where $\beta=\alpha$, the pseudo-peakon solution represented in Eq. (14) for the fifth-order b-family system transforms into the following expression

$$u = \frac{c}{\alpha}(\alpha + |x-ct|)\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}}, \quad (15)$$

which retains a discontinuity at the third order of differentiation.

Case 2. A b-independent peakon

The solution Eq. (13) results in a single peakon solution of the fifth-order b-family system Eq. (9) written as

$$u = \frac{c}{\alpha^2-\beta^2}\left(\alpha^2\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}} - \beta^2\,\mathrm{e}^{-\frac{|x-ct|}{\beta}}\right), \quad (16)$$

with a discontinuity in its first derivative. The single peakon solution has not been shown in Ref. [19].

Conversely, the single peakon solution corresponding to $\beta=\alpha$ is given by

$$u = \frac{c}{2\alpha}(2\alpha + |x-ct|)\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}}. \quad (17)$$

Notably, all solutions — both the pseudo-peakon solutions expressed in Eqs. (14) and (15), as well as the single peakon solutions described in Eqs. (16) and (17) — exhibit independence from the parameter b in both scenarios.

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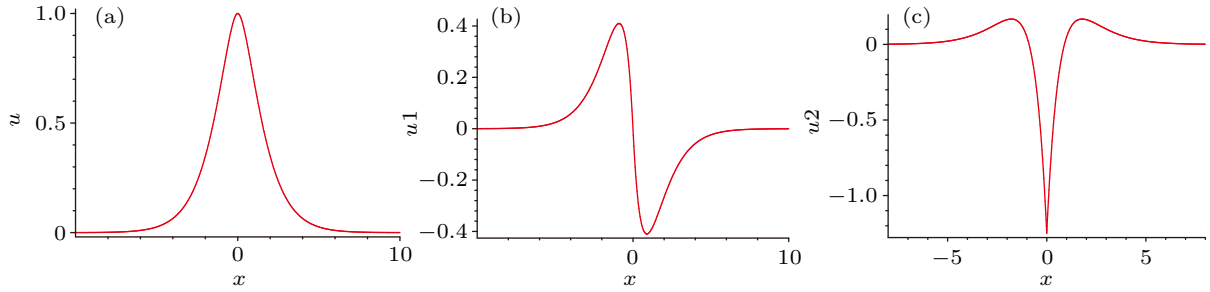


Fig. 1. (a) The third-order pseudo-peakon solution Eq. (14) with $c=1$, $\alpha=0.8$, and $\beta=1$ at $t=0$. (b) The first-order derivative of the third-order pseudo-peakon solution Eq. (14). (c) The second-order derivative of the third-order pseudo-peakon solution Eq. (14).

3. A general seventh-order *b*-family system

The general seventh-order b-family system is expressed as

$$\begin{aligned} &m_t+um_x+bmu_x=0,\\ &m=(1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)(1-\delta^2\partial_x^2)u, \end{aligned} \tag{18}$$

where b, $\alpha>0$, $\beta>0$, and $\delta>0$ are real constants. For this system described by Eq. (18), a single weak solution in the form of a peakon or pseudo-peakon can be derived

$$u=c\left(a_1\mathrm{e}^{-\frac{|X|}{\alpha}}+a_2\mathrm{e}^{-\frac{|X|}{\beta}}+a_3\mathrm{e}^{-\frac{|X|}{\delta}}\right),\quad X=x-ct. \tag{19}$$

Substituting the ansatz Eq. (19) into Eq. (18) and imposing the condition that the result vanishes in the sense of distributions yields a set of constraint equations for the coefficients a_1, a_2, and a_3

$$\begin{aligned} &(a_1+a_2+a_3-1)(a_1\beta^3\delta^3+a_2\alpha^3\delta^3+a_3\alpha^3\beta^3)=0,\\ &30bc_6(a_1\beta\delta+a_2\alpha\delta+a_3\alpha\beta)(a_1\beta^3\delta^3+a_2\alpha^3\delta^3+a_3\alpha^3\beta^3)\\ &+7c_6(a_1+a_2+a_3-1)(4a_1\alpha^2\beta^4\delta^4+4a_2\alpha^4\beta^2\delta^4\\ &+4a_3\alpha^4\beta^4\delta^2+15a_1\beta^4\delta^4+15a_2\alpha^4\delta^4+15a_3\alpha^4\beta^4)\\ &+5c_4\alpha^2\beta^2\delta^2(a_1+a_2+a_3-1)\\ &\times(a_1\beta^2\delta^2+a_2\alpha^2\delta^2+a_3\alpha^2\beta^2)=0,\\ &2b(a_1\beta\delta+a_2\alpha\delta+a_3\alpha\beta)[3c_6(3a_1\alpha^2\beta^4\delta^4+3a_2\alpha^4\beta^2\delta^4\\ &+3a_3\alpha^4\beta^4\delta^2+10a_1\beta^4\delta^4+10a_2\alpha^4\delta^4+10a_3\alpha^4\beta^4)\\ &+2c_4\alpha^2\beta^2\delta^2(a_1\beta^2\delta^2+a_2\alpha^2\delta^2+a_3\alpha^2\beta^2)]\\ &+(a_1+a_2+a_3-1)[c_6(a_1\alpha^4\beta^5\delta^5+a_2\alpha^5\beta^4\delta^5\\ &+a_3\alpha^5\beta^5\delta^4+21a_1\alpha^2\beta^5\delta^5+21a_2\alpha^5\beta^2\delta^5\\ &+21a_3\alpha^5\beta^5\delta^2+35a_1\beta^5\delta^5+35a_2\alpha^5\delta^5+35a_3\alpha^5\beta^5)\\ &+c_4\alpha^2\beta^2\delta^2(a_1\alpha^2\beta^3\delta^3+a_2\alpha^3\beta^2\delta^3+a_3\alpha^3\beta^3\delta^2\\ &+10a_1\beta^3\delta^3+10a_2\alpha^3\delta^3+10a_3\alpha^3\beta^3)\\ &+c_2\alpha^4\beta^4\delta^4(a_1\beta\delta+a_2\alpha\delta+a_3\alpha\beta)]=0, \end{aligned} \tag{20}$$

with $c_6=-\alpha^2\beta^2\delta^2$ and $c_4=\alpha^2\beta^2+\alpha^2\delta^2+\beta^2\delta^2$. Solving these constraints leads to four distinct classes of peakon and pseudo-peakon solutions, as detailed below.

Case 1. A *b*-independent third-order pseudo-peakon

In this case, the solution to the constraint system Eq. (20) is

$$a_1=-\frac{\alpha(a_0\beta-a_0\delta+\delta)}{\delta(\alpha-\beta)},\quad a_2=-\frac{\beta(a_0\alpha-a_0\delta+\delta)}{\delta(\alpha-\beta)}, \tag{21}$$

where the arbitrary constant a_3 has been relabeled as a_0. The seventh-order system Eq. (18) consequently admits a b-independent pseudo-peakon solution

$$\begin{aligned} u=&\frac{c}{\alpha-\beta}\left(\alpha\mathrm{e}^{-\frac{|x-ct|}{\alpha}}-\beta\mathrm{e}^{-\frac{|x-ct|}{\beta}}\right)\\ &+\frac{ca_0}{\beta(\alpha-\delta)}\left[\alpha(\beta-\delta)\mathrm{e}^{-\frac{|x-ct|}{\alpha}}+\beta(\delta-\alpha)\mathrm{e}^{-\frac{|x-ct|}{\beta}}\right.\\ &\left.+\delta(\alpha-\beta)\mathrm{e}^{-\frac{|x-ct|}{\delta}}\right], \end{aligned} \tag{22}$$

where c and a_0 are arbitrary constants. Direct verification confirms that u_x and u_{xx} are continuous at $x=ct$, while u_{xxx} is discontinuous, thereby establishing Eq. (22) as a third-order pseudo-peakon.

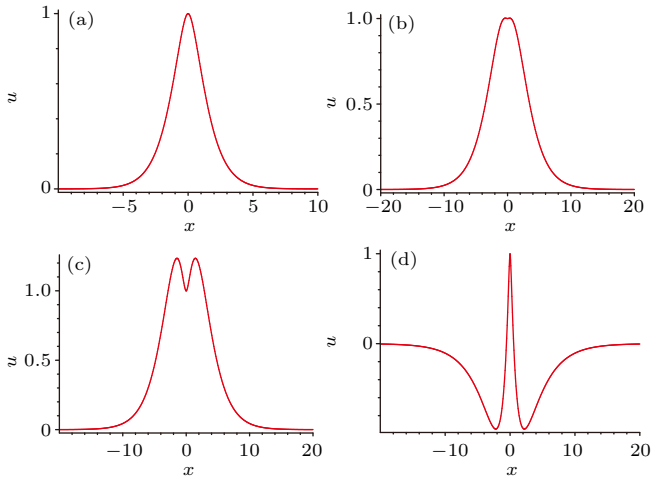


Fig. 2. Different structures of the third-order pseudo-peakon solution Eq. (22) at $t=0$ for different parameter selections: $\alpha=0.8$, $\beta=0.9$, $\delta=1$, and $a_0=0.8$ in panel (a); $\alpha=0.8$, $\beta=2$, $\delta=1$, and $a_0=-2$ in panel (b); $\alpha=0.8$, $\beta=2$, $\delta=1$, and $a_0=-5$ in panel (c); and $\alpha=0.1$, $\beta=3$, $\delta=1$, and $a_0=4$ in panel (d), respectively.

The third-order pseudo-peakon solution given by Eq. (22) admits a variety of structural profiles owing to the free parameter a_0. Figure 2 illustrates the third-order pseudo-peakon solution given by Eq. (22) at $t=0$ with $c=1$ for different selections of the parameters α, β, δ, and a_0. Depending on

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the choice of a_0, the resulting waveforms may exhibit soliton-like, plateau soliton-like, M-shaped, or W-shaped morphologies. Figure 3 further presents the first and second derivatives with respect to x of the solution under the same temporal condition, $t=0$, with $c=1$, $\alpha=0.8$, $\beta=0.9$, $\delta=1$, and $a_0=0.8$.

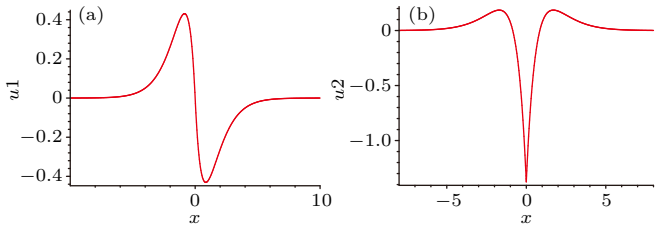


Fig. 3. The first-order derivative u_x in panel (a) and the second-order derivative u_{xx} in panel (b) of the pseudo-peakon solution Eq. (22) at $t=0$ with $c=1$, $\alpha=1.2$, $\beta=0.9$, $\delta=0.8$, and $a_0=1$.

If the parameters satisfy $\delta=\beta=\alpha$, the third-order pseudo-peakon solution Eq. (22) reduces to the simplified expression

$$u=\frac{c}{\alpha}(\alpha+|x-ct|)\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}}, \tag{23}$$

with c being an arbitrary constant.

Case 2. A b-independent fifth-order pseudo-peakon

In this case, the solution to Eq. (20) is

$$a_1=-\frac{\alpha^3(\beta+\delta)}{(\alpha-\beta)(\delta-\alpha)(\alpha\beta+\alpha\delta+\beta\delta)},$$

$$a_2=-\frac{\beta^3(\alpha+\delta)}{(\alpha-\beta)(\beta-\delta)(\alpha\beta+\alpha\delta+\beta\delta)},$$

$$a_3=-\frac{\delta^3(\alpha+\beta)}{(\beta-\delta)(\delta-\alpha)(\alpha\beta+\alpha\delta+\beta\delta)}, \tag{24}$$

which yields a fifth-order b-independent pseudo-peakon solution

$$\begin{aligned}u=\frac{c}{\alpha\beta+\alpha\delta+\beta\delta}\Bigg[&\frac{\alpha^3(\beta+\delta)}{(\alpha-\beta)(\alpha-\delta)}\mathrm{e}^{-\frac{|x-ct|}{\alpha}}\\&+\frac{\beta^3(\alpha+\delta)}{(\beta-\alpha)(\beta-\delta)}\mathrm{e}^{-\frac{|x-ct|}{\beta}}\\&+\frac{\delta^3(\alpha+\beta)}{(\delta-\beta)(\delta-\alpha)}\mathrm{e}^{-\frac{|x-ct|}{\delta}}\Bigg],\end{aligned} \tag{25}$$

with c being an arbitrary constant. This solution Eq. (25) is continuous up to u_{xxxx}, with a discontinuity appearing in the fifth derivative u_{xxxxx} at $x=ct$.

Solution Eq. (25) can be viewed as a specialization of the third-order pseudo-peakon Eq. (22). Specifically, when the free parameter a_0 is fixed as

$$a_0=-\frac{\delta^2(\alpha+\beta)}{(\alpha-\beta)(\beta-\delta)(\delta-\alpha)(\alpha\beta+\alpha\delta+\beta\delta)}, \tag{26}$$

the third-order pseudo-peakon Eq. (22) elevates to the fifth-order pseudo-peakon Eq. (25).

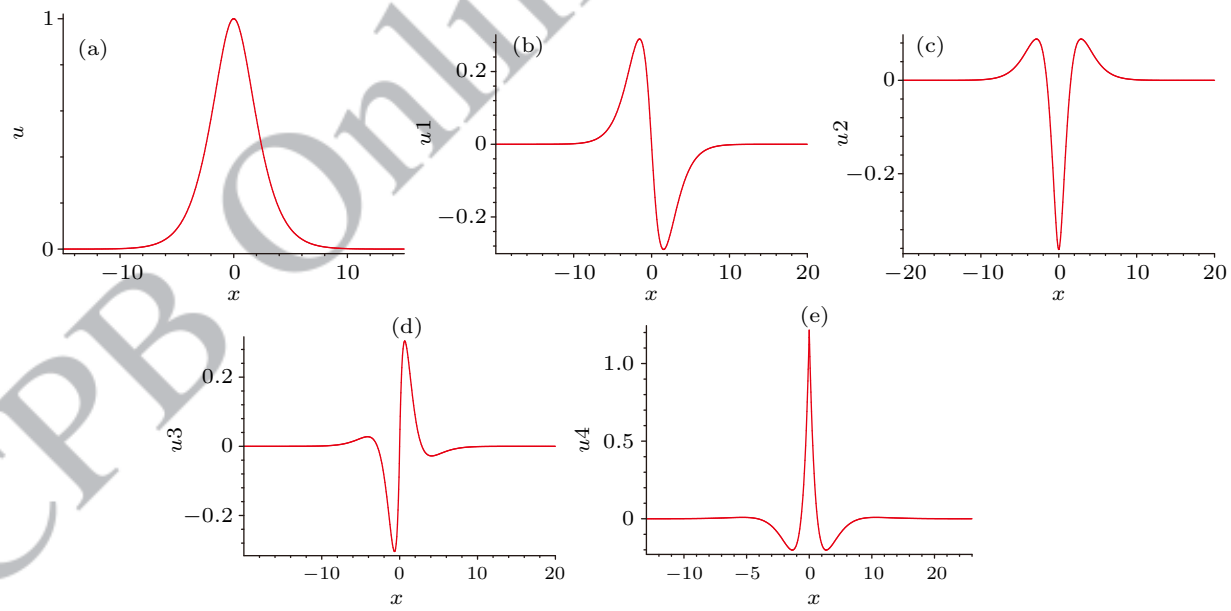


Fig. 4. Structures of the fifth-order pseudo-peakon solution Eq. (25) and its first, second, third, and fourth derivatives, u_x, u_{xx}, u_{xxx}, and u_{xxxx}, at $t=0$ with the parameter values $\alpha=0.8$, $\beta=0.9$, $\delta=1.2$, and $c=1$.

Figure 4 displays the pseudo-peakon solution given by Eq. (25) in panel (a), along with its successive derivatives u_x in panel (b), u_{xx} in panel (c), u_{xxx} in panel (d), and u_{xxxx} in panel (e) at $t=0$, under the parameter values $\alpha=0.8$, $\beta=0.9$, $\delta=1.2$, and $c=1$. The plots confirm that both the solution and its derivatives up to the fourth order remain continuous across the domain shown.

Case 3. A b-independent peakon

For this case, solving Eq. (20) gives

$$a_1=-\frac{A\alpha^3}{\Delta(\alpha-\beta)(\delta-\alpha)},\quad a_2=-\frac{B\beta^3}{\Delta(\alpha-\beta)(\beta-\delta)},$$

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$$a_3 = -\frac{C\delta^3}{\Delta(\beta-\delta)(\delta-\alpha)}, \tag{27}$$

where

$$\begin{aligned}
\Delta &= 3\alpha^3\beta^2\delta^2 + 3\alpha^2\beta^3\delta^2 + 3\alpha^2\beta^2\delta^3 - 10\alpha^3\beta\delta \\
&\quad - 10\alpha\beta^3\delta - 10\alpha\beta\delta^3 - 2\alpha^3\beta^2 - 2\alpha^3\delta^2 - 2\alpha^2\beta^3 \\
&\quad - 2\alpha^2\delta^3 + 2\beta^3\delta^2 + 2\beta^2\delta^3 + 12\alpha^2\beta^2\delta + 12\alpha^2\beta\delta^2 \\
&\quad + 12\alpha\beta^2\delta^2, \\
A &= 3\alpha^2\beta^2\delta^2 - 2\alpha^2\beta^2 - 10\alpha^2\beta\delta - 2\alpha^2\delta^2 - 2\beta^2\delta^2, \\
B &= 3\alpha^2\beta^2\delta^2 - 2\alpha^2\beta^2 - 10\alpha\beta^2\delta - 2\alpha^2\delta^2 - 2\beta^2\delta^2, \\
C &= 3\alpha^2\beta^2\delta^2 - 2\alpha^2\beta^2 - 10\alpha\beta\delta^2 - 2\alpha^2\delta^2 - 2\beta^2\delta^2,
\end{aligned} \tag{28}$$

leading to a b-independent peakon solution for system Eq. (18)

$$\begin{aligned}
u = \frac{c}{(\alpha-\beta)(\beta-\delta)(\delta-\alpha)\Delta}\Big[&(\beta-\delta)A\alpha^3 \mathrm{e}^{-\frac{|x-ct|}{\alpha}} \\
&+(\delta-\alpha)B\beta^3 \mathrm{e}^{-\frac{|x-ct|}{\beta}} + (\alpha-\beta)C\delta^3 \mathrm{e}^{-\frac{|x-ct|}{\delta}}\Big].
\end{aligned} \tag{29}$$

The solution Eq. (29) is characterized by a discontinuity in its first derivative u_x at $x = ct$, confirming its peakon nature.

Interestingly, under the parameter restriction $\delta = \beta = \alpha$, the peakon solution Eq. (29) for the seventh-order b-family system simplifies to the following form

$$\begin{aligned}
u = c\left[1 + \frac{2(\alpha^2-7)}{\alpha(3\alpha^2-26)}|x-ct| + \frac{3\alpha^2-26}{6\alpha^2(3\alpha^2-26)}|x-ct|^2\right] \\
\times \mathrm{e}^{-\frac{|x-ct|}{\alpha}}.
\end{aligned} \tag{30}$$

Case 4. A b-dependent peakon

The final case yields a constraint solution of the form

$$\begin{aligned}
a_1 &= -\frac{\omega A_1\alpha^3}{\Delta_1(\alpha-\beta)(\delta-\alpha)}, \; a_2 = -\frac{\omega B_1\beta^3}{\Delta_1(\alpha-\beta)(\beta-\delta)}, \\
a_3 &= -\frac{\omega C_1\delta^3}{\Delta_1(\beta-\delta)(\delta-\alpha)},
\end{aligned} \tag{31}$$

with the parameters being

$$\begin{aligned}
A_1 &= 28\alpha^2\beta^2\delta^2 - 5\alpha^2\beta^2 - 105\alpha^2\beta\delta - 5\alpha^2\delta^2 - 5\beta^2\delta^2, \\
B_1 &= 28\alpha^2\beta^2\delta^2 - 5\alpha^2\beta^2 - 5\alpha^2\delta^2 - 105\alpha\beta^2\delta - 5\beta^2\delta^2, \\
C_1 &= 28\alpha^2\beta^2\delta^2 - 5\alpha^2\beta^2 - 5\alpha^2\delta^2 - 105\alpha\beta\delta^2 - 5\beta^2\delta^2, \\
\omega &= 28\alpha^4\beta^4\delta^4 - 33\alpha^4\beta^4\delta^2 - 105\alpha^4\beta^3\delta^3 - 33\alpha^4\beta^2\delta^4 \\
&\quad - 105\alpha^3\beta^4\delta^3 - 105\alpha^3\beta^3\delta^4 - 33\alpha^2\beta^4\delta^4 + 5\alpha^4\beta^4 \\
&\quad + 105\alpha^4\beta^3\delta + 38\alpha^4\beta^2\delta^2 + 105\alpha^4\beta\delta^3 + 5\alpha^4\delta^4 \\
&\quad + 105\alpha^3\beta^4\delta + 1085\alpha^3\beta^3\delta^2 + 1085\alpha^3\beta^2\delta^3 \\
&\quad + 105\alpha^3\beta\delta^4 + 38\alpha^2\beta^4\delta^2 + 1085\alpha^2\beta^3\delta^3 \\
&\quad + 38\alpha^2\beta^2\delta^4 + 105\alpha\beta^4\delta^3 + 105\alpha\beta^3\delta^4 + 5\beta^4\delta^4 \\
&\quad - 5\alpha^4\beta^2 - 105\alpha^4\beta\delta - 5\alpha^4\delta^2 - 175\alpha^3\beta^3 \\
&\quad - 280\alpha^3\beta^2\delta - 280\alpha^3\beta\delta^2 - 175\alpha^3\delta^3 - 5\alpha^2\beta^4 \\
&\quad - 280\alpha^2\beta^3\delta - 3690\alpha^2\beta^2\delta^2 - 280\alpha^2\beta\delta^3 \\
&\quad - 5\alpha^2\delta^4 - 105\alpha\beta^4\delta - 280\alpha\beta^3\delta^2 - 280\alpha\beta^2\delta^3 \\
&\quad - 105\alpha\beta\delta^4 - 5\beta^4\delta^2 - 175\beta^3\delta^3 - 5\beta^2\delta^4, \\
\Delta_1 &= -10\alpha\beta\delta b(7\alpha^2\beta^2\delta^2 - 32\alpha^2\beta^2 - 32\alpha^2\delta^2 - 32\beta^2\delta^2) \\
&\quad \times(28\alpha^2\beta^2\delta^2 - 5\alpha^2\beta^2 - 105\alpha^2\beta\delta - 5\alpha^2\delta^2 \\
&\quad - 105\alpha\beta^2\delta - 105\alpha\beta\delta^2 - 5\beta^2\delta^2) + (28\alpha^3\beta^2\delta^2 \\
&\quad + 28\alpha^2\beta^3\delta^2 + 28\alpha^2\beta^2\delta^3 - 5\alpha^3\beta^2 - 105\alpha^3\beta\delta \\
&\quad - 5\alpha^3\delta^2 - 5\alpha^2\beta^3 - 110\alpha^2\beta^2\delta - 110\alpha^2\beta\delta^2 \\
&\quad - 5\alpha^2\delta^3 - 105\alpha\beta^3\delta - 110\alpha\beta^2\delta^2 - 105\alpha\beta\delta^3 \\
&\quad - 5\beta^3\delta^2 - 5\beta^2\delta^3)\omega,
\end{aligned} \tag{32}$$

which corresponds to a b-dependent peakon solution

$$\begin{aligned}
u = -\frac{c\omega}{\Delta_1(\alpha-\beta)(\beta-\delta)(\delta-\alpha)} \\
\times\Big[A_1\alpha^3(\beta-\delta)\mathrm{e}^{-\frac{|x-ct|}{\alpha}} + B_1\beta^3(\delta-\alpha)\mathrm{e}^{-\frac{|x-ct|}{\beta}} \\
+C_1\delta^3(\alpha-\beta)\mathrm{e}^{-\frac{|x-ct|}{\delta}}\Big].
\end{aligned} \tag{33}$$

The property $u_x(X = 0^+) \neq u_x(X = 0^-)$ confirms its peakon character and its dependence on the parameter b.

Figure 5 displays the b-dependent peakon solution given by Eq. (33) with parameters $\alpha = 1$, $\beta = 2$, $\delta = 3$, and $c = 1$ at $t = 0$, plotted for different values of b. The blue curve corresponds to $b = 5$, the red curve to $b = 1$, and the green curve to $b = -2$. It can be observed that the parameter b significantly influences the peak amplitude. Specifically, as the value of b increases, the peak height decreases.

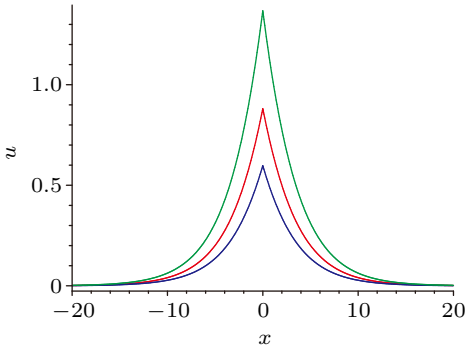


Fig. 5. Structure of the b-dependent peakon solution given by Eq. (33) with the parameter values $\alpha = 1$, $\beta = 2$, $\delta = 3$, and $c = 1$ at $t = 0$, with $b = -2$, $b = 1$, and $b = 5$ shown in green, red, and blue, respectively.

Notably, when either of the conditions

$$\begin{aligned}
28\alpha^2\beta^2\delta^2 - 5\alpha^2\beta^2 - 5\beta^2\delta^2 - 5\delta^2\alpha^2 - 105\alpha^2\beta\delta \\
-105\alpha\beta^2\delta - 105\alpha\beta\delta^2 = 0,
\end{aligned} \tag{34}$$

or

$$7\alpha^2\beta^2\delta^2 - 32\alpha^2\beta^2 - 32\alpha^2\delta^2 - 32\beta^2\delta^2 = 0, \tag{35}$$

is satisfied, the peakon solution given by Eq. (33) reduces to a form that is independent of the parameter b. Furthermore, it can be verified that under condition Eq. (34), the solution

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further simplifies to the fifth-order pseudo-peakon solution Eq. (25), which is also independent of b.

Finally, in the particular case where $\beta=\delta=\alpha$, the b-dependent peakon given by Eq. (33) simplifies to the following special form:

$$u=\frac{cB_0\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}}}{B_0-10b(14\alpha^2-165)(7\alpha^2-96)}\times\left[1+\frac{56\alpha^2-345}{3\alpha(28\alpha^2-255)}|x-ct|+\frac{2(7\alpha^2-30)}{3\alpha^2(28\alpha^2-255)}|x-ct|^2\right],\tag{36}$$

where

$$B_0=3(28\alpha^2-225)(14\alpha^6-207\alpha^4+2007\alpha^2-3120).\tag{37}$$

It can be verified that under the specific parameter constraint

$$\alpha^2=\frac{165}{14},\tag{38}$$

the b-dependent peakon solution Eqs. (36) and (37) further reduce to a b-independent fifth-order pseudo-peakon solution

$$u=c\left[1+\frac{42\alpha}{495}|x-ct|+\frac{14}{495}|x-ct|^2\right]\mathrm{e}^{-\frac{|x-ct|}{\alpha}},\tag{39}$$

where c is an arbitrary constant.

This section has systematically derived and classified weak solutions, specifically peakons and pseudo-peakons, for the seventh-order b-family system. Four distinct solution families were identified: (i) a third-order b-independent pseudo-peakon, (ii) a fifth-order b-independent pseudo-peakon (which can be obtained from the first via a parameter restriction), (iii) a b-independent peakon, and (iv) a b-dependent peakon. The analysis demonstrates how parameter constraints (particularly relations among α, β, and δ) can alter the solution type, induce transitions between peakons and higher-order pseudo-peakons, and modulate the dependence on the parameter b.

4. A general ninth-order b-family system

The general ninth-order b-family system can be written as

$$\begin{aligned}&m_t+um_x+bmu_x=0,\\&m=(1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)(1-\delta^2\partial_x^2)(1-\theta^2\partial_x^2)u,\end{aligned}\tag{40}$$

with $\alpha>0$, $\beta>0$, $\delta>0$, $\theta>0$, and b being arbitrary real constants. To search for solutions of the ninth-order b-family system Eq. (40), substituting the single weak peakon and/or pseudo-peakon solution ansatz

$$\begin{aligned}&u=c\left(a_1\,\mathrm{e}^{-\frac{|X|}{\alpha}}+a_2\,\mathrm{e}^{-\frac{|X|}{\beta}}+a_3\,\mathrm{e}^{-\frac{|X|}{\delta}}+a_4\,\mathrm{e}^{-\frac{|X|}{\theta}}\right),\\&X=x-ct,\end{aligned}\tag{41}$$

into the system and requiring the result to be a zero distribution, we can obtain the determining equations for a_i, $i=1,2,\ldots,4$, which read as

$$\begin{aligned}&(a_1+a_2+a_3+a_4-1)(a_1\beta^4\delta^4\theta^4+a_2\alpha^4\delta^4\theta^4+a_3\alpha^4\beta^4\theta^4+a_4\alpha^4\beta^4\delta^4)=0,\\&15C_6\alpha^2\beta^2\delta^2\theta^2(a_1+a_2+a_3+a_4-1)(a_1\beta^3\delta^3\theta^3+a_2\alpha^3\delta^3\theta^3+a_3\alpha^3\beta^3\theta^3+a_4\alpha^3\beta^3\delta^3)-120bC_8(a_1\beta^4\delta^4\theta^4\\&\quad+a_2\alpha^4\delta^4\theta^4+a_3\alpha^4\beta^4\theta^4+a_4\alpha^4\beta^4\delta^4)(a_1\beta\delta\theta+a_2\alpha\delta\theta+a_3\alpha\beta\theta+a_4\alpha\beta\delta)-2C_8(a_1+a_2+a_3+a_4\\&\quad-1)(137a_1\alpha^2\beta^5\delta^5\theta^5+137a_2\alpha^5\beta^2\delta^5\theta^5+137a_3\alpha^5\beta^5\delta^2\theta^5+137a_4\alpha^5\beta^5\delta^5\theta^2+270a_1\beta^5\delta^5\theta^5\\&\quad+270a_2\alpha^5\delta^5\theta^5+270a_3\alpha^5\beta^5\theta^5+270a_4\alpha^5\beta^5\delta^5)=0,\\&5C_4\alpha^4\theta^4\beta^4\delta^4(a_1+a_2+a_3+a_4-1)(a_1\beta^2\delta^2\theta^2+a_2\alpha^2\delta^2\theta^2+a_3\alpha^2\beta^2\theta^2+a_4\alpha^2\beta^2\delta^2)-C_6\alpha^2\beta^2\delta^2\theta^2\\&\quad\times\left[30b(a_1\beta^3\delta^3\theta^3+a_2\alpha^3\delta^3\theta^3+a_3\alpha^3\beta^3\theta^3+a_4\alpha^3\beta^3\delta^3)(a_1\beta\delta\theta+a_2\alpha\delta\theta+a_3\alpha\beta\theta+a_4\alpha\beta\delta)+7(a_1\right.\\&\quad+a_2+a_3+a_4-1)(4a_1\alpha^2\beta^4\delta^4\theta^4+4a_2\alpha^4\beta^2\delta^4\theta^4+4a_3\alpha^4\beta^4\delta^2\theta^4+4a_4\alpha^4\beta^4\delta^4\theta^2+15a_1\beta^4\delta^4\theta^4\\&\quad\left.+15a_2\alpha^4\delta^4\theta^4+15a_3\alpha^4\beta^4\theta^4+15a_4\alpha^4\beta^4\delta^4)\right]+C_8\left[56b\,(a_1\beta\delta\theta+a_2\alpha\delta\theta+a_3\alpha\beta\theta+a_4\alpha\beta\delta)\right.\\&\quad\times(4a_1\alpha^2\beta^5\delta^5\theta^5+4a_2\alpha^5\beta^2\delta^5\theta^5+4a_3\alpha^5\beta^5\delta^2\theta^5+4a_4\alpha^5\beta^5\delta^5\theta^2+5a_1\beta^5\delta^5\theta^5+5a_2\alpha^5\delta^5\theta^5\\&\quad+5a_3\alpha^5\beta^5\theta^5+5a_4\alpha^5\beta^5\delta^5)+3(a_1+a_2+a_3+a_4-1)(41a_1\alpha^4\beta^6\delta^6\theta^6+41a_2\alpha^6\beta^4\delta^6\theta^6\\&\quad+41a_3\alpha^6\beta^6\delta^4\theta^6+41a_4\alpha^6\beta^6\delta^6\theta^4+336a_1\alpha^2\beta^6\delta^6\theta^6+336a_2\alpha^6\beta^2\delta^6\theta^6+336a_3\alpha^6\beta^6\delta^2\theta^6\\&\quad\left.+336a_4\alpha^6\beta^6\delta^6\theta^2+210a_1\beta^6\delta^6\theta^6+210a_2\alpha^6\delta^6\theta^6+210a_3\alpha^6\beta^6\theta^6+210a_4\alpha^6\beta^6\delta^6)\right]=0,\\&C_2\alpha^6\beta^6\delta^6\theta^6(a_1+a_2+a_3+a_4-1)\,(a_1\beta\delta\theta+a_2\alpha\delta\theta+a_3\alpha\beta\theta+a_4\alpha\beta\delta)-C_4\alpha^4\beta^4\delta^4\theta^4\,[4b(a_1\beta\delta\theta\\&\quad+a_2\alpha\delta\theta+a_3\alpha\beta\theta+a_4\alpha\beta\delta)(a_1\beta^2\delta^2\theta^2+a_2\alpha^2\delta^2\theta^2+a_3\alpha^2\beta^2\theta^2+a_4\alpha^2\beta^2\delta^2)+(a_1+a_2+a_3\\&\quad+a_4-1)(a_1\alpha^2\beta^3\delta^3\theta^3+a_2\alpha^3\beta^2\delta^3\theta^3+a_3\alpha^3\beta^3\delta^2\theta^3+a_4\alpha^3\beta^3\delta^3\theta^2+10a_1\beta^3\delta^3\theta^3+10a_2\alpha^3\delta^3\theta^3\\&\quad+10a_3\alpha^3\beta^3\theta^3+10a_4\alpha^3\beta^3\delta^3)]+C_6\alpha^2\beta^2\delta^2\theta^2\,[2b(a_1\beta\delta\theta+a_2\alpha\delta\theta+a_3\alpha\beta\theta+a_4\alpha\beta\delta)(3a_1\alpha^2\beta^4\delta^4\theta^4\end{aligned}$$

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$$+3a_2\alpha^4\beta^2\delta^4\theta^4+3a_3\alpha^4\beta^4\delta^2\theta^4+3a_4\alpha^4\beta^4\delta^4\theta^2+10a_1\beta^4\delta^4\theta^4+10a_2\alpha^4\delta^4\theta^4+10a_3\alpha^4\beta^4\theta^4$$
$$+10a_4\alpha^4\beta^4\delta^4)+(a_1+a_2+a_3+a_4-1)(a_1\alpha^4\beta^5\delta^5\theta^5+a_2\alpha^5\beta^4\delta^5\theta^5+a_3\alpha^5\beta^5\delta^4\theta^5+a_4\alpha^5\beta^5\delta^5\theta^4$$
$$+21a_1\alpha^2\beta^5\delta^5\theta^5+21a_2\alpha^5\beta^2\delta^5\theta^5+21a_3\alpha^5\beta^5\delta^2\theta^5+21a_4\alpha^5\beta^5\delta^5\theta^2+35a_1\beta^5\delta^5\theta^5+35a_2\alpha^5\delta^5\theta^5$$
$$+35a_3\alpha^5\beta^5\theta^5+35a_4\alpha^5\beta^5\delta^5)\Big]-C_8\Big[b(a_1\beta\delta\theta+a_2\alpha\delta\theta+a_3\alpha\beta\theta+a_4\alpha\beta\delta)(a_1\alpha^4\beta^6\delta^6\theta^6+a_2\alpha^6\beta^4\delta^6\theta^6$$
$$+a_3\alpha^6\beta^6\delta^4\theta^6+a_4\alpha^6\beta^6\delta^6\theta^4+7a_1\alpha^2\beta^6\delta^6\theta^6+7a_2\alpha^6\beta^2\delta^6\theta^6+7a_3\alpha^6\beta^6\delta^2\theta^6+7a_4\alpha^6\beta^6\delta^6\theta^2$$
$$+7a_1\beta^6\delta^6\theta^6+7a_2\alpha^6\delta^6\theta^6+7a_3\alpha^6\beta^6\theta^6+7a_4\alpha^6\beta^6\delta^6)\Big]=0, \tag{42}$$

where

$$C_2=\alpha^2+\beta^2+\delta^2+\theta^2,$$
$$C_4=\alpha^2\beta^2+\alpha^2\delta^2+\alpha^2\theta^2+\beta^2\delta^2+\beta^2\theta^2+\delta^2\theta^2,$$
$$C_6=\alpha^2\beta^2\delta^2+\alpha^2\beta^2\theta^2+\alpha^2\delta^2\theta^2+\beta^2\delta^2\theta^2,$$
$$C_8=\alpha^2\beta^2\delta^2\theta^2. \tag{43}$$

Solving these equations, the single weak peakon and/or pseudo-peakon solutions for the ninth-order b-family system Eq. (40) can be obtained. For simplicity, we list the results here.

Case 1. the b-independent pseudo-peakon

The solution to Eqs. (42) and (43) is

$$a_1=\frac{\alpha}{\alpha-\beta}\left[1+\frac{a_3(\beta-\delta)}{\delta}+\frac{a_4(\beta-\theta)}{\theta}\right],$$
$$a_2=-\frac{\beta}{\alpha-\beta}\left[1+\frac{a_3(\alpha-\delta)}{\delta}+\frac{a_4(\alpha-\theta)}{\theta}\right], \tag{44}$$

with a_3 and a_4 being arbitrary constants. Substituting the solution into the solution ansatz Eq. (41), we can find different types of b-independent pseudo-peakon solutions for the ninth-order b-family system Eq. (40).

Solution I: third-order pseudo-peakon solution The pseudo-peakon solution for the ninth-order b-family system Eq. (40) is

$$u=\frac{c}{\alpha-\beta}\left(\alpha\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}}-\beta\,\mathrm{e}^{-\frac{|x-ct|}{\beta}}\right)+\frac{ca_3}{\delta(\alpha-\beta)}\left[\alpha(\beta-\delta)\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}}+\beta(\delta-\alpha)\,\mathrm{e}^{-\frac{|x-ct|}{\beta}}+\delta(\alpha-\beta)\,\mathrm{e}^{-\frac{|x-ct|}{\delta}}\right]$$
$$+\frac{ca_4}{\theta(\alpha-\beta)}\left[\alpha(\beta-\theta)\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}}+\beta(\theta-\alpha)\,\mathrm{e}^{-\frac{|x-ct|}{\beta}}+\theta(\alpha-\beta)\,\mathrm{e}^{-\frac{|x-ct|}{\theta}}\right], \tag{45}$$

where c, a_3, and a_4 are arbitrary constants. The solution Eq. (45) has continuous derivatives up to the second order, i.e., $u_x(X=0^+)=u_x(X=0^-)$ and $u_{xx}(X=0^+)=u_{xx}(X=0^-)$, but does not possess a third-order derivative, with $u_{xxx}(X=0^+)\neq u_{xxx}(X=0^-)$ at $x=ct$. Thus, Eq. (45) is a third-order b-independent pseudo-peakon solution.

Owing to the presence of free parameters a_3 and a_4, the third-order pseudo-peakon solution described by Eq. (45) can exhibit a diverse set of structural profiles. Figure 6 presents several wave shapes of this solution at $t=0$ with $c=1$, obtained under different parameter selections. A comparison with Eq. (22) reveals that the additional degrees of freedom introduced by these parameters significantly enrich the morphological variety of the solution. Furthermore, Fig. 7 illustrates the continuity of the first and second derivatives u_x and u_{xx} of this solution, confirming the smoothness of the wave profile within the considered domain.

For the special parameter selections $\beta=\alpha\neq\delta\neq\theta$, the pseudo-peakon solution Eq. (45) simplifies to

$$u=c\left[\left(1-\frac{a_3(\delta-\alpha)}{\delta}-\frac{a_4(\theta-\alpha)}{\theta}\right)\frac{|x-ct|}{\alpha}+(1-a_3-a_4)\right]\mathrm{e}^{-\frac{|x-ct|}{\alpha}}+ca_3\,\mathrm{e}^{-\frac{|x-ct|}{\delta}}+ca_4\,\mathrm{e}^{-\frac{|x-ct|}{\theta}}. \tag{46}$$

Equation (46) is also a third-order pseudo-peakon solution because the derivative of the solution Eq. (46) with respect to x is continuous up to the second order at $x=ct$.

Solution II: fifth-order pseudo-peakon solution If the free parameters a_3 and a_4 satisfy the following condition

$$a_3=-\frac{\delta^3\left[(\beta-\theta)(\alpha-\theta)(\alpha\beta+\theta\alpha+\theta\beta)a_4-\theta^3(\alpha+\beta)\right]}{\theta^3(\beta-\delta)(\alpha-\delta)(\alpha\beta+\delta\alpha+\delta\beta)}, \tag{47}$$

the solution Eq. (45) becomes a fifth-order pseudo-peakon solution

$$u=\frac{c}{\alpha\beta+\alpha\delta+\beta\delta}\left[\frac{\alpha^3(\beta+\delta)}{(\alpha-\delta)(\alpha-\beta)}\mathrm{e}^{-\frac{|x-ct|}{\alpha}}+\frac{\beta^3(\alpha+\delta)}{(\alpha-\beta)(\delta-\beta)}\mathrm{e}^{-\frac{|x-ct|}{\beta}}+\frac{\delta^3(\alpha+\beta)}{(\alpha-\delta)(\beta-\delta)}\mathrm{e}^{-\frac{|x-ct|}{\delta}}\right]$$
$$+\frac{ca_4}{\theta^3(\alpha\beta+\alpha\delta+\beta\delta)}\left[\frac{\alpha^3(\theta-\delta)(\beta-\theta)(\beta\delta+\beta\theta+\delta\theta)}{(\alpha-\delta)(\alpha-\beta)}\mathrm{e}^{-\frac{|x-ct|}{\alpha}}+\frac{\beta^3(\delta-\theta)(\alpha-\theta)(\alpha\delta+\alpha\theta+\delta\theta)}{(\alpha-\beta)(\beta-\delta)}\mathrm{e}^{-\frac{|x-ct|}{\beta}}\right.$$

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$$+\frac{\delta^3(\theta-\beta)(\alpha-\theta)(\alpha\beta+\alpha\theta+\beta\theta)}{(\beta-\delta)(\alpha-\delta)}\,\mathrm{e}^{-\frac{|x-ct|}{\delta}}+\theta^3(\alpha\beta+\alpha\delta+\beta\delta)\,\mathrm{e}^{-\frac{|x-ct|}{\theta}}\Bigg]. \tag{48}$$

The derivative of u Eq. (48) with respect to x at $x=ct$ is continuous up to the fourth order, i.e., $u_x(X=0^+)=u_x(X=0^-)$, $u_{xx}(X=0^+)=u_{xx}(X=0^-)$, $u_{xxx}(X=0^+)=u_{xxx}(X=0^-)$, and $u_{xxxx}(X=0^+)=u_{xxxx}(X=0^-)$, whereas the fifth-order derivative is discontinuous, with $u_{xxxxx}(X=0^+)\neq u_{xxxxx}(X=0^-)$.

The presence of the free parameter a_4 allows the fifth-order pseudo-peakon solution Eq. (48) to generate a diverse array of structural profiles. This parametric richness is illustrated in Fig. 8, which depicts the solution at $t=0$ with $c=1$ for several distinct parameter sets: (a) $\alpha=0.6$, $\beta=0.8$, $\delta=0.9$, $\theta=1$, and $a_4=0.8$; (b) $\alpha=0.6$, $\beta=0.5$, $\delta=0.9$, $\theta=5.5$, and $a_4=1.5$; (c) $\alpha=0.6$, $\beta=0.5$, $\delta=0.9$, $\theta=6$, and $a_4=2$; (d) $\alpha=0.6$, $\beta=0.5$, $\delta=0.9$, $\theta=6$, and $a_4=10$; (e) $\alpha=0.6$, $\beta=0.5$, $\delta=0.9$, $\theta=6$, and $a_4=-4$. Correspondingly, the continuity of the higher-order derivatives is verified in Fig. 9, which displays (a) u_x, (b) u_{xx}, (c) u_{xxx}, and (d) u_{xxxx} with $c=1$, $\alpha=0.6$, $\beta=2.8$, $\delta=0.9$, $\theta=1$, and $a_4=0.8$ at $t=0$.

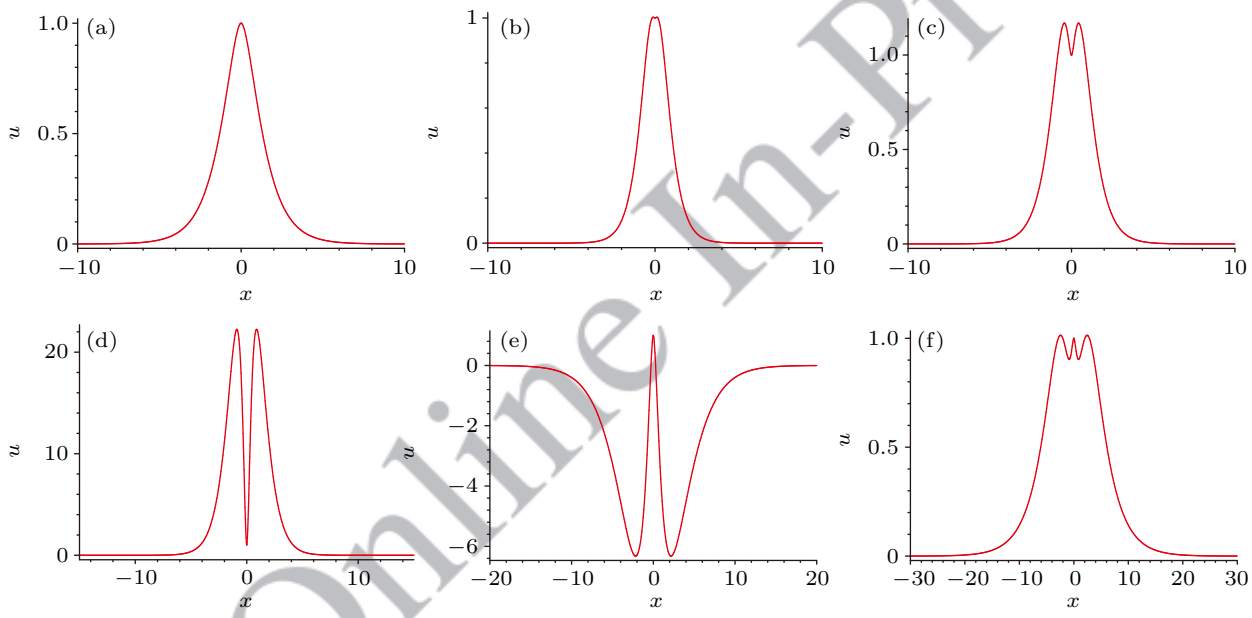


Fig. 6. The third-order pseudo-peakon solution Eq. (45) at t with $c=1$ and $\alpha=0.6$, $\beta=0.8$, $\delta=1$, $\theta=1.2$, $a_3=0.5$, and $a_4=0.8$ in panel (a); $\alpha=0.5$, $\beta=0.3$, $\delta=0.2$, $\theta=0.6$, $a_3=1.5$, and $a_4=2$ in panel (b); $\alpha=0.5$, $\beta=0.45$, $\delta=0.2$, $\theta=0.8$, $a_3=1.5$, and $a_4=2.5$ in panel (c); $\alpha=0.5$, $\beta=0.8$, $\delta=0.2$, $\theta=0.4$, $a_3=30$, and $a_4=50$ in panel (d); $\alpha=1.5$, $\beta=2$, $\delta=0.3$, $\theta=1.2$, $a_3=-5$, and $a_4=2$ in panel (e); and $\alpha=0.9$, $\beta=0.8$, $\delta=0.5$, $\theta=3$, $a_3=-3.5$, and $a_4=4$ in panel (f).

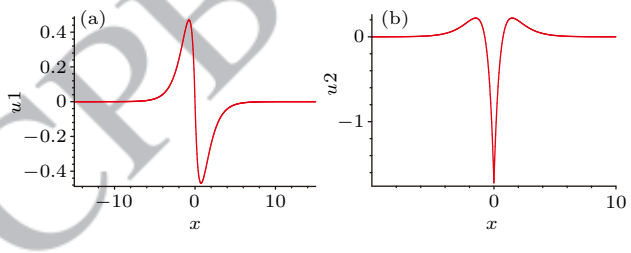


Fig. 7. The continuity of u_x and u_{xx} for the solution Eq. (45) at $t=0$ with $c=1$, $\alpha=0.5$, $\beta=0.8$, $\delta=1$, $\theta=1.2$, $a_3=0.5$, and $a_4=0.8$.

For the special parameter selection $\beta=\delta=\alpha\neq\theta$, we find that the fifth-order pseudo-peakon solution Eq. (48) simplifies to

$$u = c\left(\frac{|x-ct|^2}{3\alpha^2}+\frac{|x-ct|}{\alpha}+1\right)\mathrm{e}^{-\frac{|x-ct|}{\alpha}} -ca_4\left[\frac{(\alpha+2\theta)(\alpha-\theta)^2|x-ct|^2}{6\alpha^2\theta^3}+\frac{(\theta-\alpha)|x-ct|}{\theta-\alpha}+1\right] +ca_4\,\mathrm{e}^{-\frac{|x-ct|}{\theta}}. \tag{49}$$

Another fifth-order pseudo-peakon solution is obtained from the special solution Eq. (46) under the parameter constraint $\beta=\delta=\alpha\neq\theta$. Owing to the arbitrariness of the parameters a_3 and a_4, appropriate values may be assigned to these coefficients to satisfy the continuity condition $u_{xxx}(X=0^+)=u_{xxx}(X=0^-)$. One admissible choice for the free parameter a_3 is

$$a_3=\frac{a_4\delta^3(\alpha+2\theta)(\theta-\alpha)^2}{\theta^3(\alpha+2\delta)(\alpha-\delta)^2}+\frac{2\delta^3}{(\alpha+2\delta)(\alpha-\delta)^2}, \tag{50}$$

which, when substituted into Eq. (46), yields a simplified form that is independent of the parameter b. This reduced expression can be written as

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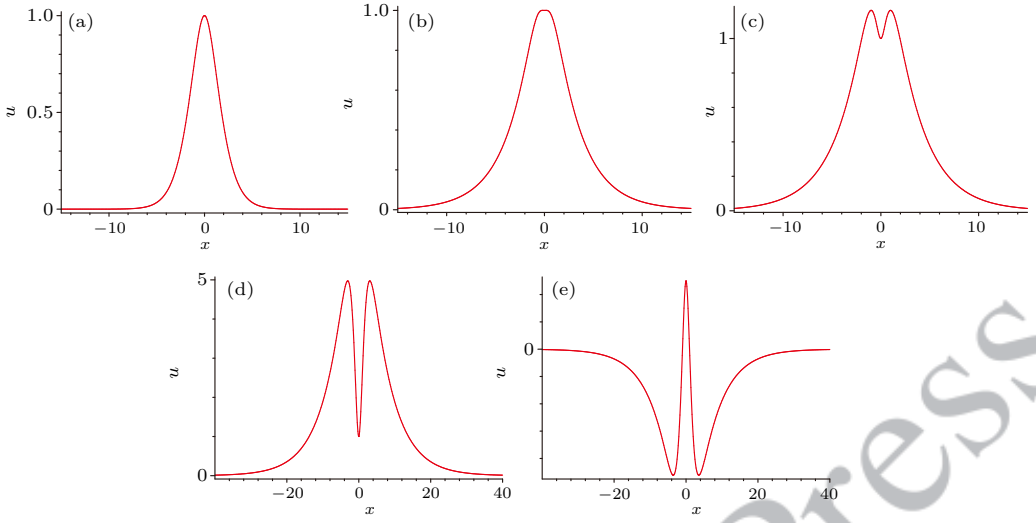


Fig. 8. The fifth-order pseudo-peakon solution given by Eq. (48) with different parameter selections.

$$u = \frac{c}{\theta^3(\alpha-\delta)(\alpha+2\delta)}\left[\left(\theta^3(\alpha+\delta)+ca_4(\delta-\theta)(\theta-\alpha)(\alpha\delta+\alpha\theta+\delta\theta)\right)|x-ct|+\frac{\alpha\theta^3(\alpha^2-3\delta^2)}{\alpha-\delta}\right.$$
$$\left.+\frac{a_4\alpha}{\alpha-\delta}(\delta-\theta)(\alpha^2\delta^2+\alpha^2\delta\theta+\alpha^2\theta^2-3\delta^2\theta^2)\right]\mathrm{e}^{-\frac{|x-ct|}{\alpha}}+\frac{c\delta^3}{\theta^3(\alpha+2\delta)(\alpha-\delta)^2}$$
$$\times\left[2\theta^3-a_4(\alpha+2\theta)(\alpha-\theta)^2\right]\mathrm{e}^{-\frac{|x-ct|}{\delta}}+ca_4\,\mathrm{e}^{-\frac{|x-ct|}{\theta}}, \tag{51}$$

obtained by direct substitution of Eq. (50) into Eq. (46). By computing the derivative of u as given in Eq. (51) with respect to x at the point $x = ct$, it can be verified that the solution is continuous up to the fourth-order derivative but has a discontinuity in the fifth-order derivative. This behavior confirms that the resulting solution Eq. (51) represents a fifth-order pseudo-peakon.

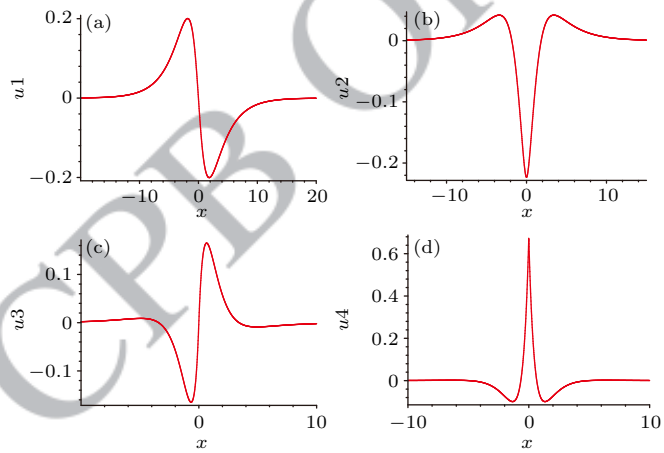


Fig. 9. The continuity of the higher-order derivatives (a) u_x, (b) u_{xx}, (c) u_{xxx}, and (d) u_{xxxx} of the fifth-order pseudo-peakon solution Eq. (48) with $c = 1$, $\alpha = 0.6$, $\beta = 2.8$, $\delta = 0.9$, $\theta = 1$, and $a_4 = 0.8$ at $t = 0$.

Solution III: seventh-order pseudo-peakon solution In particular, if the free parameters a_3 and a_4 are assigned as

$$a_3 = \frac{\delta^5(\alpha+\beta)(\beta+\theta)(\theta+\alpha)}{P_1(\beta-\delta)(\delta-\theta)(\alpha-\delta)},$$
$$a_4 = \frac{\theta^5(\alpha+\beta)(\beta+\delta)(\delta+\alpha)}{P_1(\beta-\theta)(\delta-\theta)(\theta-\alpha)}, \tag{52}$$

with

$$P_1 = (\alpha+\beta)(\beta+\delta)(\delta+\alpha)\theta^2+(\alpha\beta+\alpha\delta+\beta\delta)^2\theta$$
$$+\alpha\beta\delta(\alpha\beta+\alpha\delta+\beta\delta), \tag{53}$$

then the solution given in Eq. (45) reduces to a specific seventh-order pseudo-peakon of the form

$$u = \frac{c}{P_1}\left[\frac{\alpha^5(\beta+\delta)(\delta+\theta)(\theta+\beta)}{(\alpha-\beta)(\alpha-\delta)(\alpha-\theta)}\mathrm{e}^{-\frac{|x-ct|}{\alpha}}\right.$$
$$+\frac{\beta^5(\alpha+\delta)(\delta+\theta)(\theta+\alpha)}{(\beta-\alpha)(\beta-\delta)(\beta-\theta)}\mathrm{e}^{-\frac{|x-ct|}{\beta}}$$
$$+\frac{\delta^5(\alpha+\beta)(\beta+\theta)(\theta+\alpha)}{(\delta-\alpha)(\delta-\beta)(\delta-\theta)}\mathrm{e}^{-\frac{|x-ct|}{\delta}}$$
$$\left.+\frac{\theta^5(\alpha+\beta)(\beta+\delta)(\delta+\alpha)}{(\theta-\alpha)(\theta-\beta)(\theta-\delta)}\mathrm{e}^{-\frac{|x-ct|}{\theta}}\right]. \tag{54}$$

This solution is classified as a seventh-order pseudo-peakon because its derivative with respect to x remains continuous up to the sixth order, while the seventh derivative exhibits a discontinuity at $x = ct$, specifically, $u_{xxxxxxx}(X = 0^+) \neq u_{xxxxxxx}(X = 0^-)$.

Figure 10 displays the seventh-order pseudo-peakon solution described by Eq. (54) in (a), along with its higher-order derivatives — specifically, u_x in (b), u_{xx} in (c), u_{xxx} in (d), u_{xxxx} in (e), u_{xxxxx} in (f), and u_{xxxxxx} in (g) — evaluated at $t = 0$ under the parameter set $c = 1$, $\alpha = 0.6$, $\beta = 1.5$, $\delta = 0.9$, $\theta = 1$. The continuity and structural regularity evident across all plotted orders confirm the well-behaved nature of the solution within this parametric regime.

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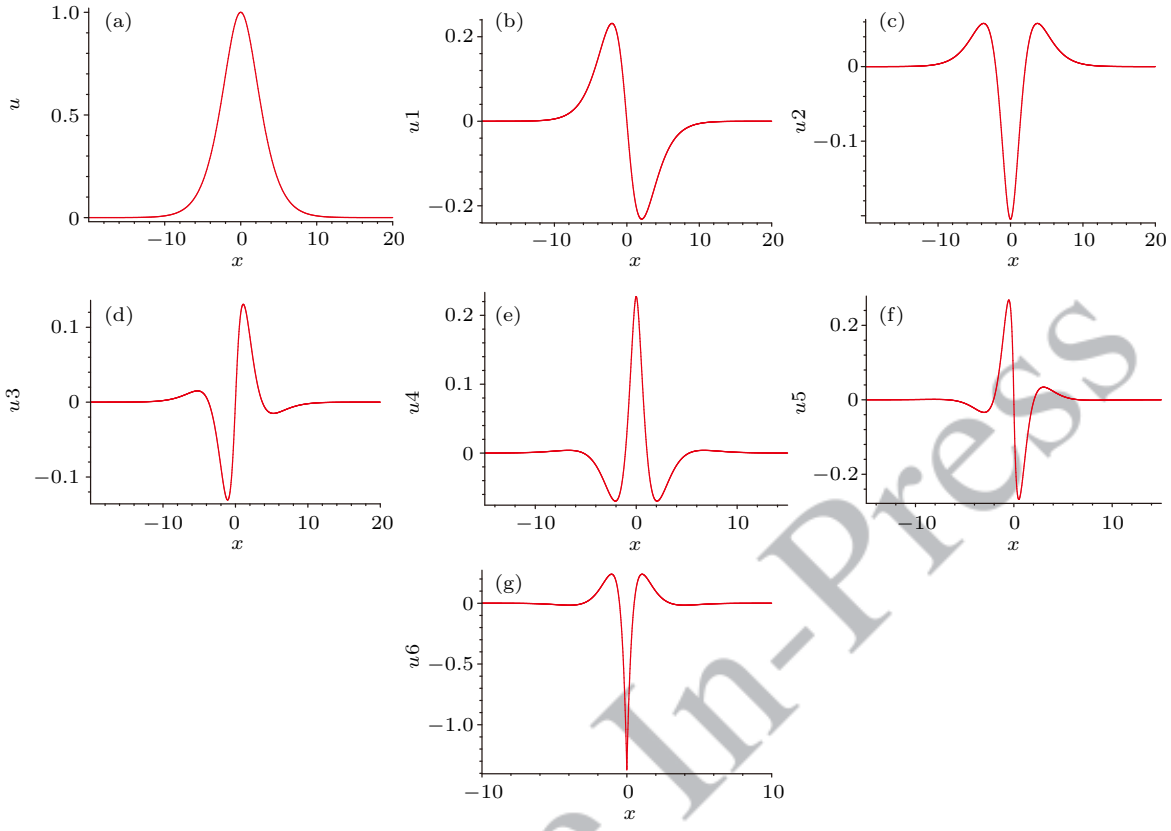


Fig. 10. The seventh-order pseudo-peakon solution and its higher-order derivatives given by Eq. (54) with $un = u_{nx}$.

For the fifth-order pseudo-peakon solutions Eq. (49) under the parameter constraints $\beta = \delta = \alpha \neq \theta$ and Eq. (51) under the constraints $\beta = \alpha \neq \delta \neq \theta$, suitable choices of the parameter a_4 can yield distinct seventh-order pseudo-peakon solutions. For example, selecting the arbitrary constant as

$$a_4 = -\frac{8\theta^5}{(3\alpha^2+9\alpha\theta+8\theta^2)(\alpha-\theta)^3},$$

reduces the solution Eq. (49) to the following special seventh-order pseudo-peakon

$$\begin{aligned} u = & \frac{c}{(3\alpha^2+9\alpha\theta+8\theta^2)(\alpha-\theta)^3}\Big[\Big(\alpha^{-1}(\alpha^2-\theta^2)^2|x-ct|^2 \\ & +(\alpha^2-\theta^2)(3\alpha^2-7\theta^2)|x-ct| \\ & +\alpha(3\alpha^4-10\alpha^2\theta^2+15\theta^4)\Big)\mathrm{e}^{-\frac{|x-ct|}{\alpha}} \\ & -8\theta^5\,\mathrm{e}^{-\frac{|x-ct|}{\theta}}\Big]. \end{aligned} \tag{55}$$

Similarly, by choosing the arbitrary constant a_4 in the fifth-order pseudo-peakon solution Eq. (51) as

$$a_4 = \frac{2\theta^5(\alpha+\delta)^2}{P_2(\theta-\delta)(\alpha-\theta)^2},$$

where

$$\begin{aligned} P_2 = & \alpha^3\delta+\alpha^3\theta+2\alpha^2\delta^2+4\alpha^2\delta\theta+2\alpha^2\theta^2 \\ & +4\alpha\delta^2\theta+4\alpha\delta\theta^2+2\delta^2\theta^2, \end{aligned}$$

an additional seventh-order pseudo-peakon solution can be obtained, expressed as

$$\begin{aligned} u = & \frac{c\alpha^2(\delta+\theta)}{P_2(\alpha-\delta)^2(\alpha-\theta)^2}\Big[(\alpha^2-\theta^2)(\alpha^2-\delta^2)|x-ct| \\ & +(\delta+\theta)(\alpha^4-3\alpha^2\delta^2-3\alpha^2\theta^2+5\delta^2\theta^2)\Big]\mathrm{e}^{-\frac{|x-ct|}{\alpha}} \\ & +\frac{2c\delta^5(\alpha+\theta)^2}{P_2(\delta-\theta)(\alpha-\delta)^2}\mathrm{e}^{-\frac{|x-ct|}{\delta}} \\ & +\frac{2c\theta^5(\alpha+\delta)^2}{P_2(\theta-\delta)(\alpha-\theta)^2}\mathrm{e}^{-\frac{|x-ct|}{\theta}}. \end{aligned} \tag{56}$$

The solutions Eqs. (55) and (56) are classified as seventh-order pseudo-peakons because their derivatives with respect to x remain continuous up to the sixth order, while the seventh-order derivative is discontinuous at $x = ct$.

Case 2. the b-independent peakon solution

In this case, the solution to Eqs. (42) and (43) is

$$\begin{aligned} a_1 &= -\frac{\alpha^6\beta^2\delta^2\theta^2\Theta_1}{\Theta(\alpha-\beta)(\alpha-\delta)(\alpha-\theta)}, \\ a_2 &= -\frac{\alpha^2\beta^6\delta^2\theta^2\Theta_2}{\Theta(\beta-\alpha)(\beta-\delta)(\beta-\theta)}, \\ a_3 &= -\frac{\alpha^2\beta^2\delta^6\theta^2\Theta_3}{\Theta(\delta-\alpha)(\delta-\beta)(\delta-\theta)}, \\ a_4 &= -\frac{\alpha^2\beta^2\delta^2\theta^6\Theta_4}{\Theta(\theta-\alpha)(\theta-\delta)(\theta-\delta)}, \end{aligned} \tag{57}$$

where the constants are given by

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$$
\begin{aligned}
\Theta_1 &= 448\alpha^4\beta^4\delta^4\theta^4 - 396\alpha^4\beta^4\delta^4\theta^2 - 560\alpha^4\beta^4\delta^3\theta^3 - 396\alpha^4\beta^4\delta^2\theta^4 - 560\alpha^4\beta^3\delta^4\theta^3 - 560\alpha^4\beta^3\delta^3\theta^4 - 396\alpha^4\beta^2\delta^4\theta^4 \\
&\quad -396\alpha^2\beta^4\delta^4\theta^4 + 45\alpha^4\beta^4\delta^4 + 420\alpha^4\beta^4\delta^3\theta + 314\alpha^4\beta^4\delta^2\theta^2 + 420\alpha^4\beta^4\delta\theta^3 + 45\alpha^4\beta^4\theta^4 + 420\alpha^4\beta^3\delta^4\theta \\
&\quad +3556\alpha^4\beta^3\delta^3\theta^2 + 3556\alpha^4\beta^3\delta^2\theta^3 + 420\alpha^4\beta^3\delta\theta^4 + 314\alpha^4\beta^2\delta^4\theta^2 + 3556\alpha^4\beta^2\delta^3\theta^3 + 314\alpha^4\beta^2\delta^2\theta^4 \\
&\quad +420\alpha^4\beta\delta^4\theta^3 + 420\alpha^4\beta\delta^3\theta^4 + 45\alpha^4\delta^4\theta^4 + 314\alpha^2\beta^4\delta^4\theta^2 + 420\alpha^2\beta^4\delta^3\theta^3 + 314\alpha^2\beta^4\delta^2\theta^4 + 420\alpha^2\beta^3\delta^4\theta^3 \\
&\quad +420\alpha^2\beta^3\delta^3\theta^4 + 314\alpha^2\beta^2\delta^4\theta^4 + 45\beta^4\delta^4\theta^4 - 30\alpha^4\beta^4\delta^2 - 280\alpha^4\beta^4\delta\theta - 30\alpha^4\beta^4\theta^2 - 420\alpha^4\beta^3\delta^3 \\
&\quad -700\alpha^4\beta^3\delta^2\theta - 700\alpha^4\beta^3\delta\theta^2 - 420\alpha^4\beta^3\theta^3 - 30\alpha^4\beta^2\delta^4 - 700\alpha^4\beta^2\delta^3\theta - 4010\alpha^4\beta^2\delta^2\theta^2 - 700\alpha^4\beta^2\delta\theta^3 \\
&\quad -30\alpha^4\beta^2\theta^4 - 280\alpha^4\beta\delta^4\theta - 700\alpha^4\beta\delta^3\theta^2 - 700\alpha^4\beta\delta^2\theta^3 - 280\alpha^4\beta\delta\theta^4 - 30\alpha^4\delta^4\theta^2 - 420\alpha^4\delta^3\theta^3 - 30\alpha^4\delta^2\theta^4 \\
&\quad -30\alpha^2\beta^4\delta^4 - 280\alpha^2\beta^4\delta^3\theta - 90\alpha^2\beta^4\delta^2\theta^2 - 280\alpha^2\beta^4\delta\theta^3 - 30\alpha^2\beta^4\theta^4 - 280\alpha^2\beta^3\delta^4\theta - 700\alpha^2\beta^3\delta^3\theta^2 \\
&\quad -700\alpha^2\beta^3\delta^2\theta^3 - 280\alpha^2\beta^3\delta\theta^4 - 90\alpha^2\beta^2\delta^4\theta^2 - 700\alpha^2\beta^2\delta^3\theta^3 - 90\alpha^2\beta^2\delta^2\theta^4 - 280\alpha^2\beta\delta^4\theta^3 - 280\alpha^2\beta\delta^3\theta^4 \\
&\quad -30\alpha^2\delta^4\theta^4 - 30\beta^4\delta^4\theta^2 - 30\beta^4\delta^2\theta^4 - 30\beta^2\delta^4\theta^4, \\
\Theta_2 &= 448\alpha^4\beta^4\delta^4\theta^4 - 396\alpha^4\beta^4\delta^4\theta^2 - 560\alpha^4\beta^4\delta^3\theta^3 - 396\alpha^4\beta^4\delta^2\theta^4 - 396\alpha^4\beta^2\delta^4\theta^4 - 560\alpha^3\beta^4\delta^4\theta^3 - 560\alpha^3\beta^4\delta^3\theta^4 \\
&\quad -396\alpha^2\beta^4\delta^4\theta^4 + 45\alpha^4\beta^4\delta^4 + 420\alpha^4\beta^4\delta^3\theta + 314\alpha^4\beta^4\delta^2\theta^2 + 420\alpha^4\beta^4\delta\theta^3 + 45\alpha^4\beta^4\theta^4 + 314\alpha^4\beta^2\delta^4\theta^2 \\
&\quad +420\alpha^4\beta^2\delta^3\theta^3 + 314\alpha^4\beta^2\delta^2\theta^4 + 45\alpha^4\delta^4\theta^4 + 420\alpha^3\beta^4\delta^4\theta + 3556\alpha^3\beta^4\delta^3\theta^2 + 3556\alpha^3\beta^4\delta^2\theta^3 + 420\alpha^3\beta^4\delta\theta^4 \\
&\quad +420\alpha^3\beta^2\delta^4\theta^3 + 420\alpha^3\beta^2\delta^3\theta^4 + 314\alpha^2\beta^4\delta^4\theta^2 + 3556\alpha^2\beta^4\delta^3\theta^3 + 314\alpha^2\beta^4\delta^2\theta^4 + 314\alpha^2\beta^2\delta^4\theta^4 + 420\alpha\beta^4\delta^4\theta^3 \\
&\quad +420\alpha\beta^4\delta^3\theta^4 + 45\beta^4\delta^4\theta^4 - 30\alpha^4\beta^4\delta^2 - 280\alpha^4\beta^4\delta\theta - 30\alpha^4\beta^4\theta^2 - 30\alpha^4\beta^2\delta^4 - 280\alpha^4\beta^2\delta^3\theta - 90\alpha^4\beta^2\delta^2\theta^2 \\
&\quad -280\alpha^4\beta^2\delta\theta^3 - 30\alpha^4\beta^2\theta^4 - 30\alpha^4\delta^4\theta^2 - 30\alpha^4\delta^2\theta^4 - 420\alpha^3\beta^4\delta^3 - 700\alpha^3\beta^4\delta^2\theta - 700\alpha^3\beta^4\delta\theta^2 - 420\alpha^3\beta^4\theta^3 \\
&\quad -280\alpha^3\beta^2\delta^4\theta - 700\alpha^3\beta^2\delta^3\theta^2 - 700\alpha^3\beta^2\delta^2\theta^3 - 280\alpha^3\beta^2\delta\theta^4 - 30\alpha^2\beta^4\delta^4 - 700\alpha^2\beta^4\delta^3\theta - 4010\alpha^2\beta^4\delta^2\theta^2 \\
&\quad -700\alpha^2\beta^4\delta\theta^3 - 30\alpha^2\beta^4\theta^4 - 90\alpha^2\beta^2\delta^4\theta^2 - 700\alpha^2\beta^2\delta^3\theta^3 - 90\alpha^2\beta^2\delta^2\theta^4 - 30\alpha^2\delta^4\theta^4 - 280\alpha\beta^4\delta^4\theta \\
&\quad -700\alpha\beta^4\delta^3\theta^2 - 700\alpha\beta^4\delta^2\theta^3 - 280\alpha\beta^4\delta\theta^4 - 280\alpha\beta^2\delta^4\theta^3 - 280\alpha\beta^2\delta^3\theta^4 - 30\beta^4\delta^4\theta^2 - 420\beta^4\delta^3\theta^3 \\
&\quad -30\beta^4\delta^2\theta^4 - 30\beta^2\delta^4\theta^4, \\
\Theta_3 &= 448\alpha^4\beta^4\delta^4\theta^4 - 396\alpha^4\beta^4\delta^4\theta^2 - 396\alpha^4\beta^4\delta^2\theta^4 - 560\alpha^4\beta^3\delta^4\theta^3 - 396\alpha^4\beta^2\delta^4\theta^4 - 560\alpha^3\beta^4\delta^4\theta^3 - 560\alpha^3\beta^3\delta^4\theta^4 \\
&\quad -396\alpha^2\beta^4\delta^4\theta^4 + 45\alpha^4\beta^4\delta^4 + 314\alpha^4\beta^4\delta^2\theta^2 + 45\alpha^4\beta^4\theta^4 + 420\alpha^4\beta^3\delta^4\theta + 420\alpha^4\beta^3\delta^2\theta^3 + 314\alpha^4\beta^2\delta^4\theta^2 \\
&\quad +314\alpha^4\beta^2\delta^2\theta^4 + 420\alpha^4\beta\delta^4\theta^3 + 45\alpha^4\delta^4\theta^4 + 420\alpha^3\beta^4\delta^4\theta + 420\alpha^3\beta^4\delta^2\theta^3 + 3556\alpha^3\beta^3\delta^4\theta^2 + 420\alpha^3\beta^3\delta^2\theta^4 \\
&\quad +3556\alpha^3\beta^2\delta^4\theta^3 + 420\alpha^3\beta\delta^4\theta^4 + 314\alpha^2\beta^4\delta^4\theta^2 + 314\alpha^2\beta^4\delta^2\theta^4 + 3556\alpha^2\beta^3\delta^4\theta^3 + 314\alpha^2\beta^2\delta^4\theta^4 \\
&\quad +420\alpha\beta^4\delta^4\theta^3 + 420\alpha\beta^3\delta^4\theta^4 + 45\beta^4\delta^4\theta^4 - 30\alpha^4\beta^4\delta^2 - 30\alpha^4\beta^4\theta^2 - 280\alpha^4\beta^3\delta^2\theta - 30\alpha^4\beta^2\delta^4 \\
&\quad -90\alpha^4\beta^2\delta^2\theta^2 - 30\alpha^4\beta^2\theta^4 - 280\alpha^4\beta\delta^4\theta - 280\alpha^4\beta\delta^2\theta^3 - 30\alpha^4\delta^4\theta^2 - 30\alpha^4\delta^2\theta^4 - 280\alpha^3\beta^4\delta^2\theta \\
&\quad -420\alpha^3\beta^3\delta^4 - 700\alpha^3\beta^3\delta^2\theta^2 - 700\alpha^3\beta^2\delta^4\theta - 700\alpha^3\beta^2\delta^2\theta^3 - 700\alpha^3\beta\delta^4\theta^2 - 280\alpha^3\beta\delta^2\theta^4 - 420\alpha^3\delta^4\theta^3 \\
&\quad -30\alpha^2\beta^4\delta^4 - 90\alpha^2\beta^4\delta^2\theta^2 - 30\alpha^2\beta^4\theta^4 - 700\alpha^2\beta^3\delta^4\theta - 700\alpha^2\beta^3\delta^2\theta^3 - 4010\alpha^2\beta^2\delta^4\theta^2 \\
&\quad -90\alpha^2\beta^2\delta^2\theta^4 - 700\alpha^2\beta\delta^4\theta^3 - 30\alpha^2\delta^4\theta^4 - 280\alpha\beta^4\delta^4\theta - 280\alpha\beta^4\delta^2\theta^3 - 700\alpha\beta^3\delta^4\theta^2 \\
&\quad -280\alpha\beta^3\delta^2\theta^4 - 700\alpha\beta^2\delta^4\theta^3 - 280\alpha\beta\delta^4\theta^4 - 30\beta^4\delta^4\theta^2 - 30\beta^4\delta^2\theta^4 - 420\beta^3\delta^4\theta^3 - 30\beta^2\delta^4\theta^4, \\
\Theta_4 &= 448\alpha^4\beta^4\delta^4\theta^4 - 396\alpha^4\beta^4\delta^4\theta^2 - 396\alpha^4\beta^4\delta^2\theta^4 - 560\alpha^4\beta^3\delta^3\theta^4 - 396\alpha^4\beta^2\delta^4\theta^4 - 560\alpha^3\beta^4\delta^3\theta^4 - 560\alpha^3\beta^3\delta^4\theta^4 \\
&\quad -396\alpha^2\beta^4\delta^4\theta^4 + 45\alpha^4\beta^4\delta^4 + 314\alpha^4\beta^4\delta^2\theta^2 + 45\alpha^4\beta^4\theta^4 + 420\alpha^4\beta^3\delta^3\theta^2 + 420\alpha^4\beta^3\delta\theta^4 + 314\alpha^4\beta^2\delta^4\theta^2 \\
&\quad +314\alpha^4\beta^2\delta^2\theta^4 + 420\alpha^4\beta\delta^3\theta^4 + 45\alpha^4\delta^4\theta^4 + 420\alpha^3\beta^4\delta^3\theta^2 + 420\alpha^3\beta^4\delta\theta^4 + 420\alpha^3\beta^3\delta^4\theta^2 + 3556\alpha^3\beta^3\delta^2\theta^4 \\
&\quad +3556\alpha^3\beta^2\delta^3\theta^4 + 420\alpha^3\beta\delta^4\theta^4 + 314\alpha^2\beta^4\delta^4\theta^2 + 314\alpha^2\beta^4\delta^2\theta^4 + 3556\alpha^2\beta^3\delta^3\theta^4 + 314\alpha^2\beta^2\delta^4\theta^4 \\
&\quad +420\alpha\beta^4\delta^3\theta^4 + 420\alpha\beta^3\delta^4\theta^4 + 45\beta^4\delta^4\theta^4 - 30\alpha^4\beta^4\delta^2 - 30\alpha^4\beta^4\theta^2 - 280\alpha^4\beta^3\delta\theta^2 - 30\alpha^4\beta^2\delta^4 - 90\alpha^4\beta^2\delta^2\theta^2 \\
&\quad -30\alpha^4\beta^2\theta^4 - 280\alpha^4\beta\delta^3\theta^2 - 280\alpha^4\beta\delta\theta^4 - 30\alpha^4\delta^4\theta^2 - 30\alpha^4\delta^2\theta^4 - 280\alpha^3\beta^4\delta\theta^2 - 700\alpha^3\beta^3\delta^2\theta^2 - 420\alpha^3\beta^3\theta^4 \\
&\quad -700\alpha^3\beta^2\delta^3\theta^2 - 700\alpha^3\beta^2\delta\theta^4 - 280\alpha^3\beta\delta^4\theta^2 - 700\alpha^3\beta\delta^2\theta^4 - 420\alpha^3\delta^3\theta^4 - 30\alpha^2\beta^4\delta^4 - 90\alpha^2\beta^4\delta^2\theta^2 \\
&\quad -30\alpha^2\beta^4\theta^4 - 700\alpha^2\beta^3\delta^3\theta^2 - 700\alpha^2\beta^3\delta\theta^4 - 90\alpha^2\beta^2\delta^4\theta^2 - 4010\alpha^2\beta^2\delta^2\theta^4 - 700\alpha^2\beta\delta^3\theta^4 - 30\alpha^2\delta^4\theta^4 \\
&\quad -280\alpha\beta^4\delta^3\theta^2 - 280\alpha\beta^4\delta\theta^4 - 280\alpha\beta^3\delta^4\theta^2 - 700\alpha\beta^3\delta^2\theta^4 - 700\alpha\beta^2\delta^3\theta^4 - 280\alpha\beta\delta^4\theta^4 \\
&\quad -30\beta^4\delta^4\theta^2 - 30\beta^4\delta^2\theta^4 - 420\beta^3\delta^3\theta^4 - 30\beta^2\delta^4\theta^4, \\
\Theta &= C_4\left[30C_6\alpha^2\beta^2\delta^2\theta^2(\alpha+\beta+\delta+\theta) - 56C_8\alpha\beta\delta\theta(4\alpha^2\beta\delta\theta + 4\alpha\beta^2\delta\theta + 4\alpha\beta\delta^2\theta + 4\alpha\beta\delta\theta^2 - 5\alpha^2\beta - 5\alpha^2\delta - 5\alpha^2\theta \right. \\
&\quad \left. -5\alpha\beta^2 - 10\alpha\beta\delta - 10\alpha\beta\theta - 5\alpha\delta^2 - 10\alpha\delta\theta - 5\alpha\theta^2 - 5\beta^2\delta - 5\beta^2\theta - 5\beta\delta^2 - 10\beta\delta\theta - 5\beta\theta^2 - 5\delta^2\theta - 5\delta\theta^2)\right]
\end{aligned}
$$

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$$
\begin{aligned}
&-45C_6^2\alpha^2\beta^2\delta^2\theta^2(\alpha+\beta+\delta+\theta)+C_6C_8\left(396\alpha^3\beta^2\delta^2\theta^2+396\alpha^2\beta^3\delta^2\theta^2+396\alpha^2\beta^2\delta^3\theta^2+396\alpha^2\beta^2\delta^2\theta^3\right.\\
&-420\alpha^3\beta^2\delta\theta-420\alpha^3\beta\delta^2\theta-420\alpha^3\beta\delta\theta^2-420\alpha^2\beta^3\delta\theta-840\alpha^2\beta^2\delta^2\theta-840\alpha^2\beta^2\delta\theta^2-420\alpha^2\beta\delta^3\theta\\
&-840\alpha^2\beta\delta^2\theta^2-420\alpha^2\beta\delta\theta^3-420\alpha\beta^3\delta^2\theta-420\alpha\beta^3\delta\theta^2-420\alpha\beta^2\delta^3\theta-840\alpha\beta^2\delta^2\theta^2-420\alpha\beta^2\delta\theta^3\\
&-420\alpha\beta\delta^3\theta^2-420\alpha\beta\delta^2\theta^3+420\alpha^3\beta\delta+420\alpha^3\beta\theta+420\alpha^3\delta\theta+420\alpha^2\beta^2\delta+420\alpha^2\beta^2\theta+420\alpha^2\beta\delta^2\\
&+1260\alpha^2\beta\delta\theta+420\alpha^2\beta\theta^2+420\alpha^2\delta^2\theta+420\alpha^2\delta\theta^2+420\alpha\beta^3\delta+420\alpha\beta^3\theta+420\alpha\beta^2\delta^2+1260\alpha\beta^2\delta\theta\\
&+420\alpha\beta^2\theta^2+420\alpha\beta\delta^3+1260\alpha\beta\delta^2\theta+1260\alpha\beta\delta\theta^2+420\alpha\beta\theta^3+420\alpha\delta^3\theta+420\alpha\delta^2\theta^2+420\alpha\delta\theta^3\\
&+420\beta^3\delta\theta+420\beta^2\delta^2\theta+420\beta^2\delta\theta^2+420\beta\delta^3\theta+420\beta\delta^2\theta^2+420\beta\delta\theta^3\left.\right)+C_8^2\left(-448\alpha^3\beta^2\delta^2\theta^2-448\alpha^2\beta^3\delta^2\theta^2\right.\\
&-448\alpha^2\beta^2\delta^3\theta^2-448\alpha^2\beta^2\delta^2\theta^3+560\alpha^3\beta^2\delta\theta+560\alpha^3\beta\delta^2\theta+560\alpha^3\beta\delta\theta^2+560\alpha^2\beta^3\delta\theta+1120\alpha^2\beta^2\delta^2\theta\\
&+1120\alpha^2\beta^2\delta\theta^2+560\alpha^2\beta\delta^3\theta+1120\alpha^2\beta\delta^2\theta^2+560\alpha^2\beta\delta\theta^3+560\alpha\beta^3\delta^2\theta+560\alpha\beta^3\delta\theta^2+560\alpha\beta^2\delta^3\theta\\
&+1120\alpha\beta^2\delta^2\theta^2+560\alpha\beta^2\delta\theta^3+560\alpha\beta\delta^3\theta^2+560\alpha\beta\delta^2\theta^3-3136\alpha^3\beta\delta-3136\alpha^3\beta\theta-3136\alpha^3\delta\theta\\
&-3136\alpha^2\beta^2\delta-3136\alpha^2\beta^2\theta-3136\alpha^2\beta\delta^2-9408\alpha^2\beta\delta\theta-3136\alpha^2\beta\theta^2-3136\alpha^2\delta^2\theta-3136\alpha^2\delta\theta^2\\
&-3136\alpha\beta^3\delta-3136\alpha\beta^3\theta-3136\alpha\beta^2\delta^2-9408\alpha\beta^2\delta\theta-3136\alpha\beta^2\theta^2-3136\alpha\beta\delta^3-9408\alpha\beta\delta^2\theta\\
&-9408\alpha\beta\delta\theta^2-3136\alpha\beta\theta^3-3136\alpha\delta^3\theta-3136\alpha\delta^2\theta^2-3136\alpha\delta\theta^3-3136\beta^3\delta\theta-3136\beta^2\delta^2\theta\\
&-3136\beta^2\delta\theta^2-3136\beta\delta^3\theta-3136\beta\delta^2\theta^2-3136\beta\delta\theta^3+3920\alpha^3+3920\alpha^2\beta+3920\alpha^2\delta+3920\alpha^2\theta\\
&+3920\alpha\beta^2+3920\alpha\beta\delta+3920\alpha\beta\theta+3920\alpha\delta^2+3920\alpha\delta\theta+3920\alpha\theta^2+3920\beta^3+3920\beta^2\delta\\
&\left.+3920\beta^2\theta+3920\beta\delta^2+3920\beta\delta\theta+3920\beta\theta^2+3920\delta^3+3920\delta^2\theta+3920\delta\theta^2+3920\theta^3\right), \qquad (58)
\end{aligned}
$$

with

$$
\begin{aligned}
C_4&=\alpha^2\beta^2+\alpha^2\delta^2+\alpha^2\theta^2+\beta^2\delta^2+\beta^2\theta^2+\delta^2\theta^2,\\
C_6&=\alpha^2\beta^2\delta^2+\alpha^2\beta^2\theta^2+\alpha^2\delta^2\theta^2+\beta^2\delta^2\theta^2,\\
C_8&=\alpha^2\beta^2\delta^2\theta^2. \qquad (59)
\end{aligned}
$$

This solution leads to a single peakon solution to the ninth-order b-family system Eq. (40), expressed as

$$
\begin{aligned}
u=\frac{cC_8}{\Theta}\left[\frac{\alpha^4\Theta_1\,\mathrm{e}^{-\frac{|x-ct|}{\alpha}}}{(\beta-\alpha)(\delta-\alpha)(\theta-\alpha)}-\frac{\beta^4\Theta_2\,\mathrm{e}^{-\frac{|x-ct|}{\beta}}}{(\beta-\alpha)(\beta-\delta)(\beta-\theta)}\right.\\
\left.-\frac{\delta^4\Theta_3\,\mathrm{e}^{-\frac{|x-ct|}{\delta}}}{(\delta-\alpha)(\delta-\beta)(\delta-\theta)}+\frac{\theta^4\Theta_4\,\mathrm{e}^{-\frac{|x-ct|}{\theta}}}{(\beta-\theta)(\delta-\theta)(\alpha-\theta)}\right], \qquad (60)
\end{aligned}
$$

where c is a free constant and Θ_i, $i=1,\ 2,\ 3,\ 4$, and Θ are provided by Eqs. (58) and (59). The solution Eq. (60) is a b-independent peakon solution for general model parameters because u_x is discontinuous, i.e., $u_x(X=0^+)\neq u_x(X=0^-)$ at $x=ct$.

Due to the arbitrariness of the parameters, the peakon solution given by Eq. (60) together with Eqs. (58) and (59) can degenerate into a pseudo-peakon solution. Specifically, under appropriate parameter constraints, it reduces to a third-order pseudo-peakon that is independent of b. For simplicity, the explicit form of this special case is not presented in detail here.

Case 3. b-dependent peakon solution

The explicit solution of Eqs. (42) and (43) is cumbersome; therefore, to illustrate the structure of the solution, we present its form for the representative parameter choice $\alpha=1$, $\beta=2$, $\delta=3$, $\theta=4$. For these parameter values, the ninth-order b-family system Eq. (40) admits the peakon solution

$$
\begin{aligned}
u\approx\frac{2c}{2.433b-650.32}\left(0.69743\,\mathrm{e}^{-|x-ct|}-78.755\,\mathrm{e}^{-\frac{|x-ct|}{2}}\right.\\
\left.+614.97\,\mathrm{e}^{-\frac{|x-ct|}{3}}-862.07\,\mathrm{e}^{-\frac{|x-ct|}{4}}\right), \qquad (61)
\end{aligned}
$$

with c being an arbitrary constant.

A direct examination of the spatial derivative u_x shows that u is continuous while u_x has a jump discontinuity at the crest; hence, the solution Eq. (61) satisfies the usual definition of a peakon.

Figure 11 displays the b-dependent peakon Eq. (61) with $c=1$ at $t=0$ for three representative values of b (blue: $b=60$; red: $b=20$; green: $b=-30$). The plot shows that the peak height increases monotonically with b over the plotted range, indicating that the parameter b controls the amplitude of the peakon in this family.

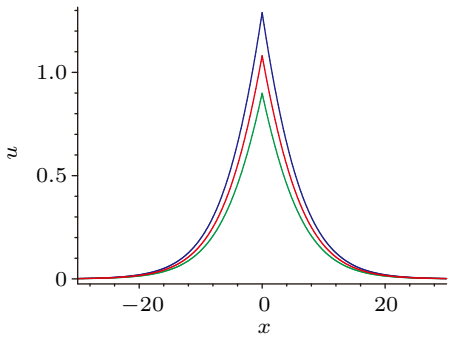


Fig. 11. The b-dependent peakon solution Eq. (61) with $\alpha=1$, $\beta=2$, $\delta=3$, $\theta=4$, and $c=1$ at $t=0$, with the blue curve corresponding to $b=60$, the red curve to $b=20$, and the green curve to $b=-30$.

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This section examines the ninth-order system and shows that it admits b-independent pseudo-peakon solutions (of third, fifth, and seventh order) as well as both b-independent and b-dependent peakon solutions. The third- and fifth-order pseudo-peakons contain arbitrary parameters, which give rise to a wide variety of spatial profiles and, by appropriate parameter choices, allow third-order solutions to be transformed into fifth- and seventh-order forms. Peakon solutions occur in both b-independent and b-dependent branches; in the b-dependent branch, the value of b controls the peak amplitude of the solution. These findings emphasize the enriched solution structure introduced at higher differential orders and highlight the central role of parameter tuning in generating and connecting distinct coherent states.

5. A special ninth-order b-family

Given that the ninth-order b-family system exhibits a rich variety of solution structures involving peakons and pseudo-peakons, it is pertinent to examine the special case where $\beta = \delta = \theta = \alpha$. Under this condition, the ninth-order b-family system reduces to

$$m_t + um_x + bmu_x = 0,$$
$$m = (1-\alpha^2\partial_x^2)(1-\alpha^2\partial_x^2)(1-\alpha^2\partial_x^2)(1-\alpha^2\partial_x^2)u, \quad (62)$$

with $\alpha > 0$ and b being an arbitrary real constant. The procedure for obtaining solutions of Eq. (62) follows an approach analogous to those described earlier. For brevity, only the final results are presented here.

I. b-independent pseudo-peakon solution

The ninth-order b-family Eq. (62) admits a pseudo-peakon solution of the form

$$u = \left(z_1|x-ct|^3 + z_2|x-ct|^2 + \frac{c|x-ct|}{\alpha} + c\right) \mathrm{e}^{-\frac{|x-ct|}{\alpha}}, \quad (63)$$

where c, z_1, and z_2 are arbitrary constants. Solution Eq. (63) represents a b-independent third-order pseudo-peakon because its third derivative u_{xxx} exhibits a discontinuity at the point $x = ct$.

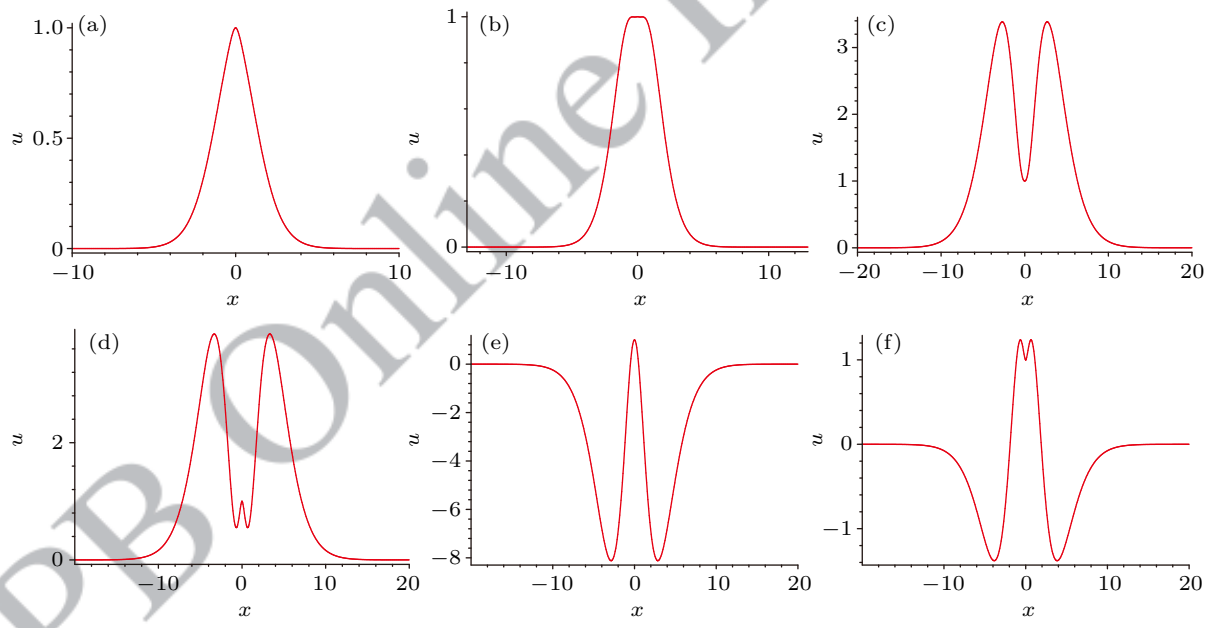


Fig. 12. The third-order pseudo-peakon solution Eq. (63) with $c = 1$ at $t = 0$ for (a) $\alpha = 0.5$, $z_1 = 1$, $z_2 = 1$; (b) $\alpha = 0.5$, $z_1 = 2$, $z_2 = 1.85$; (c) $\alpha = 1$, $z_1 = 2$, $z_2 = 1$; (d) $\alpha = 1$, $z_1 = 4$, $z_2 = -4$; (e) $\alpha = 1$, $z_1 = -5$, $z_2 = -3.5$; and (f) $\alpha = 1$, $z_1 = -2$, $z_2 = 31$.

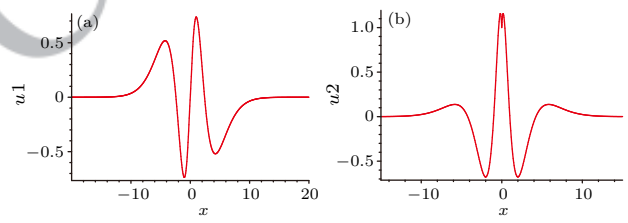


Fig. 13. (a) The third-order pseudo-peakon solution Eq. (63) with $c = 1$, $\alpha = 0.8$, and $\beta = 1$ at $t = 0$. (b) The structure of the first-order derivative of the third-order pseudo-peakon solution Eq. (14). (c) The structure of the first-order derivative of the third-order pseudo-peakon solution Eq. (14).

Figure 12 displays spatial profiles of the third-order pseudo-peakon solution Eq. (63) at $t = 0$ with $c = 1$ for several values of the free parameters. The plotted curves cover a range of waveform morphologies, from single localized peakon-like pulses to more complex multi-peaked structures; small changes in the parameters substantially affect the amplitude, width, and number of extrema. Figure 13 presents the corresponding first and second spatial derivatives, u_x and u_{xx}, thereby verifying the continuity of its derivatives across the domain for the parameter choices shown. These results illustrate the pronounced parametric flexibility of this solution family and suggest a rich repertoire of dynamical behaviors

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worthy of further analytical and numerical study.

If the free parameters c, z_1, and z_2 satisfy the relation

$$3z_1\alpha^3 - 3z_2\alpha^2 + c = 0, \tag{64}$$

then the solution Eq. (63) reduces to a b-independent fifth-order pseudo-peakon given by

$$u = \left(\frac{3z_2\alpha^2 - c}{3\alpha^3}|x-ct|^3 + z_2\alpha|x-ct|^2 + \frac{c|x-ct|}{\alpha} + c\right) \times \mathrm{e}^{-\frac{|x-ct|}{\alpha}}. \tag{65}$$

Here, continuity extends through the fourth derivative u_{xxxx} at $x = ct$, with the first discontinuity occurring in the fifth derivative.

Choosing

$$z_2 = \frac{2c}{5\alpha^2}, \tag{66}$$

together with Eq. (64), further simplifies the solution Eq. (63) to a seventh-order pseudo-peakon expressed by

$$u = \frac{c}{15\alpha^3}\left(|x-ct|^3 + 6\alpha|x-ct|^2 + 15\alpha^2|x-ct| + 15\alpha^3\right) \times \mathrm{e}^{-\frac{|x-ct|}{\alpha}}. \tag{67}$$

For this solution, derivatives up to the sixth order are continuous at $x = ct$, while a first discontinuity appears in $u_{xxxxxxx}$.

II. b-independent peakon solution

A b-independent peakon solution of the ninth-order b-family Eq. (62) is given by

$$u = c\left[\frac{1}{12\gamma_0\alpha^3}\left(7\alpha^6 - 51\alpha^4 + 258\alpha^2 - 230\right)|x-ct|^3 + \frac{1}{8\gamma_0\alpha^2}\left(28\alpha^6 - 239\alpha^4 + 1529\alpha^2 - 1480\right)|x-ct|^2 + \frac{1}{8\gamma_0\alpha}\left(84\alpha^6 - 787\alpha^4 + 5973\alpha^2 - 6260\right)|x-ct| + 1\right] \times \mathrm{e}^{-\frac{|x-ct|}{\alpha}}, \tag{68}$$

where

$$\gamma_0 = 14\alpha^6 - 137\alpha^4 + 1209\alpha^2 - 1370, \tag{69}$$

with c being an arbitrary parameter. The first derivative u_x is discontinuous at $x = ct$, which characterizes the solution as a peakon.

However, when α satisfies the condition

$$28\alpha^6 - 309\alpha^4 + 3699\alpha^2 - 4700 = 0, \text{ i.e., } \alpha \sim 1.19, \tag{70}$$

the solution undergoes a structural transition: u_x and u_{xx} become continuous, while u_{xxx} exhibits a discontinuity. Thus, under condition Eq. (70), the peakon Eq. (68) becomes a third-order pseudo-peakon.

III. b-dependent peakon solution

The ninth-order b-family equation Eq. (62) admits a b-dependent peakon solution that is not easily expressed in closed form. For illustrative purposes, we present a special case with $\alpha = 1$, which takes the form

$$u \approx \frac{0.31613c}{2.3130b + 83.183}\left(263.15 + 110.92|x-ct| + 17.045|x-ct|^2 + |x-ct|^3\right)\mathrm{e}^{-|x-ct|}, \tag{71}$$

where c is an arbitrary constant. Although Eq. (71) appears simple, it constitutes a genuine b-dependent peakon because its first derivative u_x generally exhibits a discontinuity at $x = ct$ for arbitrary values of b.

Figure 14 displays the peakon solution Eq. (71) with $c = 1$ at $t = 0$ for different values of the parameter b. The blue, red, and green curves correspond to $b = 10$, $b = 2$, and $b = -8$, respectively. The results indicate that b significantly affects the peak amplitude of the solution. Specifically, as b increases, the peak height decreases.

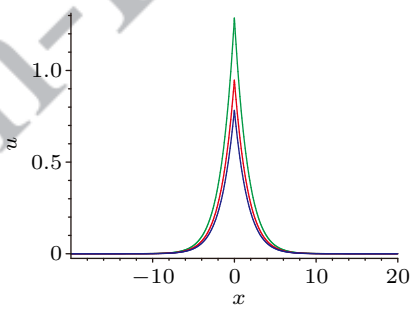


Fig. 14. The b-dependent peakon solution Eq. (71) with $c = 1$ at $t = 0$, with the blue curve corresponding to $b = 10$, the red curve to $b = 2$, and the green curve to $b = -8$.

This section has presented a systematic classification of peakon and pseudo-peakon solutions for the special ninth-order b-family equation in the degenerate case $\beta = \delta = \theta = \alpha$. Three families of solutions were identified: (i) b-independent pseudo-peakons of third, fifth, and seventh order, whose smoothness is controlled by the free parameters z_1 and z_2; (ii) a b-independent peakon that can transition into a third-order pseudo-peakon under a specific algebraic condition on α; and (iii) a b-dependent peakon with b affecting the peak amplitude. Overall, the analysis demonstrates that the special ninth-order b-family system admits pseudo-peakon solutions up to the seventh order.

6. Summary and discussion

We have systematically investigated the fifth-, seventh-, and ninth-order b-family systems and derived their pseudo-peakon and peakon solutions. Each examined system admits localized pseudo-peakon solutions and, in some parameter regimes, genuine peakon solutions; a concise summary of these results is given in Table 1. In particular, the ninth-order system supports third-, fifth-, and seventh-order pseudo-peakons that are independent of b and, under specific parameter choices, also produces b-dependent peakons.

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Table 1. Results for the b-family systems and solutions.

b-family systems \ solutions	b-independent		b-dependent	
	pseudo-peakon	peakon	pseudo-peakon	peakon
5th-order	✓(3rd-order)	✓	×	×
7th-order	✓(3rd-, 5th-order)	✓	×	✓
9th-order	✓(3rd-, 5th-, 7th-order)	✓	×	✓

A clear structural distinction between lower- and higher-order systems emerges: the fifth-order system does not admit b-dependent peakon solutions, whereas both the seventh- and ninth-order systems do. This distinction reflects the additional degrees of freedom and structural complexity introduced by higher-order terms. Furthermore, the sequence of admissible pseudo-peakon orders follows a regular pattern: the fifth-order system admits third-order pseudo-peakons; the seventh-order system admits third- and fifth-order pseudo-peakons; and the ninth-order system admits third-, fifth-, and seventh-order pseudo-peakons. Correspondingly, the number of free parameters appearing in the third-order pseudo-peakon solutions increases with the system order (one free parameter in the seventh-order case and two in the ninth-order case), enabling a broad range of solution profiles (M-shaped, W-shaped, double-peakon, and soliton-like) and parameter-driven transformations between solution types.

The arbitrary constants appearing in the pseudo-peakon solutions originate as integration constants from the reduced ordinary differential equation system. They are not independent; their admissible values are constrained by consistency conditions arising from the continuity requirements and jump conditions of the partial differential equation, as well as from the compatibility of the dynamical system. Consequently, while the number of constants increases with the order of the solution, only a subset of combinations yields valid solutions to the original b-family equations.

Motivated by these observations, we propose the following generalized $(2n+1)$th-order b-family system

$$m_t + um_x + bmu_x = 0, \qquad m = \prod_{i=1}^{n}(1-\alpha_i^2\partial_x^2)u,$$

with $\alpha_i > 0$ $(i = 1,2,\ldots,n)$ and conjecture the third-order pseudo-peakon ansatz given by

$$\begin{aligned} u = {} & \frac{c}{\alpha_1-\alpha_2}\left(\alpha_1 \mathrm{e}^{-\frac{|x-ct|}{\alpha_1}} - \alpha_2 \mathrm{e}^{-\frac{|x-ct|}{\alpha_2}}\right) \\ & + \sum_{i=3}^{n} a_i \left[\alpha_1(\alpha_2-\alpha_i)\mathrm{e}^{-\frac{|x-ct|}{\alpha_1}} + \alpha_2(\alpha_i-\alpha_1)\mathrm{e}^{-\frac{|x-ct|}{\alpha_2}} \right. \\ & \left. + \alpha_i(\alpha_1-\alpha_2)\mathrm{e}^{-\frac{|x-ct|}{\alpha_i}}\right], \end{aligned}$$

with a_i being arbitrary constants. This conjecture has been verified for the fifth-, seventh-, ninth-, and eleventh-order systems (the eleventh-order calculations are omitted here for brevity), and the consistent structure across these orders suggests that the conjectured form may extend to higher orders.

It is worth noting that the b-family includes several integrable members: the CH equation ($b = 2$) and the DP equation ($b = 3$) are both known to possess a Lax pair and a bi-Hamiltonian structure. For these special values of b, additional constraints (such as conservation laws or soliton solutions) may further restrict the admissible free parameters. However, the analysis presented here does not rely on integrability; the multiplicity of solutions arises from the structure of the equations themselves and persists across the entire b-family. A deeper exploration of how integrability affects parameter selection or reduces solution multiplicity is an interesting direction for future research.

Future work should focus on a rigorous proof of the general conjecture, a systematic classification of b-dependent versus b-independent solution branches, and numerical stability and interaction studies to assess the persistence and physical relevance of the obtained peakon and pseudo-peakon solutions. These directions will further clarify the role of higher-order terms and the parameter b in shaping coherent structures in nonlinear dispersive systems.

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