

Localized nonlinear excitation on the doubly periodic backgrounds governed byWeierstrass type elliptic waves of the fifth-order KdV equation

Zheyuan Li(李哲渊), Yunqing Yang(杨云青)
Citation: Chin. Phys. B. 35, 090501 (2026). DOI: 10.1088/1674-1056/ae5c7b | CSTR: 32038.14.CPB.ae5c7b
http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Propagation of kinks embedded in nonlocal periodic waves in the inhomogeneous fiber with higher-order effect

Xueping Cheng(程雪苹), Yuqin Li(李瑜琴), Ziyong Ding(丁自勇)
Chin. Phys. B. 35, 090502 (2026). DOI: 10.1088/1674-1056/ae617e | CSTR: 32038.14.CPB.ae617e

Periodic lump, soliton, and some mixed solutions of the (2+1)-dimensional generalized coupled nonlinear Schrödinger equations

Xiao-Min Wang(王晓敏), Ji Li(李吉), Xiao-Xiao Hu(胡霄骁)
Chin. Phys. B. 34, 110502 (2025). DOI: 10.1088/1674-1056/ae1019 | CSTR: 32038.14.CPB.ae1019

The N-periodic wave solutions to the supersymmetric Sawada–Kotera–Ramani equation

Pengcheng Xin(辛鹏程), Zhonglong Zhao(赵忠龙), Yu Wang(王宇)
Chin. Phys. B. 34, 020502 (2025). DOI: 10.1088/1674-1056/ad9912 | CSTR: 32038.14.CPB.ad9912

Analytical three-periodic solutions of Korteweg–de Vries-type equations

Mi Chen(陈觅), Zhen Wang(王振)
Chin. Phys. B. 32, 090504 (2023). DOI: 10.1088/1674-1056/acd9c4

Unstable periodic orbits analysis in the Qi system

Lian Jia(贾莲), Chengwei Dong(董成伟), Hantao Li(李瀚涛), Xiaohong Sui(眭晓红)
Chin. Phys. B. 32, 040502 (2023). DOI: 10.1088/1674-1056/aca9c3

SPECIAL TOPIC — Nonlinear localized waves: Theory, methods and applications

Localized nonlinear excitation on the doubly periodic backgrounds governed by Weierstrass type elliptic waves of the fifth-order KdV equation

Zheyuan Li(李哲渊) and Yunqing Yang(杨云青)[†]

School of Science, Zhejiang University of Science and Technology, Hangzhou 310023, *China*

(Received 7 March 2026; revised manuscript received 3 April 2026; accepted manuscript online 8 April 2026)

In this work, the linear spectral problems, that is, the Lax pairs of the fifth-order Korteweg–de Vries (KdV) model with traveling and static Weierstrass functions as coefficients, are investigated using the Lamé equation, and the nonlinear localized waves on the Weierstrass elliptic periodic function backgrounds are derived using Darboux transformations, whose propagation and corresponding nonlinear dynamics are also discussed. Furthermore, two types of degenerate nonlinear waves are studied in the limiting cases of the half-periods ω_1 and ω_2 of the Weierstrass-$\wp(x)$ function. The findings are beneficial for understanding and forecasting some nonlinear phenomena in related physical fields, including oceanography and hydrodynamics.

Keywords: periodic-background solution, fifth-order KdV model, linear spectral problems, Darboux transformation

PACS: 05.45.Yv, 02.30.Ik

DOI: 10.1088/1674-1056/ae5c7b **CSTR:** 32038.14.CPB.ae5c7b

1. Introduction

Nonlinear evolution equations (NEEs), as theoretical prototypes for describing various nonlinear phenomena, are fundamental in various areas of nonlinear science, including condensed matter physics,[1] nonlinear optics,[2,3] fluid dynamics,[4] plasma,[5] financial markets,[6] and water waves.[7] Constructing physically meaningful nonlinear waves and studying their corresponding underlying nonlinear dynamical properties have always been fundamental and core tasks in investigating NEEs, which has attracted increasing attention.

Integrable systems, as a special class of NEEs that can be equivalent to the compatibility conditions of linear spectral problems, have long served as the core governing equations for characterizing the evolution and dynamical behaviors of nonlinear waves. Consequently, a large number of powerful and effective techniques have been developed to investigate integrable models, including inverse scattering,[8–10] Darboux transformation (DT),[11–15] Hirota bilinear,[16] Riemann–Hilbert problem[17,18] methods, and the recently emerging deep learning approach.[19–28]

The majority of current investigations concentrate mainly on constant backgrounds, which mainly involve solitons with zero background, as well as rogue waves and breathers on plane wave backgrounds. The relevant nonlinear wave theories[29,30] and laboratory experiments[31] demonstrate that non-constant backgrounds are more adept at describing physical reality. Therefore, nonlinear local excitation on non-constant backgrounds has always been a fascinating and challenging topic in various physical fields.

Investigations on non-constant backgrounds concentrate mainly on periodic wave backgrounds represented by Jacobi elliptic forms, and several effective techniques have been proposed, including DT and numerical methods,[32,33] nonlocal symmetry and the dressing method,[34,35] DT and nonlinearization technique,[36–40] DT and theta functions,[41–43] DT and Baker-Akhiezer function,[44] deep learning[45] and so on.[46,47] As another typical representative of periodic background waves, nonlinear localized excitation on Weierstrass periodic wave backgrounds has recently attracted increasing attention. Zhang *et al.*[48] constructed the bilinear system scheme for Weierstrass elliptic solitons in 2020. Nijhoff *et al.*[49] proposed the elliptic N-th roots of unity, which provides a powerful tool for analyzing nonlinear waves on periodic backgrounds governed by Weierstrass functions. Subsequently, studies on nonlinear wave solutions on Weierstrass elliptic periodic wave backgrounds have been gradually presented.[50–53]

Despite extensive prior work, nonlinear waves propagating on periodic wave backgrounds governed by Weierstrass-type elliptic functions of the fifth-order Korteweg–de Vries equation (fKdV)[54] have not been studied. Motivated by this critical omission, this paper is mainly devoted to the following

[†]Corresponding author. E-mail: yqyang@amss.ac.cn

© 2026 Chinese Physical Society and IOP Publishing Ltd. All rights, including for text and data mining, AI training, and similar technologies, are reserved.

fKdV equation:

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(10u^3+5\left(\frac{\partial u}{\partial x}\right)^2+10u\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}\right)=0, \quad (1)$$

where $u\equiv u(x,t)$, which has been widely applied in fluid mechanics, such as shallow water waves with surface tension, as well as in plasma physics. The structure of this paper is organized in the following manner. In Section 2, the first and n-th DT for the fKdV Eq. (1) are given directly. In Sections 3 and 4, the $\wp$-periodic functions, including the stationary and traveling cases, are selected to be the seed solutions, yielding various nonlinear wave solutions on the Weierstrass-type periodic backgrounds for the fKdV Eq. (1), and the corresponding dynamic properties are analyzed. Finally, in Section 5, the main findings and conclusions are summarized.

2. DT for the fifth-order KdV model

Through the existence of Lax pairs, the fKdV Eq. (1) is integrable, that is, the fKdV Eq. (1) is equivalent to the compatibility condition of the linear spectral system

$$\begin{aligned} L\kappa &= \left(-\partial_x^2-u\right)\kappa=\lambda\kappa, && (2a)\\ \kappa_t &= \Big[\left(16\lambda^2-8\lambda u+6u^2+2u_{xx}\right)\partial_x \\ &\quad +\left(-6uu_x+4\lambda u_x-u_{xxx}\right)\Big]\kappa=T\kappa. && (2b)\end{aligned}$$

Based on Matveev's theorem, the first and n-th forms of the DT for the fKdV Eq. (1) are presented explicitly in the theorems.

Theorem 1 The functions $\left(u^{(1)},\kappa^{(1)}\right)$ satisfy the linear spectral system

$$\begin{aligned} &\kappa_{xx}^{(1)}+u^{(1)}\kappa^{(1)}+\lambda\kappa^{(1)}=0, && (3a)\\ &\kappa_t^{(1)}=\left(16\lambda^2-8\lambda u^{(1)}+6(u^{(1)})^2+2u_{xx}^{(1)}\right)\kappa_x^{(1)} \\ &\qquad +\left(4\lambda u_x^{(1)}-6u^{(1)}u_x^{(1)}-u_{xxx}^{(1)}\right)\kappa^{(1)}, && (3b)\end{aligned}$$

in which u is a solution of the fKdV Eq. (1), κ_1 is a solution of the linear spectral system (2) under $\lambda=\lambda_1$, and

$$\begin{aligned} \kappa^{(1)} &= \kappa_x-(\ln\kappa_1)_x\kappa, && (4a)\\ u^{(1)} &= u+2(\ln\kappa_1)_{xx}. && (4b)\end{aligned}$$

In other words, the form of the linear spectral system (2) is invariant under $(u,,\kappa)\to\left(u^{(1)},,\kappa^{(1)}\right)$, representing the DT of the fKdV Eq. (1).

It is worth noting that this process allows for successive iteration, yielding the n-th DT of the fKdV Eq. (1). To this end, we first introduce n solutions of the system (2) and denote $u^{(\mu)},\kappa^{(\mu)},\kappa_j{}^{(\mu)}$ as the functions obtained after μ iterations of the DT on the corresponding variables u,κ,κ_j, that is

$$\begin{aligned} &\kappa_{j_{xx}}^{(\mu)}+u^{(\mu)}\kappa_j{}^{(\mu)}+\lambda\kappa_j{}^{(\mu)}=0, && (5a)\\ &\kappa_{j_t}^{(\mu)}=\left(16\lambda^2-8\lambda u^{(\mu)}+6(u^{(\mu)})^2+2u_{xx}^{(\mu)}\right)\kappa_{j_x}^{(\mu)} \\ &\qquad +\left(-6u^{(\mu)}u_x^{(\mu)}+4\lambda u_x^{(\mu)}-u_{xxx}^{(\mu)}\right)\kappa_j{}^{(\mu)}. && (5b)\end{aligned}$$

A straightforward verification yields that for arbitrary integers μ,k ($\mu,\in[1,\ldots,n-1],,k,\in[1,\ldots,\mu-1]$)

$$\begin{aligned} &\mathcal{W}\left[\kappa_{\mu+1}^{(\mu)},\ldots,\kappa_{\mu+k}^{(\mu)}\right] \\ &=\frac{\mathcal{W}\left[\kappa^{(\mu-1)}*\mu,\kappa^{(\mu-1)}*\mu+1,\ldots,\kappa_{\mu+k}^{(\mu-1)}\right]}{\kappa_\mu^{(\mu-1)}}, && (6)\\ &\mathcal{W}[\kappa_{\mu+1}^{(\mu)},\ldots,\kappa_{\mu+k}^{(\mu)},\kappa^{(\mu)}] \\ &=\frac{\mathcal{W}[\kappa_\mu^{(\mu-1)},\kappa_{\mu+1}^{(\mu-1)},\ldots,\kappa_{\mu+k}^{(\mu-1)},\kappa^{(\mu-1)}]}{\kappa_\mu^{(\mu-1)}}. && (7)\end{aligned}$$

where $\mathcal{W}$ denotes the Wronskian determinant. Subsequently, in the following theorem, we obtain the n-th DT of the fKdV Eq. (1).

Theorem 2 The functions $\left(u^{(n)},\kappa^{(n)}\right)$ satisfy

$$\begin{aligned} &\kappa_{xx}^{(n)}+u^{(n)}\kappa^{(n)}+\lambda\kappa^{(n)}=0, && (8a)\\ &\kappa_t^{(n)}=\left(16\lambda^2-8\lambda u^{(n)}+6(u^{(n)})^2+2u_{xx}^{(n)}\right)\kappa_x^{(n)} \\ &\qquad +\left(4\lambda u_x^{(n)}-6u_x^{(n)}u^{(n)}-u_{xxx}^{(n)}\right)\kappa^{(n)}, && (8b)\end{aligned}$$

in which

$$\begin{aligned} \kappa^{(n)} &= \frac{\mathcal{W}[\kappa_1,\kappa_2,\ldots,\kappa_n,\kappa]}{\mathcal{W}[\kappa_1,\kappa_2,\ldots,\kappa_n]}, && (9a)\\ u^{(n)} &= u+2\left(\ln\mathcal{W}[\kappa_1,\kappa_2,\ldots,\kappa_n]\right)_{xx}, && (9b)\end{aligned}$$

where u is a solution of Eq. (1), and $\kappa_1,\kappa_2,\ldots,\kappa_n$ are solutions of the Lax pair (2) of Eq. (1) with potential u. In other words, $\left(u^{(n)},\kappa^{(n)}\right)$ and (u,κ) satisfy the same linear spectral problem (2), and therefore $u^{(n)}$ satisfies

$$u_t^{(n)}-\partial_x\left(10(u^{(n)})^3+5(u_x^{(n)})^2+10u_{xx}^{(n)}u^{(n)}+u_{xxxx}^{(n)}\right)=0. \quad (10)$$

Proof Using Eqs. (4), (6) and (7),

$$\begin{aligned} \kappa^{(n)} &= \kappa_x^{(n-1)}-\frac{\kappa_{nx}^{(n-1)}}{\kappa_n^{(n-1)}}\kappa^{(n-1)} \\ &= \frac{\mathcal{W}[\kappa_n^{(n-1)},\kappa^{(n-1)}]}{\mathcal{W}[\kappa_n^{(n-1)}]} \\ &= \frac{\mathcal{W}[\kappa_{n-1}^{(n-2)},\kappa_n^{(n-2)},\kappa^{(n-2)}]}{\mathcal{W}[\kappa_{n-1}^{(n-2)},\kappa_n^{(n-2)}]} \\ &= \cdots \\ &= \frac{\mathcal{W}[\kappa_1,\kappa_2,\ldots,\kappa_n,\kappa]}{\mathcal{W}[\kappa_1,\kappa_2,\ldots,\kappa_n]}, \\ u^{(n)} &= u^{(n-1)}+2\frac{\partial^2\ln\left(\mathcal{W}\left[\kappa_n^{(n-1)}\right]\right)}{\partial x^2} \\ &= u^{(n-2)}+2\frac{\partial^2(\ln\kappa_{n-1}^{(n-2)})}{\partial x^2}\end{aligned}$$

$$
\begin{aligned}
&+2\frac{\partial^2}{\partial x^2}\left(\ln\frac{\mathcal{W}\left[\kappa_{n-1}^{(n-2)},\kappa_n^{(n-2)}\right]}{\kappa_{n-1}^{(n-2)}}\right)\\
&=u^{(n-2)}+2\frac{\partial^2\left(\ln\mathcal{W}\left[\kappa_{n-1}^{(n-2)},\kappa_n^{(n-2)}\right]\right)}{\partial x^2}\\
&=\cdots\\
&=u+2\left(\ln\mathcal{W}[\kappa_1,\kappa_2,\ldots,\kappa_n]\right)_{xx}. \qquad (11a)
\end{aligned}
$$

3. Nonlinear waves on the periodic background governed by static $\wp(x)$ function

To be brief, some properties associated with the Weierstrass functions are listed without proof

$$\wp''''(x)=-18\wp'(x)(g_2-20\wp(x)^2), \qquad (12a)$$

$$\zeta(x+\theta)-(\zeta(x)+\zeta(\theta))=\frac{1}{2}\frac{\wp'(\theta)-\wp'(x)}{\wp(\theta)-\wp(x)}, \qquad (12b)$$

$$\wp(x+\theta)+\wp(\theta)+\wp(x)=(\zeta(\theta)+\zeta(x)-\zeta(x+\theta))^2, \qquad (12c)$$

in which $\zeta(x)$ represents the Weierstrass zeta-function.[55] Employing the function expansion technique, the Weierstrass elliptic function form solution of Eq. (1) is readily derived as

$$u=-2\wp(x)+\sqrt{-\frac{g_2}{15}}. \qquad (13)$$

Here, the periodic wave solution (13) is selected to serve as the seed solution, from which some new nonlinear waves can be constructed by DT Eq. (4) on this periodic background, and a thorough discussion is conducted on its corresponding evolutionary characteristics and nonlinear dynamical behaviors.

3.1. Solutions of the Lax pair (2) with static $\wp(x)$ coefficients

Inserting the periodic solution Eq. (13) into Eq. (2) leads to a new linear spectral problem whose coefficients are given by elliptic functions

$$
\begin{aligned}
&\kappa_{xx}-2\wp(x)\kappa+\eta\kappa=0,\\
&\kappa_t+8\left[\frac{\sqrt{-15g_2}}{5}\left(\wp(x)+\frac{\lambda}{3}\right)-2\lambda\wp(x)-2\lambda^2-\frac{g_2}{5}\right]\kappa_x\\
&+4\left(-\frac{\sqrt{-15g_2}}{5}\wp'(x)+2\lambda\wp'(x)\right)\kappa=0, \qquad (14a)
\end{aligned}
$$

where $\eta=\lambda+\sqrt{-\frac{g_2}{15}}$. Here, Eq. (14a) is precisely the Lamé equation[55,56] with the Weierstrass function $\wp(x)$ as the coefficient, whose solution takes the form

$$\kappa=\Phi_\rho(x)\,\mathrm{e}^{-\zeta(\rho)x+\omega t}, \qquad (15)$$

where $\rho,,\omega$ are arbitrary constants, and $\Phi_\rho(x)$ denotes the Lamé equation[55]

$$\Phi_\rho(x)=\frac{\sigma(x+\rho)}{\sigma(x)\sigma(\rho)}, \qquad (16)$$

where $\sigma(x)$ stands for the Weierstrass σ-function. By means of the properties Eq. (12) associated with the Weierstrass elliptic function, one can derive

$$\kappa_x=\left(\zeta(x+\rho)-\zeta(\rho)-\zeta(x)\right)\kappa, \qquad (17a)$$

$$\kappa_{xx}=\left(\wp(\rho)+2\wp(x)\right)\kappa, \qquad (17b)$$

$$\kappa_t=\omega\kappa. \qquad (17c)$$

Substituting Eqs. (16) and (17) into Eq. (14) yields

$$
\begin{cases}
\lambda=-\wp(\rho)-\sqrt{-\dfrac{g_2}{15}},\\
\omega-\dfrac{4\sqrt{-15g_2}\wp'(\rho)}{3}-8\wp(\rho)\wp'(\rho)=0,
\end{cases} \qquad (18)
$$

which implies that Eq. (15) solves Eq. (14) with the constraint conditions Eq. (18). Consequently, the solution of the linear spectral system (14) reduces to

$$\kappa_\lambda=\Phi_\rho(x)\,\mathrm{e}^{-\zeta(\rho)x+\left(\frac{4\sqrt{-15g_2}\wp'(\rho)}{3}+8\wp(\rho)\wp'(\rho)\right)t}. \qquad (19)$$

Nonetheless, only a translated form of seed solution (13) can be derived by substituting Eq. (19) into the DT (4). Therefore, it is necessary to derive the general solution of the linear spectral system Eq. (14), which means providing linearly independent solutions of Eq. (19) associated with the given eigenvalue λ under the constraints Eq. (18), which can be achieved through the elliptic square roots of unity.[49]

Theorem 3 For the given eigenvalue λ_l, the general solution for the new linear spectral system Eq. (14) under the constraint conditions Eq. (18) can be given as

$$\kappa_{\lambda_l}=\sum_{j=1}^{2}C_{lj}\kappa_{\lambda_{lj}}=\sum_{j=1}^{2}C_{lj}\Phi_{\rho_{lj}}(x)\,\mathrm{e}^{\iota_{lj}}, \qquad (20)$$

in which, for $j=1,2$

$$
\begin{aligned}
&\iota_{lj}=-\zeta\left(\rho_{lj}\right)x+\omega_{lj}t,\\
&\omega_{lj}=\frac{4\sqrt{-15g_2}\wp'(\rho_{lj})}{3}+8\wp(\rho_{lj})\wp'(\rho_{lj}), \qquad (21)
\end{aligned}
$$

with ρ_{lj} being the two roots of the elliptic square-type Eq. (18) for $\lambda=\lambda_l$, and C_{lj} being arbitrary constants.

3.2. Nonlinear wave solutions on the static $\wp(x)$ background

Taking $l=1$ in Theorem 3 yields

$$\kappa_{\lambda_1}=C_{1,1}\kappa_{\lambda_{1,1}}+C_{1,2}\kappa_{\lambda_{1,2}}, \qquad (22)$$

and the corresponding properties of the Weierstrass elliptic function can be simplified as

$$
\begin{aligned}
&\rho_1+\rho_2\equiv 0\ \ (\mathrm{mod\ period\ lattice}),\\
&\zeta(\rho_1)+\zeta(\rho_2)=0,\\
&\wp(\rho_1)=\wp(\rho_2). \qquad (23)
\end{aligned}
$$

Accordingly, the first nonlinear wave solution associated with the Weierstrass $\wp$-function (13) of the fKdV Eq. (1) is written in the form

$$u^{(1)} = u + 2\left[\frac{\Delta_{xx}}{\Delta} - \left(\frac{\Delta_x}{\Delta}\right)^2\right], \tag{24}$$

in which $C_{1,1} = C_{1,2} = 1$ is adopted for brevity, with the difference between them being nothing more than separate phase terms, and $\Delta = \left(\kappa_{\lambda_{1,1}} + \kappa_{\lambda_{1,2}}\right)$.

Based on the relationships Eqs. (17a) and (17b), and the constraint condition $\lambda_1 = -\wp(\rho_{1,j}) - \sqrt{-\frac{g_2}{15}}$ $(j = 1,2)$, the following simplifications can be obtained as

$$\frac{\Delta_{xx}}{\Delta} = 2\wp(x) - \lambda_1 - \sqrt{-\frac{g_2}{15}}, \tag{25}$$

and

$$\left(\frac{\Delta_x}{\Delta}\right)^2 = \frac{G_1^2\kappa_{\lambda_{1,1}}^2 + G_2^2\kappa_{\lambda_{1,2}}^2 + 2G_1G_2\kappa_{\lambda_{1,1}}\kappa_{\lambda_{1,2}}}{\kappa_{\lambda_{1,1}}^2 + \kappa_{\lambda_{1,2}}^2 + 2\kappa_{\lambda_{1,1}}\kappa_{\lambda_{1,2}}}, \tag{26}$$

in which, for $j = 1,,2$, $G_j = \zeta\left(x+\rho_{1,j}\right) - \left(\zeta(x) + \zeta\left(\rho_{1,j}\right)\right)$. Simultaneously, it can be derived from the properties Eqs. (12b) and (12c) that

$$G_j^2 = \wp(x) + \wp(\rho_{1,j}) + \wp(x+\rho_{1,j}), \quad j = 1,2, \tag{27a}$$

$$G_1G_2 = \frac{\wp(x)'^2 - \wp(\rho_{1,1})'^2}{4\left(\wp(x) - \wp(\rho_{1,1})\right)^2}. \tag{27b}$$

The characteristics of Lamé function[55] demonstrate that

$$\kappa_{\lambda_{1,1}}\kappa_{\lambda_{1,2}} = \wp(x) - \wp(\rho_{1,1}), \tag{28a}$$

$$\kappa_{\lambda_{1,j}}^2 = \left[\left(\zeta(\rho_{1,j}) + 2\zeta(x)\right) - \zeta(2x+\rho_{1,j})\right] \times \Phi_{\rho_{1,j}}(2x)\,\mathrm{e}^{2\iota_{1j}}, \tag{28b}$$

where ι_{1j} is defined in Eq. (21).

Combining the relationships given in Eqs. (25)–(28), the first nonlinear wave $u^{(1)}$ (see Eq. (24)) can be written as

$$u^{(1)} = 2\wp(x) - 2\lambda_1 - \sqrt{-\frac{g_2}{15}} - \frac{G1}{D1}. \tag{29}$$

in which

$$\begin{aligned} G_1(x,t) &= 2M_1N_1\Phi_{\rho_{1,1}}(2x)\,\mathrm{e}^{2\iota_{11}} \\ &\quad +2M_2N_2\Phi_{\rho_{1,2}}(2x)\,\mathrm{e}^{2\iota_{12}} \\ &\quad +2\sqrt{M_1}\left(\wp'(x) + \wp'(\rho_{1,1})\right), \end{aligned} \tag{30a}$$

$$\begin{aligned} D_1(x,t) &= N_1\Phi_{\rho_{1,1}}(2x)\,\mathrm{e}^{2\iota_{11}} + N_2\Phi_{\rho_{1,2}}(2x)\,\mathrm{e}^{2\iota_{12}} \\ &\quad +2\left(\wp(x) - \wp(\rho_{1,1})\right), \end{aligned} \tag{30b}$$

and

$$M_j = \wp(x) + \wp(\rho_{1,j}) + \wp(x+\rho_{1,j}), \tag{31a}$$

$$N_j = \zeta(\rho_{1,j}) + 2\zeta(x) - \zeta(2x+\rho_{1,j}), \tag{31b}$$

$$\iota_{1j} = -\zeta\left(\rho_{1j}\right)x + \left(\frac{4\sqrt{-15g_2}\wp'(\rho_{1j})}{3} + 8\wp(\rho_{1j})\wp'(\rho_{1j})\right)t, \tag{31c}$$

for $j = 1,2$.

Various types of nonlinear waves on the $\wp(x)$-type periodic background Eq. (13) for the fKdV Eq. (1) can be constructed by choosing different parameter values of λ_1. It can be indicated from the form of solution $u^{(1)}$ that this solution belongs to a modified breather structure. Figure 1 depicts the 3D and corresponding density propagation under the parameters $g_2 = 3$, $g_3 = 1$ and $g_2 = \frac{27}{4}$, $g_3 = \frac{27}{8}$. It is evident from the figure that the dark breather wave propagates along a linear path against the periodic wave background. Additionally, by selecting distinct parameter sets, nonlinear waves with varying amplitude characteristics can be generated.

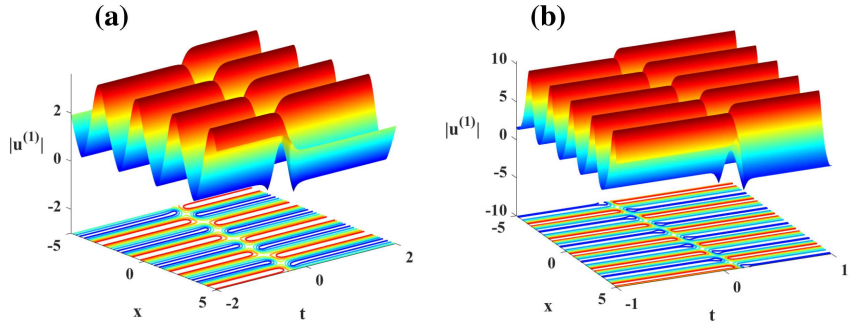


Fig. 1. The 3D evolution and corresponding contour propagation of the nonlinear wave Eq. (29), with (a) $\lambda_1 = -0.3 - \mathrm{i}\left(0.9 + \frac{\sqrt{5}}{5}\right)$, $g_2 = 3$, $g_3 = 1$, and (b) $\lambda_1 = 0.3 - \mathrm{i}\left(2 + \frac{3\sqrt{5}}{10}\right)$, $g_2 = \frac{27}{4}$, $g_3 = \frac{27}{8}$.

3.3. Degenerate cases

The degenerate situations of the nonlinear wave Eq. (29) are studied in this section. For convenience, two limiting situations of the triple Weierstrass elliptic functions are displayed in Table 1, in which ω_1 and ω_2 denote the half-periods of $\wp(x)$.

Case 1 $\omega_1 \to \infty, \omega_2 = \frac{\pi \mathrm{i}}{2p}$

Under this condition, the seed solution Eq. (13) reduces

to the static nonlinear solution

$$u \to -\frac{2p^2}{3} - 2p^2 \sinh^{-2}(px) + \sqrt{-\frac{4p^4}{45}}, \tag{32}$$

and the corresponding nonlinear wave solution Eq. (29) on the periodic background Eq. (13) reduces to the deformed solitary wave solution

$$u^{(1)} \to \frac{2p^2}{3} + \frac{2p^2}{\sinh^2(px)} - 2\lambda_1 - \sqrt{-\frac{4p^4}{45}} - \frac{G_2(x,t)}{D_2(x,t)}, \tag{33}$$

where

$$\begin{aligned} G_2(x,t) &= 2\tilde{M}_1\tilde{N}_1\tilde{\Phi}_{\rho_{1,1}}(2x)\,\mathrm{e}^{2\tilde{\imath}_{11}} + 2\tilde{M}_2\tilde{N}_2\tilde{\Phi}_{\rho_{1,2}}(2x)\,\mathrm{e}^{2\tilde{\imath}_{12}} \\ &\quad + 4p^3\sqrt{\tilde{M}_1}\left(\frac{\cosh(px)}{\sinh^3(px)} + \frac{\cosh(p\rho_{1,1})}{\sinh^3(p\rho_{1,1})}\right), \\ D_2(x,t) &= \tilde{N}_1\tilde{\Phi}_{\rho_{1,1}}(2x)\,\mathrm{e}^{2\tilde{\imath}_{11}} + \tilde{N}_2\tilde{\Phi}_{\rho_{1,2}}(2x)\,\mathrm{e}^{2\tilde{\imath}_{12}} \\ &\quad + 2p^2\left(\frac{1}{\sinh^2(px)} - \frac{1}{\sinh^2(p\rho_{1,1})}\right), \end{aligned} \tag{34}$$

where for $j = 1,2$

$$\begin{aligned} &\tilde{\Phi}_{\rho_{1,j}}(x) = p\frac{\sinh(px+p\rho_{1,j})}{\sinh(px)\sinh(p\rho_{1,j})}\,\mathrm{e}^{-\frac{p^2x\rho_{1,j}}{3}}, \\ &\tilde{M}_j = p^2\sinh^{-2}(px) + p^2\sinh^{-2}(p\rho_{1,j}) \\ &\qquad + p^2\sinh^{-2}(px+p\rho_{1,j}) + p^2, \\ &\tilde{N}_j = 2p\coth(px) + p\coth(p\rho_{1,j}) - p\coth(2px+p\rho_{1,j}), \\ &\tilde{\imath}_{1j} = \frac{p^2}{3}\rho_{1j}x - p\coth(p\rho_{1j})x - \frac{16p^5\cosh(p\rho_{1j})}{3\sinh^5(p\rho_{1j})} \\ &\qquad \times\left[(\sqrt{-5}+1)\sinh^2(p\rho_{1j}) + 3\right]t. \end{aligned} \tag{35}$$

To give a clear demonstration, Fig. 2 displays the evolutionary characteristics of the deformed soliton solution Eq. (33) under the parameter settings $p = 1$ ($g_2 = \frac{4}{3}$, $g_3 = -\frac{8}{27}$).

Table 1. Degenerate cases of the Weierstrass functions (p is positive).

Cases	$\omega_1 \to \infty, \omega_2 = \frac{\pi \mathrm{i}}{2p}$	$\omega_1 \to \infty, \omega_2 \to \mathrm{i}\infty$
$g_j\ (j=2,3)$	$g_2 = \frac{4p^4}{3}, g_3 = -\frac{8p^6}{27}$	$g_2 = 0, g_3 = 0$
$\sigma(x)$	$\frac{\sinh(px)}{p}\,\mathrm{e}^{-\frac{p^2x^2}{6}}$	x
$\zeta(x)$	$-\frac{p^2}{3}x + p\coth(px)$	$\frac{1}{x}$
$\wp(x)$	$\frac{p^2}{3} + p^2\operatorname{csch}^2(px)$	$\frac{1}{x^2}$

Case 2 $\omega_1 \to \infty$, $\omega_2 \to \mathrm{i}\infty$

Under this condition, the seed solution Eq. (13) reduces to the rational solution

$$u \to -\frac{2}{x^2}, \tag{36}$$

and the nonlinear wave Eq. (29) degenerates to the rational solution, written as

$$u^{(1)} \to \frac{2}{x^2} - 2\lambda_1 - \frac{G_3(x,t)}{D_3(x,t)}, \tag{37}$$

where

$$\begin{aligned} G_3(x,t) &= 2\bar{M}_1\bar{N}_1\bar{\Phi}_{\rho_{1,1}}(2x)\,\mathrm{e}^{2\bar{\imath}_{11}} + 2\bar{M}_2\bar{N}_2\bar{\Phi}_{\rho_{1,2}}(2x)\,\mathrm{e}^{2\bar{\imath}_{12}} \\ &\quad - 2\sqrt{\bar{M}_1}\left(2x^{-3} + 2\rho_{1,1}^{-3}\right), \\ D_3(x,t) &= \bar{N}_1\bar{\Phi}_{\rho_{1,1}}(2x)\,\mathrm{e}^{2\bar{\imath}_{11}} + \bar{N}_2\bar{\Phi}_{\rho_{1,2}}(2x)\,\mathrm{e}^{2\bar{\imath}_{12}} \\ &\quad + 2\left(x^{-2} - \rho_{1,1}^{-2}\right), \end{aligned} \tag{38}$$

in which, for $j = 1,2$,

$$\begin{aligned} &\bar{\Phi}_{\rho_{1,j}}(x) = \frac{x+\rho_{1,j}}{x\rho_{1,j}}, \\ &\bar{M}_j = \frac{1}{x^2} + \frac{1}{\rho_{1,j}^2} + \frac{1}{(x+\rho_{1,j})^2}, \\ &\bar{N}_j = \frac{2}{x} + \frac{1}{\rho_{1,j}} - \frac{1}{2x+\rho_{1,j}}, \\ &\bar{\imath}_{1j} = -\frac{1}{\rho_{1j}}x - \frac{16}{\rho_{1j}^5}t. \end{aligned} \tag{39}$$

By selecting some given parameter values, Fig. 3 demonstrates the time evolution and associated density propagation of the rational wave Eq. (37), which depicts a soliton.

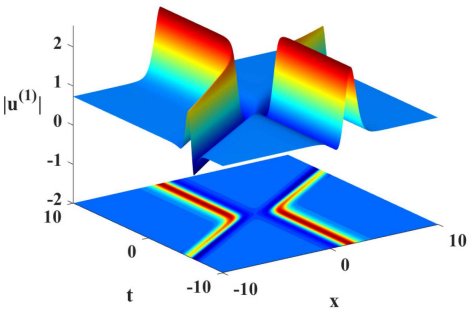


Fig. 2. The 3D evolution and corresponding density propagation of the soliton Eq. (33) with $\lambda_1 = 0.3 + (0.1 - \frac{2\sqrt{5}}{15})\mathrm{i}$, $p = 1$.

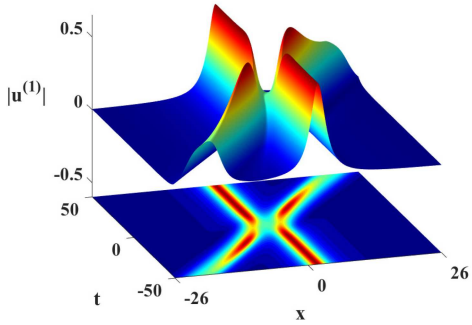


Fig. 3. The 3D evolution and corresponding density plot of the deformed soliton solution Eq. (37) with $\lambda_1 = 0.01 - 0.15\mathrm{i}$.

4. Nonlinear waves on the periodic background governed by traveling $\wp(x+vt)$ function

We further explore the seed solution of DT (4) in the form of the traveling Weierstrass-type periodic function, that is

$$u = \hat{u}(x+vt) = \hat{u}(\xi), \tag{40}$$

where $\xi = x+vt$ with v denoting the traveling speed. Substituting Eq. (40) into the fKdV Eq. (1) results in

$$\begin{aligned}&\hat{u}''''(\xi)+10\hat{u}(\xi)\hat{u}''(\xi)\\&+\left(30\hat{u}(\xi)^2-v+20\hat{u}''(\xi)\right)\hat{u}'(\xi)=0,\end{aligned} \tag{41}$$

whose solution can be obtained by the Weierstrass elliptic function expansion technique as

$$\hat{u}(\xi) = -2\wp(\xi), \tag{42}$$

with $v = 2g_2$.

4.1. Solutions of the Lax pair (2) with traveling $\wp(x+vt)$ coefficients

Inserting the traveling wave Eq. (42) into Eq. (2), the variable-coefficient linear spectral system can be obtained as

$$\hat{\kappa}_{xx} - 2\wp(\xi)\hat{\kappa} + \lambda\hat{\kappa} = 0, \tag{43a}$$

$$\hat{\kappa}_t + \left(-16\lambda^2 - 16\wp(\xi)\lambda - 2g_2\right)\hat{\kappa}_x + 8\lambda\wp'(\xi)\hat{\kappa} = 0, \tag{43b}$$

whose solution is similar to the previous section and can be expressed as

$$\hat{\kappa} = \Phi_\rho(\xi)\, \mathrm{e}^{-\zeta(\rho)\xi+\omega t}, \tag{44}$$

and the corresponding relationships of variable $\hat{\kappa}$ can be deduced as

$$\hat{\kappa}_\xi = \hat{\kappa}_x = \left(\zeta(\xi+\rho)-\zeta(\xi)-\zeta(\rho)\right)\hat{\kappa}, \tag{45a}$$

$$\hat{\kappa}_{\xi\xi} = \hat{\kappa}_{xx} = \left(\wp(\rho)+2\wp(\xi)\right)\hat{\kappa}, \tag{45b}$$

$$\hat{\kappa}_t = \omega\hat{\kappa} + v\left(\zeta(\xi+\rho)-\zeta(\xi)-\zeta(\rho)\right)\hat{\kappa}. \tag{45c}$$

Using the characteristics of the Weierstrass elliptic function, it is easy to obtain from Eq. (43) by substituting Eqs. (44) and (45) that

$$\lambda = -\wp(\rho),\ \omega - 8\wp(\rho)\wp'(\rho) = 0. \tag{46}$$

Then, two linearly independent solutions to Eq. (43) that satisfy the constraint condition Eq. (46) can be expressed as

$$\hat{\kappa}_j = \Phi_{\rho_j}(\xi)\, \mathrm{e}^{-\zeta(\rho_j)\xi+(8\wp(\rho_j)\wp'(\rho_j))t}, \tag{47}$$

where ρ_j denote the two elliptic square roots of Eq. (46). We thus conclude with the following theorem.

Theorem 4 For the given spectral parameter $\lambda = \lambda_l$, the generalized solution of the linear spectral system Eq. (43) with constraint conditions Eq. (46) can be expressed as

$$\hat{\kappa}_{\lambda_l} = \sum_{j=1}^{2} C_{lj}\Phi_{\rho_{lj}}(\xi)\, \mathrm{e}^{\hat{\imath}_{lj}}, \tag{48}$$

where, for $j = 1,2$

$$\begin{aligned}\hat{\imath}_{lj} &= -\zeta\left(\rho_{lj}\right)\xi + \omega_{lj}t,\\ \omega_{lj} &= 8\wp(\rho_{lj})\wp'(\rho_{lj}),\end{aligned} \tag{49}$$

with ρ_{lj} being the two roots of the elliptic square-type Eq. (46) for $\lambda = \lambda_l$, and C_{lj} being arbitrary constants.

4.2. Nonlinear wave solutions on the $\wp(x+vt)$ background

Similarly to Subsection 3.2, the new first-order nonlinear wave solution on the $\wp(x+vt)$ background is constructed. By setting $l = 1$ in Theorem 4, we obtain

$$\hat{\kappa}_{\lambda_1} = C_{1,1}\hat{\kappa}_{\lambda_{1,1}} + C_{1,2}\hat{\kappa}_{\lambda_{1,2}}. \tag{50}$$

Under the simplification $C_{1,1} = C_{1,2} = 1$, the first-order nonlinear wave of the fKdV Eq. (1) simplifies to

$$\begin{aligned}u^{(1)} &= \hat{u} + 2\left(\ln\hat{\kappa}_{\lambda_1}\right)_{xx}\\ &= 2\wp(\xi) - 2\lambda_1 - \frac{G_4(x,t)}{D_4(x,t)},\end{aligned} \tag{51}$$

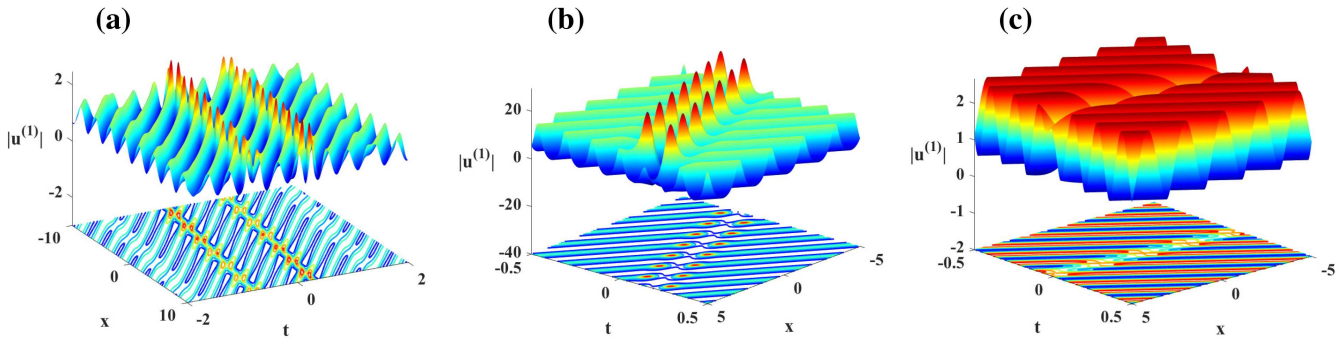


Fig. 4. The 3D evolution and corresponding contour plot of the solution Eq. (51) with (a) $\lambda_1 = -0.7+0.02\mathrm{i}$, $g_2 = 3$, $g_3 = 1$; (b) $\lambda_1 = 1-0.65\mathrm{i}$, $g_2 = \frac{27}{4}$, $g_3 = \frac{27}{8}$; (c) $\lambda_1 = -0.9+0.5\mathrm{i}$, $g_2 = \frac{27}{4}$, $g_3 = \frac{27}{8}$.

with

$$\begin{aligned} G_4(x,t) &= 2P_1Q_1\Phi_{\rho_{1,1}}(2\xi)\mathrm{e}^{2\hat{\imath}_{11}}+2P_2Q_2\Phi_{\rho_{1,2}}(2\xi)\mathrm{e}^{2\hat{\imath}_{12}} \\ &\quad +2\sqrt{P_1}\big(\wp'(\xi)+\wp'(\rho_{1,1})\big), \\ D_4(x,t) &= Q_1\Phi_{\rho_{1,1}}(2\xi)\mathrm{e}^{2\hat{\imath}_{11}}+Q_2\Phi_{\rho_{1,2}}(2\xi)\mathrm{e}^{2\hat{\imath}_{12}} \\ &\quad +2\big(\wp(\xi)-\wp(\rho_{1,1})\big), \end{aligned} \tag{52}$$

where, for $j=1,2$,

$$\begin{aligned} &P_j=\wp(\xi+\rho_{1,j})+\wp(\xi)+\wp(\rho_{1,j}), \\ &Q_j=2\zeta(\xi)-\zeta(2\xi+\rho_{1,j})+\zeta(\rho_{1,j}), \\ &\hat{\imath}_{1j}=-\zeta\left(\rho_{1j}\right)\xi+8\wp(\rho_{1j})\wp'(\rho_{1j})t. \end{aligned} \tag{53}$$

Selecting various parameter values for ω, λ_1, and $C_{1,j}$ $(j=1,2)$, many different nonlinear waves can be obtained. As an example, Fig. 4 illustrates the 3D, contour, and density evolution for the nonlinear wave Eq. (51) under three different sets of parameters.

4.3. Degenerate solutions

It is worth noting that the wave velocity v in the traveling wave seed solution Eq. (42) is related to g_2. Therefore, the nonlinear wave solution Eq. (51) does not completely degenerate into solution (29) when $v=0$, but is the degeneration of solution Eq. (29) with $g_2=0$. If $g_3=0$ is further required, which means $\omega_1\to\infty$ and $\omega_2\to \mathrm{i}\infty$, it is just the degenerate solution Eq. (37). Then, we only need to consider the degenerate case of solution Eq. (51) under the condition of $\omega_1\to\infty$ and $\omega_2=\frac{\pi \mathrm{i}}{2p}$, which exhibits distinct behaviors from solution Eq. (29).

Under this condition, the seed solution Eq. (42) reduces to a singular solution

$$\hat{u}(\xi)\to -2\left(\frac{p^2}{3}+\frac{p^2}{\sinh^2(p\xi)}\right), \tag{54}$$

and the corresponding nonlinear wave solution Eq. (51) reduces to the deformed solitary wave

$$u^{(1)}\to\frac{2p^2}{3}+\frac{2p^2}{\sinh^2(p\xi)}-2\lambda_1-\frac{G_5(x,t)}{D_5(x,t)}, \tag{55}$$

in which

$$\begin{aligned} G_5(x,t) &= 2\tilde{P}_1\tilde{Q}_1\tilde{\Phi}_{\rho_{1,1}}(2\xi)\mathrm{e}^{2\tilde{\imath}_{11}}+2\tilde{M}_2\tilde{Q}_2\tilde{\Phi}_{\rho_{1,2}}(2\xi)\mathrm{e}^{2\tilde{\imath}_{12}} \\ &\quad +4p^3\sqrt{\tilde{P}_1}\left(\frac{\cosh(p\xi)}{\sinh^3(p\xi)}+\frac{\cosh(p\rho_{1,1})}{\sinh^3(p\rho_{1,1})}\right), \\ D_5(x,t) &= \tilde{Q}_1\tilde{\Phi}_{\rho_{1,1}}(2\xi)\mathrm{e}^{2\tilde{\imath}_{11}}+\tilde{Q}_2\tilde{\Phi}_{\rho_{1,2}}(2\xi)\mathrm{e}^{2\tilde{\imath}_{12}} \\ &\quad +2p^2\left(\frac{1}{\sinh^2(p\xi)}-\frac{1}{\sinh^2(p\rho_{1,1})}\right), \end{aligned} \tag{56}$$

where, for $j=1,2$

$$\tilde{\Phi}_{\rho_{1,j}}(\xi)=p\frac{\sinh(p\xi+p\rho_{1,j})}{\sinh(p\xi)\sinh(p\rho_{1,j})}\mathrm{e}^{-\frac{p^2\xi\rho_{1,j}}{3}},$$

$$\begin{aligned} &\tilde{P}_j=\frac{p^2}{\sinh^2(p\xi)}+\frac{p^2}{\sinh^2(p\rho_{1,j})}+\frac{p^2}{\sinh^2(p\xi+p\rho_{1,j})}+p^2, \\ &\tilde{Q}_j=2p\coth(p\xi)+p\coth(p\rho_{1,j})-p\coth(2p\xi+p\rho_{1,j}), \\ &\tilde{\imath}_{1j}=\left(\frac{p^2}{3}\rho_{1j}-p\coth(p\rho_{1j})\right)\xi \\ &\qquad -\frac{16p^5\cosh(p\rho_{1j})}{3\sinh^5(p\rho_{1j})}\left(\sinh^2(p\rho_{1j})+3\right)t. \end{aligned} \tag{57}$$

For illustration, Fig. 5 depicts the evolution of the deformed soliton solution Eq. (55) with parameters $p=1$ ($g_2=\frac{4}{3}$, $g_3=-\frac{8}{27}$).

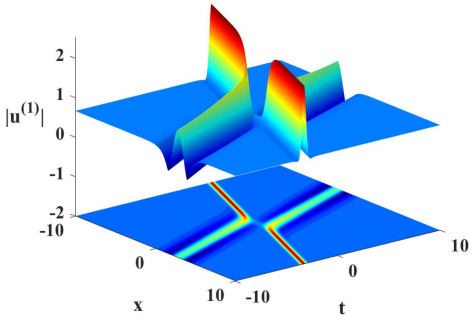


Fig. 5. The propagation of the deformed soliton solution Eq. (55) with $\lambda_1=0.2+0.3\mathrm{i}$, $p=1$.

5. Conclusions and discussions

Constructing physically meaningful nonlinear localized waves on periodic backgrounds and investigating their underlying nonlinear dynamic behaviors remain popular and challenging topics in integrable systems. Most existing studies have focused on periodic wave backgrounds described by Jacobi elliptic functions, while relatively few investigations have concerned $\wp$-function periodic backgrounds. In this paper, based on the eigenfunctions of the spectral system of the fKdV equation with the Weierstrass elliptic $\wp$-functions as coefficients and the DT, the nonlinear localized waves on two types of Weierstrass $\wp$-function backgrounds are derived, and their evolution and corresponding degeneration are also discussed. The techniques and findings proposed in this paper open a new path for studying nonlinear waves on Weierstrass elliptic periodic wave backgrounds for other integrable equations. At the same time, many issues that still need to be further studied have also been proposed, such as: Can we systematically construct various nonlinear waves, including soliton, rogue, and breather waves, on the Weierstrass $\wp$-periodic wave background and theoretically provide the transformation mechanism between different types of nonlinear waves? Can we establish the relationship between nonlinear waves on two types of elliptic periodic wave backgrounds based on the relationship between Weierstrass and Jacobi elliptic functions? Can we provide the determinant forms of high-order nonlinear waves on periodic wave backgrounds and analyze their dy-

namic properties? These are also the issues of our ongoing studies.

Acknowledgment

Project supported by the National Natural Science Foundation of China (NSFC) (Grant No. 12475007).

References

[1] Pitaevskii L and Stringari S 2016 *Bose-Einstein Condensation and Superfluidity* (Oxford: Oxford University Press)
[2] Hasegawa A 2002 *Optical Soliton Theory and Its Applications in Communication* (Berlin: Springer)
[3] Malomed B A 2015 *Nat. Photonics* **9** 287
[4] Craik A D 1988 *Wave Interactions and Fluid Flows* (London: Cambridge University Press)
[5] Bailung H, Sharma S K and Nakamura Y 2011 *Phys. Rev. Lett.* **107** 255005
[6] Yan Z Y 2011 *Phys. Lett. A* **375** 4274
[7] Madsen P A, Bingham H B and Liu H 2002 *J. Fluid Mech.* **462** 1
[8] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249
[9] Zhang G Q and Yan Z Y 2020 *J. Nonlinear Sci.* **30** 3089
[10] Li Y and Tian S F 2022 *Commun. Pure Appl. Anal.* **21** 293
[11] Matveev V B and Salle M A 1991 *Darboux Transformation and Solitons* (Berlin: Springer-Verlag)
[12] Wen X Y, Yang Y Q and Yan Z Y 2015 *Phys. Rev. E* **92** 012917
[13] Yuan C L, Liu W J, Yang H J and Tian Y 2025 *Nonlinear Dyn.* **113** 15273
[14] Zhang Y S, Qiu D Q, Cheng Y and He J S 2017 *Theor. Math. Phys.* **191** 710
[15] Guo B L, Ling L M and Liu Q P 2012 *Phys. Rev. E* **85** 026607
[16] Hirota R 2004 *Direct Methods in Soliton Theory* (Berlin: Springer)
[17] Xu J and Fan E G 2016 *Stud. Appl. Math.* **136** 321
[18] Huang L, Wang D S and Zhu X 2025 *Stud. Appl. Math.* **155** e70075
[19] Pu J C and Chen Y 2023 *Commun. Nonlinear Sci. Numer. Simul.* **118** 107051
[20] Lin S N and Chen Y 2024 *Chaos* **34** 033139
[21] Li J H, Chen J C and Li B 2022 *Nonlinear Dyn.* **107** 781
[22] Zhong M, Yan Z Y and Tian S F 2023 *Commun. Theor. Phys.* **75** 025001
[23] Zhang Z Y, Liu M W and Liu W J 2025 *J. Syst. Sci. Complex.* **38** 2732
[24] Xu S Y, Zhou Q and Liu W 2023 *Nonlinear Dyn.* **111** 18401
[25] Si Z Z, Wang D L, Zhu B W, Ju Z T, Wang X P, Liu W, Malomed B A, Wang Y Y and Dai C Q 2024 *Laser Photon. Rev.* **18** 2400097
[26] Jia Y H, Wang D L, Shen B, Ju Z T, Huang Q Y, Wang X P, Zhang J H, Wang Y Y, Liu W and Dai C Q 2026 *Laser Photon. Rev.* **20** 2503212
[27] Jia Y H, Si Z Z, Ju Z T, Feng H Y, Zhang J H, Yan X and Dai C Q 2025 *Sci. China Phys. Mech. Astron.* **68** 284211
[28] Wang J Y and Yang Y Q 2025 *Eur. Phys. J. Spec. Top.* **234** 5637
[29] Davis R E and Acrivos A 1967 *J. Fluid Mech.* **29** 593
[30] Farmer D M and Smith J D 1980 *Deep-Sea Res. A* **27** 239
[31] Xu G, Chabchoub A, Pelinovsky D E and Kibler B 2020 *Phys. Rev. Res.* **2** 033528
[32] Kedziora D J, Ankiewicz A and Akhmediev N 2014 *Eur. Phys. J. Spec. Top.* **223** 43
[33] Chin S A, Ashour O A, Nikolić S N and Belić M R 2017 *Phys. Rev. E* **95** 012211
[34] Lou S Y, Cheng X P and Tang X Y 2014 *Chin. Phys. Lett.* **31** 070201
[35] Lin J, Jin X W, Gao X L and Lou S Y 2018 *Commun. Theor. Phys.* **70** 119
[36] Chen J B and Pelinovsky D E 2018 *Proc. R. Soc. A* **474** 20170814
[37] Chen J B, Pelinovsky D E and White R E 2019 *Phys. Rev. E* **100** 052219
[38] Chen J B, Pelinovsky D E and White R E 2020 *Physica D* **405** 132378
[39] Zhang H Q and Chen F 2021 *Chaos* **31** 023129
[40] Huang Y H and Guo R 2024 *Phys. Fluids* **36** 045107
[41] Mucalica A and Pelinovsky D E 2024 *Lett. Math. Phys.* **114** 100
[42] Hoefer M A, Mucalica A and Pelinovsky D E 2023 *J. Phys. A-Math. Theor.* **56** 185701
[43] Feng B F, Ling L M and Takahashi D A 2020 *Stud. Appl. Math.* **144** 46
[44] Li R M, Geng J R and Geng X G 2023 *Appl. Math. Lett.* **142** 108661
[45] Peng W Q, Pu J C and Chen Y 2022 *Commun. Nonlinear Sci.* **105** 106067
[46] Yang Y Q, Dong H H and Chen Y 2021 *Wave Motion* **106** 102787
[47] Xu X M and Yang Y Q 2024 *Appl. Math. Lett.* **157** 109208
[48] Li X and Zhang D J 2020 *J. Nonlinear Sci.* **32** 70
[49] Nijhoff F W, Sun Y Y and Zhang D J 2023 *Commun. Math. Phys.* **399** 599
[50] Li J B, Yang Y Q and Sun W Y 2024 *Chaos* **34** 023141
[51] Sun W Y and Sun Y Y 2022 *Appl. Math. Lett.* **128** 107884
[52] Wang Y Q, Yang Y Q, Sun W Y and Li J B 2025 *J. Math. Phys.* **66** 083502
[53] He Q, Li J B, Yang Y Q and Zhang Y S 2025 *Commun. Theor. Phys.* **77** 115002
[54] Lax P D 1968 *Commun. Pure Appl. Math.* **21** 467
[55] Pastras G 2020 *The Weierstrass Elliptic Function and Applications in Classical and Quantum Mechanics: A Primer for Advanced Undergraduates* (Berlin: Springer Nature)
[56] Ince E L 1956 *Ordinary Differential Equations* (Chicago: Courier Corporation)