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SPECIAL TOPIC — Nonlinear localized waves: Theory, methods and applications

Exact solutions for a $(2+1)$-dimensional B-type Kadomtsev–Petviashvili equation

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We study a $(2+1)$-dimensional B-type Kadomtsev–Petviashvili (BKP) equation. Using the Hirota bilinear method, we construct multi-soliton solutions and analyse their propagation and interaction properties. By appropriately tuning the solution parameters, we identify resonant interactions. On the basis of the Bäcklund transformation (BT), we obtain another class of soliton solutions, from which breather solutions are further derived. In addition, a double-pole solution is obtained through an appropriate limiting procedure. These solutions reveal an underlying periodic mechanism associated with parameter tuning and phase matching, providing insight into the emergence of breather oscillations and resonant energy exchange in the BKP system.

Keywords: (2+1)-dimensional BKP equation, Bäcklund transformation, solitons, breather solution, double-pole solution

PACS: 05.45.Yv, 94.05.Fg, 47.35.Fg

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1. Introduction

The Kadomtsev–Petviashvili (KP) equation is a basic model for weakly dispersive, small amplitude waves in quasi two dimensional media.[1–9] It is a completely integrable nonlinear partial differential equation with a Lax pair, so it can be studied by methods such as the inverse scattering transform[10,11] and neural networks.[12] This integrable structure also extends to the KP hierarchy, which is an infinite dimensional system with infinitely many conservation laws, commuting flows, and rich algebraic and geometric symmetries.[13–15] In this setting, the B-type Kadomtsev–Petviashvili (BKP) hierarchy, introduced by Date *et al.*[16] is an important reduction of the KP hierarchy. The BKP equation keeps the main integrable features of the KP theory, including Lax representations, Hirota's bilinear form, and τ-function structures, but it also has different algebraic properties because of the reduction constraint. This reduction further connects the BKP hierarchy to several well-known soliton equations, such as the Sawada–Kotera equation, Ramani's equation, and model equations describing shallow water waves.[17,18]

In the study of integrable equations such as the BKP equation, Hirota's bilinear method has grown from a useful solution method into a general way to construct and classify integrable systems.[19,20] It not only gives concise bilinear forms for many nonlinear models, but also helps discover new integrable structures and extends the theory of soliton equations. Hietarinta reported several Hirota-integrable bilinear equations in 1987, and some of them did not have explicit nonlinear forms for a long time.[21–24] Recent studies show that Bell polynomials provide an effective way to connect bilinear equations with nonlinear equations.[25] In particular, Zhang[26] used this method to derive the nonlinear forms of several bilinear equations given by Hietarinta.

Here, we investigate the $(2+1)$-dimensional nonlinear BKP equation

$$\begin{aligned}&(w_y+15w_xw_{xxx}+15w_x^3+15w_xw_t+w_{xxxxx})_x\\&+5w_{xxxt}-5w_{tt}=0.\end{aligned}\tag{1}$$

Introducing $w=u_x$ and differentiating with respect to x, we obtain[16,26–30]

$$\begin{aligned}&(u_{xxxxx}+15u_xu_{xx}+15uu_{xxx}+45u^2u_x+15uu_t+5u_{xxt}+u_y)_x\\&-5u_{tt}+15u_xu_t+15u_{xx}\int u_t\,\mathrm{d}x=0,\end{aligned}\tag{2}$$

where the subscripts denote partial derivatives with respect to the corresponding variables. u is the wave amplitude function of the scaled space coordinate x, y and time coordinate t.

A broad range of exact solutions to the BKP equation has been reported in recent studies. Soliton solutions under general dispersion relations were studied in Ref. [27], and

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resonant solitons were given in Ref. [28]. Lump and mixed lump-kink solutions were obtained in Refs. [29,30]. Soliton molecules were studied in Ref. [31], where bound states of solitons were discussed. Breather and rogue wave were considered in Ref. [32], and periodic and travelling wave solutions were given in Ref. [33].

However, BT and double-pole solutions have not yet been studied. The paper is organized as follows. In Section 2, we derive a set of one soliton and two soliton solutions under general dispersion relations and analyze their interactions. In Section 3, by adjusting the parameters, we derive the second set of soliton solutions, which allows us to further investigate inelastic collisions. In Section 4, we derive a BT of the equation to obtain the third type of soliton solution, which depends on a single parameter. In Sections 5 and 6, we obtain the double-pole solution and breathers, respectively, based on the soliton solutions.

2. The first set of multi-soliton solutions for Eq. (2)

It is known that Eq. (2) can be converted into the Hirota bilinear representation[26]

$$\left(D_x^6+5D_x^3D_t-5D_t^2+D_xD_y\right)f\cdot f=0, \tag{3}$$

where $f(x,y,t)$ is an auxiliary function, introduced here through a perturbation expansion

$$f(x,y,t)=1+\sum_{n=1}^{\infty}\varepsilon^n f_n(x,y,t), \tag{4}$$

with ε as a parameter. The functions $f_n(x,y,t)$, where $n=1,2,\ldots$, are initially unknown. We derive them step by step using a perturbation method. We substitute this expansion into the bilinear form of the equation. Then we collect the terms of the same order in ε. We set each coefficient of ε^k to zero. This process gives a recursive set of linear equations. We then solve the functions f_n in order.

For the one soliton solution of Eq. (2), we assume

$$f_1=\mathrm{e}^{\xi_1}, \tag{5}$$

where ξ_1 is given by $\xi_1=k_1x+n_1y+\omega_1,t+\xi_1^0$, and the parameters k_1, n_1, ω_1, and ξ_1^0 are real constants.

After these steps, we obtain the relation

$$n_1=\frac{-k_1^6-5k_1^3\omega_1+5\omega_1^2}{k_1}. \tag{6}$$

If we set $f_j=0$ for $j=2,3,\ldots$ and set $\varepsilon=1$, then Eq. (4) becomes

$$f=1+\mathrm{e}^{\xi_1}. \tag{7}$$

Thus, the one soliton solution to Eq. (2) is given by

$$u=2\left[\ln(1+\mathrm{e}^{\xi_1})\right]_{xx}=\frac{k_1^2}{2}\mathrm{sech}^2\left(\frac{\xi_1}{2}\right). \tag{8}$$

For the two soliton solution of Eq. (2), we take the form

$$f=1+\mathrm{e}^{\xi_1}+\mathrm{e}^{\xi_2}+a_{12}\mathrm{e}^{\xi_1+\xi_2}, \tag{9}$$

where $\xi_j=k_jx+m_jy+\omega_jt+\xi_j^0$ for $j=1,2$, and k_j, m_j, ω_j, ξ_j^0, and a_{12} are real constants. The two soliton solution of Eq. (2) is

$$u=2\left[\ln(1+\mathrm{e}^{\xi_1}+\mathrm{e}^{\xi_2}+a_{12}\mathrm{e}^{\xi_1+\xi_2})\right]_{xx}, \tag{10}$$

with

$$\begin{aligned}
n_2&=\frac{-k_2^6-5k_2^3\omega_2+5\omega_2^2}{k_2},\\
a_{12}&=\frac{\Theta_{12}-\Delta_{12}}{\Theta_{12}+\Delta_{12}},\\
\Theta_{12}&=k_1^6k_2^2+4k_1^4k_2^4+k_1^2k_2^6+2k_1^3k_2^2\omega_1+k_1k_2^4\omega_1\\
&\quad+k_1^4k_2\omega_2+2k_1^2k_2^3\omega_2+(k_2\omega_1-k_1\omega_2)^2,\\
\Delta_{12}&=3k_1^2k_2^2\left(k_1^3k_2+k_1k_2^3+k_1\omega_2+k_2\omega_1\right).
\end{aligned}$$

3. The interaction between soliton solutions for Eq. (2)

Next, we study the asymptotic behavior of solution (10). We consider its dependence on the spatial variables x and y, and on the parameters k_1, k_2, ω_1, and ω_2. To keep the discussion simple, we use the following conditions. We assume that k_1, k_2, ω_1, and ω_2 are positive. We also assume that $k_1\omega_1>k_2\omega_2$ and $k_1>k_2$.

If we keep ξ_1 fixed, then ξ_2 can be written as

$$\begin{aligned}
\xi_2&=k_2\left[\frac{\xi_1}{k_1}+\left(\frac{n_2}{k_2}-\frac{n_1}{k_1}\right)y+\left(\frac{\omega_2}{k_2}-\frac{\omega_1}{k_1}\right)t\right.\\
&\quad\left.+\left(\frac{\xi_2^0}{k_2}-\frac{\xi_1^0}{k_1}\right)\right].
\end{aligned} \tag{11}$$

The asymptotic behavior of the solution depends on the direction of the limit. When $(y,t)\to(-\infty,-\infty)$, corresponding to the pre-interaction region, we have $\mathrm{e}^{\xi_2}\to 0$. When $(y,t)\to(\infty,\infty)$, corresponding to the post-interaction region, we have $\mathrm{e}^{-\xi_2}\to 0$. We then obtain the following asymptotic limits

$$\begin{aligned}
\lim_{y\to-\infty\ t\to-\infty}w&=\frac{k_1^2}{2},\mathrm{sech}^2\left(\frac{\xi_1}{2}\right),\\
\lim_{y\to\infty\ t\to\infty}w&=\frac{k_1^2}{2},\mathrm{sech}^2\left(\frac{\xi_1+a_{12}}{2}\right).
\end{aligned} \tag{12}$$

Similarly, when ξ_2 is held constant, ξ_1 can be written as

$$\begin{aligned}
\xi_1&=k_1\left[\frac{\xi_2}{k_2}+\left(\frac{n_1}{k_1}-\frac{n_2}{k_2}\right)y+\left(\frac{\omega_1}{k_1}-\frac{\omega_2}{k_2}\right)t\right.\\
&\quad\left.+\left(\frac{\xi_1^0}{k_1}-\frac{\xi_2^0}{k_2}\right)\right].
\end{aligned} \tag{13}$$

The asymptotic behavior in this case also depends on the direction of the limit: as $(y,t)\to(-\infty,-\infty)$ (pre-interaction), $e^{-\xi_2}\to 0$; whereas as $(y,t)\to(\infty,\infty)$ (post-interaction), $e^{\xi_2}\to 0$. Consequently, we obtain the following asymptotic limits:

$$\lim_{\substack{y\to\infty\\ t\to\infty}} u=\frac{k_1^2}{2},\operatorname{sech}^2\left(\frac{\xi_2}{2}\right),$$
$$\lim_{\substack{y\to-\infty\\ t\to-\infty}} u=\frac{k_1^2}{2},\operatorname{sech}^2\left(\frac{\xi_2+a_{12}}{2}\right). \tag{14}$$

The interactions between solitons can be classified into three types: overtaking, head-on collisions, and parallel interactions.

In Fig. 1, the two solitons move in the same direction. The red soliton is the faster one, and the orange soliton is the slower one. After the faster soliton passes the slower one, it retains its original shape and speed.

In Fig. 2, the two solitons collide head-on. At the center of the collision, the wave amplitude becomes very small and forms a low-energy region. The energy is not lost. Instead, it is redistributed through a change in phase. After the collision, the solitons exhibit a phase shift and continue to move in their original directions.

In Fig. 3, the two solitons retain their shapes and move in parallel. They do not show clear distortion during the interaction. This interaction does not cause energy loss or merging of the solitons.

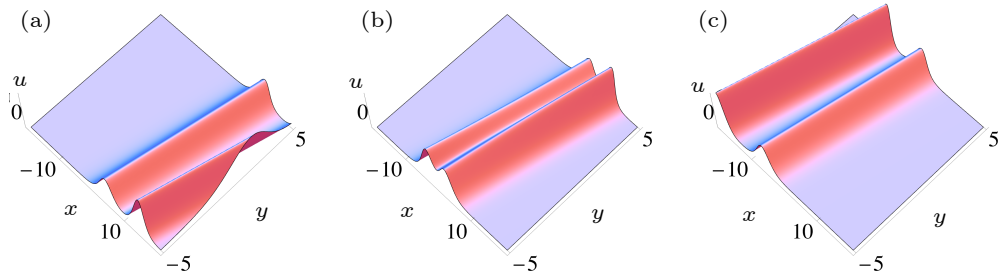


Fig. 1. Overtaking interacting solitons governed by Eq. (10) for parameters $k_1=1$, $k_2=0.8$, $\omega_1=1$, and $\omega_2=0.4$ are depicted at three times: (a) $t=-15$; (b) $t=-2$; (c) $t=15$.

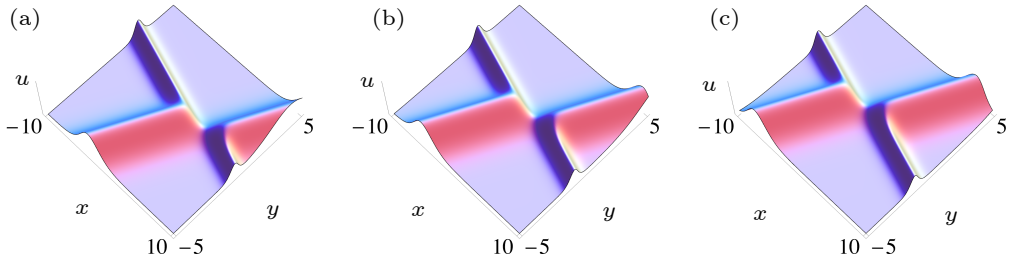


Fig. 2. Head-on interacting solitons governed by Eq. (10) for parameters $k_1=1$, $k_2=0.8$, $\omega_1=1$, and $\omega_2=1.2$ are illustrated at the following times: (a) $t=-2$; (b) $t=0$; (c) $t=2$.

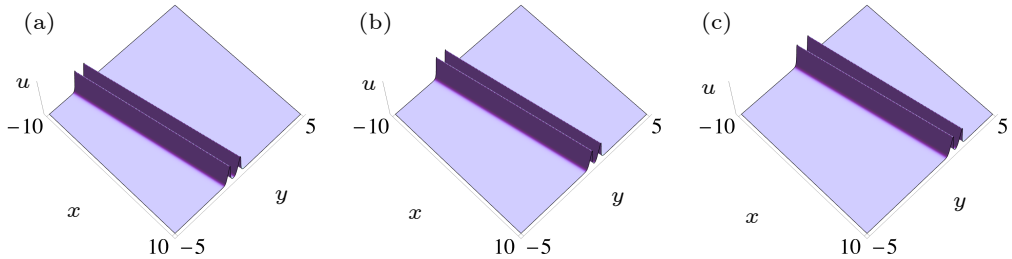


Fig. 3. Parallel interacting solitons governed by Eq. (10) for parameters $k_1=1.2$, $k_2=1.2$, $\omega_1=1.2$, and $\omega_2=0.8$ are illustrated at the following times: (a) $t=-5$; (b) $t=0$; (c) $t=5$.

4. The second set of multi-soliton solutions for Eq. (2)

From Eq. (6), we see that the system has two free parameters. We write them as follows:

$$k_i=p_i-q_i,\ \omega_i=-(p_i^3-q_i^3),\ n_i=9(p_i^5-q_i^5)\ \ (i=1,2).$$

This gives another form of the one soliton solution

$$u=2\left[\ln\left(1+e^{\theta_1}\right)\right]_{xx}=\frac{(p_1-q_1)^2}{2}\operatorname{sech}^2\left(\frac{\theta_1}{2}\right), \tag{15}$$

where $\theta_1=(p_1-q_1)x+9(p_1^5-q_1^5)y-(p_1^3-q_1^3)t+\theta_1^0$, and p_1,q_1 are real constants.

Similarly, another form of the two soliton solution can be derived

$$u=2\left[\ln\left(1+e^{\theta_1}+e^{\theta_2}+A_{12}e^{\theta_1+\theta_2}\right)\right]_{xx}, \tag{16}$$

with

$$A_{12}=-\frac{(p_1-p_2)(p_2+q_1)(q_1-q_2)(p_1+q_2)}{(p_1+p_2)(p_2-q_1)(p_1-q_2)(q_1+q_2)}.$$

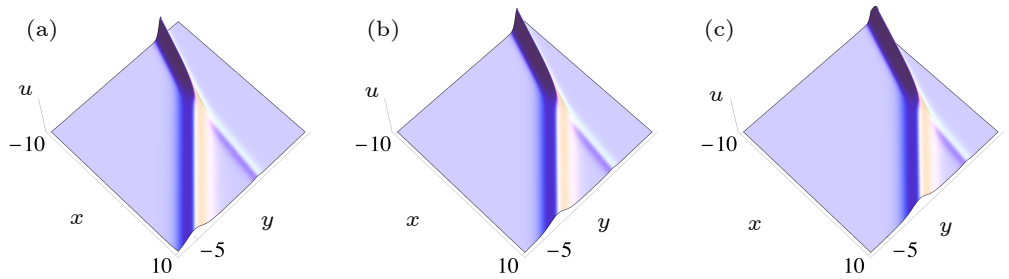


Fig. 4. Inelastic interacting solitons governed by Eq. (10) for parameters $p_1 = 1$, $p_2 = -0.8$, $q_1 = 0.5$, and $q_2 = 0.5$ are illustrated at the following times: (a) $t = -5$; (b) $t = 0$; (c) $t = 5$.

Soliton interactions can also be classified into elastic and inelastic collisions. When $A_{12} = 0$, the phase shift disappears, indicating the occurrence of inelastic interactions, as shown in Fig. 4.

In Fig. 4, a large-amplitude soliton collides with a small-amplitude soliton. The large soliton transfers energy to the small one. As a result, the amplitude of the small soliton increases. At the same time, the large soliton changes its shape because it loses energy. This interaction represents an inelastic collision. The two solitons exchange energy, and their amplitudes change after the collision.

5. The third set of multi-soliton solutions for Eq. (2)

If we take h as another solution of the bilinear form of Eq. (3), then we consider

$$\left(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y\right)h\cdot h = 0. \tag{17}$$

Based on this method, we derive a bilinear BT for the original equation Eq. (1). We introduce the following function, which links two solutions, f and h, of the bilinear equation Eq. (3)

$$\begin{aligned} Q \;=\; & \left[\left(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y\right)f\cdot f\right]h^2 \\ & - f^2\left[\left(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y\right)h\cdot h\right]. \end{aligned} \tag{18}$$

Then, the condition $Q = 0$ shows that both f and h satisfy Eq. (2).

First, we list five basic exchange formulas for Hirota's bilinear operators

$$\begin{aligned} & b^2(D_x^3D_ta\cdot a) - a^2(D_x^3D_tb\cdot b) = 2D_t(D_x^3a\cdot b)\cdot ab \\ & \qquad\qquad + 6D_x(D_xD_ta\cdot b)(D_xa\cdot b), \\ & b^2(D_xD_ya\cdot b) - a^2(D_xD_yb\cdot b) = 2D_x(D_ya\cdot b)\cdot ab, \\ & b^2(D_x^2a\cdot b) - a^2(D_x^2b\cdot b) = 2D_x(D_xa\cdot b)\cdot ab, \\ & b^2D_x^6a\cdot a - a^2D_x^6b\cdot b = 3[a\cdot b(D_x^5a\cdot b) + 5(D_x^3a\cdot b)\cdot(D_x^2a\cdot b)] \\ & \qquad\qquad + 5D_x^3(D_x^3a\cdot b)\cdot ab, \\ & D_x^3(D_ta\cdot b)\cdot ab = D_t(D_x^3a\cdot b)\cdot ab \\ & \qquad\qquad + 3D_t(D_xa\cdot b)\cdot(D_x^2a\cdot b), \end{aligned} \tag{19}$$

where a and b stand for arbitrary functions.

Using these formulas and substituting Eq. (19) into Eq. (18) yields

$$\begin{aligned} Q \;=\; & \left[\left(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y\right)f\cdot f\right]h^2 \\ & - f^2\left[\left(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y\right)h\cdot h\right] \\ =\; & 3[f\cdot h(D_x^5f\cdot h) + 5(D_x^3f\cdot h)\cdot(D_x^2f\cdot h)] \\ & + 5D_x^3(D_x^3f\cdot h)\cdot fh + 10D_t(D_x^3f\cdot h)\cdot fh \\ & + 30D_x(D_xD_tf\cdot h)\cdot(D_xf\cdot h) - 10D_t(D_tf\cdot h)\cdot fh \\ & + 2D_x(D_yf\cdot h)\cdot fh \\ =\; & 15D_x[(D_x^3f\cdot h)\cdot(D_x^2f\cdot h)] - 3D_x(D_x^5f\cdot h)\cdot fh \\ & - 15D_x(D_tf\cdot h)\cdot(D_x^2f\cdot h) + 5D_x^3[(D_x^3 + D_t)f\cdot h]\cdot fh \\ & - 5D_x^3(D_tf\cdot h)\cdot fh + 10D_t(D_x^3f\cdot h)\cdot fh \\ & + 30D_x(D_xf\cdot h)\cdot(D_xf\cdot h) - 10D_t(D_tf\cdot h)\cdot fh \\ & + 2D_x(D_yf\cdot h)\cdot fh \\ =\; & D_x[(-3D_x^5 + 15D_x^2D_t + 2D_y)f\cdot h]\cdot fh \\ & + 15D_x[(D_x^3 + D_t)\cdot fh\cdot(D_x^2f\cdot h)] \\ & + 5D_x^3[(D_x^3 + D_t)f\cdot h]\cdot fh \\ & - 10D_t[(D_x^3 + D_t)f\cdot h]\cdot fh = 0. \end{aligned} \tag{20}$$

Thus, a bilinear BT for Eq. (2) can be obtained

$$\begin{aligned} & \left(-3D_x^5 + 15D_x^2D_t + 2D_y\right)f\cdot h = 0, \\ & \left(D_x^3 + D_t\right)f\cdot h = 0. \end{aligned} \tag{21}$$

Setting $h = 1 + \mathrm{e}^{kx+ny+\omega t}$ and $f = 1$, which correspond to the solution $u = 0$, and then substituting them into Eq. (21), we can obtain a solution for Eq. (2)

$$u = 2(\ln h)_{xx} = \frac{k^2}{2}\mathrm{sech}^2\left(\frac{kx+ny+\omega t}{2}\right), \tag{22}$$

with

$$n = 9k^5, \quad \omega = -k^3.$$

It is worth mentioning that we have obtained the third type of soliton solution.

Based on the one-soliton solution (22), its amplitude $\mathcal{A}$, which measures its peak height, and its constant velocities $\mathcal{V}_x$ and $\mathcal{V}_y$ can be written as

$$\mathcal{A} = \left|\frac{k^2}{2}\right|, \quad \mathcal{V}_x = k^2, \quad \mathcal{V}_y = \frac{1}{9k^2}. \tag{23}$$

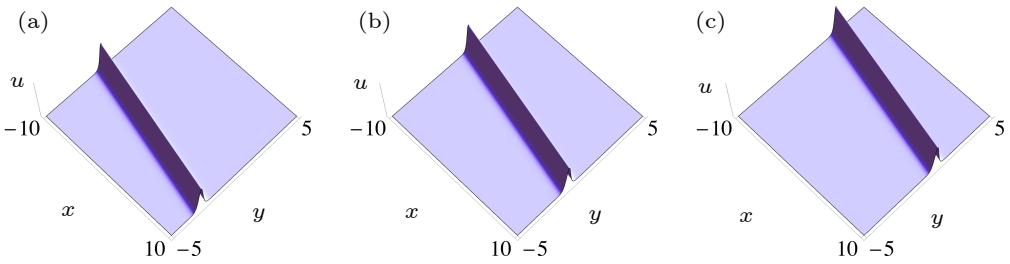


Fig. 5. One-soliton solution obtained from solution (22) with $k_1 = 1$: (a) $t = -15$; (b) $t = 0$; (c) $t = 15$.

Figure 5 shows the one-soliton solution in the x–y plane at three times: $t = -2$, $t = 0$, and $t = 2$. During this time, the wave maintains a constant amplitude of $\frac{1}{2}$. It also maintains the same shape as it moves. From Eq. (23), we obtain the velocity in the x-direction as $\mathcal{V}_x = 1$. We also obtain the velocity in the y-direction as $\mathcal{V}_y = \frac{1}{9}$. These two values are both positive. Therefore, the wave moves in the positive x- and y-directions.

6. Double-pole solution for Eq. (2)

Multi-pole solutions arise when simple poles in the reflection coefficient merge into a multiple pole in the inverse scattering transform.[34,35] One common way to derive multiple-pole solutions is to take the coalescence limit of wave numbers in multi-soliton solutions.[36–38] By choosing proper phase factors and taking the limit of the wave numbers,[39,40] we obtain the double-pole solution of the BKP equation from the two-soliton solution. This result also shows its singular behavior.

To derive the two-soliton solution, we first expand f and h

$$\begin{aligned} f &= 1 + f_1 + f_2 + f_3 + \cdots, \\ h &= 1 + h_1 + h_2 + h_3 + \cdots. \end{aligned} \tag{24}$$

Substituting Eq. (24) into Eq. (21) results in a system of linear equations whose solution provides the soliton solutions for the BKP equation

$$\begin{aligned} &(-3\partial_x^5 + 15\partial_x^2\partial_t + 2\partial_y)(f_1 - h_1) = 0, \\ &(-3\partial_x^5 + 15\partial_x^2\partial_t + 2\partial_y)(f_2 - h_2) \\ &\quad = -(-3D_x^5 + 15D_x^2 D_t + 2D_y)(f_1 \cdot h_1), \\ &(-3\partial_x^5 + 15\partial_x^2\partial_t + 2\partial_y)(f_3 - h_3) \\ &\quad = -(-3D_x^5 + 15D_x^2 D_t + 2D_y)(f_1 \cdot h_2 - f_2 \cdot h_1), \end{aligned} \tag{25}$$

and

$$\begin{aligned} &(\partial_t + \partial_x^3)(f_1 - h_1) = 0, \\ &(\partial_t + \partial_x^3)(f_2 - h_2) = -(D_t + D_x^3)(f_1 \cdot h_1), \\ &(\partial_t + \partial_x^3)(f_3 - h_3) = -(D_t + D_x^3)(f_1 \cdot h_2 - f_2 \cdot h_1). \end{aligned} \tag{26}$$

Considering the simplest basic solution $h_1 = \mathrm{e}^{\xi_1}$, we obtain the following for f_1

$$f_1 = -\frac{k_1 + k_2}{k_1 - k_2}\mathrm{e}^{\xi_1} + \frac{k_1 + k_2}{k_1 - k_2}\mathrm{e}^{\xi_2}. \tag{27}$$

Substituting Eq. (27) into Eq. (26) and solving for f_2 yields

$$f_2 = -\mathrm{e}^{\xi_1 + \xi_2}. \tag{28}$$

Thus, the two-soliton solution is obtained

$$f = 1 - \frac{k_1 + k_2}{k_1 - k_2}\mathrm{e}^{\xi_1} + \frac{k_1 + k_2}{k_1 - k_2}\mathrm{e}^{\xi_2} - \mathrm{e}^{\xi_1 + \xi_2}. \tag{29}$$

To obtain the double-pole solution, we set

$$k_1 = m_1 + \varepsilon, \quad k_2 = m_1 - \varepsilon, \tag{30}$$

and take the limit as $\varepsilon \to 0$. This yields the double-pole solution

$$\begin{aligned} u &= 2(\ln f)_{xx}, \\ f &= 1 + (2 + 3m_1^2 t - \frac{45}{2}m_1^4 y)\,\mathrm{e}^{m_1 x + \frac{m_1^3}{2}t - \frac{9m_1^5}{4}y} \\ &\quad - \frac{1}{m_1^2}\,\mathrm{e}^{2m_1 x + 2m_1^3 t - \frac{9m_1^5}{2}y}. \end{aligned} \tag{31}$$

From this expression, we can see that the double-pole solution is a mixed expression composed of linear polynomial and exponential functions.

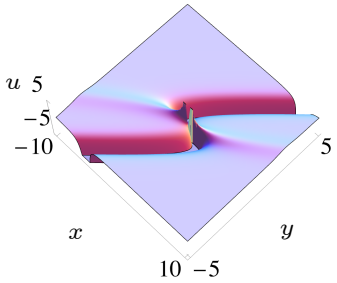


Fig. 6. Double-pole solution (31) with $k_1 = 1$.

When the poles in the scattering data get close to each other, the solution of the BKP equation may show singular behavior. This becomes more pronounced when the poles merge and form a multi-pole structure. At this time, the solution may become unbounded at some points, and singularities may appear. When the poles merge, wave breaking may also occur. The solution can then develop local discontinuities and highly concentrated structures, as shown in Fig. 6.

7. Breather solution for Eq. (2)

The BKP equation has real coefficients, so it is closed under complex conjugation. For this reason, a pair of complex-conjugate exponential phases can give real solutions with oscillatory behavior in space and time. This is the basic idea for constructing breather solutions[39,41–43]

$$f = 1 - \frac{k_1+k_2}{k_1-k_2}\mathrm{e}^{\xi_1} + \frac{k_1+k_2}{k_1-k_2}\mathrm{e}^{\xi_2} - \mathrm{e}^{\xi_1+\xi_2}, \tag{32}$$

$$\xi_j = k_j x + 9k_j^5 y - k_j^3 t. \tag{33}$$

By choosing the complex-conjugate pair $k_1 = a + \mathrm{i}b$ and $k_2 = a - \mathrm{i}b$, where $a, b \in \mathbb{R}$, we obtain the breather solution of Eq. (2)

$$u = 4\left[\ln(\sinh\theta - \frac{a}{b}\sin\omega)\right]_{xx}, \tag{34}$$

with

$$\begin{aligned}\theta &= ax + 9(a^5 - 10a^3b^2 + 5ab^4)y - (a^3 - 3ab^2)t,\\ \omega &= bx + 9(5a^4b - 10a^2b^3 + b^5)y - (3a^2b - b^3)t.\end{aligned} \tag{35}$$

Here, θ gives the localized envelope and the propagation path of the wave center, while ω describes the periodic oscillation within the envelope.

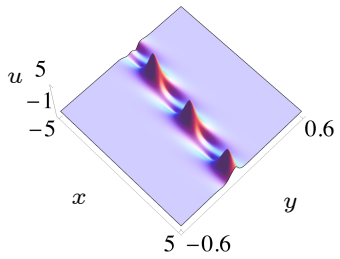


Fig. 7. A breather solution obtained from Solution (32) with $k_1 = 0.6 + 1.2\mathrm{i}$ and $k_2 = 0.6 - 1.2\mathrm{i}$.

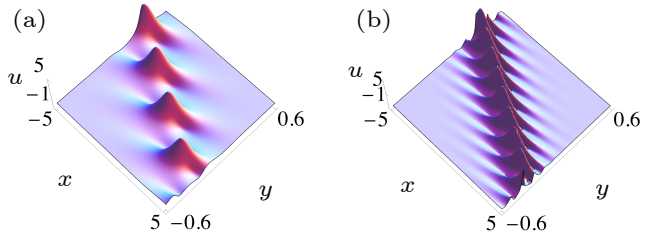


Fig. 8. Breather solutions obtained from Solution (32) with (a) $k_1 = 0.6 + \mathrm{i}$, $k_2 = 0.6 - \mathrm{i}$; (b) $k_1 = 0.8 + 1.2\mathrm{i}$, $k_2 = 0.8 - 1.2\mathrm{i}$.

The propagation of the breather, shown in Fig. 7, follows the propagation path $ax + 9(a^5 - 10a^3b^2 + 5ab^4)y - (a^3 - 3ab^2)t = 0$. As detailed in solution (34), the dynamics of the breather are governed by the real parameters a and b. By analyzing the asymptotic behavior as $x \to \pm\infty$ with fixed t, an oscillation frequency of $\frac{\omega}{2\pi}$ is obtained. In the regime where $|b| \ll |a|$, the breather exhibits a rapid decay in amplitude while maintaining a regular wave profile. When $|a|$ dominates $|b|$, the breather's frequency increases sharply, leading to a spindle-shaped deformation in its profile.

When $a < b$, the breather moves forward and exhibits periodic oscillations, as shown in Fig. 8. This behavior differs from that shown in Fig. 7. In that case, the collision period becomes shorter as the real part a of the wavenumber increases. This means that the breather frequency changes with the real part a. The real part a affects the time interval between two nearby collisions. Also, when the imaginary part b becomes smaller, the breather contains more wave packets. This occurs because the breather becomes more tightly confined. At the same time, a smaller imaginary part causes the amplitude to decay more slowly, allowing more wave packets to remain within the localized structure. These results show that the breather dynamics are very sensitive to the magnitudes of the real and imaginary parts of the wavenumber.

8. Conclusions

We derive three classes of multi-soliton solutions for the $(2+1)$-dimensional BKP equation using the Hirota bilinear method. Although these solutions involve different numbers of free parameters, each can be used to analyze soliton propagation and interaction dynamics. For clarity, different solution classes are adopted to describe different interaction scenarios. In particular, the second class of soliton solutions is obtained by parameter tuning, whereas the third class, constructed via the BT, provides the basis for deriving both breather and double-pole solutions.

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