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Competitive higher-order network evolution model

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A simplicial complex that evolves over time is called a higher-order network, which has recently found applications in social, technological, and biological contexts. The higher-order network is a generalization of traditional networks, and its evolution models are generally more difficult to analyze than those of traditional networks. This paper proposes a higher-order network evolution model with competitiveness. We introduce the method of difference equation analysis into the study of higher-order networks to enable more rigorous analysis. This approach avoids relying on the assumption of continuity in node degree, which is commonly made in traditional network analysis. Applying Poisson process theory, we not only prove the existence of a stationary higher-order degree distribution but also derive an analytical expression for this distribution. Our results show that the scale-free behavior of $(d-1)$-dimensional simplex is controlled by the competitiveness in the d-dimensional simplicial complex. As competitiveness increases, the d-order degree distribution of $(d-1)$-dimensional simplices exhibits curvature in log–log coordinates, whereas for d-order degree of the e-dimensional simplices (where $e \leq d-2$) follows a scale-free distribution. We further extract competitiveness parameters under different probability distributions to simulate the model, and find that simulation results agree with the theoretical analysis.

Keywords: scale-free, simplicial complex, higher-order network, hypernetwork

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1. Introduction

The real world is intricate and complex. The network can describe complex systems very well and is ubiquitous in the real world. Before this century, people usually use the ER (Erdös–Rényi) model to describe networks in the real world. Since Watts and Strogatz proposed the WS model[1] and Barabasi and Albert proposed the BA model[2] at the end of the 20th century, a wave of research on complex networks has been triggered. Over the past two decades, research on complex networks has been very active, involving scholars from many different fields, such as physics, computer science, mathematics, economics, and so on. The relevant research scope ranges from the Internet to the WWW network, from power networks to transportation networks, and from academic cooperation networks to various economic and social relationship networks. After more than two decades of efforts, scholars have constructed a large number of network models and introduced many methods for analyzing these models. The theory of complex networks has gradually transitioned from extensive research to rigorous research, and the related theories have also gradually matured.

Over the past two decades, complex network theory has become a powerful tool for studying complex systems. However, traditional networks can only describe the binary relationships between nodes. With the advancement of science and technology, the complexity of real-world systems is constantly increasing. Traditional networks can no longer fully depict real-world systems with multiple interactions. In the brain network, the signal transmission of neurons does not merely rely on the connections (edges) between individual neurons (nodes), but also involves "functional clusters" formed by three or more neurons. The description of multi-element synergy requires the topological structure of "simplicial complex". Simplicial complexes are formed by a set of simplices. In this framework, simplices indicate the interactions between two or more nodes. A node is a 0-simplex, a link is a 1-dimensional simplex, a triangle is a 2-dimensional simplex, a tetrahedron is a 3-dimensional simplex, and so on.

Higher-order structured networks refer to hypernetworks based on hypergraphs and higher-order networks based on simplices.[3] Boccaletti *et al.* discussed the relationship between hypergraphs and simplicial complexes.[4] In recent years, higher-order structured networks have gradually become a research hotspot.[3–20] Most of the research findings have been published in important journals in the field of physics.[3–9,12–16,19,20] Empirical research on higher-order structured networks mainly focuses on determining the network characteristics of real systems. Hu *et al.* constructed a hypernetwork with proteins as nodes and complexes as hyperedges, analyzed the structural characteristics of the network, and identified the key proteins.[10] In Ref. [11], the authors collected the patent cooperation data of new energy vehicles

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and verified that the distribution of higher-order degrees of new energy vehicle patent cooperation weighted hypernetwork follows a power law.

In the field of evolution hypernetworks, Wang *et al.* first extended the BA model in complex networks to hypernetworks.[15] Subsequently, Hu *et al.*[16] proposed the dual model of Ref. [15]. Guo and Suo, taking into account the brand effect and competitiveness in hypernetworks, proposed an evolution model based on brand effect and competitiveness.[5]

In the aspect of evolution models of higher-order networks based on simplices, Bianconi made pioneering work.[12] She innovatively proposed the NGF (network geometry with flavor) model.[17] The NGF model is an important model in higher-order networks. Courtney conducted in-depth research on higher-order networks based on Bianconi's work.[18] Courtney and Bianconi proposed a weighted simplicial complex evolution model.[7] Remarkable advancements were achieved in expanding complex network models to simplicial complexes, such as growing simplicial complex with face dimension selection and preferential attachment,[21] random simplicial complexes,[22,23] configuration models,[6,24,25] and an activity-driven model.[26] Simplicial complex evolution models are fundamental to studying information or disease spreading in higher-order networks. In recent years, researchers have focused on spreading processes in higher-order networks.[27–29] You *et al.* introduce an activity-driven networks with attractiveness model and develop a coevolution framework for network structures.[28,29] The attractiveness can be regarded as the competitiveness of the simplex. References [28,29] mainly focus on the study of information-disease coupled transmission on high-order networks. However, the higher-degree distribution of simplices was not studied in Refs. [28,29].

The NGF model starts from a d-dimensional simplex and adds one node at each time step to form a new d-dimensional simplex together with a $(d-1)$-dimensional simplex selected from the network. Since the NGF model generates only one new d-dimensional simplex at each time step, strictly speaking, the NGF model does not extend the BA model in complex networks to higher-order networks. For this purpose, Kovalenko *et al.* proposed an NGF model with m new triangles added at each time step in a 2-dimensional simplicial complex,[9] extending the BA model to 2-dimensional simplicial complex. However, neither the NGF model nor the model by Kovalenko *et al.* took into account the competitiveness of simplices. The purpose of the paper is to propose a higher-order network evolution model with competitiveness and, at the same time, extend the well-known BA model to higher-order networks.

2. Higher-order networks

Let $V=\{v_1,v_2,\ldots,v_n\}$ be a finite set. A network is a graph $G=(V,E)$ formed by a set of nodes V and a set of links E that represent the elements of a complex system and their interactions, respectively. Networks are pivotal to capturing the architecture of complex systems; however, they have the important limitation that they cannot be used to capture the higher-order interactions. In order to encode for the many-body interactions between the elements of a complex system, higher-order networks need to be used. A powerful mathematical framework to describe higher-order networks is provided by simplicial complexes. Simplicial complexes are formed by a set of simplices.[17] The simplices indicate the interactions existing between two or more nodes and are defined as follows:

A d-dimensional simplex α (also indicated as a d-simplex α) is formed by a set of $(d+1)$ interacting nodes

$$\alpha=[v_{\alpha,0},v_{\alpha,1},\ldots,v_{\alpha,d-1},v_{\alpha,d}].$$

It describes a many-body interaction between the nodes.

A face of a d-dimensional simplex α is a simplex α' formed by a proper subset of nodes of the simplex, i.e., $\alpha'\subset\alpha$.

A simplicial complex K is formed by a set of simplices that is closed under the inclusion of the faces of each simplex.

The simplicial complex K that evolves over time is called a higher-order network based on the simplex, abbreviated as a higher-order network. The dimension $\dim K$ of a simplicial complex is the largest dimension of its simplices. K is abbreviated as the $\dim K$-order network. The d-order degree $h^{\alpha}_{d,m}$ of an m-dimensional simplex (face) α indicates the number of d-dimensional simplices incident to the simplex α.

3. Model description

In traditional network analysis, it is assumed that nodes arrive at the system at equal time intervals. Such an assumption not only does not conform to the continuous arrival of nodes in the actual network,[30,31] but also leads to logical contradictions when analyzing the network.[31] Introducing stochastic processes in network modeling can depict the continuous arrival of actual network nodes, providing a basis for analyzing the distribution of the arrival time of nodes in the network. In this section, we propose a competitive Poisson growing higher-order network evolution model. If a higher-order network meets the following two conditions, it is called a competition-driven higher-order network.

(i) Poisson growth: Starting with a small number m_0 of d-dimensional simplexes of the d-order network. The arrival process of nodes follows a Poisson process with rate λ. At time t, when one new node arrives, m new d-dimensional simplices formed by the new node and previously existing $(d-1)$-

dimensional faces are added to the network; and no repeating simplex is allowed. Meanwhile, the competitiveness ξ is drawn according to the probability distribution $F(x)$. Here, the average competitiveness $a = \int x \mathrm{d}F(x)$ is finite.

(ii) Preferential attachment: The probability $\boldsymbol{\Theta}$ that the new d-dimensional simplex is glued to a $(d-1)$-dimensional face α depends on the d-order degree of face α and the competitiveness ξ_α on the α, is given by

$$\boldsymbol{\Theta}\left(h^{\alpha}_{d,d-1}\right) = \frac{h^{\alpha}_{d,d-1} + \xi_\alpha}{\sum\limits_{\beta \in S_{d-1}(t)} \left(h^{\beta}_{d,d-1} + \xi_\beta\right)}, \tag{1}$$

where $S_{d-1}(t)$ represents all $(d-1)$-dimensional simplexes in the network at time t.

As shown in Fig. 1, at each time step, one tetrahedron (3-dimensional simplex), three triangles (2-dimensional simplex), three line segments (1-dimensional simplex), and one node (0-dimensional simplex) are added.

If $\xi_\alpha \equiv 0$ is taken in the model, then the competition-driven higher-order network evolution model is the BA model in the higher-order network.

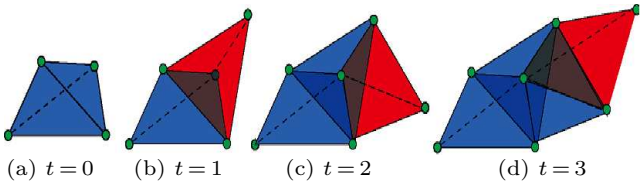


Fig. 1. Schematic diagram of the first three evolution steps of the model.

Bollobás and Riordan pointed out that the BA model may exhibit the phenomenon where isolated nodes never acquire connections.[32] However, in real-world networks, there exists the possibility for new nodes to connect to isolated ones. Indeed, Price introduced the concept of node attractiveness in his study of citation networks,[33] and Dorogovtsev *et al.* conducted in-depth research on networks with node attractiveness.[34] The competitiveness parameter is a generalization of node attractiveness in complex network models to higher-order networks, with the difference being that the attractiveness in Dorogovtsev *et al.*'s model is a constant A,[34] which fails to reflect competition among nodes. In our model, the competitiveness ξ_α of a $(d-1)$-dimensional simplex α is related to α, highlighting competition among simplices. Therefore, ξ_α is referred to as the competition factor. In practical systems, whether in traditional networks or higher-order networks, both nodes and simplices possess their own competitiveness. In real-world systems, both nodes in traditional networks and simplices in higher-order networks possess their own competitiveness. For example, a "simplex" could represent a research team (multiple authors collaborating on a paper). The "competitiveness" of such a team could be its funding strength or the prominence of its core scholars. Alternatively, a simplex might represent a coexisting species group (such as a plant–pollinator interaction set). Its "competitiveness" could be interpreted as the community's resource-use efficiency or overall fitness.

4. Model analysis and simulation

Without loss of generality, let $N(t)$ be the number of nodes in the network at time t. Since the arrival process of network nodes is a Poisson process of intensity λ, it is known from the properties of the Poisson process[32] that $E[N(t)] = \lambda t$. Let t_α be the time when the simplex α enters the network, and $h^{\alpha}_{d,d-1}(N(t))$ be the d-order degree of the $(d-1)$-dimensional simplex α when the $N(t)$-th node enters the system at time t. From the conditions of the model, it can be obtained that $h^{\alpha}_{d,d-1}(N(t))$ satisfies the following difference equation:

$$\begin{aligned} &h^{\alpha}_{d,d-1}(N(t)) - h^{\alpha}_{d,d-1}(N(t)-1) \\ &= m\frac{h^{\alpha}_{d,d-1}(N(t)-1) + \xi_\alpha}{\sum\limits_{\beta}\left(h^{\beta}_{d,d-1}(N(t)-1) + \xi_\beta\right)}. \end{aligned} \tag{2}$$

At each time step, the network generates m new d-dimensional simplexes, each of which has d $(d-1)$-dimensional faces. Ignoring the overlapping surfaces, thus

$$\sum_{\beta \in S_{d-1}(t)} 1 = c_{m,1}(d-1)\,dN(t),$$

where

$$c_{m,1}(e) = \begin{cases} 1, & \text{if } e = 0, \\ m, & \text{if } e \neq 0. \end{cases}$$

For $t \gg 1$, we have

$$\sum_{\beta \in S_{d-1}(t)} h^{\beta}_{d,d-1} = m(d+1)N(t).$$

Since $c_{m,1}(d-1)\,d$ $(d-1)$-dimensional simplexes are generated each time, and

$$\frac{1}{N(t)}\sum_{i=0}^{N(t)} \xi_i = a,$$

therefore,

$$\frac{1}{N(t)}\sum_{\beta \in S_{d-1}(t)} \xi_\beta = a c_{m,1}(d-1)\,d.$$

For sufficiently large t, we have

$$\sum_{\beta \in S_{d-1}(t)} \left(h^{\beta}_{d,d-1} + \xi_\beta\right) \approx m(d+1)N(t) + a c_{m,1}(d-1)\,dN(t). \tag{3}$$

Substituting Eq. (3) into Eq. (2), we have

$$\begin{aligned} &h^{\alpha}_{d,d-1}(N(t)) - h^{\alpha}_{d,d-1}(N(t)-1) \\ &= \frac{h^{\alpha}_{d,d-1}(N(t)-1) + \xi_\alpha}{\left(\left(\frac{a}{m}c_{m,1}(d-1)+1\right)d+1\right)N(t)}. \end{aligned} \tag{4}$$

For $e < d-1$, the probability of an increase in the d-order degree of the e-dimensional simplex δ is

$$\begin{aligned}\Theta_e(\delta) &= \sum_{\alpha\in S_{d-1}|\alpha\supseteq\delta}\Theta(h^{\alpha}_{d,d-1}) \\ &= \frac{\sum_{\alpha\in S_{d-1}|\alpha\supseteq\delta}(h^{\alpha}_{d,d-1}(N(t)-1)+\xi_\alpha)}{\sum_{\beta\in S_{d-1}(t)}(h^{\beta}_{d,d-1}(N(t)-1)+\xi_\beta)}.\end{aligned} \tag{5}$$

For sufficiently large t

$$\begin{aligned}&\sum_{\alpha\in S_{d-1}|\alpha\supseteq\delta}(h^{\alpha}_{d,d-1}(N(t)-1)+\xi_\alpha) \\ &= \sum_{\alpha\in S_{d-1}|\alpha\supseteq\delta}h^{\alpha}_{d,d-1}(N(t)-1)+a\sum_{\alpha\in S_{d-1}|\alpha\supseteq\delta}1.\end{aligned} \tag{6}$$

When $e_1 > e$, from Ref. [7], we know

$$h^{\delta}_{d,e} = \frac{1}{\binom{d-e}{e_1-e}}\sum_{\alpha\in S_{e_1}|\alpha\supseteq\delta}h^{\alpha}_{d,e_1}. \tag{7}$$

From Eq. (7), we obtain

$$\sum_{\alpha\in S_{d-1}|\alpha\supseteq\delta}h^{\alpha}_{d,d-1} = (d-e)h^{\delta}_{d,e}. \tag{8}$$

Since $h^{\delta}_{d,e}(t)$ is the d-order degree of the e-dimensional simplex δ, from Ref. [18], we have

$$\begin{aligned}\sum_{\alpha\in S_{d-1}|\alpha\supseteq\delta}1 &= \begin{cases} m+(d-1)h^{\delta}_{d,e}, & \text{if } e=0 \\ 1+(d-e-1)h^{\delta}_{d,e}, & \text{if } e>0 \end{cases} \\ &= c_{1,m}(e)+(d-e-1)h^{\delta}_{d,e},\end{aligned} \tag{9}$$

where

$$c_{1,m}(e) = \begin{cases} m, & \text{if } e=0, \\ 1, & \text{if } e>0. \end{cases}$$

Substituting Eqs. (8) and (9) into Eq. (6), we obtain

$$\begin{aligned}&\sum_{\alpha\in S_{d-1}|\alpha\supseteq\delta}\left(h^{\alpha}_{d,d-1}(N(t)-1)+\xi_\alpha\right) \\ &\approx (d-e)h^{\delta}_{d,e}(N(t)-1) \\ &\quad +a\begin{cases} m+(d-1)h^{\delta}_{d,e}(N(t)-1), & \text{if } e=0 \\ 1+(d-e-1)h^{\delta}_{d,e}(N(t)-1), & \text{if } e>0 \end{cases} \\ &= ac_{1,m}(e)+((a+1)(d-e)-a)h^{\delta}_{d,e}(N(t)-1).\end{aligned} \tag{10}$$

Since the rate of change of $h^{\delta}_{d,e}(t)$ is proportional to the probability $\Theta_e(\delta)$, we have

$$\begin{aligned}&h^{\delta}_{d,e}(N(t))-h^{\delta}_{d,e}(N(t)-1) \\ &= m\frac{\sum_{\alpha\in S_{d-1}|\alpha\supseteq\delta}\left(h^{\alpha}_{d,d-1}(N(t)-1)+\xi_\alpha\right)}{\sum_{\beta\in S_{d-1}(t)}\left(h^{\beta}_{d,d-1}(N(t)-1)+\xi_\beta\right)}.\end{aligned} \tag{11}$$

Substituting Eqs. (3) and (10) into Eq. (11), we obtain

$$\begin{aligned}&h^{\delta}_{d,e}(N(t))-h^{\delta}_{d,e}(N(t)-1) \\ &= \frac{ac_{1,m}(e)+[(a+1)(d-e)-a]h^{\delta}_{d,e}(N(t)-1)}{\left(\left(\frac{a}{m}c_{m,1}(d-1)+1\right)d+1\right)N(t)}.\end{aligned} \tag{12}$$

From Eq. (12), we have

$$\begin{aligned}&ac_{1,m}(e)+[(a+1)(d-e)-a]h^{\delta}_{d,e}(N(t)) \\ &= \left(1+\frac{(a+1)(d-e)-a}{\left(\left(\frac{a}{m}c_{m,1}(d-1)+1\right)d+1\right)N(t)}\right) \\ &\quad\times\left\{ac_{1,m}(e)+[(a+1)(d-e)-a]h^{\delta}_{d,e}(N(t)-1)\right\}.\end{aligned} \tag{13}$$

Let $\gamma = \frac{\left(\frac{a}{m}c_{m,1}(d-1)+1\right)d+1}{(a+1)(d-e)-a}$, then

$$\begin{aligned}&\ln\frac{ac_{1,m}(e)+[(a+1)(d-e)-a]h^{\delta}_{d,e}(N(t))}{ac_{1,m}(e)+[(a+1)(d-e)-a]h^{\delta}_{d,e}(N(t_\delta))} \\ &= \sum_{j=N(t_\delta)+1}^{N(t)}\ln\left(1+\frac{1}{\gamma j}\right).\end{aligned} \tag{14}$$

We ignore the overlapping simplex adjacency surfaces. Therefore,

$$h^{\delta}_{d,e}(N(t_\delta)) = c_{1,m}(e) = \begin{cases} m, & \text{if } e=0, \\ 1, & \text{if } e>0, \end{cases} \tag{15}$$

$$\begin{aligned}&\ln\frac{ac_{1,m}(e)+[(a+1)(d-e)-a]h^{\delta}_{d,e}(N(t))}{ac_{1,m}(e)+[(a+1)(d-e)-a]c_{1,m}(e)} \\ &= \frac{1}{\gamma}\left(\sum_{j=1}^{N(t)}\frac{1}{j}-\sum_{j=1}^{N(t_\delta)}\frac{1}{j}\right).\end{aligned} \tag{16}$$

Since $E[N(t)] = \lambda t$, $E[N(t_\delta)] = \lambda t_\delta$ and

$$\begin{aligned}&\sum_{j=1}^{N(t)}\frac{1}{j}-\sum_{j=1}^{N(t_\delta)}\frac{1}{j} \\ &= C+\ln N(t)+\varepsilon_{N(t)}-\left[C+\ln N(t_\delta)+\varepsilon_{N(t_\delta)}\right],\end{aligned} \tag{17}$$

where C is an Euler constant, and $\lim\limits_{n\to\infty}\varepsilon_n = 0$, therefore, for sufficiently large t, we have

$$\frac{ac_{1,m}(e)+[(a+1)(d-e)-a]h^{\delta}_{d,e}(N(t))}{ac_{1,m}(e)+[(a+1)(d-e)-a]c_{1,m}(e)} = \left(\frac{t}{t_\delta}\right)^{\frac{1}{\gamma}}.$$

Therefore,

$$\begin{aligned}h^{\delta}_{d,e}(N(t)) &= \frac{(ac_{1,m}(e)+[(a+1)(d-e)-a]c_{1,m}(e))}{[(a+1)(d-e)-a]}\left(\frac{t}{t_\delta}\right)^{\frac{1}{\gamma}} \\ &\quad -\frac{ac_{1,m}(e)}{[(a+1)(d-e)-a]}.\end{aligned} \tag{18}$$

From Eq. (18), we have

$$\begin{aligned}&P\left\{h^{\delta}_{d,e}(N(t))\geq k\right\} \\ &= P\left\{t_\delta\leq\left(\frac{(a+1)(d-e)c_{1,m}(e)}{[(a+1)(d-e)-a]k+ac_{1,m}(e)}\right)^{\gamma}t\right\}.\end{aligned} \tag{19}$$

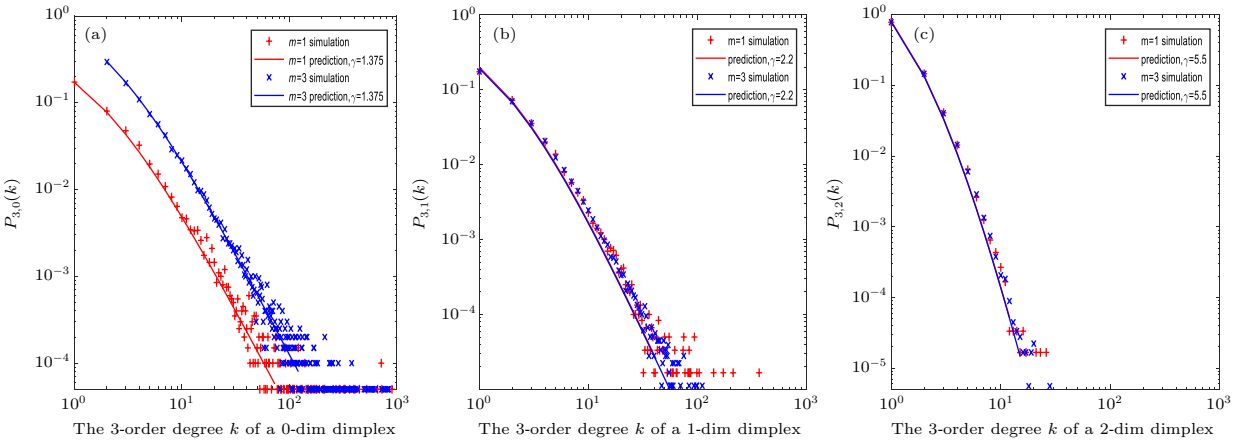


Fig. 2. The simulation of the model with competitiveness randomly selected in [0,1]. (a) is the distribution diagram of the 3-order degree of the 0-dimensional simplex; the red plus sign + is the simulated value of the model with only one tetrahedron added at each time step, and the red line − is the theoretical predicted value. The blue cross × is the simulated value of the model with 3 tetrahedra added at each time step, and the blue line − is the theoretical predicted value. (b) is the distribution diagram of the 3-order degree of the 1-dimensional simplex. (c) is the distribution diagram of the 3-order degree of the 2-dimensional simplex. In figures (b) and (c), the red plus sign + is the simulated value of the model with only one tetrahedron added at each time step, the blue cross × is the simulated value of the model with three tetrahedra added at each time step, and the red line − and the blue line − are the theoretical prediction values of the model with one and three tetrahedra added at each time step, respectively. From figures (a)–(c), it can be seen that the newly generated number of 3-dimensional simplex m only affects the position of the distribution of the 3-order degree of 0-dimensional simplex, and has no effect on the power-law exponent γ. The theoretical curves in (b) and (c) coincide.

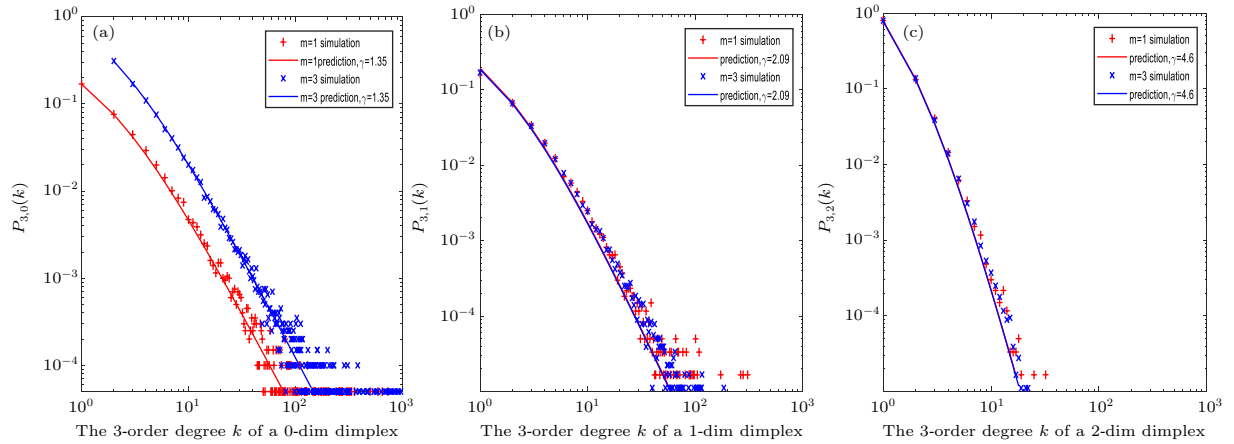


Fig. 3. The simulation of the model with competitiveness extracted from Poisson distribution with parameter $\lambda = 0.2$. The symbols and their descriptions are the same as those in Fig. 2.

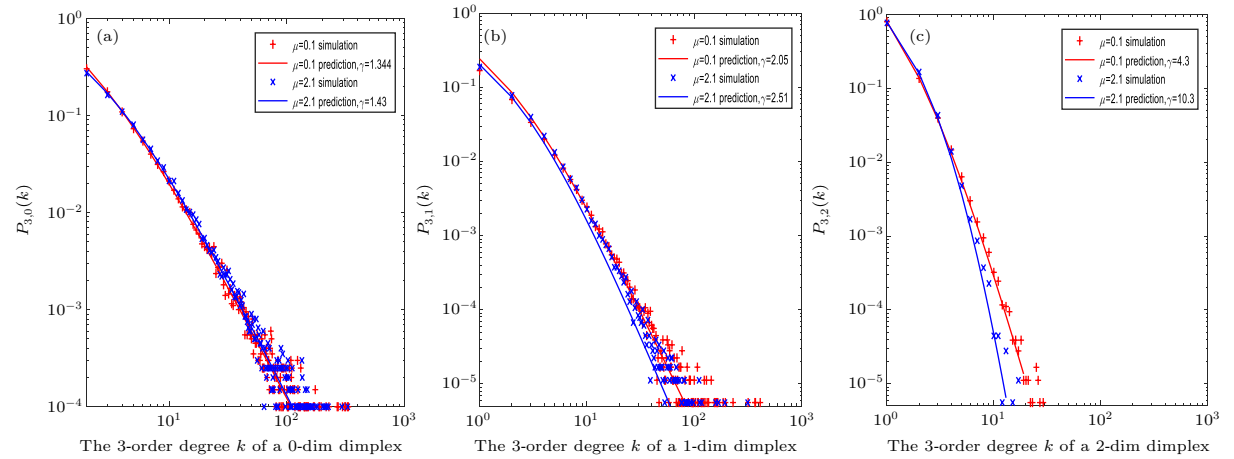


Fig. 4. The simulation of the model with competitiveness extracted from the normal distribution $N(\mu, 0.1)$ and with 3 tetrahedrons added each time step. The red plus sign + is the simulation value of the model with $\mu = 0.1$, and the red line − is its theoretical predicted value. The blue cross × is the simulation value of the model with $\mu = 2.1$, and the blue line − is its theoretical predicted value.

Without loss of generality, $\delta = n$ $(n = 0,1,2,\ldots,)$ represents the e-dimensional simplex of the n-th batch entering the network, then

$$\begin{aligned} & P\left\{h_{d,e}^{n}\left(N\left(t\right)\right) \geq k\right\} \\ = & P\left\{t_n \leq \left(\frac{\left(a+1\right)\left(d-e\right)c_{1,m}\left(e\right)}{\left[\left(a+1\right)\left(d-e\right)-a\right]k+ac_{1,m}\left(e\right)}\right)^{\gamma} t\right\}. \end{aligned} \tag{20}$$

According to the theory of Poisson process,[32] it is known that $t_0 = 0$, $t_1 - t_0$, $t_2 - t_1, \ldots, t_n - t_{n-1}$ are independent identically distributed exponential random variables having mean $1/\lambda$. The arrival time of the e-dimensional simplex n arrival time is $t_n = t_n - t_{n-1} + t_{n-1} - t_{n-2} + \ldots + t_2 - t_1 + t_1 - t_0$, then

$$P\left(t_n \leq x\right) = F_n\left(x\right),$$

where $F_n(x)$ is an n-fold convolution of an exponential distribution with the parameter λ.

$$\begin{aligned} & P\left\{t_n \leq \left(\frac{\left(a+1\right)\left(d-e\right)c_{1,m}\left(e\right)}{\left[\left(a+1\right)\left(d-e\right)-a\right]k+ac_{1,m}\left(e\right)}\right)^{\gamma} t\right\} \\ = & F_n\left(\left(\frac{\left(a+1\right)\left(d-e\right)c_{1,m}\left(e\right)}{\left[\left(a+1\right)\left(d-e\right)-a\right]k+ac_{1,m}\left(e\right)}\right)^{\gamma} t\right). \end{aligned} \tag{21}$$

Since

$$\begin{aligned} & \sum_{n=1}^{+\infty} F_n\left(\left(\frac{\left(a+1\right)\left(d-e\right)c_{1,m}\left(e\right)}{\left[\left(a+1\right)\left(d-e\right)-a\right]k+ac_{1,m}\left(e\right)}\right)^{\gamma} t\right) \\ = & E\left[N\left(\left(\frac{\left(a+1\right)\left(d-e\right)c_{1,m}\left(e\right)}{\left[\left(a+1\right)\left(d-e\right)-a\right]k+ac_{1,m}\left(e\right)}\right)^{\gamma} t\right)\right], \end{aligned} \tag{22}$$

therefore, we know that the cumulative distribution of the d-order of the $e(<d)$-dimensional simplex is

$$\begin{aligned} & P_{d,e}^{\mathrm{cum}}\left(k\right) \\ = & \lim_{t\to\infty} \frac{1}{E\left[N\left(t\right)\right]} \\ & \times \sum_{n=1}^{\infty} F_n\left(\left(\frac{\left(a+1\right)\left(d-e\right)c_{1,m}\left(e\right)}{\left[\left(a+1\right)\left(d-e\right)-a\right]k+ac_{1,m}\left(e\right)}\right)^{\gamma} t\right) \\ = & \left(\frac{\left(a+1\right)\left(d-e\right)c_{1,m}\left(e\right)}{\left[\left(a+1\right)\left(d-e\right)-a\right]k+ac_{1,m}\left(e\right)}\right)^{\gamma} \propto \frac{1}{k^{\gamma}}, \end{aligned} \tag{23}$$

where

$$\gamma = \frac{\left(\frac{a}{m}c_{m,1}\left(d-1\right)+1\right)d+1}{\left(a+1\right)\left(d-e\right)-a}. \tag{24}$$

From Eq. (24), it can be obtained that the d-order distribution of the $e(<d)$-dimensional simplex is

$$\begin{aligned} P_{d,e}\left(k\right) = & \left(\frac{\left(a+1\right)\left(d-e\right)c_{1,m}\left(e\right)}{\left[\left(a+1\right)\left(d-e\right)-a\right]k+ac_{1,m}\left(e\right)}\right)^{\gamma} \\ & - \left(\frac{\left(a+1\right)\left(d-e\right)c_{1,m}\left(e\right)}{\left[\left(a+1\right)\left(d-e\right)-a\right]\left(k+1\right)+ac_{1,m}\left(e\right)}\right)^{\gamma} \\ \propto & \frac{1}{k^{\gamma+1}}. \end{aligned} \tag{25}$$

The higher-order network model we simulate is a 3-order network, that is, a higher-order network formed by tetrahedrons. The number of nodes is $N = 20000$, and the newly added simplex numbers are $m = 1$ and $m = 3$ respectively. In Fig. 2, competitiveness values are randomly selected in [0,1]. In Fig. 3, competitiveness values are sampled from a Poisson distribution with mean $\lambda = 0.2$. Figure 4 compares the higher-order distributions of the competitiveness models, which were sampled from the normal distributions $N(0.1,0.1)$ and $N(2.1,0.1)$, on a double-logarithmic plot. The simulation results show that the simulation values of the competition-driven higher-order network evolution model are in good agreement with the theoretical analysis results. It can be seen from Figs. 2 to 3 that the number m of newly generated d-dimensional simplices only affects the position of the d-order distribution of the 0-dimensional simplex (i.e., the node) in the figures, and has no effect on the power-law exponent γ. As can be seen from Fig. 4, the d-order distribution of the $(d-1)$-dimensional simplex is curved in the double logarithmic coordinate system, while the d-order distribution of the $e(\leq d-2)$-dimensional simplex is relatively straight. The lower the dimension of the simplex, the straighter the d-order distribution. Computer simulation verified the above theoretical analysis. It can be seen from this that in higher-order networks, the scale-free phenomenon occurs in lower-dimensional simplices. The greater the competitiveness, the more curved the d-order distribution of the $(d-1)$-dimensional simplex in the double logarithmic coordinate system.

5. Conclusion

In traditional network analysis, a common assumption is that nodes arrive discretely while their degrees are treated as continuous real-valued variables. Such an assumption not only fails to conform to the often continuous nature of node arrival in real-world networks, but also lacks rigor in analytical treatment. To address this, we introduce stochastic processes into the model and propose the Poisson continuous growth higher-order network evolution model driven by competitiveness. This model not only conforms to the continuous arrival of nodes in the actual network, but also provides a basis for analyzing the distribution of node arrival times in the network. We introduced the difference equation analysis method in network analysis, avoiding the unreasonable assumption of node degree continuity in traditional network analysis. We conduct a rigorous theoretical analysis of the competition-driven higher-order network evolution model by using the difference equation method and Poisson process theory, and obtain the expression of the higher-order degree distribution of the model. From the perspective of network dynamics, the higher-order degree distribution of lower-dimensional simplices exhibits scale-free characteristics over time, while

the scale-free property of sub-higher-order simplices is influenced by competitiveness. Competition-driven evolution models will play an important role in the research of higher-order networks. Although there is some literature on the evolution models of higher-order networks, the research on the topological structure of higher-order networks has only just begun. There is still a lot of work to be done regarding the evolution models of higher-order networks.

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