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Kinteic models of wealth distribution incorporating non-Maxwellian collision kernel and value function

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A kinetic model of wealth incorporating a non-Maxwellian collision kernel and a value function is investigated. The non-Maxwellian collision kernel describes the variable transaction frequency between agents, which is related to their wealth. A value function describing psychological characteristics is embedded into the interaction rules, and the resulting Boltzmann equation, incorporating the non-Maxwellian collision kernel, is used to analyze the evolutionary dynamics of wealth distribution.

Keywords: kinetic model, non-Maxwellian collision kernel, value function, wealth distribution

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1. Introduction

The key points in constructing a wealth distribution model using the theory of rarefied gas dynamics are determining the interaction rules and the collision kernel. On one hand, the interaction rules simulate the laws followed by agents participating in market transactions, which include two typical characteristics: the savings effect and randomness. Savings mean that an agent in the market does not invest all their wealth at once in each transaction. This idea is introduced by Chakraborti and Chakrabarti[1] and Chatterjee *et al.*,[2] respectively. Randomness indicates that an agent's wealth is affected by market risks, and their returns may no longer be a definite value. Cordier *et al.*[3] develop this idea when studying a wealth distribution model of agents in a risky market. Based on such binary interaction rules, Pareschi and Toscani[4] point out that agents' knowledge helps reduce transaction risks and examine the implications of knowledge for wealth distribution. Bisi *et al.*[5] embed a taxation mechanism into the framework of binary transactions and adopt an appropriate redistribution operator to analyze how taxation and redistribution affect wealth distribution. On the other hand, the collision kernel is used to describe the interaction frequency of agent transactions. Due to the tractability of the Maxwell collision kernel (constant collision kernel), it is often used in wealth distribution models (see Refs. [6–10]).

Furioli *et al.*[11] introduce a class of variable transaction frequencies, known as non-Maxwellian collision kernels, namely

$$K_1(v,w) = \kappa_1 (vw)^{\delta_0}, \tag{1}$$

where $\kappa > 0$, $0 < \delta_0 \leq 1$. v and w represent the wealth holdings of the two agents, respectively. If $\delta_0 = 0$, then $K(v,w) = \kappa$ is a constant, that is, Eq. (1) becomes the Maxwell collision kernel. The research findings of Furioli *et al.*[11] illustrate that the parameter δ_0 affects the Pareto index corresponding to the steady-state wealth distribution solution.[12] Employing the non-Maxwellian collision kernel Eq. (1), Furioli *et al.*[11] demonstrate that the steady-state solution of the associated Fokker–Planck equation converges exponentially to the equilibrium solution.

Del-Mul[13] gives the non-Maxwellian collision kernel in the following mathematical form

$$K_2(v,w) = \kappa_2 v^{\alpha_0} w^{\beta_0}, \tag{2}$$

where $\kappa > 0$, $0 < \alpha_0 \leq 1$ and $0 < \beta_0 \leq 1$. The parameters α_0 and β_0 describe the trading intentions of the two agents, respectively.

Zhou *et al.*[14] study the following non-Maxwellian collision kernel

$$K_3(v,w) = \alpha_1 v + \beta_1 w, \tag{3}$$

where $0 < \alpha_1 \leq 1$ and $0 < \beta_1 \leq 1$. An agent with wealth v invests $\alpha_1 v$ of their wealth in trading, and an agent with wealth w invests $\beta_1 w$ of their wealth in trading.

In the collision kernel Eq. (1), the trading intentions of the two agents are the same. However, in the collision kernels Eqs. (2) and (3), the trading intentions of different agents are different.

This paper studies the influence of non-Maxwellian collision kernels and value functions on wealth distribution.

(i) Considering the inconsistency in the trading willingness intensity of the two trading parties, we use the following

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non-Maxwellian collision kernel

$$K(v,w) = \kappa(w^{\theta} + \theta w^{\theta-1} v), \tag{4}$$

where $\kappa > 0$ and $\theta \geq 0$. θ describes the trading willingness intensity of the agent with wealth w. When the variable κ remains fixed, the larger the θ, the more frequent the transactions; the smaller the θ, the lower the transaction frequency. When the variable θ is constant, the larger the κ, the more frequent the transactions; the smaller the κ, the lower the transaction frequency.

(ii) We describe not only the wealth distribution of one type of trading group, but also the wealth distribution arising from transactions between two types of groups.

(iii) Considering the different trading willingness intensities of the two counterparties, we add a value function describing the psychological characteristics of agents to the interaction rules and explore the corresponding wealth distribution.

The work in Ref. [15] investigates the impacts of agents' psychological factors and the power-law collision kernel on wealth distribution. The power-law collision kernel in that paper is characterized by the fact that an agent's transaction frequency exhibits a power-law relationship with their own wealth, whereas the transaction frequency considered here is correlated with both trading parties. In addition, our value function differs from that describing agents' psychological characteristics in Ref. [15]. Meng *et al.*[16] explore the wealth distribution dynamics of two agent groups, where the non-Maxwellian collision kernel they utilize has the form given in Eq. (2); however, we adopt a distinctly different collision kernel. The collision kernel and value function in Ref. [17] are also different from ours.

The structure of this paper is presented below. In Section 2, we adopt the non-Maxwellian collision kernel Eq. (4) to construct a Boltzmann system describing wealth evolution. By using the scale transformation and the method of solving ordinary differential equations, we obtain the corresponding steady-state wealth density function under specific conditions. In Section 3, we discuss the wealth evolution processes of two types of trading groups. In Section 4, we take into account both the non-Maxwellian collision kernel specified in Eq. (4) and the psychological characteristics of agents, and study the corresponding wealth distribution. In Section 5, the influence of variable trading frequency on the dynamics and patterns of wealth distribution is numerically investigated. Section 6 concludes the paper.

2. Boltzmann kinetic model with non-Maxwellian collision kernels

The evolution of wealth distribution hinges on agents' interaction rules. During the trading process, agents use their own wealth to participate in market transactions. We assume that the wealth values before and after each transaction are non-negative.

2.1. Model setting and property analysis

We define v as an agent's pre-transaction wealth and v_* as their post-transaction wealth. The following interaction rule is defined:

$$v_* = (1-\lambda)v + \lambda w + \eta_1 v, \tag{5}$$

where the savings propensity parameter satisfies $0 < \lambda < 1$, λv denotes the wealth used by the agent for transactions, and w represents the market wealth. We assume that the market wealth density $h(w)$ is known. λw denotes the wealth that the agent acquires from the market. The risk parameter η_1 is a random variable, and $\eta_1 v$ represents the wealth change caused by the uncertain factors associated with market risk. To ensure non-negative wealth after the interaction, we assume that $\eta_1 > -1+\lambda$, the expectation $\mathbb{E}(\eta_1) = 0$, and the variance $\mathbb{E}(\eta_1^2) = \sigma > 0$.

We let $f(v,t)$ represent the density function of agents possessing wealth v at time t. Then

$$\int_{R_+} f(v,t)\,\mathrm{d}v = 1.$$

Suppose that the trading frequency of agents follows the non-Maxwellian collision kernel Eq. (4). In the context of rarefied gas dynamics,[18] agents continuously update their wealth, such that the evolution of the density function $f(v,t)$ is governed by the following Boltzmann equation[19]

$$\frac{\partial f(v,t)}{\partial t} = Q(f,f)(v,t),$$

where the collision operator $Q(f,f)$ characterizes the evolution of the density function (see Ref. [20]). Generally, its weak form is adopted, that is, for any smooth function $\varphi(v)$ with compact support, $f(v,t)$ satisfies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+} \varphi(v) f(v,t)\,\mathrm{d}v \\ &= \mathbb{E}\left[\int_{R_+^2} K(v,w)(\varphi(v_*) - \varphi(v)) h(w) f(v,t)\,\mathrm{d}v\mathrm{d}w\right], \end{aligned} \tag{6}$$

where $\mathbb{E}$ denotes the mathematical expectation. The transaction frequency of an agent mainly depends on the trading willingness of the counterparty.

Substituting $\varphi(v) = v$ and the non-Maxwellian collision kernel $K(v,w)$ into Eq. (6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} m_1(t) = \lambda\kappa \int_{R_+^2} (w^{\theta} + \theta w^{\theta-1} v)(w-v) h(w) f(v,t)\,\mathrm{d}v\mathrm{d}w, \tag{7}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} m_1(t) = \int_{R_+} v f(v,t)\,\mathrm{d}v.$$

Suppose that the first N moments of $h(w)$ are bounded, namely

$$\int_{R_+} w^i h(w)\,\mathrm{d}w = M_i < \infty, \quad i = 1,2,\ldots,N, \quad \theta \le N. \tag{8}$$

Furthermore, $h(w)$ satisfies

$$\int_{R_+} h(w)\,\mathrm{d}w = 1.$$

Substituting Eq. (8) into Eq. (7), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}m_1(t)\\
&= \lambda\kappa\int_{R_+^2}(w^{\theta+1} - w^\theta v + \theta w^\theta v - \theta w^{\theta-1}v^2)h(w)f(v,t)\,\mathrm{d}v\mathrm{d}w\\
&= \lambda\kappa\int_{R_+}(M_{\theta+1} - M_\theta v + \theta M_\theta v - \theta M_{\theta-1}v^2)f(v,t)\,\mathrm{d}v\\
&= \lambda\kappa\int_{R_+}[M_{\theta+1} + (\theta-1)M_\theta v - \theta M_{\theta-1}v^2]f(v,t)\,\mathrm{d}v. \qquad (9)
\end{aligned}$$

According to Jensen's inequality,

$$\left(\int_{R_+} xf(x,t)\,\mathrm{d}x\right)^{1+\alpha} \le \int_{R_+} x^{1+\alpha}f(x,t)\,\mathrm{d}x,$$
$$\left(\int_{R_+} x^\alpha f(x,t)\,\mathrm{d}x\right)^{1/\alpha} \le \int_{R_+} xf(x,t)\,\mathrm{d}x.$$

Applying Jensen's inequality to Eq. (9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}m_1(t) \le \lambda\kappa[M_{\theta+1} + (\theta-1)M_\theta m_1(t) - \theta M_{\theta-1}(m_1(t))^2]. \tag{10}$$

We note that the right-hand side of inequality (10) is a quadratic polynomial in $m_1(t)$ with a negative coefficient for the quadratic term. Using the same method as in Ref. [11], we derive from Eq. (10)

$$m_1(t) \le \max\{m_1(0), \bar{m_1}\},$$

where

$$\begin{aligned}
\bar{m_1} &= \frac{-(\theta-1)M_\theta - \sqrt{(\theta-1)^2M_\theta^2 - 4(-\theta M_{\theta-1})M_{\theta+1}}}{2(-\theta M_{\theta-1})}\\
&= \frac{(\theta-1)M_\theta + \sqrt{(\theta-1)^2M_\theta^2 + 4\theta M_{\theta-1}M_{\theta+1}}}{2\theta M_{\theta-1}}.
\end{aligned}$$

Therefore, the average wealth of agents remains bounded.

2.2. The Fokker–Planck equation and the steady-state wealth distribution function

A direct solution of Eq. (6) is infeasible, so we employ a scaling transformation[17] to derive the associated Fokker–Planck equation. Introduce the parameter ε with $0 < \varepsilon \ll 1$, and perform the following scaling transformation on the savings propensity and risk parameters:

$$\lambda \to \varepsilon\lambda, \quad \eta_1 \to \sqrt{\varepsilon}\eta_1. \tag{11}$$

Applying this scaling transformation to the interaction rule, Eq. (9) becomes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}m_1(t)\\
&= \varepsilon\lambda\kappa\int_{R_+}\left[M_{\theta+1} + (\theta-1)M_\theta v - \theta M_{\theta-1}v^2\right]f(v,t)\,\mathrm{d}v.
\end{aligned}$$

Let $\tau = \varepsilon t$. Then, the density function $f(v,t)$ is expressed as $g(v,\tau)$. Accordingly, the average wealth $m_1(\tau)$ satisfies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}m_1(\tau)\\
&= \lambda\kappa\int_{R_+}[M_{\theta+1} + (\theta-1)M_\theta v - \theta M_{\theta-1}v^2]g(v,\tau)\,\mathrm{d}v. \qquad (12)
\end{aligned}$$

Equation (12) does not contain the parameter ε, indicating that ε does not affect the evolution of the average wealth.

According to Eq. (11), the interaction rule Eq. (5) becomes

$$\mathbb{E}(v_* - v) = \varepsilon\lambda(w - v), \tag{13}$$
$$\mathbb{E}((v_* - v)^2) = \varepsilon^2\lambda^2(w-v)^2 + \varepsilon\sigma v^2. \tag{14}$$

Using Taylor's formula and expanding the smooth function $\varphi(v)$ at v, we have

$$\begin{aligned}
\varphi(v_*) - \varphi(v) &= \varphi'(v)(v_* - v) + \frac{1}{2}\varphi''(v)(v_* - v)^2\\
&\quad + \frac{1}{6}\varphi'''(\tilde{v})(v_* - v)^3, \qquad (15)
\end{aligned}$$

where $\tilde{v} = \vartheta v_* + (1-\vartheta)v$, $0 \le \vartheta \le 1$.

Substituting Eqs. (13) and (14) into Eq. (15) yields

$$\begin{aligned}
\mathbb{E}(\varphi(v_*) - \varphi(v)) &= \mathbb{E}\Big[\varphi'(v)\varepsilon\lambda(w-v)\\
&\quad + \frac{1}{2}\varphi''(v)(\varepsilon^2\lambda^2(w-v)^2 + \varepsilon\sigma v^2)\\
&\quad + \frac{1}{6}\varphi'''(\tilde{v})(v_* - v)^3\Big]\\
&= \varepsilon\left[\lambda(w-v)\varphi'(v) + \frac{1}{2}\sigma v^2\varphi''(v)\right] + R_\varepsilon, \qquad (16)
\end{aligned}$$

where

$$R_\varepsilon = \mathbb{E}\left[\frac{1}{2}\varepsilon^2\lambda^2(w-v)^2\varphi''(v) + \frac{1}{6}\varphi'''(\tilde{v})(v_* - v)^3\right].$$

Combined with the relation $\tau = \varepsilon t$, the density function is expressed as $g(v,\tau)$. When $\tau = t = 0$, we have $g(v,0) = f(v,0)$. Substituting Eq. (16) into the Boltzmann equation (6), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g(v,\tau)\,\mathrm{d}v = \kappa\int_{R_+}\int_{R_+}\left(w^\theta + \theta w^{\theta-1}v\right)\\
&\times\left[\lambda(w-v)\varphi'(v) + \frac{1}{2}\sigma v^2\varphi''(v)\right]h(w)g(v,\tau)\,\mathrm{d}v\mathrm{d}w + \tilde{R_\varepsilon}, \qquad (17)
\end{aligned}$$

where

$$\begin{aligned}\tilde{R}_\varepsilon &= \mathbb{E}\Big\{\frac{1}{\varepsilon}\kappa\int_{R_+}\int_{R_+}\left(w^\theta+\theta w^{\theta-1}v\right)\Big[\frac{1}{2}\varepsilon^2\lambda^2(w-v)^2\varphi''(v)\\&\quad+\frac{1}{6}\varphi'''(\tilde{v})(v_*-v)^3\Big]h(w)g(v,\tau)\,\mathrm{d}v\mathrm{d}w\Big\}\\&= \kappa\int_{R_+}\int_{R_+}(w^\theta+\theta w^{\theta-1}v)\Big[\frac{1}{2}\varepsilon\lambda^2(w-v)^2\varphi''(v)\,\mathrm{d}v\mathrm{d}w\\&\quad+\mathbb{E}\Big[\frac{1}{\varepsilon}\kappa\int_{R_+}\int_{R_+}\frac{1}{6}(w^\theta+\theta w^{\theta-1}v)\varphi'''(\tilde{v})(v_*-v)^3\Big]\\&\quad\times h(w)g(v,\tau)\,\mathrm{d}v\mathrm{d}w\Big].\end{aligned}$$

When $\varepsilon\to 0$,

$$\lim_{\varepsilon\to 0}\tilde{R}_\varepsilon = 0.$$

Consequently, Eq. (17) reduces to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g(v,\tau)\,\mathrm{d}v\\&= \kappa\int_{R_+}\int_{R_+}\left(w^\theta+\theta w^{\theta-1}v\right)\Big[\lambda(w-v)\varphi'(v)+\frac{1}{2}\sigma v^2\varphi''(v)\Big]\\&\quad\times h(w)g(v,\tau)\,\mathrm{d}v\mathrm{d}w\\&= \kappa\int_{R_+}\int_{R_+}\Big(\lambda w^{\theta+1}\varphi'(v)-\lambda w^\theta v\varphi'(v)+\frac{1}{2}\sigma w^\theta v^2\varphi''(v)\\&\quad+\lambda\theta w^\theta v\varphi'(v)-\lambda\theta w^{\theta-1}v^2\varphi'(v)+\frac{1}{2}\theta\sigma w^{\theta-1}v^3\varphi''(v)\Big)\\&\quad\times h(w)g(v,\tau)\,\mathrm{d}v\mathrm{d}w.\end{aligned}\tag{18}$$

From Eq. (18), we obtain the following Fokker–Planck equation:

$$\begin{aligned}\frac{1}{\kappa}\frac{\partial g(v,\tau)}{\partial\tau} &=\\&\lambda\frac{\partial}{\partial v}\Big[\left(-M_{\theta+1}+M_\theta v-\theta M_\theta v+\theta M_{\theta-1}v^2\right)g(v,\tau)\Big]\\&+\frac{1}{2}\sigma\frac{\partial^2}{\partial v^2}\Big(M_\theta v^2 g(v,\tau)+\theta M_{\theta-1}v^3 g(v,\tau)\Big).\end{aligned}\tag{19}$$

Although the existence and uniqueness of the solution corresponding to Eq. (19) have been verified in Ref. [21], obtaining its analytical solution remains a nontrivial task.

We consider the steady-state solution as time tends to infinity. In the steady state, the left-hand side of Eq. (19) becomes 0. At this stage, $g(v,\tau)$ is denoted by $g_\infty(v)$, and Eq. (19) becomes

$$\begin{aligned}0 &= \lambda\frac{\partial}{\partial v}[(-M_{\theta+1}+M_\theta v-\theta M_\theta v+\theta M_{\theta-1}v^2)g_\infty(v)]\\&\quad+\frac{1}{2}\sigma\frac{\partial^2}{\partial v^2}(M_\theta v^2 g_\infty(v)+\theta M_{\theta-1}v^3 g_\infty(v)).\end{aligned}\tag{20}$$

In the case where $\theta=0$, the non-Maxwellian collision kernel reduces to the Maxwellian collision kernel, and the associated steady-state solution is governed by the following equation:

$$\lambda M_1\frac{\partial g_\infty(v)}{\partial v}=\lambda\frac{\partial(vg_\infty(v))}{\partial v}+\frac{1}{2}\sigma\frac{\partial^2(v^2g_\infty(v))}{\partial v^2}.\tag{21}$$

We find that Eq. (21) is consistent with the steady-state equation derived in Ref. [11].

From Eq. (20), we obtain

$$\begin{aligned}&-\lambda[(-M_{\theta+1}+M_\theta v-\theta M_\theta v+\theta M_{\theta-1}v^2)g_\infty(v)]\\&= \frac{1}{2}\sigma\frac{\partial}{\partial v}(M_\theta v^2 g_\infty(v)+\theta M_{\theta-1}v^3 g_\infty(v)).\end{aligned}\tag{22}$$

Accordingly, Eq. (22) becomes

$$a(v)g_\infty(v)=\frac{\mathrm{d}(B(v)g_\infty(v))}{\mathrm{d}v},\tag{23}$$

where

$$\begin{aligned}a(v)&=-\lambda(-M_{\theta+1}+M_\theta v-\theta M_\theta v+\theta M_{\theta-1}v^2),\\B(v)&=\frac{1}{2}\sigma(M_\theta v^2+\theta M_{\theta-1}v^3).\end{aligned}\tag{24}$$

Let $G(v)=B(v)g_\infty(v)$. Then $g_\infty(v)=G(v)/B(v)$. Solving Eq. (23) yields

$$\frac{a(v)}{B(v)}\mathrm{d}v=\frac{\mathrm{d}G(v)}{G(v)}.\tag{25}$$

Integrating both sides of Eq. (25), we obtain

$$G(v)=c\times\exp\left(\int_0^v\frac{a(x)}{B(x)}\mathrm{d}x\right).$$

According to $G(v)=B(v)g_\infty(v)$, we obtain

$$g_\infty(v)=c\frac{1}{B(v)}\exp\left(\int_0^v\frac{a(x)}{B(x)}\mathrm{d}x\right),\tag{26}$$

where the constant c satisfies the normalization condition $\int_{R_+}g_\infty(v)\,\mathrm{d}v=1$, $a(x)=-\lambda(-M_{\theta+1}+M_\theta x-\theta M_\theta x+\theta M_{\theta-1}x^2)$, and $B(x)=\frac{1}{2}\sigma(M_\theta x^2+\theta M_{\theta-1}x^3)$. As $v\to\infty$, we have $g_\infty(v)\sim\frac{1}{B(v)}$. The exponent of v does not contain θ, indicating that the wealth decay rate of an agent is only weakly affected by the trading willingness of the counterparty. Since wealth decay is only weakly affected by the trading willingness of counterparties, economic policy adjustments should focus more on optimizing the objective trading environment than on intervening in the subjective decisions of agents. For instance, to stabilize the wealth levels of market entities, irrational wealth fluctuations can be reduced from an objective perspective by improving trading supervision rules, lowering transaction costs, and preventing systemic risks, without excessive intervention in agents' willingness to trade.

Considering $\theta=0$, the collision kernel Eq. (4) becomes

$$K(v,w)=\kappa.$$

This equation indicates that the trading frequency is a fixed constant. When the trading frequency is fixed, the steady-state wealth distribution $g_\infty^*(v)$ satisfies the following equation:

$$\lambda(v-M_1)g_\infty^*(v)+\frac{\sigma}{2}\frac{\partial}{\partial v}(v^2g_\infty^*(v))=0,\tag{27}$$

Solving Eq. (27), we obtain

$$g_\infty^*(v)=\frac{\left(\frac{2\lambda M_1}{\sigma}\right)^{1+\frac{2\lambda}{\sigma}}}{\Gamma(1+\frac{2\lambda}{\sigma})}v^{-2-\frac{2\lambda}{\sigma}}\exp\left(-\frac{2\lambda M_1}{\sigma v}\right). \quad (28)$$

When the variable v is very large (namely, the amount of wealth is very large), the exponential term $\exp\left(-\frac{2\lambda M_1}{\sigma v}\right)$ approaches 1. Therefore, the steady-state wealth distribution $g_\infty^*(v)$ exhibits a Pareto tail, and the associated Pareto index is $1+\frac{2\lambda}{\sigma}$. A direct comparison of Eqs. (26) and (28) shows that the tail structures of the corresponding steady-state distributions are different, indicating that changes in trading frequency have a moderate impact on the steady-state wealth distribution.

3. The evolution of the wealth of two trading groups and the wealth distribution function

In the wealth trading process of agents, some agents have a high trading frequency, while others have a low trading frequency. According to the different trading frequencies during the trading process, we divide agents into two different groups. Let $f_1(v,t)$ and $f_2(v,t)$ represent the wealth density functions of the two trading groups, respectively. Then the total wealth density $f(v,t)$ satisfies

$$f_t(v,t)=f_1(v,t)+f_2(v,t). \quad (29)$$

Assume that the total wealth density function satisfies

$$\int_{R_+} f_t(v,t)\,\mathrm{d}v=1.$$

Therefore,

$$\int_{R_+} f_1(v,t)\,\mathrm{d}v+\int_{R_+} f_2(v,t)\,\mathrm{d}v=1.$$

It is assumed that the interaction rules are given by

$$v_*=(1-\lambda)v+\lambda w+\eta_1 v, \quad (30)$$

$$w_*=(1-\lambda)w+\lambda v+\eta_2 w, \quad (31)$$

where the parameter $0<\lambda<1$. The expectations of η_1 and η_2 satisfy $\mathbb{E}(\eta_1)=0$ and $\mathbb{E}(\eta_2)=0$, and their variances are given by $\mathbb{E}(\eta_1^2)=\sigma$ and $\mathbb{E}(\eta_2^2)=\sigma$. In Eq. (30), v_* describes the wealth of the agents in the first group after the transaction, λv represents the wealth invested in the transaction by the agents in the first group, $(1-\lambda)v$ corresponds to the wealth retained by these agents before the transaction, λw denotes the wealth obtained from the transaction, and $\eta_1 v$ represents the wealth variation caused by other random factors. In Eq. (31), w_* describes the wealth of the agents in the second group after the transaction, λw represents the wealth invested in the transaction by these agents, and the complementary term $(1-\lambda)w$ corresponds to the wealth retained before the transaction. Moreover, λv denotes the wealth obtained from the transaction, and $\eta_2 w$ represents the wealth variation caused by other random factors. Equations (30) and (31) describe the asset exchange process between paired agents. Agents in the first group contribute the asset amount λv, while agents in the second group contribute the asset amount λw for the reciprocal exchange. The parameter λ represents the savings propensity of the agents.

Adding Eqs. (30) and (31), we have

$$w_*+v_*=(1-\lambda)v+\lambda w+\eta_1 v+(1-\lambda)w+\lambda v+\eta_2 w.$$

Since the expectations of η_1 and η_2 are zero, we obtain

$$\mathbb{E}(w_*+v_*)=v+w. \quad (32)$$

Equation (32) shows that the total wealth of the system is conserved in the statistical sense.

Assume that the non-Maxwell collision kernels take the following forms:

$$K_{12}(v,w)=\kappa(w^\theta+\theta w^{\theta-1}v),$$
$$K_{21}(v,w)=\kappa(v^\theta+\theta v^{\theta-1}w). \quad (33)$$

Assume that there is no trading among agents in the same group. According to the interaction rules Eqs. (30) and (31), the evolution of the density function $f_i(v,t)$ $(i=1,2)$ satisfies the Boltzmann equation

$$\frac{\partial f_i(v,t)}{\partial t}=Q(f_i,f_j)(v),\quad i,j=1,2,\quad i\neq j,$$

where the collision operator $Q(f_i,f_j)(v)$ describes wealth transactions between different trading groups. For any smooth function $\varphi(v)$ with compact support, the wealth density $f_i(v,t)$ $(i=1,2)$ satisfies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+}\varphi(v)f_i(v,t)\,\mathrm{d}v\\&=\mathbb{E}\left[\frac{1}{2}\int_{R_+}\int_{R_+}K_{ij}(v,w)(\varphi(w_*)+\varphi(v_*)\right.\\&\quad\left.-\varphi(w)-\varphi(v))f_i(v,t)f_j(w,t)\,\mathrm{d}v\mathrm{d}w\right]\\&=\mathbb{E}\left[\int_{R_+}\int_{R_+}K_{ij}(v,w)(\varphi(v_*)-\varphi(v))\right.\\&\quad\left.\times f_i(v,t)f_j(w,t)\,\mathrm{d}v\mathrm{d}w\right], \quad (34)\end{aligned}$$

where $i\neq j$.

In Eq. (34), taking $\varphi(v)=1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+}f_i(v,t)\,\mathrm{d}v=0. \quad (35)$$

Equation (35) indicates that the density functions $f_1(v,t)$ and $f_2(v,t)$ satisfy the conservation property, namely,

$$\int_{R_+}f_1(v,t)\,\mathrm{d}v=\int_{R_+}f_1(v,0)\,\mathrm{d}v=\rho_1,$$

$$\int_{R_+} f_2(v,t)\,\mathrm{d}v = \int_{R_+} f_2(v,0)\,\mathrm{d}v = \rho_2,$$

where $\rho_1+\rho_2=1$. In Eq. (34), choosing $\varphi(v)=v$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+} v f_i(v,t)\,\mathrm{d}v \\
&= \mathbb{E}\left[\frac{1}{2}\int_{R_+}\int_{R_+} K_{ii}(v,w)(w_*+v_*-w-v)\right. \\
&\quad \left.\times f_i(w,t)f_i(v,t)\,\mathrm{d}v\mathrm{d}w\right] \\
&\quad +\mathbb{E}\left[\frac{1}{2}\int_{R_+}\int_{R_+} K_{ij}(v,w)(w_*+v_*-w-v)\right. \\
&\quad \left.\times f_i(w,t)f_j(v,t)\,\mathrm{d}v\mathrm{d}w\right] = 0. \qquad (36)
\end{aligned}$$

Equation (36) shows that the average wealth remains unchanged over time.

For the purpose of deriving the Fokker–Planck equation associated with Eq. (34), we continue to employ the scaling approach. Introducing the parameter ε with $0<\varepsilon\ll 1$, we adopt the following scaling:

$$\lambda\to\varepsilon\lambda,\quad \eta_1\to\sqrt{\varepsilon}\eta_1,\quad \eta_2\to\sqrt{\varepsilon}\eta_2. \qquad (37)$$

According to Eq. (37), the interaction rule Eq. (30) satisfies

$$\mathbb{E}(v_*-v)=\varepsilon\lambda(w-v),\ \ \mathbb{E}[(v_*-v)^2]=\varepsilon^2\lambda^2(w-v)^2+\varepsilon\sigma v^2.$$

Setting $\tau=\varepsilon t$, $g_1(v,\tau)=f_1(v,t)$, and $g_2(v,\tau)=f_2(v,t)$, we obtain

$$\begin{aligned}
&\int_{R_+} w^n g_1(w,\tau)\,\mathrm{d}w = m_{n,1}(\tau), \\
&\int_{R_+} w^n g_2(w,\tau)\,\mathrm{d}w = m_{n,2}(\tau). \qquad (38)
\end{aligned}$$

To derive the Fokker–Planck equation associated with Eq. (34), we continue to employ the scaling approach. Introducing the parameter ε with $0<\varepsilon\ll 1$, we adopt the following scaling

$$\lambda\to\varepsilon\lambda,\quad \eta_1\to\sqrt{\varepsilon}\eta_1,\quad \eta_2\to\sqrt{\varepsilon}\eta_2. \qquad (39)$$

According to the scaling in Eq. (37), the interaction rule Eq. (30) satisfies

$$\mathbb{E}(v_*-v)=\varepsilon\lambda(w-v),\ \ \mathbb{E}((v_*-v)^2)=\varepsilon^2\lambda^2(w-v)^2+\varepsilon\sigma v^2.$$

Setting $\tau=\varepsilon t$, $g_1(v,\tau)=f_1(v,t)$ and $g_2(v,\tau)=f_2(v,t)$, we obtain

$$\begin{aligned}
&\int_{R_+} w^n g_1(w,\tau)\,\mathrm{d}w = m_{n,1}(\tau), \\
&\int_{R_+} w^n g_2(w,\tau)\,\mathrm{d}w = m_{n,2}(\tau). \qquad (40)
\end{aligned}$$

Substituting the collision kernel Eq. (33) into Eq. (34), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g_1(v,\tau)\,\mathrm{d}v \\
&= \int_{R_+}\int_{R_+}\kappa\left(w^\theta+\theta w^{\theta-1}v\right)\left(\lambda(w-v)\varphi'(v)\right. \\
&\quad \left.+\frac{1}{2}\sigma v^2\varphi''(v)\right)g_1(v,\tau)g_2(w,\tau)\,\mathrm{d}v\mathrm{d}w+R_\varepsilon^1, \qquad (41)
\end{aligned}$$

where

$$\begin{aligned}
R_\varepsilon^1 &= \int_{R_+}\int_{R_+}\kappa\left(w^\theta+\theta w^{\theta-1}v\right)\left[\frac{1}{2}\varepsilon^2\lambda^2(w-v)^2\varphi''(v)\right. \\
&\quad \left.+\frac{1}{6}\varphi'''(\tilde{v})(v_*-v)^3\right]g_1(v,\tau)g_2(w,\tau)\,\mathrm{d}v\mathrm{d}w. \qquad (42)
\end{aligned}$$

Under the scaling transformation Eq. (39), the interaction rule Eq. (30) gives

$$\mathbb{E}[(v_*-v)^3]=\varepsilon^3\lambda^3(w-v)^3+\varepsilon^2 3\lambda(w-v)\sigma v^2. \qquad (43)$$

Substituting Eqs. (40) and (43) into Eq. (42), we obtain

$$\begin{aligned}
R_\varepsilon^1 &= \frac{1}{\varepsilon}\int_{R_+}\kappa\Big[\frac{1}{2}\varepsilon^2\lambda^2(m_{2+\theta,2}(\tau)-2m_{1+\theta,2}(\tau)v+m_{\theta,2}(\tau)v^2 \\
&\quad +\theta m_{1+\theta,2}(\tau)v-2\theta m_{\theta,2}(\tau)v^2+\theta m_{\theta-1,2}(\tau)v^3)\Big] \\
&\quad \times\varphi''(v)g_1(v,\tau)\,\mathrm{d}v+\frac{1}{\varepsilon}\int_{R_+}\kappa\frac{1}{6}\varphi'''(\tilde{v})[\varepsilon^3\lambda^3(m_{3+\theta,2}(\tau) \\
&\quad -3m_{2+\theta,2}(\tau)v+3m_{1+\theta,2}(\tau)v^2-m_{\theta,2}(\tau)v^3 \\
&\quad +\theta m_{2+\theta,2}(\tau)v-3m_{1+\theta,2}(\tau)v^2+3\theta m_{\theta,2}(\tau)v^3 \\
&\quad -\theta m_{\theta-1,2}(\tau)v^4)+\varepsilon^2 3\lambda(m_{\theta+1,2}(\tau)-m_{\theta,2}(\tau)v \\
&\quad +\theta m_{\theta,2}(\tau)v-\theta m_{\theta,2}(\tau)v^2)\sigma v^2]g_1(v,\tau)\,\mathrm{d}v.
\end{aligned}$$

Since $\varphi(v)$ is a smooth function with compact support, all moments of the wealth distribution $g_1(v,\tau)$ remain bounded. Therefore, in the limit $\varepsilon\to 0$, we obtain

$$\lim_{\varepsilon\to 0} R_\varepsilon^1 = 0.$$

Consequently, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g_1(v,\tau)\,\mathrm{d}v \\
&= \int_{R_+}\int_{R_+}\kappa\left(w^\theta+\theta w^{\theta-1}v\right)[\lambda(w-v)\varphi'(v) \\
&\quad +\frac{1}{2}\sigma v^2\varphi''(v)]g_1(v,\tau)g_2(w,\tau)\,\mathrm{d}v\mathrm{d}w. \qquad (44)
\end{aligned}$$

Combining Eqs. (40) and (44), we obtain

$$\begin{aligned}
&\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g_1(v,\tau)\,\mathrm{d}v \\
&= \int_{R_+}\lambda(m_{\theta+1,2}(\tau)+\theta m_{\theta,2}(\tau)v-m_{\theta,2}(\tau)v \\
&\quad -\theta m_{\theta-1,2}(\tau)v^2)\varphi'(v)g_1(v,\tau)\,\mathrm{d}v \\
&\quad +\int_{R_+}\frac{1}{2}\sigma(m_{\theta,2}(\tau)v^2+\theta m_{\theta-1,2}(\tau)v^3) \\
&\quad \times\varphi''(v)g_1(v,\tau)\,\mathrm{d}v. \qquad (45)
\end{aligned}$$

Applying integration by parts to the right-hand side of Eq. (45), we obtain

$$\begin{aligned}
&\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g_1(v,\tau)\mathrm{d}v\\
&=\lambda(m_{\theta+1,2}(\tau)+\theta m_{\theta,2}(\tau)v-m_{\theta,2}(\tau)v-\theta m_{\theta-1,2}(\tau)v^2)\\
&\quad\times\varphi(v)g_1(v,\tau)\big|_0^{+\infty}\\
&\quad-\int_{R_+}\varphi(v)\frac{\partial[\lambda(m_{\theta+1,2}(\tau)+\theta m_{\theta,2}(\tau)v)g_1(v,\tau)]}{\partial v}\mathrm{d}v\\
&\quad-\int_{R_+}\varphi(v)\frac{\partial[\lambda(-m_{\theta,2}(\tau)v-\theta m_{\theta-1,2}(\tau)v^2)g_1(v,\tau)]}{\partial v}\mathrm{d}v\\
&\quad+\frac{1}{2}\sigma(m_{\theta,2}(\tau)v^2+\theta m_{\theta-1,2}(\tau)v^3)\varphi'(v)g_1(v,\tau)\big|_0^{+\infty}\\
&\quad-\int_{R_+}\varphi'(v)\frac{\partial[\frac{1}{2}\sigma(m_{\theta,2}(\tau)v^2+\theta m_{\theta-1,2}(\tau)v^3)g_1(v,\tau)]}{\partial v}\mathrm{d}v,
\end{aligned}$$

where

$$\begin{aligned}
&\int_{R_+}\varphi'(v)\frac{\partial[\frac{1}{2}\sigma(m_{\theta,2}(\tau)v^2+\theta m_{\theta-1,2}(\tau)v^3)g_1(v,\tau)]}{\partial v}\mathrm{d}v\\
&=\varphi(v)\frac{\partial[\frac{1}{2}\sigma(m_{\theta,2}(\tau)v^2+\theta m_{\theta-1,2}(\tau)v^3)g_1(v,\tau)]}{\partial v}\Big|_0^{+\infty}\\
&\quad-\int_{R_+}\varphi(v)\frac{\partial^2[\frac{1}{2}\sigma(m_{\theta,2}(\tau)v^2+\theta m_{\theta-1,2}(\tau)v^3)g_1(v,\tau)]}{\partial v^2}\mathrm{d}v.
\end{aligned}$$

The resulting Fokker–Planck equation for $g_1(v,\tau)$ is

$$\begin{aligned}
&\frac{1}{\kappa}\frac{g_1(v,\tau)}{\mathrm{d}\tau}\\
&=-\frac{\partial[\lambda(m_{\theta+1,2}(\tau)+\theta m_{\theta,2}(\tau)v)g_1(v,\tau)]}{\partial v}\\
&\quad-\frac{\partial[\lambda(-m_{\theta,2}(\tau)v-\theta m_{\theta-1,2}(\tau)v^2)g_1(v,\tau)]}{\partial v}\\
&\quad+\frac{\partial^2[\frac{1}{2}\sigma(m_{\theta,2}(\tau)v^2+\theta m_{\theta-1,2}(\tau)v^3)g_1(v,\tau)]}{\partial v^2}.
\end{aligned}\tag{46}$$

Equation (46) can be simplified as

$$\begin{aligned}
&\frac{1}{\kappa}\frac{g_1(v,\tau)}{\mathrm{d}\tau}\\
&=-\frac{\lambda\partial[m_{\theta+1,2}(\tau)+(\theta-1)m_{\theta,2}(\tau)v-\theta m_{\theta-1,2}(\tau)v^2]g_1(v,\tau)}{\partial v}\\
&\quad+\frac{\sigma}{2}\frac{\partial^2[m_{\theta,2}(\tau)v^2+\theta m_{\theta-1,2}(\tau)v^3]g_1(v,\tau)}{\partial v^2}.
\end{aligned}\tag{47}$$

For the steady-state solution in the long-time limit, the left-hand side of Eq. (47) vanishes. In this case, $g_1(v,\tau)$ is denoted as $g_{1,\infty}(v)$, and Eq. (47) reduces to

$$\begin{aligned}
&\lambda[m_{\theta+1,2}(\infty)+(\theta-1)m_{\theta,2}(\infty)v-\theta m_{\theta-1,2}(\infty)v^2]g_{1,\infty}(v)\\
&=\frac{\sigma}{2}\frac{\partial[m_{\theta,2}(\infty)v^2+\theta m_{\theta-1,2}(\infty)v^3]g_{1,\infty}(v)}{\partial v}.
\end{aligned}\tag{48}$$

Equation (48) can be rewritten as

$$a_1(v)g_{1,\infty}(v)=\frac{\mathrm{d}[b_1(v)g_{1,\infty}(v)]}{\mathrm{d}v},\tag{49}$$

where $a_1(v)=\lambda[m_{\theta+1,2}(\infty)+(\theta-1)m_{\theta,2}(\infty)v-\theta m_{\theta-1,2}(\infty)v^2]$ and $b_1(v)=\frac{\sigma}{2}[m_{\theta,2}(\infty)v^2+\theta m_{\theta-1,2}(\infty)v^3]$.

To solve the differential equation (49), we define $G_1(v)=b_1(v)g_{1,\infty}(v)$. Then $g_{1,\infty}(v)=G_1(v)/b_1(v)$, which gives

$$\frac{a_1(v)}{b_1(v)}\mathrm{d}v=\frac{dG_1(v)}{G_1(v)}.\tag{50}$$

Integrating both sides, we obtain

$$G_1(v)=\exp\left(\int_0^v\frac{a_1(x)}{b_1(x)}\mathrm{d}x\right).\tag{51}$$

Using $G_1(v)=b_1(v)g_{1,\infty}(v)$, we obtain

$$g_{1,\infty}(v)=c_1\frac{1}{b_1(v)}\exp\left(\int_0^v\frac{a_1(x)}{b_1(x)}\mathrm{d}x\right),\tag{52}$$

where the constant c_1 satisfies $\int_{\mathbb{R}_+}g_{1,\infty}(v)\,\mathrm{d}v=\rho_1$, $a_1(x)=\lambda\left[m_{\theta+1,2}(\infty)+(\theta-1)m_{\theta,2}(\infty)x-\theta m_{\theta-1,2}(\infty)x^2\right]$, and $b_1(x)=\frac{\sigma}{2}\left(m_{\theta,2}(\infty)x^2+\theta m_{\theta-1,2}(\infty)x^3\right)$. In the steady-state solution $g_{1,\infty}(v)$, $b_1(v)$ is a polynomial, and the decay rate in the large wealth limit is related to $\frac{1}{b_1(v)}$.

Following the same procedure, the evolution equation for $g_2(v,\tau)$ is given by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g_2(v,\tau)\mathrm{d}v\\
&=\int_{R_+}\int_{R_+}\kappa[v^\theta+\theta v^{\theta-1}w]\left[\lambda(w-v)\varphi'(v)+\frac{1}{2}\sigma v^2\varphi''(v)\right]\\
&\quad\times g_2(v,\tau)g_1(w,\tau)\mathrm{d}v\mathrm{d}w.
\end{aligned}\tag{53}$$

Further calculation of Eq. (53) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g_2(v,\tau)\mathrm{d}v\\
&=\kappa\int_{R_+}\lambda\left(m_{1,1}(\tau)v^\theta-v^{\theta+1}\right)\varphi'(v)g_2(v,\tau)\mathrm{d}v\\
&\quad+\kappa\int_{R_+}\lambda\left(\theta m_{2,1}(\tau)v^{\theta-1}-\theta m_{1,1}(\tau)v^\theta\right)\varphi'(v)g_2(v,\tau)\mathrm{d}v\\
&\quad+\kappa\int_{R_+}\frac{\sigma}{2}\left(v^{\theta+2}+\theta m_{1,1}(\tau)v^{\theta+1}\right)\varphi''(v)g_2(v,\tau)\mathrm{d}v.
\end{aligned}\tag{54}$$

Applying integration by parts to the right-hand side of Eq. (54), we obtain

$$\begin{aligned}
&\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g_2(v,\tau)\mathrm{d}v\\
&=-\int_{R_+}\varphi(v)\frac{\partial}{\partial v}\left[\lambda\left(m_{1,1}(\tau)v^\theta-v^{\theta+1}\right)g_2(v,\tau)\right]\mathrm{d}v\\
&\quad-\int_{R_+}\varphi(v)\frac{\partial}{\partial v}\left[\lambda\left(\theta m_{2,1}(\tau)v^{\theta-1}-\theta m_{1,1}(\tau)v^\theta\right)g_2(v,\tau)\right]\mathrm{d}v\\
&\quad+\int_{R_+}\varphi(v)\frac{\partial^2}{\partial v^2}\left[\frac{\sigma}{2}\left(v^{\theta+2}+\theta m_{1,1}(\tau)v^{\theta+1}\right)g_2(v,\tau)\right]\mathrm{d}v.
\end{aligned}\tag{55}$$

Further computation of Eq. (55) gives

$$\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g_2(v,\tau)\mathrm{d}v$$

$$
\begin{aligned}
&= -\int_{R_+} \varphi(v)\frac{\partial}{\partial v}\left[\lambda\left(m_{1,1}(\tau)v^{\theta} - v^{\theta+1}\right)g_2(v,\tau)\right]\mathrm{d}v \\
&\quad -\int_{R_+} \varphi(v) \\
&\quad \times\frac{\partial}{\partial v}\left[\lambda\left(\theta m_{2,1}(\tau)v^{\theta-1} - \theta m_{1,1}(\tau)v^{\theta}\right)g_2(v,\tau)\right]\mathrm{d}v \\
&\quad +\int_{R_+} \varphi(v)\frac{\sigma}{2}\frac{\partial^2}{\partial v^2}\left[\left(v^{\theta+2} + \theta m_{1,1}(\tau)v^{\theta+1}\right)g_2(v,\tau)\right]\mathrm{d}v.
\end{aligned}
$$

Therefore, the Fokker–Planck equation satisfied by $g_2(v,\tau)$ is obtained as

$$
\begin{aligned}
\frac{1}{\kappa}\frac{\partial g_2(v,\tau)}{\partial \tau} &= -\frac{\partial}{\partial v}\left[\lambda\left(m_{1,1}(\tau)v^{\theta} - v^{\theta+1}\right)g_2(v,\tau)\right] \\
&\quad -\frac{\partial}{\partial v}\left[\lambda\left(\theta m_{2,1}(\tau)v^{\theta-1} - \theta m_{1,1}(\tau)v^{\theta}\right)g_2(v,\tau)\right] \\
&\quad +\frac{\sigma}{2}\frac{\partial^2}{\partial v^2}\left[\left(v^{\theta+2} + \theta m_{1,1}(\tau)v^{\theta+1}\right)g_2(v,\tau)\right]. \quad (56)
\end{aligned}
$$

Equation (56) can be rewritten as

$$
\begin{aligned}
\frac{1}{\kappa}\frac{\partial g_2(v,\tau)}{\partial \tau} &= -\frac{\partial}{\partial v}\left\{\lambda\left[\theta m_{2,1}(\tau)v^{\theta-1}\right.\right. \\
&\quad \left.\left.+(1-\theta)m_{1,1}(\tau)v^{\theta} - v^{\theta+1}\right]g_2(v,\tau)\right\} \\
&\quad +\frac{\sigma}{2}\frac{\partial^2}{\partial v^2}\left\{\left[v^{\theta+2} + \theta m_{1,1}(\tau)v^{\theta+1}\right]g_2(v,\tau)\right\}. \quad (57)
\end{aligned}
$$

At the steady state, the left-hand side of Eq. (57) vanishes. Denoting $g_2(v,\tau)$ by $g_{2,\infty}(v)$, we obtain

$$
\begin{aligned}
&\lambda\left[\theta m_{2,1}(\infty)v^{\theta-1} + (1-\theta)m_{1,1}(\infty)v^{\theta} - v^{\theta+1}\right]g_{2,\infty}(v) \\
&= \frac{\sigma}{2}\frac{\partial}{\partial v}\left[\left(v^{\theta+2} + \theta m_{1,1}(\infty)v^{\theta+1}\right)g_{2,\infty}(v)\right]. \quad (58)
\end{aligned}
$$

Introducing

$$
a_2(v) = \lambda\left[\theta m_{2,1}(\infty)v^{\theta-1} + (1-\theta)m_{1,1}(\infty)v^{\theta} - v^{\theta+1}\right],
$$

and

$$
b_2(v) = \frac{\sigma}{2}\left(v^{\theta+2} + \theta m_{1,1}(\infty)v^{\theta+1}\right),
$$

the steady-state solution can be expressed as

$$
g_{2,\infty}(v) = c_2\frac{1}{b_2(v)}\exp\left(\int_0^v \frac{a_2(x)}{b_2(x)}\mathrm{d}x\right). \quad (59)
$$

where c_2 satisfies

$$
\int_{\mathbb{R}_+} g_{2,\infty}(v)\mathrm{d}v = \rho_2.
$$

Here, $b_2(v)$ is a power-law function whose exponent depends on the collision-kernel parameter θ. Therefore, unlike the first trading group, the steady-state wealth distribution of the second group is directly influenced by the trading frequency associated with its own interaction process. Moreover, since $b_1(v)$ and $b_2(v)$ have different functional forms, the two trading groups exhibit different sensitivity to the collision kernel, implying that agents' wealth distributions are more strongly affected by their own trading behavior than by that of their counterparties.

To optimize wealth distribution and mitigate excessive wealth concentration, policy interventions should focus on guiding agents' own trading behaviors rather than directly regulating the interaction frequency between counterparties. Specific measures include strengthening investor education to promote rational trading decisions; improving market information disclosure mechanisms to reduce the impact of information asymmetry on individual decision-making; and influencing agents' risk-taking incentives and transaction costs through taxation, subsidies, and other policy instruments, thereby indirectly improving the overall structure of wealth distribution.

4. Wealth distribution functions incorporating psychological characteristics

We consider the following value function, which differs from that proposed in Ref. [22]

$$
\Phi(s) = \mu\frac{s^{\delta} - 1}{\gamma s^{\delta} + \beta}, \quad s = v/v_{\mathrm{L}}, \quad (60)
$$

where $\gamma > 0$, $0 < \delta < 1$, $0 < \mu < 1$, and $\beta > 0$. These parameters characterize the psychological characteristics of agents, allowing the value function to take both positive and negative values. The properties of this function are consistent with the value function in prospect theory.[23] Specifically, $\Phi(s)$ is negative when the wealth v is below the reference wealth level v_{L}, and becomes positive when $v > v_{\mathrm{L}}$. Therefore, the range of the value function (60) is given by

$$
-\frac{\mu}{\beta} \le \Phi(s) \le \frac{\mu}{\gamma} \le 1.
$$

From the agents' perspective, we consider the following interaction rule incorporating psychological effects

$$
v_* = v - \Phi(s)v + \eta v, \quad (61)
$$

where $\Phi(s)$ represents the psychological characteristics of agents. The term $\Phi(s)v$ describes the wealth adjustment induced by psychological valuation, whereas ηv represents wealth fluctuations caused by random external factors. Here, η is a random variable satisfying $\mathbb{E}(\eta) = 0$ and $\mathbb{E}(\eta^2) = \sigma$. The wealth density function $f(v,t)$ satisfies Eq. (6).

We introduce the following scaling transformation:

$$
\delta \to \varepsilon\delta, \quad \eta \to \sqrt{\varepsilon}\eta. \quad (62)
$$

Under this scaling, the value function Eq. (60) becomes

$$
\Phi_{\varepsilon}(s) = \frac{\mu s^{\varepsilon\delta} - \mu}{\gamma s^{\varepsilon\delta} + \beta}.
$$

Since

$$
s^{\varepsilon\delta} = 1 + \varepsilon\delta\log s + O(\varepsilon^2),
$$

we obtain

$$\lim_{\varepsilon\to 0}\frac{\Phi_\varepsilon(s)}{\varepsilon}=\frac{\mu\delta\log s}{\gamma+\beta}.$$

Therefore, $\Phi_\varepsilon(s)$ is of the same order as ε.

Based on Eq. (61), we obtain

$$\mathbb{E}(v_*-v)=-\varepsilon\frac{\mu\log s}{\gamma+\beta}, \tag{63}$$

$$\mathbb{E}[(v_*-v)^2]=\varepsilon^2\left(\frac{\mu\log s}{\gamma+\beta}\right)^2+\varepsilon\sigma v^2. \tag{64}$$

For any smooth function $\varphi(v)$, applying the Taylor expansion gives

$$\begin{aligned}&\mathbb{E}[\varphi(v_*)-\varphi(v)]\\ =&-\varepsilon\frac{\mu\log s}{\gamma+\beta}\varphi'(v)+\frac{1}{2}\left[\varepsilon^2\left(\frac{\mu\log s}{\gamma+\beta}\right)^2+\varepsilon\sigma v^2\right]\varphi''(v)\\ &+\frac{1}{6}\varphi'''(\tilde v)(v_*-v)^3\\ =&\ \varepsilon\left(-\frac{\mu\log s}{\gamma+\beta}\varphi'(v)+\frac{1}{2}\sigma v^2\varphi''(v)\right)+r_\varepsilon,\end{aligned} \tag{65}$$

where

$$r_\varepsilon=\frac{1}{2}\varepsilon^2\left(\frac{\mu\log s}{\gamma+\beta}\right)^2\varphi''(v)+\frac{1}{6}\varphi'''(\tilde v)(v_*-v)^3.$$

Introducing the rescaled time $\tau=\varepsilon t$, the density function $f(v,t)$ is denoted by $g_*(v,\tau)$. When $\tau=t=0$, we have $g_*(v,0)=f(v,0)$.

Substituting Eq. (65) into the Boltzmann equation (6), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g_*(v,\tau)\mathrm{d}v\\ =&\ \kappa\int_{R_+}\int_{R_+}\left(w^\theta+\theta w^{\theta-1}v\right)\left[-\frac{\mu\log s}{\gamma+\beta}\varphi'(v)+\frac{1}{2}\sigma v^2\varphi''(v)\right]\\ &\times h(w)g_*(v,\tau)\mathrm{d}v\mathrm{d}w+\tilde r_\varepsilon,\end{aligned} \tag{66}$$

where

$$\begin{aligned}\tilde r_\varepsilon=&\ \mathbb{E}\left\{\kappa\int_{R_+}\int_{R_+}\left(w^\theta+\theta w^{\theta-1}v\right)\left[\frac{1}{2}\varepsilon^2\left(\frac{\mu\log s}{\gamma+\beta}\right)^2\varphi''(v)\right.\right.\\ &\left.\left.+\frac{1}{6}\varphi'''(\tilde v)(v_*-v)^3\right]h(w)g_*(v,\tau)\mathrm{d}v\mathrm{d}w\right\}.\end{aligned}$$

Since $\tilde r_\varepsilon$ contains terms of order ε or higher, we have

$$\lim_{\varepsilon\to 0}\tilde r_\varepsilon=0.$$

Applying integration by parts to Eq. (66), we obtain the following Fokker–Planck equation:

$$\begin{aligned}\frac{1}{\kappa}\frac{\partial g_*(v,\tau)}{\partial\tau}=&-\frac{\partial}{\partial v}\left[(M_\theta+\theta M_{\theta-1}v)\left(-\frac{\mu\log s}{\gamma+\beta}\right)vg_*(v,\tau)\right]\\ &+\frac{\partial^2}{\partial v^2}\left[\frac{\sigma}{2}(M_\theta+\theta M_{\theta-1}v)v^2g_*(v,\tau)\right].\end{aligned} \tag{67}$$

Obtaining an analytical solution to Eq. (67) is difficult. Therefore, we consider the steady-state solution as $\tau\to\infty$. In this case, the left-hand side of Eq. (67) vanishes. Denoting the steady-state distribution by $g_*^\infty(v)$, Eq. (67) becomes

$$\begin{aligned}&(M_\theta+\theta M_{\theta-1}v)\left(-\frac{\mu\log s}{\gamma+\beta}\right)vg_*^\infty(v)\\ =&\ \frac{\partial}{\partial v}\left[\frac{\sigma}{2}(M_\theta+\theta M_{\theta-1}v)v^2g_*^\infty(v)\right].\end{aligned} \tag{68}$$

Equation (68) can be rewritten as

$$a_*(v)g_*^\infty(v)=\frac{\mathrm{d}[b_*(v)g_*^\infty(v)]}{\mathrm{d}v}, \tag{69}$$

where

$$\begin{aligned}a_*(v)&=(M_\theta+\theta M_{\theta-1}v)\left(-\frac{\mu\log s}{\gamma+\beta}\right)v,\\ b_*(v)&=\frac{\sigma}{2}(M_\theta+\theta M_{\theta-1}v)v^2.\end{aligned}$$

Let $G_*(v)=b_*(v)g_*^\infty(v)$. Then $g_*^\infty(v)=G_*(v)/b_*(v)$, and Eq. (69) yields

$$\frac{a_*(v)}{b_*(v)}\mathrm{d}v=\frac{\mathrm{d}G_*(v)}{G_*(v)}. \tag{70}$$

After integration,

$$G_*(v)=c_*\exp\left(\int_0^v\frac{a_*(x)}{b_*(x)}\mathrm{d}x\right).$$

Therefore,

$$g_*^\infty(v)=c_*\frac{1}{b_*(v)}\exp\left(\int_0^v\frac{a_*(x)}{b_*(x)}\mathrm{d}x\right), \tag{71}$$

where c_* satisfies

$$\int_{R_+}g_*^\infty(v)\mathrm{d}v=1,$$

and

$$a_*(x)=(M_\theta+\theta M_{\theta-1}x)\left(-\frac{\mu\log s}{\gamma+\beta}\right)x,$$

$$b_*(x)=\frac{\sigma}{2}(M_\theta+\theta M_{\theta-1}x)x^2.$$

In the steady-state solution $g_*^\infty(v)$, $b_*(v)$ is a polynomial, and the decay behavior of the wealth distribution for large v is determined by the factor $1/b_*(v)$. Since $b_*(v)$ depends explicitly on θ, the collision kernel, which characterizes the trading frequency, influences the steady-state wealth distribution.

5. Numerical experiments

This section presents numerical analyses of the steady-state wealth distribution functions from three perspectives. First, we investigate the case where only one type of agent exists in the market. Second, we analyze the scenario involving two types of agents with different trading behaviors. Finally, we examine the effects of trading frequency and psychological factors on the wealth distribution.

(1) Discussion on the steady-state wealth distribution with a single type of agent. We investigate the influence of different model parameters on the steady-state wealth distribution. First, we analyze the role of the non-Maxwellian collision kernel Eq. (4) in determining the wealth distribution. In this case, the steady-state distribution function $g_\infty(v)$ is given by Eq. (26). The parameters are fixed as $\lambda = 0.5$ and $\sigma = 1$.

When θ takes the values 2, 3, and 4, respectively, the corresponding steady-state wealth density functions are presented in Fig. 1. It can be observed that each steady-state distribution curve has a single peak. As θ increases, the peak becomes higher and the tail decays more rapidly, indicating that a larger trading-frequency parameter results in a more concentrated wealth distribution.

Second, we investigate the influence of different saving propensities on the steady-state wealth distribution. The parameters are chosen as $\theta = 2$ and $\sigma = 1$.

When λ takes the values 0.4, 0.5, and 0.6, respectively, the corresponding steady-state wealth density functions are shown in Fig. 2. It is observed that all distribution curves exhibit a single peak. With increasing λ, the peak becomes lower and the tail becomes heavier, indicating that a larger saving propensity leads to a broader wealth distribution.

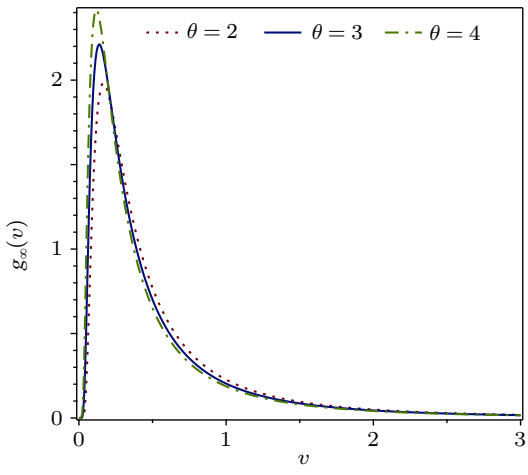


Fig. 1. Steady-state wealth distribution functions for $\theta = 2$, 3, and 4.

(2) Discussion on the steady-state wealth distribution with two types of agents. We investigate the wealth distributions when two different types of agents participate in market transactions. First, we analyze the influence of the parameter θ on the steady-state wealth distribution $g_{1,\infty}(v)$ given by Eq. (50) for the first type of agents.

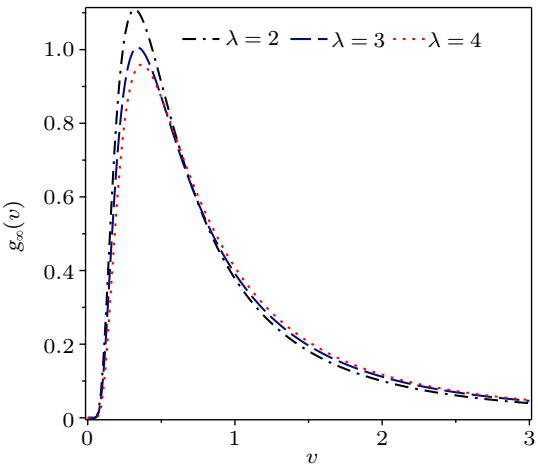


Fig. 2. Steady-state wealth distribution functions for $\lambda = 0.4$, 0.5, and 0.6.

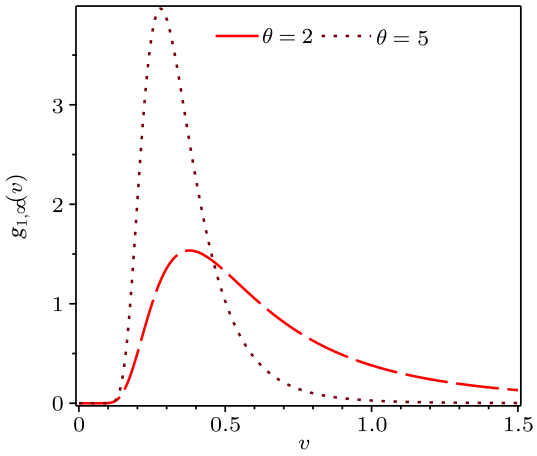


Fig. 3. Steady-state wealth distribution functions of the first type of agents for $\theta = 2$ and 5.

The parameters are set as $\lambda = 0.5$, $\sigma = 1$, and $\rho_1 = 0.5$. When θ takes the values 2 and 5, the corresponding steady-state wealth distribution functions are shown in Fig. 3. It can be observed that each steady-state distribution curve has a single peak. As θ increases, the peak becomes higher and the tail decays faster, indicating that an increase in the trading-frequency parameter leads to a more concentrated wealth distribution.

Next, we investigate the influence of the parameter λ on the steady-state wealth distribution $g_{1,\infty}(v)$ of the first type of agents.

The parameters are chosen as $\theta = 2$, $\sigma = 1$, and $\rho_1 = 0.5$. When λ takes the values 0.4 and 0.5, the corresponding steady-state wealth distribution functions are presented in Fig. 4. It is observed that each steady-state curve has a single peak. As λ decreases, the peak becomes higher and the tail decays faster.

Secondly, when θ varies, we investigate the influence of the collision-kernel parameter on the steady-state wealth

distribution $g_{2,\infty}(v)$ given by Eq. (59) for the second type of agents.

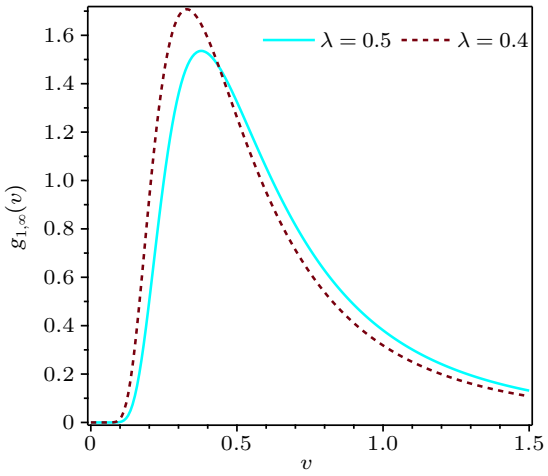


Fig. 4. Steady-state wealth distribution functions of the first type of agents for $\lambda = 0.4$ and 0.5.

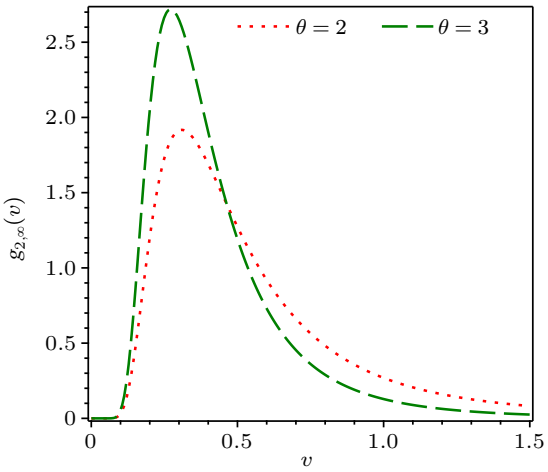


Fig. 5. Behaviors of the steady-state wealth distribution function for $\theta = 2$ and $\theta = 3$.

We set the parameters as $\lambda = 0.5$, $\sigma = 1$, and $\rho_2 = 0.5$. When θ takes the values 2 and 3, the corresponding steady-state wealth density functions are shown in Fig. 5. It can be observed that each steady-state distribution exhibits a single peak. Moreover, as θ increases, the peak becomes higher and the tail decays faster.

When λ changes, we analyze the variation trend of the steady-state wealth curve $g_{2,\infty}(v)$ for the second type of agents in the market. We set the parameters as $\theta = 2$, $\sigma = 1$, $\rho_2 = 0.5$. When λ takes values 0.4 and 0.5 respectively, the plots of the agents' steady-state wealth density functions are shown in Fig. 6. Observations reveal that each steady-state curve has only one vertex: the smaller the parameter λ, the higher the vertex, and the thinner the curve's tail.

(3) We investigate the influence of agents' psychological characteristics on the wealth distribution. In this case, the steady-state solution $g_*^{\infty}(v)$ is given by Eq. (71). We examine the effects of the psychological parameters μ and γ on the steady-state wealth distribution.

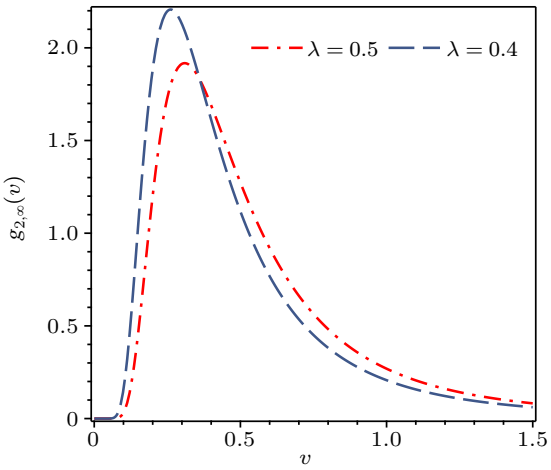


Fig. 6. Behaviors of the steady-state wealth distribution function for $\lambda = 0.4$ and $\lambda = 0.5$.

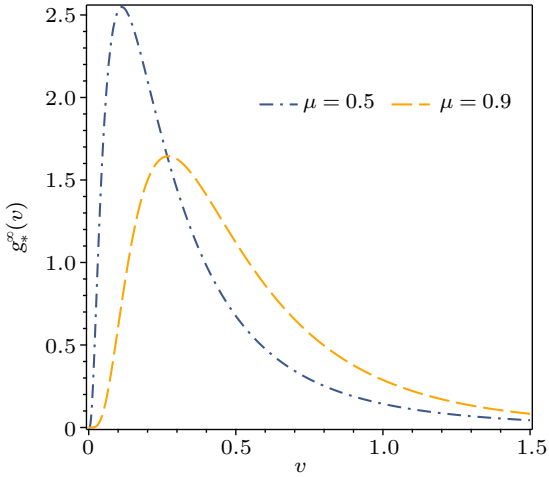


Fig. 7. Behaviors of the steady-state wealth distribution function for $\mu = 0.5$ and $\mu = 0.9$.

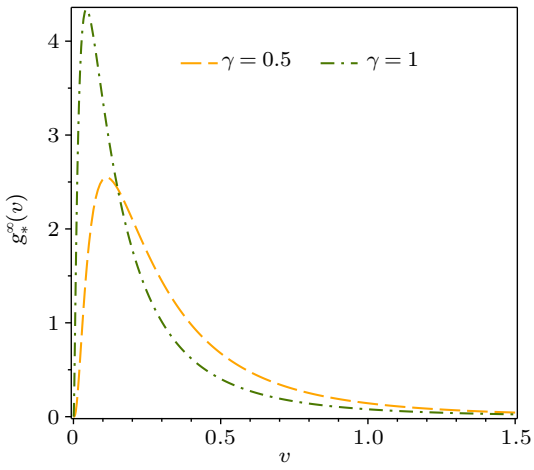


Fig. 8. Behaviors of the steady-state wealth distribution function for $\gamma = 0.5$ and $\gamma = 1$.

First, we consider the influence of μ on the steady-state wealth distribution given by Eq. (71).

We set the parameters as follows: $\theta = 2$, $\sigma = 1$, $\gamma = 0.5$, $\beta = 0.5$, and $v_{\mathrm{L}} = 1$. When μ takes the values 0.5 and 0.9, the corresponding steady-state wealth density functions are shown in Fig. 7. It can be observed that each steady-state distribution

exhibits a single peak. Moreover, a smaller value of μ results in a higher peak and a faster decay of the distribution tail.

Next, we investigate the influence of γ on the steady-state wealth distribution.

We set the parameters as follows: $\theta = 2$, $\sigma = 1$, $\mu = 0.5$, $\beta = 0.5$, and $v_{\mathrm{L}} = 1$. When γ takes the values 0.5 and 1, the corresponding steady-state wealth density functions are shown in Fig. 8. It can be observed that each steady-state distribution exhibits a single peak. Furthermore, a larger value of γ leads to a higher peak and a faster decay of the distribution tail.

6. Conclusion

This work employs a non-Maxwellian collision kernel to characterize the trading frequency among agents and investigates its effects, together with psychological factors, on the wealth distribution. We consider both single-group and two-group trading scenarios and establish the corresponding Boltzmann equations based on the proposed trading rules and collision kernel. Through a scaling transformation, the Boltzmann equations are further reduced to the corresponding Fokker–Planck equations.

For the single-group case, a univariate wealth exchange rule is introduced to describe the wealth evolution of agents. For the two-group case, an interaction rule between different trading groups is constructed to characterize the wealth exchange process. Furthermore, a value function derived from prospect theory is incorporated into the trading mechanism to capture the psychological characteristics of agents, and its influence on the wealth distribution is investigated. Under the assumptions of the proposed model, the steady-state solutions of the resulting Fokker–Planck equations are obtained, followed by a comprehensive numerical analysis of their properties.

Appendix A: The derivation process of the Fokker–Planck equation

Assume Eq. (18)

$$\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+}\varphi(v)g(v,\tau)\mathrm{d}v = I_1 - I_2 + I_3 + I_4 - I_5 + I_6. \quad \text{(A1)}$$

Since $\varphi(v)$ is a smooth function with compact support, it follows that

$$\begin{aligned} I_1 &= \int_{R_+}\int_{R_+}\lambda w^{\theta+1}\varphi'(v)h(w)g(v,\tau)\mathrm{d}v\mathrm{d}w \\ &= \lambda\int_{R_+}M_{\theta+1}\varphi'(v)g(v,\tau)\mathrm{d}v \\ &= \lambda M_{\theta+1}\int_{R_+}g(v,\tau)\mathrm{d}\varphi(v) \\ &= \lambda M_{\theta+1}g(v,\tau)\varphi(v)\big|_0^{+\infty} - \lambda M_{\theta+1}\int_{R_+}\varphi(v)\frac{\partial g(v,\tau)}{\partial v}\mathrm{d}v \\ &= -\lambda M_{\theta+1}\int_{R_+}\varphi(v)\frac{\partial g(v,\tau)}{\partial v}\mathrm{d}v. \end{aligned} \quad \text{(A2)}$$

Using integration by parts for I_2 gives

$$\begin{aligned} I_2 &= \int_{R_+}\int_{R_+}\lambda w^{\theta}v\varphi'(v)h(w)g(v,\tau)\mathrm{d}v\mathrm{d}w \\ &= \lambda\int_{R_+}M_{\theta}v\varphi'(v)g(v,\tau)\mathrm{d}v \\ &= \lambda M_{\theta}vg(v,\tau)\varphi(v)\big|_0^{+\infty} - \lambda M_{\theta}\int_{R_+}\varphi(v)\frac{\partial vg(v,\tau)}{\partial v}\mathrm{d}v \\ &= -\lambda M_{\theta}\int_{R_+}\varphi(v)\frac{\partial vg(v,\tau)}{\partial v}\mathrm{d}v. \end{aligned} \quad \text{(A3)}$$

Calculating I_3 yields

$$\begin{aligned} I_3 &= \int_{R_+}\int_{R_+}\frac{1}{2}\sigma w^{\theta}v^2\varphi''(v)h(w)g(v,\tau)\mathrm{d}v\mathrm{d}w \\ &= \frac{1}{2}\sigma M_{\theta}\int_{R_+}v^2\varphi''(v)g(v,\tau)\mathrm{d}v \\ &= \frac{1}{2}\sigma M_{\theta}\int_{R_+}v^2g(v,\tau)d\varphi'(v) \\ &= \frac{1}{2}\sigma M_{\theta}v^2g(v,\tau)\varphi'(v)\big|_0^{+\infty} \\ &\quad -\frac{1}{2}\sigma M_{\theta}\int_{R_+}\frac{\partial(v^2g(v,\tau))}{\partial v}\varphi'(v)\mathrm{d}v \\ &= -\frac{1}{2}\sigma M_{\theta}v^2g(v,\tau)\frac{\partial(v^2g(v,\tau))}{\partial v}\big|_0^{+\infty} \\ &\quad +\frac{1}{2}\sigma M_{\theta}\int_{R_+}\frac{\partial^2(v^2g(v,\tau))}{\partial v^2}\varphi(v)\mathrm{d}v \\ &= \frac{1}{2}\sigma M_{\theta}\int_{R_+}\frac{\partial^2(v^2g(v,\tau))}{\partial v^2}\varphi(v)\mathrm{d}v. \end{aligned} \quad \text{(A4)}$$

The calculation of I_4 is as follows:

$$\begin{aligned} I_4 &= \int_{R_+}\int_{R_+}\lambda\theta w^{\theta}v\varphi'(v)h(w)g(v,\tau)\mathrm{d}v\mathrm{d}w \\ &= \lambda\theta\int_{R_+}M_{\theta}v\varphi'(v)g(v,\tau)\mathrm{d}v \\ &= \lambda\theta M_{\theta}\int_{R_+}vg(v,\tau)d\varphi(v) \\ &= \lambda\theta M_{\theta}vg(v,\tau)\varphi(v)\big|_0^{+\infty} - \lambda\theta M_{\theta}\int_{R_+}\varphi(v)\frac{\partial vg(v,\tau)}{\partial v}\mathrm{d}v \\ &= -\lambda\theta M_{\theta}\int_{R_+}\varphi(v)\frac{\partial vg(v,\tau)}{\partial v}\mathrm{d}v. \end{aligned} \quad \text{(A5)}$$

Calculating I_5, we have

$$\begin{aligned} I_5 &= \int_{R_+}\int_{R_+}\lambda\theta w^{\theta-1}v^2\varphi'(v)h(w)g(v,\tau)\mathrm{d}v\mathrm{d}w \\ &= \lambda\theta\int_{R_+}M_{\theta-1}v^2\varphi'(v)g(v,\tau)\mathrm{d}v \\ &= \lambda\theta M_{\theta-1}\int_{R_+}v^2g(v,\tau)\mathrm{d}\varphi(v) \\ &= \lambda\theta M_{\theta-1}v^2g(v,\tau)\varphi(v)\big|_0^{+\infty} \\ &\quad -\lambda\theta M_{\theta-1}\int_{R_+}\varphi(v)\frac{\partial v^2g(v,\tau)}{\partial v}\mathrm{d}v \end{aligned}$$

$$= -\lambda\theta M_{\theta-1}\int_{R_+}\varphi(v)\frac{\partial v^2 g(v,\tau)}{\partial v}\mathrm{d}v, \tag{A6}$$

and

$$\begin{aligned} I_6 &= \int_{R_+}\int_{R_+}\frac{1}{2}\theta\sigma w^{\theta-1}v^3\varphi''(v)h(w)g(v,\tau)\mathrm{d}v\mathrm{d}w \\ &= \frac{1}{2}\theta\sigma M_{\theta-1}\int_{R_+} v^3\varphi''(v)g(v,\tau)\mathrm{d}v \\ &= \frac{1}{2}\theta\sigma M_{\theta-1}\int_{R_+} v^3 g(v,\tau)\mathrm{d}\varphi'(v) \\ &= \frac{1}{2}\theta\sigma M_{\theta-1}v^3 g(v,\tau)\varphi'(v)\big|_0^{+\infty} \\ &\quad -\frac{1}{2}\theta\sigma M_{\theta-1}\int_{R_+}\frac{\partial(v^3 g(v,\tau))}{\partial v}\varphi'(v)\mathrm{d}v \\ &= -\frac{1}{2}\theta\sigma M_{\theta-1}\int_{R_+}\frac{\partial(v^3 g(v,\tau))}{\partial v}\mathrm{d}\varphi(v) \\ &= \frac{1}{2}\theta\sigma M_{\theta-1}\int_{R_+}\frac{\partial^2(v^3 g(v,\tau))}{\partial v^2}\varphi(v)\mathrm{d}v. \end{aligned} \tag{A7}$$

From Eqs. (A2)–(A7), Eq. (A1) becomes

$$\begin{aligned} \frac{1}{\kappa}\frac{\partial g(v,\tau)}{\partial\tau} &= -\lambda M_{\theta+1}\frac{\partial g(v,\tau)}{\partial v}+\lambda M_\theta\frac{\partial v g(v,\tau)}{\partial v} \\ &\quad -\lambda\theta M_\theta\frac{\partial v g(v,\tau)}{\partial v}+\lambda\theta M_{\theta-1}\frac{\partial(v^2 g(v,\tau))}{\partial v} \\ &\quad +\frac{1}{2}\sigma M_\theta\frac{\partial^2(v^2 g(v,\tau))}{\partial v^2} \\ &\quad +\frac{1}{2}\theta\sigma M_{\theta-1}\frac{\partial^2(v^3 g(v,\tau))}{\partial v^2}. \end{aligned} \tag{A8}$$

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