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# Riemann–Hilbert approach to the reverse space–time nonlocal short pulse equation

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We investigate the reverse space–time nonlocal short pulse (NSP) equation. First, starting from the Lax pair of the reverse space–time NSP equation, we derive and present its isospectral deformations. The paper focuses on the spectral analysis related to the equation. It is particularly noteworthy that, due to the singular behavior of the spectral parameter, we analyze it by dividing it into two cases: $\gamma = 0$ and $\gamma = \infty$. Based on the preceding analysis results, we construct a Riemann–Hilbert problem for the reverse space–time NSP equation and develop the solution in the new scale $(z,t)$.

**Keywords:** reverse space–time nonlocal short pulse equation, Riemann–Hilbert problem, jump matrix

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## 1. Introduction

First introduced by Schäfer and Wayne,[1] the short pulse (SP) equation constitutes a fundamental model for describing ultrashort optical pulse propagation in nonlinear media such as silica fibers. The specific form of this equation is expressed as

$$q_{xt} = q + \frac{1}{6}(q^3)_{xx}, \tag{1}$$

where $q(x,t)$ is a real-valued function, and its partial derivatives are represented by the subscripts $x$ and $t$. The SP equation with initial value data

$$q(x,t)|_{t=0} = q_0(x), \quad x \in \mathbb{R}, \tag{2}$$

and assuming that $q_0(x)$ belongs to Schwartz space. Xu has conducted research on the long-time asymptotic behavior of the SP equation via the Riemann–Hilbert (RH) method in Ref. [2]. When the pulse duration is extremely short, the traditional nonlinear Schrödinger (NLS) equation suffers from reduced accuracy, whereas the SP equation can more precisely approximate the solutions to Maxwell's equations, thus making it suitable for studying soliton propagation, waveform evolution, and interactions of ultrashort optical pulses.[3] Furthermore, the theoretical framework of the SP equation is also applicable in other areas such as acoustics and plasma physics.

The NSP equation and its Lax pair were proposed by Yang in Ref. [4]. This equation takes the form

$$q_{xt}(x,t) + q(x,t) + \frac{1}{2}\left[q(x,t)q(-x,-t)q_x(x,t)\right]_x = 0, \tag{3}$$

where $q$ is a real function of $x$ and $t$, and Eq. (3) with the subsequent initial condition

$$q(x,t)|_{t=0} = q_0(x) \to 0, \quad x \to \pm\infty. \tag{4}$$

The formulation of nonlocal problems can be traced back to $\mathcal{PT}$-symmetry: this symmetry enables non-Hamiltonian operators to exhibit real spectra, and the nonlocal equations derived therefrom remain invariant under inverse spatiotemporal transformations($\mathcal{P}: x \to -x, \mathcal{T}: t \to -t$). Nonlocal equations have attracted significant research attention in recent years, yielding a large body of findings. Wu and Tian[5] have conducted research on the long-time asymptotic behavior of the NSP equation with Schwartz initial values. Xu and Li[6] investigated the Darboux transformation and $\mathcal{PT}$-symmetry of the nonlocal NLS equation. The NSP equation serves as a key model for describing the propagation behavior of ultrashort optical pulses in nonlocal nonlinear media. In typical environments such as silica fibers and plasma waveguides, this equation can accurately characterize the evolution laws of pulses — their evolution process depends not only on the local electric field but also on the field distribution within the nonlocal range.

Before conducting research on the reverse space–time NSP equation, it is essential to provide an overview of the two-component SP system[7]

$$\begin{cases} q_{xt} - 2q + (qrq_x)_x = 0, \\ r_{xt} - 2r + (rqr_x)_x = 0, \end{cases} \tag{5}$$

both variables $q$ and $r$ are complex-valued functions of the spatial variable $x$ and temporal variable $t$, and they satisfy the constraint relation $r = \sigma q^\dagger$ (where $\sigma = \pm 1$). Here, $q^\dagger$ stands for

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the complex conjugate of the function $q$. Specifically, when $\sigma = 1$, the system corresponds to the defocusing case; when $\sigma = -1$, the system corresponds to the focusing case. The RH method for the two-component modified short-pulse (mSP) system posed on the real line subject to zero boundary conditions was proposed in Ref. [8].

The authors of Ref. [8] further present the reverse space–time NSP equation. Its explicit form is given by

$$q_{xt}(x,t) - 2q(x,t) + (\sigma q(x,t)q(-x,-t)q_x(x,t))_x = 0. \quad (6)$$

Substitution $r(x,t) = \sigma q(-x,-t)$ into the two-component SP system (5) leads to Eq. (6). Wang and He[9] developed the Darboux transformation for Eq. (6) through the nonlocal symmetry reduction of the AKNS spectral problem together with the covariant hodograph transformation. They also obtained various solutions of Eq. (6) under different backgrounds, including multi-solitons, rogue waves, and breathers, categorized and analyzed the dynamic characteristics and asymptotic behaviors of these solutions. The solution structures (e.g., multi-solitons, rogue waves) can guide the design of stable optical pulse transmission and manipulation schemes, providing theoretical support for information encoding and all-optical switching technologies in high-speed optical communication. Additionally, Shan and Li expressed the multi-soliton solutions of Eq. (6) in determinant form, obtained $N$-positon solutions via the degenerate Darboux determinant, and analyzed the trajectories, decomposition, and phase shifts of positons as well as their interactions with solitons in Ref. [10].

The case with $\sigma = -1$ corresponds to the focusing case of Eq. (6); it can be expressed as

$$q_{xt}(x,t) - 2q(x,t) - (q(x,t)q(-x,-t)q_x(x,t))_x = 0. \quad (7)$$

To date, Eq. (7) has not yet been studied via the RH method. In this paper, we investigate Eq. (7) with initial data

$$q(x,0) = q_0(x) \to 0, \quad x \to \pm\infty. \quad (8)$$

In Section 2, we begin by reformulating the Lax pair for the reverse space–time NSP equation. The spectral analysis in Subsection 2.3 focuses on the singular behavior of the spectral parameter. Based on the analysis results, the corresponding RH problem and jump matrix are established in Subsection 2.4, and the approximate solution of the reverse space–time NSP equation is constructed under the new scale $(z,t)$. Finally, a summary of the study is provided in Section 3.

## 2. Fundamental Riemann–Hilbert problem

In this section, we commence with the Lax pair of Eq. (7), conduct spectral analysis on it, and obtain the jump matrix of Eq. (7) through calculations.

### 2.1. Formulas and symbols

To start with, some fundamental notations and definitions employed in this paper are outlined.

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ represents the third Pauli matrix, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$.

$X, Y$ are both $2\times 2$ matrices, matrix commutator $[X,Y] = XY - YX$.

$\hat{\sigma}_3$ represents the matrix commutator involving $\sigma_3$, with $\hat{\sigma}_3 X = [\sigma_3, X]$, then $\exp(\hat{\sigma}_3)$ admits a straightforward computation: $\mathrm{e}^{\hat{\sigma}_3} X = \mathrm{e}^{\sigma_3} X e^{-\sigma_3}$, where $X$ is a $2\times 2$ matrix.

$I = \mathrm{diag}(1,1)$ represents the $2\times 2$ identity matrix.

For any function $u(\cdot)$, $u^\dagger(\cdot)$ denotes its complex conjugate.

### 2.2. Lax pairs

When $\sigma = -1$, Eq. (6) can be rewritten as Eq. (7). Lax pairs of reverse space–time NSP equation were presented in Refs. [4,9,11]. For the spectral parameter $\gamma$, Eq. (7) satisfies the Lax pair criterion, and it is specified as follows:

$$\begin{aligned} \Psi(x,t;\gamma)_x &= \mathfrak{M}(x,t;\gamma)\Psi(x,t;\gamma), \\ \Psi(x,t;\gamma)_t &= \mathfrak{N}(x,t;\gamma)\Psi(x,t;\gamma), \end{aligned} \quad (9)$$

with

$$\begin{aligned} \mathfrak{M}(x,t;\gamma) &= \frac{1}{2\gamma}\begin{pmatrix} -\mathrm{i} & \mathrm{i}q_x \\ -\mathrm{i}r_x & \mathrm{i} \end{pmatrix}, \\ \mathfrak{N}(x,t;\gamma) &= \begin{pmatrix} \frac{\mathrm{i}qr}{2\gamma} + \mathrm{i}\gamma & -\frac{\mathrm{i}qr_x}{2\gamma} + q \\ \frac{\mathrm{i}qr_x}{2\gamma} + r & -\frac{\mathrm{i}qr}{2\gamma} - \mathrm{i}\gamma \end{pmatrix}, \end{aligned}$$

where $r(x,t) = -q(-x,-t)$ and $\Psi(x,t;\gamma)$ is a matrix function.

To construct the RH problem, we rewrite the Lax pair (9)

$$\Psi_x + \frac{\mathrm{i}}{2\gamma}\sigma_3\Psi = U\Psi, \quad \Psi_t - \mathrm{i}\gamma\sigma_3\Psi = V\Psi, \quad (10)$$

where

$$U = \frac{\mathrm{i}}{2\gamma}Q_{1x}, \quad V = \frac{\mathrm{i}}{2\gamma}A + Q_2, \quad (11)$$

with

$$\begin{aligned} Q_1 &= \begin{pmatrix} 0 & q(x,t) \\ -r(x,t) & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix}, \\ A &= \begin{pmatrix} q(x,t)r(x,t) & -q(x,t)q_x(x,t)r(x,t) \\ q(x,t)r(x,t)r_x(x,t) & -q(x,t)r(x,t) \end{pmatrix}. \end{aligned}$$

### 2.3. Spectral analysis

Since it is difficult to obtain the symmetric relationship of scattering data for Eq. (7) through spectral analysis of the spatial part, we will develop a RH approach for the reverse space–time NSP equation by performing spectral analysis on the time component of Lax pair (10).

Analysis of Lax pair (10) reveals that the two matrix functions $U$ and $V$ exhibit singular behaviors at $\gamma=0$ and $\gamma=\infty$ on the extended complex $\gamma$-plane. In light of this, we will carry out spectral analysis from the following two cases, $\gamma=0$ and $\gamma=\infty$.

### 2.3.1. Case 1: $\gamma=0$

We begin by defining the transformation as

$$\Psi(x,t;\gamma)=\nu^0(x,t;\gamma)\,\mathrm{e}^{\left(-\frac{\mathrm{i}}{2\gamma}x+\mathrm{i}\gamma t\right)\sigma_3}, \tag{12}$$

here

$$\nu^0(x,t;\gamma)\sim\mathbb{I},\quad \text{as } x\to\pm\infty. \tag{13}$$

Then we can derive a Lax pair that is equivalent to the Lax pair (9).

$$\begin{cases}\nu^0_x+\frac{\mathrm{i}}{2\gamma}[\sigma_3,\nu^0]=U^0\nu^0,\\ \nu^0_t-\mathrm{i}\gamma[\sigma_3,\nu^0]=V^0\nu^0,\end{cases} \tag{14}$$

with $U^0=U$ and $V^0=V$.

The Lax pair (14) is expressible in full derivative form

$$\mathrm{d}\left(\mathrm{e}^{\left(\frac{\mathrm{i}}{2\gamma}x-\mathrm{i}\gamma t\right)\hat{\sigma}_3}\nu^0\right)=\mathrm{e}^{\left(\frac{\mathrm{i}}{2\gamma}x-\mathrm{i}\gamma t\right)\hat{\sigma}_3}\left(U^0\mathrm{d}x+V^0\mathrm{d}t\right)\nu^0. \tag{15}$$

Integrating both sides at the same time, we obtain

$$\nu_0(x,t;\gamma)=I+\int_{(x_j,t_j)}^{(x,t)}\mathrm{e}^{-(\frac{\mathrm{i}}{2\gamma}x-\mathrm{i}\gamma t)\hat{\sigma}_3}F(\xi,\tau,\gamma), \tag{16}$$

where

$$\begin{aligned}F(x,t;\gamma)\;=\;&\mathrm{e}^{(\frac{\mathrm{i}}{2\gamma}x-\mathrm{i}\gamma t)\hat{\sigma}_3}[U^0(x,t;\gamma)\mathrm{d}x+V^0(x,t;\gamma)\mathrm{d}t]\\&\times\nu_0(x,t;\gamma).\end{aligned} \tag{17}$$

We choose two different integration paths: from $(x,0)$ to $(x,t)$ and from $(x,\infty)$ to $(x,t)$, see Fig. 1.

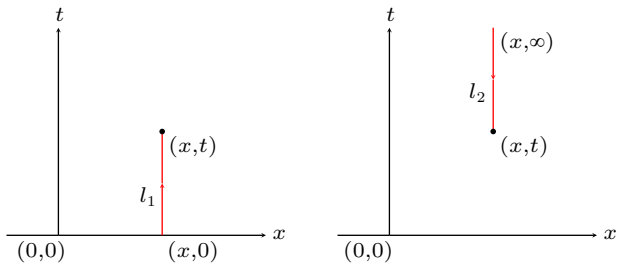



**Fig. 1.** The two contours $l_1$, $l_2$ in $(x,t)$-domaint.

The solutions $\nu^0_\pm(x,t;\gamma)$ of Eq. (16) arise directly from linear Volterra integral equations and can be expressed as

$$\nu^0_-(x,t;\gamma)=I+\int_0^t\mathrm{e}^{\mathrm{i}\gamma(t-\tau)\hat{\sigma}_3}V^0(x,\tau;\gamma)\nu^0_-(x,\tau;\gamma)\mathrm{d}\tau, \tag{18}$$

$$\nu^0_+(x,t;\gamma)=I-\int_t^\infty\mathrm{e}^{\mathrm{i}\gamma(t-\tau)\hat{\sigma}_3}V^0(x,\tau;\gamma)\nu^0_+(x,\tau;\gamma)\mathrm{d}\tau. \tag{19}$$

Depending on different integration paths, the following inequalities hold:

$$\begin{aligned}&\nu^0_-(x,t;\gamma),(x,0)\to(x,t):0<\tau<t,\\&\nu^0_+(x,t;\gamma),(x,\infty)\to(x,t):t<\tau<\infty.\end{aligned}$$

By analyzing, we find that the first column of integral equations $\nu^0_\pm(x,t;\gamma)$ contains only the exponential term $\mathrm{e}^{-2\mathrm{i}\gamma(t-\tau)}$, and with the aid of the aforementioned inequalities, this term decays in the following regions of the complex $\gamma$-plane:

$$\begin{aligned}&\nu^0_{-,1}(x,t;\gamma),(x,0)\to(x,t):\{\mathrm{Im}\gamma<0\},\\&\nu^0_{+,1}(x,t;\gamma),(x,\infty)\to(x,t):\{\mathrm{Im}\gamma>0\}.\end{aligned}$$

In addition, the second column of $\nu^0_\pm(x,t;\gamma)$ only contains the exponential term $\mathrm{e}^{2\mathrm{i}\gamma(t-\tau)}$, which decays in

$$\begin{aligned}&\nu^0_{-,2}(x,t;\gamma),(x,0)\to(x,t):\{\mathrm{Im}\gamma>0\},\\&\nu^0_{+,2}(x,t;\gamma),(x,\infty)\to(x,t):\{\mathrm{Im}\gamma<0\}.\end{aligned}$$

Therefore, we define $\nu^0_\pm=[\nu^0_{\pm,1},\nu^0_{\pm,2}]$. The analytic properties of $\nu^0_\pm$ are as follows:

$\nu^0_{-,1}$ allows analytic extensions into $D_2$, and $\nu^0_{-,2}$ allows such extensions into $D_1$;

$\nu^0_{+,1}$ allows analytic extensions into $D_1$, $\nu^0_{+,2}$ allows such extensions into $D_2$; where $D_1$ denotes $\{\gamma|\mathrm{Im}\gamma>0\}$, and $D_2$ denotes $\{\gamma|\mathrm{Im}\gamma<0\}$, see Fig. 2.

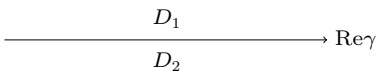



**Fig. 2.** The sets $D_j$ (where $j=1,2$) partition the complex $\gamma$-plane.

By expanding $\nu^0_\pm(x,t;\gamma)$, the following analysis can be conducted on its asymptotic properties:

$$\begin{aligned}&\nu^0_\pm(x,t;\gamma)=I+\begin{pmatrix}0&q(x,t)\\-r(x,t)&0\end{pmatrix}\gamma+O(\gamma^2),\\&\text{as }\gamma\to0.\end{aligned} \tag{20}$$

### 2.3.2. Case 2: $\gamma=\infty$

For the case where $\gamma=\infty$, prior to constructing its RH problem, it is necessary to introduce a subsequent transformation to regulate the eigenfunctions.

Given that the matrix function $U$ is the product of a matrix unrelated to the spectral parameter and another associated with it, to derive $Q_1$, the latter matrix has to be diagonalized. Initially, we introduce a $2\times2$ matrix function $W(x,t)$ in the form

$$W(x,t)=\sqrt{\frac{1+\sqrt{w}}{2\sqrt{w}}}\begin{pmatrix}1&\frac{\mathrm{i}(\sqrt{w}-1)}{q_x(x,t)}\\\frac{\mathrm{i}(\sqrt{w}-1)}{q_x(-x,-t)}&1\end{pmatrix}, \tag{21}$$

where $w=1+q_x(x,t)q_x(-x,-t)$ and $w(x,t)>0$.

Subsequently, the inverse matrix of $W(x,t)$ is readily derived

$$W^{-1}(x,t)=\sqrt{\frac{1+\sqrt{w}}{2\sqrt{w}}}\begin{pmatrix}1 & -\frac{\mathrm{i}(\sqrt{w}-1)}{q_x(x,t)}\\ -\frac{\mathrm{i}(\sqrt{w}-1)}{q_x(-x,-t)} & 1\end{pmatrix}. \tag{22}$$

It is observed that as $q_x(x,t)\to 0$, the term $\sqrt{w}-1$ vanishes faster than the denominator $q_x(x,t)$, and based on this, the matrix-valued function Eq. (21) can be defined.

Define

$$k(x,t;\gamma)=t-\int_t^{+\infty}(\sqrt{w(x,t')}-1)\,\mathrm{d}t'-\frac{x}{2\gamma^2}. \tag{23}$$

The reverse space–time NSP equation can be expressed in the form of conservation law

$$(\sqrt{w})_x=\frac{1}{2}\left[q(x,t)q(-x,-t)\sqrt{w}\right]_t. \tag{24}$$

From this, we can obtain

$$\begin{aligned}k_x&=\frac{1}{2}q(x,t)q(-x,-t)\sqrt{w}-\frac{1}{2\gamma^2},\\ k_t&=\sqrt{w}.\end{aligned} \tag{25}$$

We define the following transformation:

$$\Psi(x,t;\gamma)=W(x,t)\nu'(x,t;\gamma)\,\mathrm{e}^{\mathrm{i}\gamma k(x,t;\gamma)\sigma_3}. \tag{26}$$

By substituting Eq. (26) into Lax pair (10), the Lax pair about $\nu'(x,t;\gamma)$ is obtained

$$\begin{cases}\nu'_x-\mathrm{i}\gamma k_x[\sigma_3,\nu']=U_1\nu',\\ \nu'_t-\mathrm{i}\gamma k_t[\sigma_3,\nu']=V_1\nu',\end{cases} \tag{27}$$

where

$$\begin{aligned}U_1&=-\frac{\mathrm{i}}{2\gamma\sqrt{w}}\sigma_3-\frac{\mathrm{i}\gamma\sqrt{w}}{2}\alpha\sigma_3+\frac{\mathrm{i}}{2\gamma}\sigma_3+\frac{1}{2\gamma\sqrt{w}}\begin{pmatrix}0 & q_x(-x,-t)\\ -q_x(x,t) & 0\end{pmatrix}+\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & -a_{11}\end{pmatrix},\\ V_1&=\frac{\mathrm{i}\gamma}{\sqrt{w}}\sigma_3-\mathrm{i}\gamma\sqrt{w}\sigma_3-\frac{\gamma}{\sqrt{w}}\begin{pmatrix}0 & q_x(-x,-t)\\ -q_x(x,t) & 0\end{pmatrix}+\frac{\mathrm{i}\alpha}{4\gamma\sqrt{w}}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & -b_{11}\end{pmatrix}+\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & -c_{11}\end{pmatrix},\end{aligned}$$

with

$$\begin{aligned}\alpha&=q(x,t)q(-x,-t),\\ a_{11}&=-\frac{{q_x}^2(x,t)+{q_x}^2(-x,-t)}{4\gamma\sqrt{w}}-\frac{q_{xx}(x,t)q_x(-x,-t)+q_x(x,t)q_{xx}(-x,-t)}{4\sqrt{w}(\sqrt{w}+1)},\\ a_{12}&=\mathrm{i}\frac{(1-\sqrt{w}){q_x}^2(-x,-t)+(1+\sqrt{w}){q_x}^2(x,t)}{4\sqrt{w}q_x(x,t)}-\mathrm{i}\frac{(1-\sqrt{w})q_x(-x,-t)q_{xx}(x,t)-(1+\sqrt{w})q_{xx}(-x,-t)q_x(x,t)}{4wq_x(x,t)},\\ a_{21}&=\mathrm{i}\frac{(\sqrt{w}-1){q_x}^2(x,t)-(1+\sqrt{w}){q_x}^2(-x,-t)}{4\sqrt{w}q_x(-x,-t)}-\mathrm{i}\frac{(1+\sqrt{w})q_x(-x,-t)q_{xx}(x,t)+(\sqrt{w}-1)q_{xx}(-x,-t)q_x(x,t)}{4wq_x(-x,-t)},\\ b_{11}&=\mathrm{i}q_x^2(x,t)+\mathrm{i}q_x^2(-x,-t)-2,\\ b_{12}&=\frac{(1-\sqrt{w})q_x^2(-x,-t)}{q_x(x,t)}+(\sqrt{w}+1)q_x(x,t)-2\mathrm{i}q_x(-x,-t),\\ b_{21}&=\frac{(\sqrt{w}-1)q_x^2(x,t)}{q_x(-x,-t)}-(\sqrt{w}+1)q_x(-x,-t)+2\mathrm{i}q_x(x,t),\\ c_{11}&=\mathrm{i}\frac{q(-x,-t)q_x(-x,-t)+q(x,t)q_x(x,t)}{2\sqrt{w}}-\frac{q_{xt}(x,t)q_x(-x,-t)+q_x(x,t)q_{xt}(-x,-t)}{4\sqrt{w}(\sqrt{w}+1)},\\ c_{12}&=\frac{(1+\sqrt{w})q(x,t)q_x(x,t)-(\sqrt{w}-1)q(-x,-t)q_x(-x,-t)}{2\sqrt{w}q_x(x,t)}-\mathrm{i}\frac{(1-\sqrt{w})q_x(-x,-t)q_{xt}(x,t)-(1+\sqrt{w})q_x(x,t)q_{xt}(-x,-t)}{4wq_x(x,t)},\\ c_{21}&=\frac{(\sqrt{w}-1)q(x,t)q_x(x,t)-(1+\sqrt{w})q(-x,-t)q_x(-x,-t)}{2\sqrt{w}q_x(-x,-t)}-\mathrm{i}\frac{(1+\sqrt{w})q_x(-x,-t)q_{xt}(x,t)+(1-\sqrt{w})q_x(x,t)q_{xt}(-x,-t)}{4wq_x(-x,-t)}.\end{aligned}$$

We denote the elements of $V_1$ as $A_{11}$, $A_{12}$ and $A_{21}$ respectively, then we have

$$A_{11}=\frac{\mathrm{i}\gamma}{\sqrt{w}}-\mathrm{i}\gamma\sqrt{w}+\frac{\mathrm{i}\alpha}{4\gamma\sqrt{w}}b_{11}+c_{11}, \tag{28}$$

$$A_{12}=-\frac{\gamma}{\sqrt{w}}q_x(-x,-t)+\frac{\mathrm{i}\alpha}{4\gamma\sqrt{w}}b_{12}+c_{12}, \tag{29}$$

$$A_{21}=\frac{\gamma}{\sqrt{w}}q_x(x,t)+\frac{\mathrm{i}\alpha}{4\gamma\sqrt{w}}b_{21}+c_{21}. \tag{30}$$

We note that the elements on the main diagonal of $V_1$ do not tend to zero, so it is necessary for us to carry out the transformation

$$\nu'(x,t;\gamma)=\mathrm{e}^{d_-\sigma_3}\nu(x,t;\gamma)\,\mathrm{e}^{-d_+\sigma_3}, \tag{31}$$

where

$$d_- = -\int_0^t A_{11}\,\mathrm{d}t', \tag{32}$$

$$d_+ = -\int_t^\infty A_{11}\,\mathrm{d}t', \tag{33}$$

$$d = d_+ + d_- = -\int_0^\infty A_{11}\,\mathrm{d}t'. \tag{34}$$

Consequently, Lax pair associated with $v(x,t;\gamma)$ can be expressed as

$$\begin{cases} v_x - \mathrm{i}\gamma k_x[\sigma_3, v] = \mathrm{e}^{-d_-\sigma_3} U_1 v, \\ v_t - \mathrm{i}\gamma k_t[\sigma_3, v] = \mathrm{e}^{-d_-\sigma_3} V_2 v, \end{cases} \tag{35}$$

here $V_2 = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}$.

It is well-known that converting a Lax pair into an integral equation provides an effective method for constructing the solutions of the Lax pair, as the solution to this integral equation is a particular solution of the Lax pair. Based on this, by virtue of Volterra integrable equations, we can obtain the following two eigenfunctions:

$$\begin{aligned} v_-(x,t;\gamma) &= I + \int_0^t \mathrm{e}^{\mathrm{i}\gamma[k(x,t;\gamma)-k(x,\tau;\gamma)]\hat{\sigma}_3} \\ &\quad \times V_2(x,\tau;\gamma) v_-(x,\tau;\gamma)\,\mathrm{d}\tau, \end{aligned} \tag{36}$$

$$\begin{aligned} v_+(x,t;\gamma) &= I - \int_t^\infty \mathrm{e}^{\mathrm{i}\gamma[k(x,t;\gamma)-k(x,\tau;\gamma)]\hat{\sigma}_3} \\ &\quad \times V_2(x,\tau;\gamma) v_+(x,\tau;\gamma)\,\mathrm{d}\tau. \end{aligned} \tag{37}$$

Similar to the case when $\gamma = 0$, the analytic behaviours of $v_\pm$ are obtained as follows by analyzing its exponential parts of each column:

$v_{-,1}$ allows analytic extensions into $D_2$, and $v_{-,2}$ allows such extensions into $D_1$;

$v_{+,1}$ allows analytic extensions into $D_1$, and $v_{+,2}$ allows such extensions into $D_2$.

Moreover, the matrix functions $v_\pm(x,t;\gamma)$ fulfill the asymptotic condition

$$v_\pm(x,t;\gamma) = I + O(\gamma^{-1}), \quad \gamma \to \infty. \tag{38}$$

The symmetry characteristics of functions $v_\pm(x,t;\gamma)$ are presented as follows:

$$\begin{aligned} &v(x,t;\gamma) = \mathcal{X} v(-x,-t;\gamma)\mathcal{X}^{-1}; \\ &v(x,t;\gamma) = \mathcal{X} v^\dagger(-x,-t;-\gamma)\mathcal{X}^{-1}; \end{aligned} \tag{39}$$

with

$$\mathcal{X} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\sigma_2.$$

Through derivation, we can obtain the symmetry properties regarding each column of the Jost functions $v_\pm(x,t;\gamma)$

$$\begin{aligned} &v_{+,1}(x,t;\gamma) = \mathcal{X}^{-1} v_{-,2}(-x,-t;\gamma), \\ &v_{+,2}(x,t;\gamma) = \mathcal{X} v_{-,1}(-x,-t;\gamma), \\ &v_{+,1}(x,t;\gamma) = \mathcal{X}^{-1} v^\dagger_{-,2}(-x,-t;-\gamma^\dagger), \\ &v_{+,2}(x,t;\gamma) = \mathcal{X} v^\dagger_{-,1}(-x,-t;-\gamma^\dagger). \end{aligned} \tag{40}$$

Since eigenfunctions $\Psi_\pm$ are both the solutions of the Lax pair (9), the matrix $\mathcal{R}(\gamma) = (r_{ij})_{2\times 2}$ can be obtained accordingly

$$\Psi_-(x,t;\gamma) = \Psi_+(x,t;\gamma)\mathcal{R}(\gamma), \tag{41}$$

that is

$$v_-(x,t;\gamma) = v_+(x,t;\gamma)\,\mathrm{e}^{\mathrm{i}\gamma k(x,t;\gamma)\hat{\sigma}_3}\mathcal{R}(\gamma). \tag{42}$$

The symmetry properties of $v_\pm(x,t;\gamma)$ directly imply the corresponding characteristics of $\mathcal{R}(\gamma)$

$$\begin{aligned} &r_{11}(\gamma) = r^\dagger_{11}(-\gamma^\dagger), \\ &r_{22}(\gamma) = r^\dagger_{22}(-\gamma^\dagger), \\ &r_{12}(\gamma) = r_{21}(\gamma) = r^\dagger_{12}(-\gamma^\dagger). \end{aligned} \tag{43}$$

We next express the matrix $\mathcal{R}(\gamma)$ in the form

$$\mathcal{R}(\gamma) = \begin{pmatrix} f_1(\gamma) & z(\gamma) \\ z(\gamma) & f_2(\gamma) \end{pmatrix}, \tag{44}$$

where

$$f_1(\gamma) = \det(v_{+,1}, v_{-,2}), \quad f_2(\gamma) = \det(v_{-,1}, v_{+,2}). \tag{45}$$

By taking Eq. (42) into Eq. (40) and performing the corresponding calculations, we can easily obtain

$$\begin{cases} f_1(\gamma) v_{+,1} + z(\gamma)\,\mathrm{e}^{-2\mathrm{i}\gamma k(x,t;\gamma)} v_{+,2} = v_{-,1}, \\ z(\gamma)\,\mathrm{e}^{2\mathrm{i}\gamma k(x,t;\gamma)} v_{+,1} + f_2(\gamma) v_{+,2} = v_{-,2}. \end{cases} \tag{46}$$

Furthermore, for any $\gamma \in \mathbb{R}$, the matrix $\mathcal{R}(\gamma)$ fulfills the property

$$\det \mathcal{R}(\gamma) = f_1(\gamma) f_2(\gamma) - z^2(\gamma) = 1. \tag{47}$$

The properties of $v_\pm(x,t;\gamma)$ yield the corresponding properties of the scattering data $f_1(\gamma)$ and $f_2(\gamma)$ as follows:

$f_1(\gamma)$ is continuous in $\{\gamma | \mathrm{Im}\gamma \geq 0\}$ and analytic in $\{\gamma | \mathrm{Im}\gamma > 0\}$;

$f_2(\gamma)$ is continuous in $\{\gamma | \mathrm{Im}\gamma \leq 0\}$ and analytic in $\{\gamma | \mathrm{Im}\gamma < 0\}$.

### 2.4. The Riemann–Hilbert problem

In the present section, we convert the Lax pair of Eq. (7) into a RH problem.[12–26] Equations (40) and (44) describe the relationship between the two eigenfunctions $v_+(x,t;\gamma)$ and $v_-(x,t;\gamma)$, which can be transformed into jump conditions for a $2\times 2$ RH problem through the construction of a piecewise meromorphic function $G_\pm(x,t;\gamma)$.

Let $G(x,t;\gamma)$ be given by

$$G_+(x,t,\gamma) = (v_{+,1}(x,t,\gamma), \frac{v_{-,2}(x,t,\gamma)}{f_1(\gamma)}), \quad \gamma \in D_1; \tag{48}$$

$$G_-(x,t,\gamma)=(\frac{\nu_{-,1}(x,t,\gamma)}{f_2(\gamma)},\nu_{+,2}(x,t,\gamma)),\quad \gamma\in D_2. \tag{49}$$

These definitions imply that

$$\det G(x,t;\gamma)=1, \tag{50}$$

$$G(x,t;\gamma)\to I,\quad \text{as } \gamma\to\infty, \tag{51}$$

the two limiting values

$$G_\pm(x,t;\gamma)=\lim_{\lambda\to 0^+} G(x,t;\gamma\pm i\lambda),\ \ \lambda,\gamma\in\mathbb{R}, \tag{52}$$

and $G(x,t;\gamma)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$.

Furthermore, we introduce reflection coefficients as

$$r_1(\gamma)=\frac{z(\gamma)}{f_1(\gamma)},\quad r_2(\gamma)=\frac{z(\gamma)}{f_2(\gamma)}. \tag{53}$$

We assume that the scattering data $f_1(\gamma)$ and $f_2(\gamma)$ have no zeros in the complex plane $\mathbb{C}$.

**Proposition 2.4.1.** The matrix functions $\nu_\pm(x,t;\gamma)$ are defined by Eqs. (34) and (35), and $G(x,t;\gamma)$ is defined by Eqs. (46) and (47), which is a sectionally meromorphic function; then the matrix function $G(x,t;\gamma)$ satisfies the jump condition

$$G_+(x,t;\gamma)=G_-(x,t;\gamma)J(\gamma), \tag{54}$$

where

$$\begin{aligned} J(x,t;\gamma)&=\mathrm{e}^{\mathrm{i}\gamma k(x,t;\gamma)\hat{\sigma}_3}J_0(\gamma)\\ &\doteq \mathrm{e}^{\mathrm{i}\gamma k(x,t;\gamma)\hat{\sigma}_3}\frac{r_1}{r_2}\begin{pmatrix}1 & r_1(\gamma)\\ -r_2(\gamma) & 1-r_1(\gamma)r_2(\gamma)\end{pmatrix}. \end{aligned} \tag{55}$$

Moreover, as $\gamma\to 0$, $G(x,t;\gamma)$ exhibits the asymptotic behavior

$$G(x,t;\gamma)=\mathrm{e}^{-d_-\sigma_3}W^{-1}(x,t)\left[I+\gamma\left(-\mathrm{i}h_+\sigma_3+\begin{pmatrix}0 & q(x,t)\\ q(-x,-t) & 0\end{pmatrix}\right)+O(\gamma^2)\right]\mathrm{e}^{-d\sigma_3}, \tag{56}$$

with $h_+=\int_t^{+\infty}(\sqrt{w(x,t')}-1)\,\mathrm{d}t'$.

Based on the above analysis, focusing on singular behavior of spectral parameter $\gamma$, we have intensively studied two cases: $\gamma=0$ and $\gamma=\infty$. Specifically, for $\gamma=0$, we constructed a solution to Eq. (7); whereas for $\gamma=\infty$, we completed the construction of the solution to the RH problem. It should be noted that if we aim to represent the solution of Eq. (7) by means of the RH problem solution, the core lies in further establishing the correlation between the spectral functions $\nu_0(x,t;\gamma)$ and $\nu(x,t;\gamma)$.

Since both $\nu_0(x,t;\gamma)$ and $\nu(x,t;\gamma)$ are solutions to the Lax pair of Eq. (7), their linear relationship can be expressed as

$$\begin{aligned} \nu_\pm(x,t;\gamma)&=\mathrm{e}^{-d_-\sigma_3}W^{-1}(x,t)\nu_\pm^0(x,t;\gamma)\\ &\quad\times\mathrm{e}^{-\left(\frac{\mathrm{i}}{2\gamma}x-\mathrm{i}\gamma t\right)\hat{\sigma}_3}\mathbb{H}_\pm(\gamma)\,\mathrm{e}^{-\mathrm{i}\gamma k(x,t;\gamma)\sigma_3}\,\mathrm{e}^{d\sigma_3}, \end{aligned} \tag{57}$$

there is no dependence of $\mathbb{H}_\pm(\gamma)$ on $x$ and $t$.

In light of the asymptotic characteristics of $\nu_0(x,t;\gamma)$ and $\nu(x,t;\gamma)$, we have

$$\mathbb{H}_-(\gamma)=\mathrm{e}^{-d\sigma_3}\,\mathrm{e}^{-\mathrm{i}\gamma h\sigma_3},\quad \mathbb{H}_+(\gamma)=I. \tag{58}$$

Specifically, $h$ in Eq. (56) is an invariant quantity under the control of reverse space–time NSP equation.

The relationship between $\nu_0(x,t;\gamma)$ and $\nu(x,t;\gamma)$ is given by

$$\begin{aligned} \nu_-(x,t;\gamma)&=\mathrm{e}^{-d_-\sigma_3}W^{-1}(x,t)\nu_-^0(x,t;\gamma)\\ &\quad\times\mathrm{e}^{-\mathrm{i}\gamma\int_0^t(\sqrt{w(x,t')}-1)\,\mathrm{d}t'\sigma_3}\,\mathrm{e}^{d\sigma_3}, \end{aligned} \tag{59}$$

$$\begin{aligned} \nu_+(x,t;\gamma)&=\mathrm{e}^{-d_-\sigma_3}W^{-1}(x,t)\nu_+^0(x,t;\gamma)\\ &\quad\times\mathrm{e}^{\mathrm{i}\gamma\int_t^{+\infty}(\sqrt{w(x,t')}-1)\,\mathrm{d}t'\sigma_3}\,\mathrm{e}^{d\sigma_3}. \end{aligned} \tag{60}$$

In analogy with the case of the Camassa–Holm type equations, the jump matrix $J(\gamma)$ for the RH problem of Eq. (7) is associated with $J_0(\gamma)$ and $k(x,t;\gamma)$. Here, $J_0(\gamma)$ is uniquely determined by $q_0(x)$, whereas $k(x,t;\gamma)$ does not possess such a property and explicitly involves the unknown function $w(x,t)$. In view of this, direct analysis in the $(x,t)$-plane is not feasible. Therefore, it is necessary to introduce the following transformation to convert $(x,t)$-plane.

Here, we define a novel scale

$$\begin{aligned} z(x,t)&=x-\int_x^{+\infty}(\sqrt{w(x',t)}-1)\,\mathrm{d}x'\\ &=x-c_+, \end{aligned} \tag{61}$$

next defining

$$\mathfrak{F}(z,t;\gamma)=G(x(z,t),t;\gamma), \tag{62}$$

here $\mathfrak{F}(z,t;\gamma)$ fulfills the RH problem presented below.

**Proposition 2.4.2. (Riemann–Hilbert problem)** We construct a matrix function $\mathfrak{F}(z,t;\gamma)$ which is sectionally meromorphic and possesses the following conditions:

Analyticity: $\mathfrak{F}(z,t;\gamma)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$.

Jump condition:

$$\mathfrak{F}_+(z,t;\gamma)=\mathfrak{F}_-(z,t;\gamma)\tilde{J}(z,t;\gamma),\quad \gamma\in\mathbb{R}, \tag{63}$$

where the jump matrix is

$$\begin{aligned} \tilde{J}(x,t;\gamma)&=\mathrm{e}^{-\mathrm{i}\left(\frac{\mathrm{i}}{2\gamma}z-\mathrm{i}\gamma t\right)\hat{\sigma}_3}J_0(\gamma)\\ &\doteq \mathrm{e}^{-\mathrm{i}\left(\frac{\mathrm{i}}{2\gamma}z-\mathrm{i}\gamma t\right)\hat{\sigma}_3}\frac{r_1}{r_2}\begin{pmatrix}1 & r_1(\gamma)\\ -r_2(\gamma) & 1-r_1(\gamma)r_2(\gamma)\end{pmatrix}. \end{aligned} \tag{64}$$

Normalize condition:

$$\mathfrak{F}(z,t;\gamma)\to I,\quad \gamma\to\infty. \tag{65}$$

Through plane transformation, the solution $q(x,t)$ corresponding to the reverse space–time NSP equation can additionally be mapped into the $(z,t)$-plane, where it can be expressed in terms of $\mathfrak{F}(z,t;\gamma)$ as

$$q(x,t) = q(z(x,t),t), \tag{66}$$

where

$$x(z,t) = z + \lim_{\gamma\to 0} \frac{\mathfrak{F}^{-1}(z,t;0)\,(\mathfrak{F}(z,t;\gamma))_{11} - 1}{\mathrm{i}\gamma}, \tag{67}$$

$$\mathrm{e}^{-2d} q(z,t) = \lim_{\gamma\to 0} \frac{\big((\mathfrak{F}(z,t;0))^{-1}\mathfrak{F}(z,t;\gamma)\big)_{21}}{\mathrm{i}\gamma}. \tag{68}$$

## 3. Conclusion

In this paper, we investigate the reverse space–time NSP equation (with $\sigma = -1$) via the RH approach. Spectral analysis was conducted on the singular spectral parameter $\gamma$ of the Lax pair, leading to the construction of an equivalent Lax pair and integral equations to determine the eigenfunctions. A corresponding RH problem was formulated. By introducing a new scale $(z,t)$, the dependence of the jump matrix on the unknown function was eliminated, and the connection between the solution of the RH problem and the original equation was established. This work enriches the application of the RH methodology in the study of nonlocal integrable equations and lays a foundation for exploring the dynamics of ultrashort optical pulses in nonlocal media. Future studies could focus on soliton resolution, extensions to the defocusing regime, and long-time asymptotic analysis of solutions.

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