

Influence of memory effects on quantum Stackelberg duopoly game under amplitude damping channel

Xiang-Ping Liao(廖湘萍) and Xin-Yi Wang(王馨仪)
Citation: Chin. Phys. B, 2026, 35 (**8**): 080203. DOI: 10.1088/1674-1056/ae29ff
http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Impact of peer pressure on cooperation evolution in the networked prisoner's dilemma game with migration mechanisms

Xianjia Wang(王先甲), Yanan Li(李亚楠), and Zhipeng Yang(杨志鹏)
Chin. Phys. B, 2025, 34 (**11**): 110201. DOI: 10.1088/1674-1056/ade425

Generic stability of cooperative equilibria for multi-leader-follower-population mixed games

Wenjun Wu(武文俊), Hui Yang(杨辉), and Guanghui Yang(杨光惠)
Chin. Phys. B, 2025, 34 (**10**): 100202. DOI: 10.1088/1674-1056/ade666

Adaptive interaction driven by the learning effect in the spatial prisoner's dilemma

Jiaqi Li(李佳奇), Jianlei Zhang(张建磊), and Qun Liu(刘群)
Chin. Phys. B, 2024, 33 (**3**): 030202. DOI: 10.1088/1674-1056/acf702

Inference of interactions between players based on asynchronously updated evolutionary game data

Hong-Li Zeng(曾红丽), Bo Jing(景涛), Yu-Hao Wang(王于豪), and Shao-Meng Qin(秦绍萌)
Chin. Phys. B, 2023, 32 (**8**): 080201. DOI: 10.1088/1674-1056/acc7fd

Reputational preference and other-regarding preference based rewarding mechanism promotes cooperation in spatial social dilemmas

Huayan Pei(裴华艳), Guanghui Yan(闫光辉), and Huanmin Wang(王焕民)
Chin. Phys. B, 2021, 30 (**5**): 050203. DOI: 10.1088/1674-1056/abcf9a

---

Influence of memory effects on quantum Stackelberg duopoly game under amplitude damping channel

Xiang-Ping Liao(廖湘萍)[†] and Xin-Yi Wang(王馨仪)

College of Science, Hunan University of Technology, Zhuzhou 412008, *China*

(Received 8 October 2025; revised manuscript received 10 November 2025; accepted manuscript online 9 December 2025)

We study the influence of memory effects on the quantum Stackelberg duopoly game through an amplitude-damping channel where successive uses of the channels are correlated. It is shown that the memory effects can drastically change the Nash equilibrium and the payoffs of the two firms. As the degree of channel memory increases, there exists a Nash equilibrium for the entire range of the entanglement parameter. Similarly, the presence of memory ensures the existence of the Nash equilibrium for the whole range of the decoherence parameter both for entangled and unentangled initial states. Moreover, quantum memory leads to a 'critical point' of the damping parameter, at which both firms are equally benefited. With the maximum-memory effect, the position of the 'critical point' moves to the place of fully decohered.

Keywords: Stackelberg duopoly game, memory effects, payoff, amplitude-damping channel

PACS: 02.50.Le, 03.67.–a, 05.30.–d

DOI: 10.1088/1674-1056/ae29ff **CSTR:** 32038.14.CPB.ae29ff

1. Introduction

Game theory,[1] a tool for rational decision making in the case of conflict, is generally used in economics, physics and biology. In the past decade, quantum game theory has emerged as a new active field since the concept of quantum information was introduced into the classical game theory. Quantum games offer additional strategies and new equilibria that resolve dilemmas that occur in the classical games. Nevertheless, it is well known that quantum systems are easily influenced by ambient environments, which leads to decoherence. Decoherence effects have been studied in different quantum games such as Prisoners' Dilemma, Battle of the Sexes, Chicken game, Stackelberg duopoly, and so on. Early work on quantum games in noisy channels was mainly committed to memoryless traditional channels. And, consecutive signal transmissions were not correlated through the channel.[2–8]

Interest has then been spread to extend the theory of quantum channels to include memory effects.[9–11] There are time scales for which successive uses of the channel are correlated and memory effects need to be considered. Research involving quantum games under decoherence and correlated noise has produced interesting results. Recently, Nawaz and Toor[12] have shown for quantum games based on a quantum-correlated phase-damping channel that the quantum player only enjoys an advantage over the classical player when both the initial quantum state and the measurement basis are in entangled form. It is also shown that for maximum correlation the effects of decoherence diminish and it behaves as a noiseless game. Ramzan *et al.*[13] investigated the effects of memory on quantum games, such as Prisoner's Dilemma, Battle of the Sexes, and Chicken, through three prototype quantum-correlated channels. It is shown that the quantum player enjoys an advantage over the classical player for all nine cases considered in the paper for the maximally entangled case. The effect of entanglement and correlated noise in a four-player quantum Minority game is investigated by Ramzan and Khan.[14] Different time-correlated quantum memory channels are considered to analyze the Nash equilibrium payoff of the 1st player. The three-player Prisoner's Dilemma game under the effect of decoherence and correlated noise has been investigated in Ref. [15]. It is seen that the quantum player is always better off than the classical players. It is also seen that the game's Nash equilibrium does not change in the presence of correlated noise, in contradiction to the effect of decoherence in the multiplayer case. In Ref. [16], Khan *et al.* studied the influence of entanglement and correlated noise on quantum Stackelberg duopoly in different memory channels, including correlated amplitude damping. For the correlated amplitude damping channel, they mainly consider the situation in which the game experiences only the second decohering process while the first decohering process vanishes. And, the study is chiefly concerned with which firm becomes better off than another firm beyond the critical point.

In this paper, we study the influence of memory effects on the quantum Stackelberg duopoly game with two amplitude damping processes where successive uses of the channels are correlated. The effect of channels with memory on the moves and payoffs of the firms has been discussed in detail. The rest of this paper is organized as follows. In Section 2, preliminaries (Stackelberg duopoly game, amplitude-damping

[†]Corresponding author. E-mail: liaoxp1@126.com

© 2026 Chinese Physical Society and IOP Publishing Ltd. All rights, including for text and data mining, AI training, and similar technologies, are reserved.

channel with memory) are given. In Section 3, we show how the memory effects affect the quantum Stackelberg duopoly game, and some analytical or numerical results are presented. In Section 4, we offer our conclusions.

2. Preliminaries

2.1. Stackelberg duopoly game

As we know, the Stackelberg duopoly game[17–21] is a market game, which is a dynamic extension of the static Cournot duopoly game. In the Cournot duopoly game, both firms make their strategic moves at the same time and guess what action the opponent will take. However, the Stackelberg duopoly is a dynamic model of duopoly. The Stackelberg duopoly allows one of the firms, say firm A, to move first. The other firm, say firm B, can observe the move of firm A before making its decision. Firm A is usually called the leader and firm B the follower; on this basis, the game is also called the leader–follower model.[22]

In classical Stackelberg duopoly, it is assumed that firm B will respond optimally to the strategic decision of firm A. The sequence of events is: (i) firm A chooses a quantity q_1; (ii) firm B observes q_1 and then chooses a quantity q_2; (iii) the payoffs to firms A and B are given by their respective profit functions as $P_A(q_1,q_2)$, $P_B(q_1,q_2)$. The backwards-induction outcome[23] in the Stackelberg duopoly is found by first finding the reaction function $R_2(q_1)$ of firm B to an arbitrary quantity q_1 chosen by firm A. It is found by differentiating firm B's payoff with respect to q_2, and maximizing the result for q_1 and can be written as $R_2(q_1) = \max P_B(q_1,q_2)$. Once firm B chooses this quantity, firm A can compute its optimization problem by differentiating its own payoff with respect to q_1 and then maximizing it to find the value $q_1 = q_1^*$. Using the value of q_1^*, the value of q_2^* can be obtained. These quantities define the backwards-induction outcome of the quantum Stackelberg duopoly and represent the subgame perfect Nash equilibrium. Then, the payoffs of the firms at the subgame perfect Nash equilibrium can be found.

A number of authors have proposed various quantization protocols for observing the behavior of Stackelberg duopoly game in the quantum realm.[24–26] We consider the quantum Stackelberg duopoly (QSD) game in the classical-probability action formalism with discrete variables. The QSD game can be modelled by two qubits, one for each player of firm A and firm B who have two possible strategies: identity operator (I) and the inversion operator (C). Both I and C are unitary operators. The quantum Stackelberg duopoly has been studied under various circumstances and interesting results have been obtained.[7,8,16,23,27–31] It has been shown that quantum entanglement affects payoff of the first mover and produces an equilibrium that corresponds to classical static form of the same game.[23] The effects of decoherence produced by various prototype quantum channels with memoryless on quantum Stackelberg duopoly have been studied by Zhu *et al*.[8] The influence of Unruh effect on quantum Stackelberg duopoly has been studied in Ref. [30]. It is shown that the acceleration of noninertial frame strongly affects the payoffs of the firms and the payoffs decrease with the increasing of acceleration parameter. In Ref. [31], an efficient method has been proposed to improve the payoffs of the firms in quantum Stackelberg duopoly game under noninertial frame using partial-collapse measurement.

2.2. Amplitude-damping channel with memory

As we all know, there are two different types of quantum channels including memoryless channels and memory channels. The simplest models for quantum channels are memoryless, when environmental correlation time is smaller than the time between consecutive uses, so that at each channel use the environment back action can be negligible. Namely, the system undertakes the same quantum channel ε, in which independent noise acts on each use. Suppose N times uses of this channel, then we have $\varepsilon_N = \varepsilon^{\otimes N}$. However, real systems among subsequent channel uses exhibit some correlations; this can happen when environmental correlation time is longer than the time between consecutive uses, so that the channel acts dependently on each channel input $\varepsilon_N \neq \varepsilon^{\otimes N}$. These kinds of channels are called memory channels.

In what follows we consider N channel uses. Given an input state ρ, a quantum channel ε is defined as a completely positive, trace-preserving map from input-state density matrices to output-state density matrices,

$$\varepsilon(\rho) = \sum_i E_i \rho E_i^\dagger, \tag{1}$$

where $E_i = \sqrt{P_{i_1\cdots i_N}} A_i$ are the Kraus operators of the channel which satisfy the completeness relationship, and $\sum_i P_{i_1\cdots i_N} = 1$. Here $P_{i_1\cdots i_N}$ can be interpreted as the probability that a random sequence of operations is applied to the sequence of N qubits transmitted through the channel. For a memoryless channel, these operations A_i are independent, and we have $P_{i_1\cdots i_N} = P_{i_1} P_{i_2} \cdots P_{i_N}$. However, for a channel with memory, these operations are time-correlated, $P_{i_1\cdots i_N} = P_{i_1} P_{i_2|i_1} \cdots P_{i_N|i_{N-1}}$, where $P_{i_N|i_{N-1}}$ is the conditional probability for that operation. For simplicity, we consider the Kraus operators for two consecutive uses of a channel with partial memory as

$$E_{i,j} = \sqrt{P_i[(1-u)P_j + u\delta_{i,j}]} A_i \otimes A_j, \tag{2}$$

where $0 \leq u \leq 1$ is the memory coefficient of the channel. $u = 0$ and $u = 1$ represent no-memory and maximum-memory effects, respectively.

Following, we will focus on the amplitude damping noisy channels with time-correlated Markov noise for two consecutive uses. Based on the Kraus operator approach, for any initial state ρ, the final state under noise is given by[9,10]

$$\varepsilon(\rho) = (1-u)\sum_{i,j} E^{v}_{i,j}\rho E^{v\dagger}_{i,j} + u\sum_{k} E^{c}_{k,k}\rho E^{c\dagger}_{k,k}. \tag{3}$$

The above expression means that with probability u they are correlated while with probability $1-u$ both operations are uncorrelated.

The amplitude-damping channel, which is used to characterize spontaneous emission, describes the energy dissipation from a quantum system.

$$A_0 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{bmatrix}, \tag{4}$$

where $p = 1-\exp(-\gamma t)$ is the single-qubit damping rate. The quantum amplitude-damping channel with uncorrelated noise can similarly be defined as the following Kraus operators

$$E^{v}_{i,j} = A_i \otimes A_j, \quad (i,j=0,1). \tag{5}$$

Based on previous analysis,[9,10] an amplitude-damping channel with finite memory for two consecutive uses, the task of constructing Kraus operators $E^{c}_{k,k}$ for

$$E^{c}_{00} = \begin{bmatrix} \sqrt{1-p} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad E^{c}_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{p} & 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

3. The influence of memory effects on quantum Stackelberg duopoly game

Initially the two firms, A and B share an initial state

$$|\psi_1\rangle = \cos(Q)|00\rangle_{A,B} + \sin(Q)|11\rangle_{A,B}, \tag{7}$$

where Q is a measure of entanglement. The first entry in each ket of Eq. (7) corresponds to firm A and the second entry corresponds to firm B. Through the amplitude damping noisy channels with time-correlated Markov noise, the state is,

$$\rho_{\mathrm{G}} = \varepsilon_{\mathrm{channel}}(\rho), \tag{8}$$

where $\varepsilon_{\mathrm{channel}}$ is defined by Eq. (3).

In the quantum Stackelberg duopoly, each firm has two possible strategies: I (the identity operator) and C (the inversion operator). When two firms apply the unitary operators I and C with probabilities x and y on the two-qubit state, respectively, the state ρ_{G} evolves into the following state:

$$\begin{aligned}\rho_{\mathrm{f}} = {} & xyI_A\otimes I_B\rho_{\mathrm{G}}I_A^{\dagger}\otimes I_B^{\dagger} + x(1-y)I_A\otimes C_B\rho_{\mathrm{G}}I_A^{\dagger}\otimes C_B^{\dagger} \\ & + y(1-x)C_A\otimes I_B\rho_{\mathrm{G}}C_A^{\dagger}\otimes I_B^{\dagger} \\ & + (1-x)(1-y)C_A\otimes C_B\rho_{\mathrm{G}}C_A^{\dagger}\otimes C_B^{\dagger}.\end{aligned} \tag{9}$$

The moves of firms A and B in a classical duopoly game are given by quantities q_1 and q_2, which have values in the range $[0,\infty)$. We assume that firms A and B agree on a function that can uniquely define a real positive number in the region $(0,1]$ for every quantity q_1, q_2 in $[0,\infty)$. A simple such function is $1/(1+q_i)$. So, firms A and B find the probabilities x and y to be given as

$$\begin{aligned} x &= \frac{1}{1+q_1} \\ y &= \frac{1}{1+q_2}. \end{aligned} \tag{10}$$

Through mean values of their corresponding payoff operators with regard to the final state of the game, the payoffs of the two firms are acquired. Then, the payoffs of firms A and B are given by the following trace operations:

$$\begin{aligned} P_A(q_1,q_2) &= \mathrm{Tr}[\rho_{\mathrm{f}}P_A^{\mathrm{op}}(q_1,q_2)], \\ P_B(q_1,q_2) &= \mathrm{Tr}[\rho_{\mathrm{f}}P_B^{\mathrm{op}}(q_1,q_2)], \end{aligned} \tag{11}$$

where P_A^{op} and P_B^{op} are payoff operators of the firms and are given as[30]

$$\begin{aligned} P_A^{\mathrm{op}}(q_1,q_2) &= \frac{q_1}{q_{12}}(k\rho_{11}-\rho_{22}-\rho_{33}), \\ P_B^{\mathrm{op}}(q_1,q_2) &= \frac{q_2}{q_{12}}(k\rho_{11}-\rho_{22}-\rho_{33}), \end{aligned} \tag{12}$$

where k is a constant related to the capacity of the market and the marginal cost as given in Ref. [22], and ρ_{ii} are the diagonal elements of the final density matrix. The parameter q_{12} is introduced by

$$q_{12} = \frac{1}{(1+q_1)(1+q_2)}. \tag{13}$$

Next, we take $p_1 = p_2 = p$, $u_1 = u_2 = u$ and the value of k is set to 1, the payoffs to firms A and B can be calculated by using Eqs. (11)–(13) with the following expressions:

$$\begin{aligned} P_A = {} & -q_1[q_1+q_2+\sin(Q)^2 - 2pq_1 - 2pq_2 - 14p^2\sin(Q)^2 + 12p^3\sin(Q)^2 \\ & - 3p^4\sin(Q)^2 + 4p\sin(Q)^2 - 3p^2q_1q_2 - 7p^2u^2\sin(Q)^2 + 10p^3u^2\sin(Q)^2 \\ & - 3p^4u^2\sin(Q)^2 - 3pq_1\sin(Q)^2 - 3pq_2\sin(Q)^2 - q_1q_2\sin(Q)^2 - 6pu\sin(Q)^2 \\ & + 2pq_1q_2 + 2pq_1u + 2pq_2u + 9p^2q_1\sin(Q)^2 + 9p^2q_2\sin(Q)^2 \\ & - 9p^3q_1\sin(Q)^2 - 9p^3q_2\sin(Q)^2 + 3p^4q_1\sin(Q)^2 + 3p^4q_2\sin(Q)^2 \end{aligned}$$

$$
\begin{aligned}
&+22p^2u\sin(Q)^2-22p^3u\sin(Q)^2+6p^4u\sin(Q)^2-3pq_1q_2u-4p^2q_1q_2\sin(Q)^2\\
&+6p^3q_1q_2\sin(Q)^2-3p^4q_1q_2\sin(Q)^2-16p^2q_1u\sin(Q)^2-16p^2q_2u\sin(Q)^2\\
&+19p^3q_1u\sin(Q)^2+19p^3q_2u\sin(Q)^2-6p^4q_1u\sin(Q)^2-6p^4q_2u\sin(Q)^2\\
&+3p^2q_1q_2u+7p^2q_1u^2\sin(Q)^2+7p^2q_2u^2\sin(Q)^2-10p^3q_1u^2\sin(Q)^2\\
&-10p^3q_2u^2\sin(Q)^2+3p^4q_1u^2\sin(Q)^2+3p^4q_2u^2\sin(Q)^2+2pq_1q_2\sin(Q)^2\\
&+3pq_1u\sin(Q)^2+3pq_2u\sin(Q)^2-7p^2q_1q_2u^2\sin(Q)^2+10p^3q_1q_2u^2\sin(Q)^2\\
&-3p^4q_1q_2u^2\sin(Q)^2+10p^2q_1q_2u\sin(Q)^2-16p^3q_1q_2u\sin(Q)^2+6p^4q_1q_2u\sin(Q)^2-1],
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
P_B=&-q_2[q_1+q_2+\sin(Q)^2-2pq_1-2pq_2-14p^2\sin(Q)^2+12p^3\sin(Q)^2\\
&-3p^4\sin(Q)^2+4p\sin(Q)^2-3p^2q_1q_2-7p^2u^2\sin(Q)^2+10p^3u^2\sin(Q)^2\\
&-3p^4u^2\sin(Q)^2-3pq_1\sin(Q)^2-3pq_2\sin(Q)^2-q_1q_2\sin(Q)^2-6pu\sin(Q)^2\\
&+2pq_1q_2+2pq_1u+2pq_2u+9p^2q_1\sin(Q)^2+9p^2q_2\sin(Q)^2\\
&-9p^3q_1\sin(Q)^2-9p^3q_2\sin(Q)^2+3p^4q_1\sin(Q)^2+3p^4q_2\sin(Q)^2\\
&+22p^2u\sin(Q)^2-22p^3u\sin(Q)^2+6p^4u\sin(Q)^2-3pq_1q_2u-4p^2q_1q_2\sin(Q)^2\\
&+6p^3q_1q_2\sin(Q)^2-3p^4q_1q_2\sin(Q)^2-16p^2q_1u\sin(Q)^2-16p^2q_2u\sin(Q)^2\\
&+19p^3q_1u\sin(Q)^2+19p^3q_2u\sin(Q)^2-6p^4q_1u\sin(Q)^2-6p^4q_2u\sin(Q)^2\\
&+3p^2q_1q_2u+7p^2q_1u^2\sin(Q)^2+7p^2q_2u^2\sin(Q)^2-10p^3q_1u^2\sin(Q)^2\\
&-10p^3q_2u^2\sin(Q)^2+3p^4q_1u^2\sin(Q)^2+3p^4q_2u^2\sin(Q)^2+2pq_1q_2\sin(Q)^2\\
&+3pq_1u\sin(Q)^2+3pq_2u\sin(Q)^2-7p^2q_1q_2u^2\sin(Q)^2+10p^3q_1q_2u^2\sin(Q)^2\\
&-3p^4q_1q_2u^2\sin(Q)^2+10p^2q_1q_2u\sin(Q)^2-16p^3q_1q_2u\sin(Q)^2\\
&+6p^4q_1q_2u\sin(Q)^2-1].
\end{aligned}
\tag{15}
$$

We can find the backwards-induction outcome in the QSD game exactly in the same way as done in the classical game with the subgame perfect Nash equilibrium point[23,30] given by

$$
q_1^*=\frac{N}{D},
\tag{16}
$$

where

$$
\begin{aligned}
N=&\,3p^4u^2\sin(Q)^2-6p^4u\sin(Q)^2+3p^4\sin(Q)^2-10p^3u^2\sin(Q)^2+22p^3u\sin(Q)^2\\
&-12p^3\sin(Q)^2+7p^2u^2\sin(Q)^2-22p^2u\sin(Q)^2+14p^2\sin(Q)^2+6pu\sin(Q)^2\\
&-4p\sin(Q)^2-\sin(Q)^2+1,\\
D=&\,6p^4u^2\sin(Q)^2-12p^4u\sin(Q)^2+6p^4\sin(Q)^2-20p^3u^2\sin(Q)^2+38p^3u\sin(Q)^2\\
&-18p^3\sin(Q)^2+14p^2u^2\sin(Q)^2-32p^2u\sin(Q)^2+18p^2\sin(Q)^2+6pu\sin(Q)^2\\
&+4pu-6p\sin(Q)^2-4p+2,
\end{aligned}
\tag{17}
$$

$$
q_2^*=\frac{N}{D},
\tag{18}
$$

where

$$
\begin{aligned}
N=&[3p^4u^2\sin(Q)^2-6p^4u\sin(Q)^2+3p^4\sin(Q)^2-10p^3u^2\sin(Q)^2\\
&+22p^3u\sin(Q)^2-12p^3\sin(Q)^2+7p^2u^2\sin(Q)^2-22p^2u\sin(Q)^2\\
&+14p^2\sin(Q)^2+6pu\sin(Q)^2-4p\sin(Q)^2-\sin(Q)^2+1]\\
&[3p^4u^2\sin(Q)^2-6p^4u\sin(Q)^2+3p^4\sin(Q)^2-10p^3u^2\sin(Q)^2\\
&+19p^3u\sin(Q)^2-9p^3\sin(Q)^2+7p^2u^2\sin(Q)^2-16p^2u\sin(Q)^2\\
&+9p^2\sin(Q)^2+3pu\sin(Q)^2+2pu-3p\sin(Q)^2-2p+1],\\
D=&\,18p^8u^4\sin(Q)^4-72p^8u^3\sin(Q)^4+108p^8u^2\sin(Q)^4-72p^8u\sin(Q)^4\\
&+18p^8\sin(Q)^4-120p^7u^4\sin(Q)^4+468p^7u^3\sin(Q)^4-684p^7u^2\sin(Q)^4
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&+444\,p^7\,u\sin(Q)^4-108\,p^7\sin(Q)^4+284\,p^6\,u^4\sin(Q)^4-1120\,p^6\,u^3\sin(Q)^4\\
&+18\,p^6\,u^3\sin(Q)^2+1676\,p^6\,u^2\sin(Q)^4-54\,p^6\,u^2\sin(Q)^2-1128\,p^6\,u\sin(Q)^4\\
&+54\,p^6\,u\sin(Q)^2+288\,p^6\sin(Q)^4-18\,p^6\sin(Q)^2-280\,p^5\,u^4\sin(Q)^4\\
&+1208\,p^5\,u^3\sin(Q)^4-30\,p^5\,u^3\sin(Q)^2-2008\,p^5\,u^2\sin(Q)^4+96\,p^5\,u^2\sin(Q)^2\\
&+1500\,p^5\,u\sin(Q)^4-102\,p^5\,u\sin(Q)^2-420\,p^5\sin(Q)^4+36\,p^5\sin(Q)^2\\
&+98\,p^4\,u^4\sin(Q)^4-568\,p^4\,u^3\sin(Q)^4-58\,p^4\,u^3\sin(Q)^2+1220\,p^4\,u^2\sin(Q)^4\\
&+136\,p^4\,u^2\sin(Q)^2-1080\,p^4\,u\sin(Q)^4-108\,p^4\,u\sin(Q)^2+332\,p^4\sin(Q)^4\\
&+30\,p^4\sin(Q)^2+84\,p^3\,u^3\sin(Q)^4+70\,p^3\,u^3\sin(Q)^2-348\,p^3\,u^2\sin(Q)^4\\
&-232\,p^3\,u^2\sin(Q)^2+372\,p^3\,u\sin(Q)^4+288\,p^3\,u\sin(Q)^2-116\,p^3\sin(Q)^4\\
&-124\,p^3\sin(Q)^2+36\,p^2\,u^2\sin(Q)^4+54\,p^2\,u^2\sin(Q)^2+16\,p^2\,u^2-24\,p^2\,u\sin(Q)^4\\
&-162\,p^2\,u\sin(Q)^2-26\,p^2\,u+102\,p^2\sin(Q)^2+10\,p^2-12\,p\,u\sin(Q)^4+30\,p\,u\\
&\sin(Q)^2+10\,p\,u+4\,p\sin(Q)^4-24\,p\sin(Q)^2-12\,p+2\sin(Q)^4-2\sin(Q)^2+4. \qquad (20)
\end{aligned}
$$

When the QSD game is initially in the unentangled state, *i.e.*, $Q=0$, we obtain

$$q_1^*=\frac{1}{2\,(2\,p\,(u-1)+1)}, \qquad (21)$$

$$q_2^*=\frac{p\,(u-1)+1/2}{(8\,u^2-13\,u+5)\,p^2+(5\,u-6)\,p+2}, \qquad (22)$$

$$P_A(q_1^*,q_2^*)=\frac{1}{8\,(2\,p\,(u-1)+1)}, \qquad (23)$$

$$P_B(q_1^*,q_2^*)=\frac{p\,(u/4-1/4)+1/8}{(8\,u^2-13\,u+5)\,p^2+(5\,u-6)\,p+2}. \qquad (24)$$

When the QSD game is initially in the maximally entangled state, *i.e.*, $Q=\pi/4$, we have

$$q_1^*=\frac{1}{2}-\frac{(3/2-3\,u/2)\,p^3+(3\,u-5/2)\,p^2+(u/2-3/2)\,p+1/2}{(3\,u^2-6\,u+3)\,p^4+(-10\,u^2+19\,u-9)\,p^3+(7\,u^2-16\,u+9)\,p^2+(7\,u-7)\,p+2}, \qquad (25)$$

$$q_2^*=\frac{N}{D}, \qquad (26)$$

where

$$
\begin{aligned}
N=&\,[3\,p^4\,u^2-6\,p^4\,u+3\,p^4-10\,p^3\,u^2+19\,p^3\,u-9\,p^3+7\,p^2\,u^2-16\,p^2\,u+9\,p^2+7\,p\,u-7\,p+2]\\
&\times[3\,p^4\,u^2-6\,p^4\,u+3\,p^4-10\,p^3\,u^2+22\,p^3\,u-12\,p^3+7\,p^2\,u^2-22\,p^2\,u+14\,p^2+6\,p\,u+4\,p+1],\\
D=&\,18\,p^8\,u^4-72\,p^8\,u^3+108\,p^8\,u^2-72\,p^8\,u+18\,p^8-120\,p^7\,u^4+468\,p^7\,u^3-684\,p^7\,u^2\\
&+444\,p^7\,u-108\,p^7+284\,p^6\,u^4-1084\,p^6\,u^3+1568\,p^6\,u^2-1020\,p^6\,u+252\,p^6-280\,p^5\,u^4\\
&+1148\,p^5\,u^3-1816\,p^5\,u^2+1296\,p^5\,u-348\,p^5+98\,p^4\,u^4-684\,p^4\,u^3+1492\,p^4\,u^2-1296p^4\,u\\
&+392\,p^4+224\,p^3\,u^3-812\,p^3\,u^2+948\,p^3\,u-364p^3+208p^2u^2-452p^2u+244p^2+88pu-92p+14, \qquad (27)
\end{aligned}
$$

$$P_A(q_1^*,q_2^*)=\frac{\left(3\,p^4\,u^2-6\,p^4\,u+3\,p^4-10\,p^3\,u^2+22\,p^3\,u-12\,p^3+7\,p^2\,u^2-22\,p^2\,u+14\,p^2+6\,p\,u-4\,p+1\right)^2}{16\,(3\,p^4\,u^2-6\,p^4\,u+3\,p^4-10\,p^3\,u^2+19\,p^3\,u-9\,p^3+7\,p^2\,u^2-16\,p^2\,u+9\,p^2+7\,p\,u-7\,p+2)}, \qquad (28)$$

$$P_B(q_1^*,q_2^*)=\frac{N}{D}, \qquad (29)$$

where

$$
\begin{aligned}
N=&\,[3\,p^4\,u^2-6\,p^4\,u+3\,p^4-10\,p^3\,u^2+19\,p^3\,u-9\,p^3+7\,p^2\,u^2-16\,p^2\,u+9\,p^2+7\,p\,u-7\,p+2]\\
&\left(3\,p^4\,u^2-6\,p^4\,u+3\,p^4-10\,p^3\,u^2+22\,p^3\,u-12\,p^3+7\,p^2\,u^2-22\,p^2\,u+14\,p^2+6\,p\,u-4\,p+1\right)^2,\\
D=&\,144\,p^8\,u^4-576\,p^8\,u^3+864\,p^8\,u^2-576\,p^8\,u+144\,p^8-960\,p^7\,u^4+3744\,p^7\,u^3\\
&-5472\,p^7\,u^2+3552\,p^7\,u-864\,p^7+2272\,p^6\,u^4-8672\,p^6\,u^3+12544\,p^6\,u^2-8160\,p^6\,u\\
&+2016\,p^6-2240\,p^5\,u^4+9184\,p^5\,u^3-14528\,p^5\,u^2+10368\,p^5\,u-2784\,p^5+784\,p^4\,u^4\\
&-5472\,p^4\,u^3+11936\,p^4\,u^2-10368\,p^4\,u+3136\,p^4+1792\,p^3\,u^3-6496\,p^3\,u^2+7584\,p^3\\
&u-2912\,p^3+1664\,p^2\,u^2-3616\,p^2\,u+1952\,p^2+704\,p\,u-736\,p+112. \qquad (30)
\end{aligned}
$$

In order to see the memory effects on the subgame perfect Nash equilibrium, using the expressions of the moves of the two firms given by Eqs. (16) and (18), we plot q_1^* and q_2^* against the entanglement parameter Q with $k = 1$, $p = 0.3$ for different u in Fig. 1. The lines correspond to $u = 0$ (dashed line), $u = 0.4$ (dotted line), $u = 0.8$ (dash–dotted line), $u = 1$ (solid line). It is shown that without memory effect ($u = 0$) there are two zero-points for the moves of the two firms which can be labelled by Q_1 and Q_2 with $Q_1 < Q_2$, respectively. When $Q_1 < Q < Q_2$, both q_1^* and q_2^* are negative due to the influence of decoherence induced by the amplitude damping.[7,8] Therefore, there does not exist the subgame perfect Nash equilibrium in this regime since the existence of the equilibrium point requires both q_1^* and q_2^* to be positive. However, with memory effect and as memory effect parameter u increases, the regime of absence of the subgame perfect Nash equilibrium becomes small. When u increases to a certain value, zero-points are gone and both q_1^* and q_2^* are positive. And, there exists the subgame perfect Nash equilibrium for all the function of the entanglement angle Q.

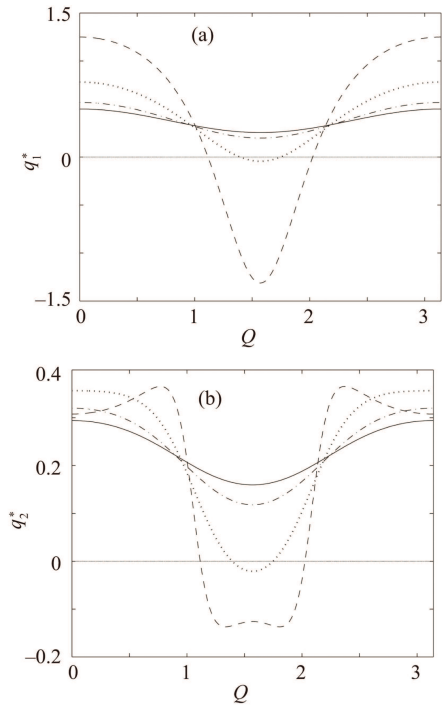


Fig. 1. The moves of firms at the subgame perfect Nash equilibrium point against the function of entanglement angle Q with $k = 1$, $p = 0.3$: (a) the moves of firm A; (b) the moves of firm B. The lines correspond to $u = 0$ (dashed line), $u = 0.4$ (dotted line), $u = 0.8$ (dash–dotted line), and $u = 1$ (solid line).

Using the expressions given by Eqs. (14) and (15), we also plot the corresponding payoffs to the two firms in Fig. 2, which indicate that payoffs of both firms can be improved with the increase of the memory effect. Especially, when u increases to a certain degree, two zero-points for the payoffs of the two firms disappear. And, the payoffs to both firms are positive for the whole range of values of the entanglement parameter.

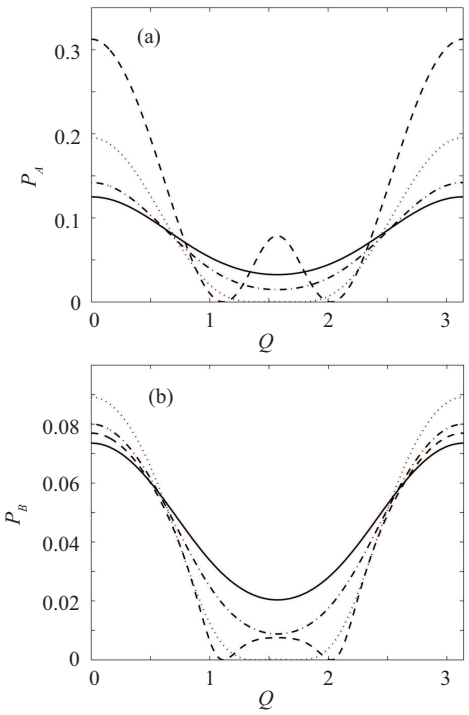


Fig. 2. The firms' payoffs against the function of entanglement angle Q with $k = 1$, $p = 0.3$: (a) the payoff of firm A; (b) the payoff of firm B. The lines correspond to $u = 0$ (dashed line), $u = 0.4$ (dotted line), $u = 0.8$ (dash–dotted line), and $u = 1$ (solid line).

The moves of firms at the subgame perfect Nash equilibrium point and corresponding payoffs against the damping rate p with the intermediate entangled state $Q = \pi/6$, $k = 1$ for different u are plotted in Fig. 3. The dashed, dotted, dash–dotted and solid lines correspond to q_1^*, q_2^*, P_A, and P_B, respectively. When $u = 0$ (without memory effect), there exists the subgame perfect Nash equilibrium in the region of $p < 0.475$, since in this region $q_1^* > 0$ and $q_2^* > 0$, and $P_{A,B}(q_1^*, q_2^*) > 0$. Moreover, the game is always first mover advantage game. The presence of quantum memory allows the existence of Nash equilibrium for the entire range of values of p, when $u \geq 0.85$.

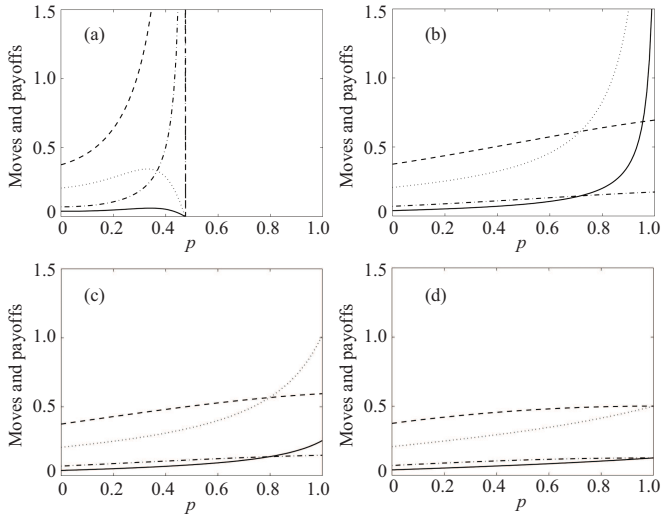


Fig. 3. The moves of firms at the subgame perfect Nash equilibrium point and corresponding payoffs against the damping rate p with $Q = \pi/6$, $k = 1$. The dashed, dotted, dash–dotted, and solid lines correspond to q_1^*, q_2^*, P_A, and P_B, respectively: (a) $u = 0$, (b) $u = 0.86$, (c) $u = 0.92$, and (d) $u = 1$.

As can be seen from Figs. 3(b), 3(c), and 3(d), the presence of quantum memory results in a 'critical point' at which both firms are equally benefited. Beyond this critical point, the game becomes a follower advantage game for a highly decohered channel. Also, the position of this 'critical point' moves to the right (*i.e.*, the direction in which the damping rate increases) as u increases. It is interesting that for maximum-memory effect ($u = 1$), the position of this 'critical point' moves to $p = 1$, where the channel is fully decohered. And, both firms are equally benefited and get a move equal to $1/2$ and a payoff equal to $1/8$ (see Fig. 3(d)).

In Fig. 4, we plot the moves of firms at the subgame perfect Nash equilibrium point and corresponding payoffs against the damping rate p with unentangled initial state $Q = 0$, $k = 1$ for two cases, the no-memory effect and the maximum-memory effect. When $u = 0$ (without memory effect), there exists the subgame perfect Nash equilibrium in the region of $p < 1/2$, since in this region $q_1^* > 0$ and $q_2^* > 0$, and $P_{A,B}(q_1^*, q_2^*) > 0$ (see Fig. 4 dashed and dash–dotted lines). When $u = 0$, from Eqs. (21)–(24), we obtain

$$\begin{aligned} q_1^* &= \frac{1}{2(1-2p)}, \\ q_2^* &= \frac{1/2-p}{5p^2-6p+2}, \\ P_A(q_1^*, q_2^*) &= \frac{1}{8(1-2p)}, \\ P_B(q_1^*, q_2^*) &= \frac{1/8-(1/4)p}{5p^2-6p+2}. \end{aligned} \tag{31}$$

Obviously when $p = 1/2$, we have $q_1^* \to \infty$, $P_A(q_1^*, q_2^*) \to \infty$, $q_2^* = 0$, $P_B(q_1^*, q_2^*) = 0$. In other words, there exists the subgame perfect Nash equilibrium in the region of $p < 1/2$.

When $u = 1$, from Eqs. (21)–(24), we obtain

$$\begin{aligned} q_1^* &= \frac{1}{2}, \\ q_2^* &= \frac{1/2}{2-p}, \\ P_A(q_1^*, q_2^*) &= \frac{1}{8}, \\ P_B(q_1^*, q_2^*) &= \frac{(1/8)}{(2-p)}, \end{aligned} \tag{32}$$

we can see that there exists the subgame perfect Nash equilibrium for all the decoherence ranges $0 \le p \le 1$. Obviously, q_1^* and P_A always maintain initial values regardless of the damping parameter with the maximum-memory effect, as shown in Fig. 4 (dotted lines). It seems that q_1^* and P_A are not affected by amplitude damping decoherence. And, q_2^* and P_B gradually increase as p increases. When $p = 1$, we obtain $q_2^* = 1/2$ and $P_B = 1/8$. That is, there is a 'special point' for the moves and payoffs to the two firms. At the 'special point', namely, maximum-memory effect ($u = 1$) and fully decohered channel ($p = 1$), both firms are equally benefited and get a move equal to $1/2$ and a payoff equal to $1/8$ (see the intersection of the dotted and the solid lines in Fig. 4).

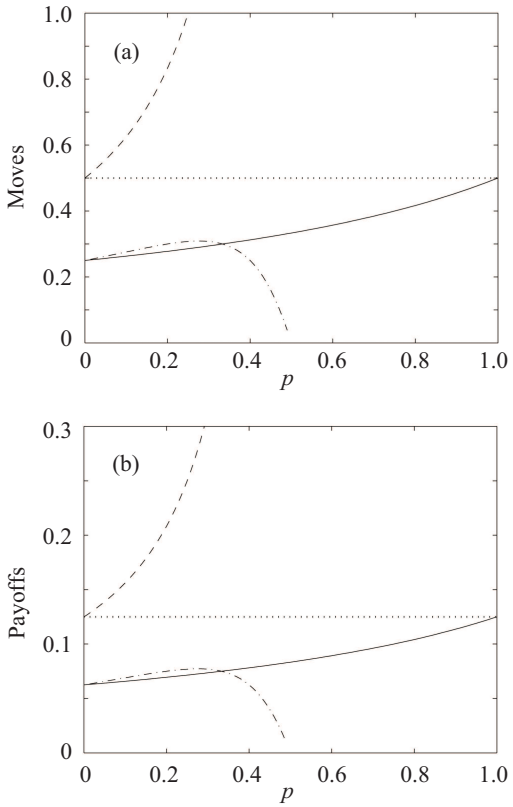


Fig. 4. The moves of firms at the subgame perfect Nash equilibrium point and corresponding payoffs against the damping rate p with $Q = 0$, $k = 1$: (a) the moves of firms. The dashed, dotted, dash–dotted, and solid lines correspond to the cases of q_1^* ($u = 0$), q_1^* ($u = 1$), q_2^* ($u = 0$), and q_2^*($u = 1$), respectively; (b) The corresponding payoffs: the dashed, dotted, dash–dotted, and solid lines correspond to the cases of P_A ($u = 0$), P_A ($u = 1$), P_B ($u = 0$), and P_B ($u = 1$), respectively.

In Fig. 5, we plot the moves of firms at the subgame perfect Nash equilibrium point and corresponding payoffs against the damping rate p with maximally entangled state $Q = \pi/4$, $k = 1$ for two cases, no-memory effect and maximum-memory effect. Without memory effect, there exists the subgame perfect Nash equilibrium in the region of $p < 0.45$, since in this region $q_1^* > 0$ and $q_2^* > 0$, and $P_{A,B}(q_1^*, q_2^*) > 0$. However, with maximum-memory effect ($u = 1$), there exists the subgame perfect Nash equilibrium for all the decoherence ranges $0 \le p \le 1$ (see Fig. 5 dotted and solid lines). Like the unentangled state with maximum-memory effect, there is a 'special point' ($p = 1$) for the moves and payoffs to the two firms. At the 'special point', the two players have the same move $1/2$ and payoff $1/8$ (see the intersection of the dotted and the solid lines in Fig. 5). Further research shows that for other general entangled states, we get the same results in the case of maximum-memory effect and fully decohered channel.

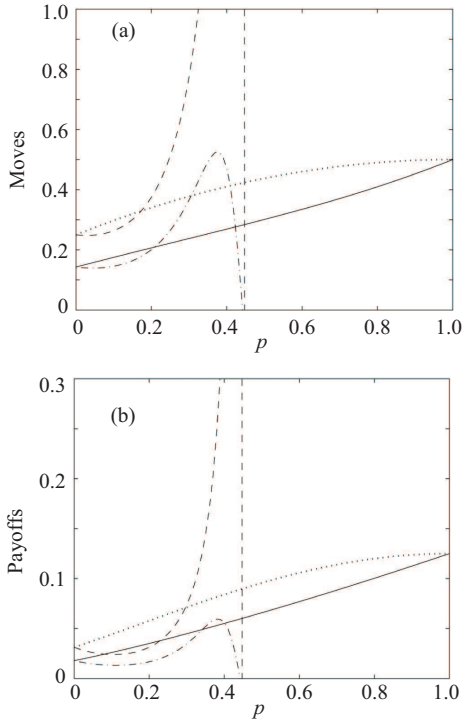


Fig. 5. The moves of firms at the subgame perfect Nash equilibrium point and corresponding payoffs against the damping rate p with $Q = \pi/4$, $k = 1$. (a) The moves of firms: the dashed, dotted, dash–dotted, and solid lines correspond to the cases of q_1^* $(u = 0)$, q_1^* $(u = 1)$, q_2^* $(u = 0)$, and $q_2^*(u = 1)$, respectively; (b) The corresponding payoffs: the dashed, dotted, dash–dotted, and solid lines correspond to the cases of P_A $(u = 0)$, P_A $(u = 1)$, P_B $(u = 0)$, and P_B $(u = 1)$, respectively.

4. Conclusion

In this work, we have investigated the influence of memory effects on quantum Stackelberg duopoly game through amplitude-damping channel where successive uses of the channels are correlated. Some analytical and numerical results are presented. The effects of channels with memory on the moves and payoffs of the firms have been discussed in detail. The results show that the memory effects can seriously change the Nash equilibrium and the payoffs of the two firms. It is observed that as the degree of channel memory increases, there exists Nash equilibrium for all the range of entanglement parameter and the payoffs of both firms are positive. Moreover, the presence of quantum memory allows the existence of Nash equilibrium for the entire range of decoherence parameter for both unentangled and entangled initial states.

It is worth mentioning that the presence of quantum memory results in a 'critical point' of the damping parameter, at which both firms are equally benefited. Beyond this critical point, the game becomes a follower advantage game. Moreover, as the memory parameter increases, the position of the 'critical point' moves in the direction in which the damping rate increases. We have also observed that for fully correlated $(u = 1)$ and fully decohered $(p = 1)$ channels, both firms are equally benefited and get a move equal to $1/2$ and a payoff equal to $1/8$ in both of the situations (for unentangled and entangled initial states). We hope that the new feature would be of interest not only to the field of quantum information but also to the applied economics.

Acknowledgement

Project supported by the National Natural Science Foundation of China (Grant No. U2031135).

References

[1] Myerson R B 1991 *Game Theory: Analysis of Conflict* (Boston: Havard University Press)
[2] Flitney A P and Abbott D 2005 *J. Phys. A* **38** 449
[3] Chen L K, Ang H, Kiang D, Kwek L C and Lo C F 2003 *Phys. Lett. A* **316** 317
[4] Chen J L, Kwek L C and Oh C H 2002 *Phys. Rev. A* **65** 052320
[5] Khan S, Ramzan M and Khan M K 2010 *Int. J. Theor. Phys.* **49** 31
[6] Cao S, Fang M F and Zheng X J 2007 *Chin. Phys.* **16** 915
[7] Zhu X and Kuang L M 2008 *Commun. Theor. Phys.* **49** 111
[8] Zhu, X and Kuang L M 2007 *J. Phys. A: Math. Theor.* **40** 7729
[9] Macchiavello C and Palma G M 2002 *Phys. Rev. A* **65** 050301
[10] Yeo Y and Skeen A 2003 *Phys. Rev. A* **67** 064301
[11] Mazhar Ali 2024 *Chin. Phys. B* **33** 020307
[12] Nawaz A and Toor A H 2006 *J. Phys. A: Math. Gen.* **39** 9321
[13] Ramzan M, Nawaz A, Toor A H and Khan M H 2008 *J. Phys. A: Math. Theor.* **41** 5
[14] Ramzan M and Khan M K 2013 *Fluctuation and Noise Letters* **12** 1350025
[15] Ramzan M and Khan M K 2008 *J. Phys. A: Math. Theor.* **41** 435302
[16] Khan S, Ramzan M and Khan M K 2010 *J. Phys. A: Math. Theor.* **43** 375301
[17] Lo C F and Kiang D 2003 *Phys. Lett. A* **318** 333
[18] Bierman H S and Fernandez L 1998 *Game Theory with Economic Applications* (MA: Addison–Wesley, Reading)
[19] Gravelle H and Rees R 1992 *Microeconomics* 2nd. edn. (New York: Longman Harlow)
[20] Rasmusen E 1994 *Games and Information: An Introduction to Game Theory* 2nd. edn. (Oxford: Blackwell Publishers)
[21] Zhang W Y 2004 *Game Theory and Information Economics* (Shanghai: Shanghai People's Press)
[22] Gibbons R 1992 *Game theory for Applied Economists* (Princeton: Princeton University Press)
[23] Iqbal A and Toor A H 2002 *Phys. Rev. A* **65** 052328
[24] Lo C F and Kiang D 2005 *Phys. Lett. A* **346** 65
[25] Wang X and Hu C Z 2012 *Chin. Phys. Lett.* **29** 120303
[26] Marinatto L and Weber T 2000 *Phys. Lett. A* **272** 291
[27] Kameshwari A V S and Balakrishnan S 2021 *Quantum Inf. Process.* **20** 337
[28] Lian S and Feng X b 2021 *Phys. Lett. A* **385** 126956
[29] Ramón A S and Samuel M G 2020 *Physica A: Statistical Mechanics and its Applications* **553** 124271
[30] Khan S and Khan M K 2011 *Chin. Phys. Lett.* **28** 070202
[31] Liao X P, Pan C N, Rong M S and Fang M F 2019 *Quantum Inf. Process.* **18** 91