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Nonlinearly dispersive mKdV equations and weak compactons

Jing-Qun Wang(王景群)[1,2,†], Jian-Gen Liu(刘建根)[1,‡], and Li-Xin Tian(田立新)[2,§]

[1] *School of Mathematics and Statistics, Suzhou University of Technology, Changshu* 215500, *China*
[2] *School of Mathematical Sciences, Nanjing Normal University, Nanjing* 210023, *China*

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We are concerned with the nonlinearly dispersive mKdV (mK(m,n)) equations. The conservation laws are obtained by using the rank analysis method and the transformation relationship between conservation laws and the traveling wave ordinary differential equation (ODE) is presented. We establishes precise conditions for weak compactons based on the power the nonlinearity powers in this equation via cutoff conditions, which is different from previous studies on exact compacton solutions. Moreover, the weak compactons, formulae of compactons and pattern solutions are given.

Keywords: mK(m,n) equations, weak compactons, conservation laws

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1. Introduction

The fully nonlinear KdV equations K(m,n) equation was first introduced by Rosenau and Hyman,[1] where the authors present that its compacton solution, which are solitary waves with the property that after colliding with other compacton solutions, in 1993. Based on this work, Yan and Bluman[2] introduce the following nonlinearly dispersive Boussinesq equations (B(m,n) equation)

$$u_{tt}-u_{xx}-a(u^n)_{xx}+b(u^m)_{xxxx}=0, \tag{1}$$

and they investigated the special exact solutions of B(m,n) equations by using four direct ansatze. Moreover, another new compacton solution was obtained in the special case, B$(2,2)$ equation. As is well known, a finite number of techniques are used to study the compacton structure for the K(m,n) systems,[3–8] such as, Chen and Li[9] introduced modified nonlinear dispersive mK(m,n) equation in higher dimensional spaces

$$u^{n-1}u_t+a(u^m)_x+b(u^n)_{xxx}=0 \tag{2}$$

with $m,n\neq 0$, and they obtained a new Bäcklund transformation for modified nonlinear dispersive equation mK(m,n) based on the idea of homogenous balance method and with the help of Mathematica. Further, Chen[10] and Yan[11] gave exact solutions for the general modified nonlinear dispersive equations mK(m,n) in higher dimensional spaces, by using some proper transformations and symbolic computation and some direct ansatze. As far as we know, there are several well-known equations that are closely related to the mK(m,n) equation (2), for example, the K(m,n) equation, the nonlinearly dispersive generalization of the Kadomstev–Petviashvili equation (KP(m,n) equation) and the nonlinearly dispersive R(m,n) equation. Those equations were studied on physical properties of compacton structures in special cases. In addition, considering that equation (2) is a form of conservation laws and to our knowledge, many methods have been developed to find the conservation laws, see Refs. [12–16].

Naturally, conservation laws of the mK(m,n) equation and the existing conditions of compacton solutions emerge. Based on the above works, in this paper, we give the conservation laws of the mK(m,n) equation (2) and its weak compacton structures to extend the work presented in Refs. [17–22].

The paper is organized as follows. In Section 2, we study the conservation laws and the transformation relationship between conservation laws and the traveling wave ordinary differential equation (ODE) equation is presented. The existed conditions of weak compactons via cutoff conditions in Section 3. Section 4 devotes to the compacton solutions in a special case $m=n$ and the general case $m\neq n$. In Section 5, we give the general formulae of compactons and pattern solutions for Eq. (2). The conclusions are summarized in Section 6.

2. Travelling wave ODE and conservation laws

In this section, we will give some brief discussions of travelling wave ODE and conservation laws for the mK(m,n) equation. Before providing a detailed process, we will firstly

[†]Corresponding author. E-mail: wjq@cslg.edu.cn
[‡]Corresponding author. E-mail: ljgzr557@cslg.edu.cn
[§]Corresponding author. E-mail: tianlixin@njnu.edu.cn

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introduce some related concepts.

Definition 1 Suppose for all test function $\phi(t,x)$ satisfies the integral equation

$$\iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{u^n}{n}\phi_t + au^m\phi_x + bu^n\phi_{xxx}\right) \mathrm{d}t\,\mathrm{d}x = 0, \tag{3}$$

then $u(t,x)$ is the weak formulation of the mK(m,n) equation (2).

Lemma 1 For any test function $\phi(\xi)$ and any function $U(\xi)$ that is at least C^1, one has that

$$\int_{\mathbb{R}} \phi'(\xi)U(\xi)\mathrm{d}\xi = \phi(L)U(L) - \phi(-L)U(-L) - \int_{\mathbb{R}} \phi(\xi)U'(\xi)H(L\pm\xi)\mathrm{d}\xi \tag{4}$$

is an identity, where $H(L\pm\xi)$ is a step function.

Proof This Lemma 1 can be proved with standard methods,[23] therefore, here we will omit the detailed proof process.

For Eq. (2), set $u = U(\xi)$, $\xi = x - ct$, equation (2) becomes

$$\frac{-c}{n}(U^n)_\xi + a(U^m)_\xi + b(U^n)_{\xi\xi\xi} = 0. \tag{5}$$

Let $V = U^n$, we have the following ODE

$$\frac{-c}{n}V + bV'' + aV^{m/n} = C_1. \tag{6}$$

Then, we obtain the nonlinear oscillator equation

$$\frac{1}{2}(V')^2 - \frac{C_1}{b}V - \frac{c}{2nb}(V)^2 + \frac{an}{(m+n)b}V^{(m+n)/n} = E \tag{7}$$

with $E = C_2/b$, C_2 is an arbitrary constant and its solution can be given

$$\int \frac{\mathrm{d}V}{\sqrt{2\left[E + (C_1/b)V + (c/2nb)(V)^2 - (na/(m+n)b)V^{(m+n)/n}\right]}} = \pm\xi + C_3, \tag{8}$$

where C_3 is an arbitrary constant.

A local conservation law is of the form

$$D_t\rho + D_x\phi = 0, \tag{9}$$

which is satisfied for all solutions $u(t,x)$ of Eq. (2), where ρ is the conserved density, and ϕ is the associated flux. Set $u_i = \partial^i u/\partial x^i$, substituting the following formulas:

$$\rho = \sum_{a_0,a_1,\ldots,a_p} b_i u_0^{a_0}u_1^{a_1}\cdots u_p^{a_p}, \qquad \phi = \sum_{a_0,a_1,\ldots,a_p} c_j u_0^{a_0}u_1^{a_1}\cdots u_q^{a_q}$$

into Eq. (9) with a^i being the power of u_i, b_i is the coefficient corresponding to u_i. According to all terms in the given equation having the same rank, we have the first three densities and fluxes, as follows:

$$\begin{aligned}
&\rho_1 = \tfrac{1}{n}u^n, && \phi_1 = -\left(au^m + b(u^n)_{xx}\right),\\
&\rho_2 = u^{2n}, && \phi_2 = -\frac{2amn}{m+n}u^{m+n} - 2bn^2u^{2n-1}u_{xx} - bn^2(n-2)u^{2n-2}(u_x)^2,\\
&\rho_3 = \frac{1}{2}n^2u^{2n-2}(u_x)^2 - \frac{an}{(m+n)b}u^{m+n}, && \phi_3 = n^2u^{2n-2}u_tu_x + \frac{bn}{2}\left[n(n-1)u^{n-2}(u_x)^2 + nu^{n-1}u_{xx} + \frac{a}{b}u^m\right]^2.
\end{aligned}$$

Define

$$M_1[u] = \int_{\mathbb{R}} u^n\,\mathrm{d}x, \tag{10}$$

$$M_2[u] = \int_{\mathbb{R}} u^{2n}\,\mathrm{d}x, \tag{11}$$

$$M_3[u] = \int_{\mathbb{R}} \left(\frac{1}{2}n^2u^{2n-2}(u_x)^2 - \frac{an}{(m+n)b}u^{m+n}\right)\mathrm{d}x. \tag{12}$$

Here $M_1[u]$, $M_2[u]$, and $M_3[u]$ are the first three conservation quantities of the mK(m,n) equation. Noting that if $u^n = V(\xi)$, $\xi = x - ct$, the densities and fluxes are equivalent to

$$anV^{m/n} + bnV'' - cV = K_1, \tag{13}$$

$$\frac{2amn}{m+n}V^{(m+n)/n}+2bn^2V^{(2n-1)/n}(V^{1/n})''+bn^2(n-2)V^{(2n-2)/n}((V^{1/n})')^2-cV^2=K_2, \tag{14}$$

$$\frac{bn}{2}\left[n(n-1)V^{(n-2)/n}\left[(V^{1/n})'\right]^2+nV^{(n-1)/n}\left[(V^{1/n})''\right]+\frac{a}{b}V^{m/n}\right]^2-\frac{3c}{2}n^2V^{(2n-2)/n}\left[(V^{1/n})'\right]^2+\frac{an}{b(m+n)}V^{(m+n)/n}=K_3. \tag{15}$$

Combined Eqs. (2)–(13), it is easy to obtain $cK_2+K_1^2=-2K_3$ and travelling wave ODE (8).

Alternatively,

$$\int_{\mathbb{R}}\rho\,\mathrm{d}x=\int_{\mathbb{R}}\rho\mid_{u=V^{1/n}(\xi)}\mathrm{d}\xi=\int_{V_-}^{V^+}\frac{\sqrt{2}\rho\mid_{V(\xi)}}{\sqrt{E+\dfrac{C_1}{b}V+\dfrac{c}{2nb}(V)^2-\dfrac{na}{(m+n)b}V^{(m+n)/n}}}\mathrm{d}V, \tag{16}$$

therefore, we can obtain

$$M_1[u]=\int_{V_-}^{V^+}\frac{\sqrt{2}V}{\sqrt{E+\dfrac{C_1}{b}V+\dfrac{c}{2nb}(V)^2-\dfrac{na}{(m+n)b}V^{(m+n)/n}}}\mathrm{d}V,$$

$$M_2[u]=\int_{V_-}^{V^+}\frac{\sqrt{2}V^2}{\sqrt{E+\dfrac{C_1}{b}V+\dfrac{c}{2nb}(V)^2-\dfrac{na}{(m+n)b}V^{(m+n)/n}}}\mathrm{d}V,$$

$$M_3[u]=\sqrt{2}\int_{V_-}^{V^+}\sqrt{E+\frac{C_1}{b}V+\frac{c}{2nb}(V)^2-\frac{na}{(m+n)b}V^{(m+n)/n}}\mathrm{d}V$$
$$-\frac{\sqrt{2}n}{m+n}\int_{V_-}^{V^+}\frac{V^{(m+n)/n}}{\sqrt{E+\dfrac{C_1}{b}V+\dfrac{c}{2nb}(V)^2-\dfrac{na}{(m+n)b}V^{(m+n)/n}}}\mathrm{d}V.$$

3. Conditions for existence of compacton solutions

In this section, we will present the conditions that derives a compacton solution $U_c(\xi)=U(\xi)H(L-|\xi|)$ from a traveling wave solutions $U(\xi)$, where $H(L-|\xi|)$ is the step function. Then, we have the following results.

Theorem 1 Suppose there exists a compacton profile $u=U_c(\xi)=U(\xi)H(L-|\xi|)$ satisfying

$$U(\xi)\sim U_0(L-|\xi|)^k=U_0H(L\pm\xi)^k,$$

as $\xi\sim\pm L$, where $k\neq 0$, one has that if $U_c(\xi)=U(\xi)H(L-|\xi|)$ is a compacton solution of Eq. (2), m,n,k, and U_0 obey at least one of the following expressions

(Rt1) $k>\dfrac{2}{n}$, with $U(\xi)\in C^2(\mathbb{R}$,

(Rt2) $k>\max\left(\dfrac{3}{n},\dfrac{1}{m}\right)$, with $U(\xi)\in C^3(\mathbb{R})$,

(Rt3) $k=\dfrac{2}{n}$, $mk>1$, with $U(\xi)\in C^3(\mathbb{R})$,

(Rt4) $k=\dfrac{2}{n-m}$, $n>3m$, with $U(\xi)\in C^3(\mathbb{R})$,

$$U_0=\left[-\frac{am}{bn(nk-1)(nk-2)}\right]^{1/(n-m)}$$

for $m,n>0,m\neq n$.

Proof From the Definition 1, suppose $u=U_c(\xi)=U(\xi)H(L-|\xi|)$, we obtain

$$\frac{-c}{n}(U_c)_\xi^n+a(U_c^m)_\xi+b(U_c^n)_{\xi\xi\xi}=\left[\frac{-c}{n}(U)_\xi^n+a(U^m)_\xi+b(U^n)_{\xi\xi\xi}\right]H+\left[\frac{-c}{n}(U)^n+a(U^m)+b(U^n)_{\xi\xi}\right]\delta$$
$$+\left[b(U^n)_\xi\delta\right]_\xi+\left[b(U^n)\delta\right]_{\xi\xi}. \tag{17}$$

According to Lemma 1, let $U(\xi)\sim U_0(L-|\xi|)^k=U_0H(L\pm\xi)^k$, as $\xi\sim\pm L$, equation (17) is equivalent to the following equation

$$\frac{-c}{n}(U_c)_\xi^n+a(U_c^m)_\xi+b(U_c^n)_{\xi\xi\xi}\sim A_3H+A_2\delta+(A_1\delta)_\xi+(A_0\delta)_{\xi\xi}, \tag{18}$$

where

$$A_0=bU^n\sim bU_0^n(L\pm\xi)^{nk},\qquad A_1=b(U^n)_\xi\sim bnkU_0^n(L\pm\xi)^{nk-1},$$

$$A_2 = \frac{-c}{n}U^n + aU^m + b(U^n)_{\xi\xi} \sim \frac{-c}{n}U_0^n(L\pm\xi)^{nk} + aU_0^m(L\pm\xi)^{mk} + bnk(nk-1)U_0^n(L\pm\xi)^{nk-2},$$
$$A_3 = \frac{-c}{n}(U)_\xi^n + a(U^m)_\xi + b(U^n)_{\xi\xi\xi} \sim amkU_0^m(L\pm\xi)^{mk-1} - ckU_0^n(L\pm\xi)^{nk-1} + bnk(nk-1)(nk-2)U_0^n(L\pm\xi)^{nk-3}.$$

Near $\xi = \pm L$, for $U(\xi) \in C^2(\mathbb{R})$, the weak compactons $U(\xi)$ is determined by the minimum exponents of terms from A_0, A_1, A_2, and A_3. In the case $A_0 = 0, A_1 = 0$, this is required that $U_0^n(L\pm\xi)^{nk}$ and $(U^n)_\xi \sim nkU_0^n(L\pm\xi)^{nk-1}$ go to 0, which implies $kn > 1$. Similarly, when $A_2 = 0$, which is $kn > 2$.

Thus, for $U(\xi) \in C^2(\mathbb{R})$, the existence of a weak solution $U(\xi)$ requires $kn > 2$. In the case $A_3 = 0$, $U(\xi) \in C^3(\mathbb{R})$, which implies

$$-ckU_0^n(L\pm\xi)^{nk-1}, \quad amkU_0^m(L\pm\xi)^{mk-1},$$
$$bnk(nk-1)(nk-2)U_0^n(L\pm\xi)^{nk-3}$$

go to zero. Obviously, if $nk(nk-1)(nk-2)U_0^n(L\pm\xi)^{nk-3}$ goes to 0, $amkU_0^m(L\pm\xi)^{mk-1}$ also goes to 0. Therefore, the second and third terms go to 0 that presents $k > \max\{3/n, 1/m\}$. Conversely, when $nk-3 \le 0$ that requires $k = 2/n$ and $mk > 1$ or $k = 2/(n-m)$ which is equivalent to $n \ge 3m$, and $U_0(\xi) = (-am/bn(nk-1)(nk-2))^{1/(n-m)}$. This completes the proof.

4. Weak compactons

In this section, we will give the line compactons in the special case $m = n$ and the general case $m \ne n$. Firstly, when $m = n$, equation (6) is

$$\left(a - \frac{c}{n}\right)V + bV'' = C_1, \tag{19}$$

which leads to the following solution

$$V(\xi) = -\frac{C_1}{a - c/n} + C_2\cos\left(\sqrt{a - c/n}\xi + C_3\right), \tag{20}$$

where C_2 and C_3 are arbitrary constants. According to Theorem 1 that requires $V(\pm L) = 0$, we have

$$0 = -\frac{C_1}{a - c/n} + C_2\cos\left(\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}\xi \mp C_3\right), \tag{21}$$

then

$$C_1 = \left(a - \frac{c}{n}\right)C_2\cos\left(\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}L\right)\cos C_3. \tag{22}$$

Hence, equation (20) becomes

$$V(\xi) = C_2\cos\left(\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}L\right)\cos C_3 + C_2\cos\left(\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}\xi + C_3\right). \tag{23}$$

Note that C_2 is the amplitude factor, Therefore, $C_2 \ne 0$ and $V(\xi)$ is determined by parameters C_4 and L. From the conditions that derives a compacton solution in Theorem 1, the expression (24) is a compacton solution that requires $k = 2/n$ and $V'(\pm L) = 0$, which is

$$\begin{cases} \sin\left(\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}L\right)\cos C_3 = 0, \\ \cos\left(\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}L\right)\sin C_3 = 0. \end{cases} \tag{24}$$

Through direct calculations, we have

$$C_3 = 0, \quad L = \pm\frac{\pi}{\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}}, \tag{25}$$

or

$$C_3 = \pm\frac{\pi}{2}, \quad L = \pm\frac{\pi}{2\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}}. \tag{26}$$

Substituting Eqs. (25) and (26) into the expression (24), we have the following weak-compactons.

Theorem 2 When $m = n$, the weak-compactons are given by

$$\begin{cases} u = -C_2 + C_2\cos\left(\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}\xi\right)^{1/n}, \\ L = \pm\frac{\pi}{\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}}, \end{cases} \tag{27}$$

$$\begin{cases} u = \mp C_2\sin\left(\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}\xi\right)^{1/n}, \\ L = \pm\frac{\pi}{2\sqrt{\frac{1}{b}\left(a - \frac{c}{n}\right)}}, \end{cases} \tag{28}$$

where $\xi = x - ct$ and C_2 is a positive constant.

Since Theorem 1 requires that $V(\pm L) = 0$, $V(\xi)$ can be given in the more convenient form

$$\int_0^V \frac{\mathrm{d}V}{\sqrt{2\left(E + \frac{C_1}{b}V + \frac{c}{2nb}(V)^2 - \frac{na}{(m+n)b}V^{(m+n)/n}\right)}} = L \pm \xi \tag{29}$$

with $L=C_3$. It is easy to note that the asymptotic behavior of $V(\xi)$ is determined by the minimum exponential term of Eq. (31) in E, C_1/bV, $c/2nb(V)^2$, $na/(m+n)bV^{(m+n)/n}$. According to Theorem 1, combining whether the coefficients of the exponential term to Eq. (31) are zero, we obtain the following situations:

In the case $E\neq 0$, the asymptotic behavior of $V(\xi)$ is determined by E, then we have $V(\xi)\sim\sqrt{2E}(L\pm\xi)$ as $\xi\sim\pm L$. This implies $kn=1$, according to Theorem 1, there is no compacton solution.

In the case $E=0$, $C_1/b\neq 0$, it is easy to see that the asymptotic behavior of $V(\xi)$ is determined by the term C_1/bV and $V(\xi)\sim\sqrt{C_1/2b}(L\pm\xi)$ as $\xi\sim\pm L$. This requires $C_1/2b>0$, $kn=2$, and we obtain that the compactons exist according to conditions Rt3 of Theorem 1. In addition, if there exists a smallest positive root $V_{\max}$ ($0\leq V\leq V_{\max}$) that satisfies $E+[C_1/b]V+[c/2nb](V)^2-[na/(m+n)b]V^{(m+n)/n}=0$, then the quadrature (31) can obtain the traveling wave solution that satisfies $V(\pm L)=0$ for some $L>0$. Through calculations, when $m>n$, the existence of such a root is necessary for $C_1/b>0$, $na/(m+n)b>0$, or $c/2nb<0$, $na/(m+n)b<0$, $0<C_1/b<na/(m+n)bV_*^{m/n}-[c/2nb]V_*$ with $V_*=(|c(m+n)/2amn|)^{n/(m-n)}$. When $m<n$, which requires $C_1/b>0$, $c/2nb<0$, or $c/2nb>0$, $na/(m+n)b>0$, $0<[C_1/b<na/(m+n)b]\widetilde{V}^{m/n}-[c/2nb]\widetilde{V}$ with $\widetilde{V}=(|2amn/c(m+n)|)^{n/(n-m)}$.

In the case $E=0$, $C_1/b=0$, $c/2nb\neq 0$, $na/(m+n)b\neq 0$, we can obtain that when $m>n$, the smallest exponent is given by $[c/2nb](V)^2$, and $V(\xi)$ is equivalent to the exponential type, the compacton solution does not exist. Similarly, when $m<n$, the asymptotic behavior has $V^{(n-m)/2n}\sim(n-m)/2n\sqrt{-2na/(m+n)b}(L\pm\xi)$ as $\xi\sim\pm L$. $kn=2n/(n-m)$, which requires $n>3m$. This is the condition Rt4 of Theorem 1.

In the case $E=0, C_1/b=0, c/2nb=0, na/(m+n)b\neq 0$, or $c/2nb\neq 0$, $na/(m+n)b=0$, the square root contains only a single term, there is no positive root. Hence, there is no compacton solution.

Therefore, we have the compacton solutions in the general case $m\neq n$, as follows. In the case $m\neq n$, we can deduce that the compacton solutions from the quadrature (8), as follows.

Theorem 3 In the general case $m\neq n$ with $m,n>0$, the weak-compactons are given by the quadratures,

$$\int_0^{u^n}\frac{nU^{n-1}\,\mathrm{d}U}{\sqrt{2\left(\dfrac{c}{2nb}U^{2n}-\dfrac{na}{(m+n)b}U^{m+n}\right)}}=L\pm\xi,\quad n>3m, \tag{30}$$

$$\int_0^{u^n}\frac{nU^{n-1}\,\mathrm{d}U}{\sqrt{2\left(\dfrac{C_1}{b}U^{n}+\dfrac{c}{2nb}U^{2n}-\dfrac{na}{(m+n)b}U^{m+n}\right)}}=L\pm\xi, \tag{31}$$

$$m>n:\ \frac{C_1}{b}>0,\ \ \frac{na}{(m+n)b}>0;$$

$$\frac{c}{2nb}<0,\ \ \frac{na}{(m+n)b}<0, 0<\frac{C_1}{b}<\frac{na}{(m+n)b}V_*^{m/n}-\frac{c}{2nb}V_*;$$

$$m<n:\ \frac{C_1}{b}>0,\ \ \frac{c}{2nb}<0;$$

$$\frac{c}{2nb}>0,\ \ \frac{na}{(m+n)b}>0, 0<\frac{C_1}{b}<\frac{na}{(m+n)b}\widetilde{V}^{\frac{m}{n}}-\frac{c}{2nb}\widetilde{V};$$

with

$$V_*=(|c(m+n)/2amn|)^{n/(m-n)},$$
$$\widetilde{V}=(|2amn/c(m+n)|)^{n/(n-m)}.$$

5. General formulae of solutions

For the convenience of calculation and without loss of generality, in this section, we choose $a=b=1$ to study the compacton solutions and solitary pattern solutions of Eq. (2), we first assume that equation (2) has compactons in the following forms.

Type A

$$u=\begin{cases}A\cos^B(\xi), & |\xi|\leq\pi/2,\\ 0, & \text{others.}\end{cases} \tag{32}$$

Type B

$$u=\begin{cases}A\sin^B(\xi), & 0\leq\xi\leq\pi,\\ 0, & \text{others,}\end{cases} \tag{33}$$

where A, B are constants to be determined.

Substituting Eq. (32) into Eq. (5) gives

$$A^nBn\left[b(Bn)^2+\frac{c}{n}\right]\cos^{Bn-1}(\xi)\sin(\xi)-aA^mBm\cos^{Bm-1}(\xi)\sin(\xi)-bA^nBn(Bn-1)(Bn-2)\cos^{Bn-3}(\xi)\sin(\xi)=0, \tag{34}$$

in which we obtain

$$c=-\frac{4bn^3}{(n-m)^2},\quad A=\left[-\frac{a(n-m)^2}{2bn(n+m)}\right]^{1/(n-m)},\quad B=\frac{2}{n-m}.$$

Thus, when $|x+[4bn^3/(n-m)^2]t|\leq\pi/2$, the compacton solution (as shown in Fig. 1(a)) can be expressed as

$$u(x,t)=\left[-\frac{a(n-m)^2}{2bn(n+m)}\right]^{1/(n-m)}\cos^{2/(n-m)}\left(x+\frac{4bn^3}{(n-m)^2}t\right). \tag{35}$$

Similarly, substituting Eq. (33) into Eq. (5) yields sine compacton solution (as shown in Fig. 1(b)) with $|x+[4bn^3/(n-m)^2]t| \leq \pi/2$, as follows:

$$u(x,t) = \left[-\frac{a(n-m)^2}{2bn(n+m)}\right]^{1/(n-m)} \sin^{2/(n-m)}\left(x+\frac{4bn^3}{(n-m)^2}t\right). \tag{36}$$

Similar to the general formulae of compactons, suppose that equation (2) has solitary pattern solutions in the following two forms.

$$\text{Type C:} \qquad u = A\cosh^B(\xi), \tag{37}$$

$$\text{Type D:} \qquad u = A\sinh^B(\xi), \tag{38}$$

where A, B are constants to be determined.

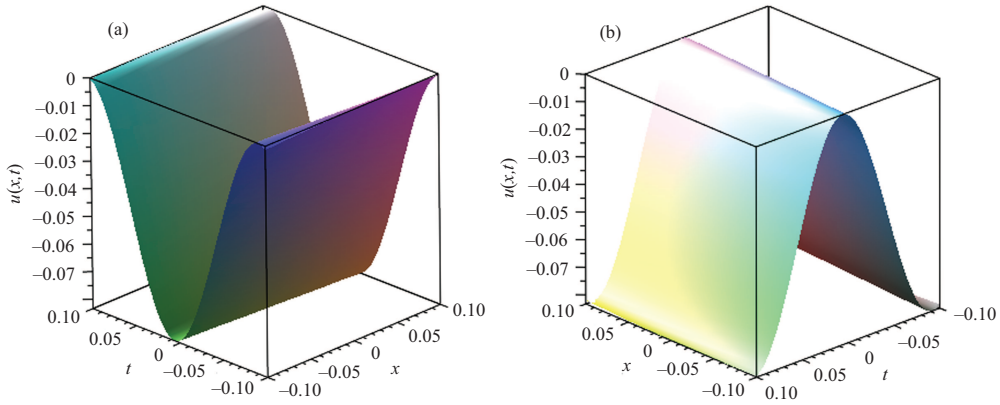

Fig. 1. (a) Compacton solution (4) and (b) compacton solution (5), with parameters $a = b = 1$, $m = 1$, and $n = 2$.

By direct calculations, we obtain the corresponding general formulae of solitary pattern solutions of Eq. (2), as shown in Fig. 2.

$$u(x,t) = \left[\frac{a(n-m)^2}{2bn(n+m)}\right]^{1/(n-m)} \cosh^{2/(n-m)}\left(x-\frac{4bn^3}{(n-m)^2}t\right), \tag{39}$$

$$u(x,t) = \left[-\frac{a(n-m)^2}{2bn(n+m)}\right]^{1/(n-m)} \sinh^{2/(n-m)}\left(x-\frac{4bn^3}{(n-m)^2}t\right). \tag{40}$$

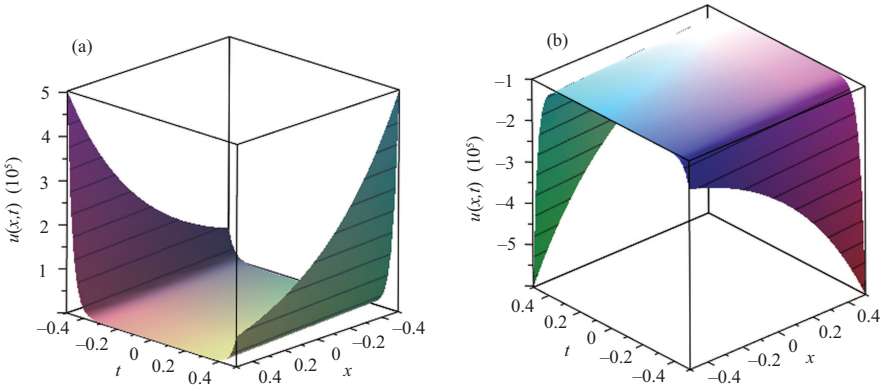

Fig. 2. (a) Solitary pattern solutions (8) and (b) solitary pattern solutions (9), with parameters $a = b = 1$, $m = 1$, and $n = 2$.

6. Conclusion

In this paper, we studied the nonlinearly dispersive mKdV (mK(m,n)) equations. The conservation laws are obtained by using the rank analysis method and the existed conditions of weak compactons based on the power (the nonlinearity powers) in this equation via cutoff conditions, which is different from previous studies on exact compacton solutions. Furthermore, we presented the details of the transformation relationship between conservation laws and the traveling wave ODE. Actually, for more generalized case, for example, the higher-

dimensional mK(m,n,a,b) equation

$$u^{m-1}u_t+\alpha(u^n)_x+\beta(u^a(u^b)_{xx})_x+\sum_{i=2}^{N}c_i(u^{a_i}(u^{b_i})_{x_ix_i})_{x_i},$$

we think it also has these solutions, and we will study it in future.

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