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Public goods game with umpire supervision: Based on complex networks and birandom geometric graph

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Using complex networks and the birandom geometric graph (BRGG) model, we first propose a construction algorithm for a two-layer network with umpire supervision (i.e., US-BRGBA). Subsequently, we introduce a reward-punishment mechanism imposed by umpires on players while considering both fair and corrupt umpires, thereby presenting the US-BRGBA game model with umpire supervision. Further, the feasibility and effectiveness of the proposed US-BRGBA model based on Monte Carlo (MC) simulations are verified, as well as the influence on players' cooperative behaviors arising from umpires' spatial distribution, the fraction of fair umpires, the reward value, the fine value, and the bribe value. Abundant simulations demonstrate that the US-BRGBA model promotes cooperation significantly. Additionally, the spatial distribution patterns of umpires do not consistently yield uniform effects on promoting cooperation, as their influences are contingent upon the fraction of fair umpires and the magnitude of the synergy factor. In resource-abundant regions, corruption does not block cooperative behaviors. Meanwhile, the introduction of umpires consistently facilitates cooperation, even in scenarios where all umpires exhibit corrupt behavior. Further, compared with the reward value and fine value, the cooperative behavior in the US-BRGBA game model is more sensitive to the bribe value.

Keywords: public goods game, two-layer network with umpire supervision, US-BRGBA game model, birandom geometric graph, complex networks

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1. Introduction

As a classic paradigm for discussing multi-person social dilemmas, the public goods game (PGG) model[1] provides an important theoretical perspective for exploring the generation, sustainability, and evolution of human cooperative behaviors. In a classic PGG model, all the players are required to decide whether to donate to the common pool simultaneously. The players who choose to donate are defined as cooperators, while those who do not donate are termed defectors. By observing players' decision-making behaviors in the PGG, we can analyze how individuals balance their self-interest and collective interests under different conditions, thereby revealing the influencing factors of cooperative behaviors. Meanwhile, the PGG can reveal the intrinsic logic of collective cooperation, providing a theoretical framework for explaining cooperation dilemmas in real-world scenarios, and offering actionable solutions to practical issues such as public resource management, public service provision, and public policy design.

In the PGG, evolutionary mechanisms play a crucial role. In 2006, Nowak[2] summarized and proposed five mechanisms for promoting and sustaining cooperative behaviors in his research on the evolution of cooperation. They are direct reciprocity,[3,4] indirect reciprocity,[5,6] group selection,[7,8] kin selection,[9,10] and network reciprocity.[11,12] Among these five mechanisms, research on network reciprocity has attracted significant attention from many scholars. Subsequently, numerous scholars proposed a series of derivative mechanisms to promote cooperative behaviors in the PGG, including punishing defectors,[13–15] rewarding cooperators,[16–18] reputation mechanisms,[5] environmental feedback,[19] optional participation,[20] and structured populations.[21] Among the mechanisms mentioned above, punishing defectors and rewarding cooperators are among the most intensively researched and scientifically intriguing mechanisms. It has been proved that punishing defectors and rewarding cooperators can not only improve cooperation levels significantly but also be widely applied in real-world societies.[13–18] For example, government departments, such as tax bureaus, generally act as third parties to punish tax evaders.

Currently, for evolutionary games, there are relatively few studies that consider the effects of reward-punishment mechanisms and third-party institutions simultaneously. Moreover, for the above results, even when third-party institutions are considered, it is assumed that they must be fair and just.

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However, in real life, corruption in third-party institutions exists across various fields. In recent years, some scholars have focused on corrupt third-party institutions in evolutionary games. Huang *et al.*[22] explored the influence of punishment on cooperation evolution in scenarios with or without corruption. Shi *et al.*[23] proposed a corrupt umpire intervention model to investigate the effects of corruption on cooperative behavior, revealing that the fraction and spatial distribution of fair umpires have a significant influence on cooperation.

Additionally, research on PGG based on complex network theory[24–26] has been one of the intense research topics in recent years. Lee *et al.*[27] proposed a tax-based peer punishment model on a square lattice. If the tax burden is negligible, cyclic strategy dominance may emerge. Bahbouhi *et al.*[28] provided a comparative model of individual decision rule and collective decision rule under reward-punishment mechanism on a scale-free network, aiming to investigate their effects on contribution evolution. Su *et al.*[29] investigated the impact of resource interaction in interdependent networks on public cooperation. Ding *et al.*[30] integrated an insurance mechanism into spatial PGG, which showed that insured defectors can foster cooperation by creating self-organized cyclic dynamics.

All the above results represent public goods and players with the same type of network nodes. However, in reality, they have an independent relationship. Their connections should revolve around spatial location. For this situation, Li *et al.*[31] proposed the birandom geometric graph (BRGG) model to study the PGG. In the model, the public good nodes and the player nodes are set separately. Each public good node has an influence zone. The player nodes within the influence zone are interconnected and participate in the PGG. Wang *et al.*[32] investigated the evolution of regional PGG. In comparison, the BRGG model proposed in Ref. [31] can reflect the geographical relationship between individuals and public resources excellently. Moreover, the interactions among the players are uniform. However, in reality, the interactions between two players are heterogeneous and have their own characteristics. Therefore, based on the BRGG model, this paper will reconfigure the connection patterns among the players.

Based on the above, in this paper, integrating complex network theory and the BRGG model, we will first propose a two-layer network US-BRGBA with umpire supervision. Then, by introducing fair and corrupt umpires and a reward-punishment mechanism, the US-BRGBA game model is proposed. Finally, the feasibility and effectiveness of the proposed US-BRGBA PGG model are verified using MC simulations. Additionally, the influence of umpires' spatial distribution, the fraction of fair umpires, the reward value, the fine value, and the bribe value on players' cooperative behaviors is explored.

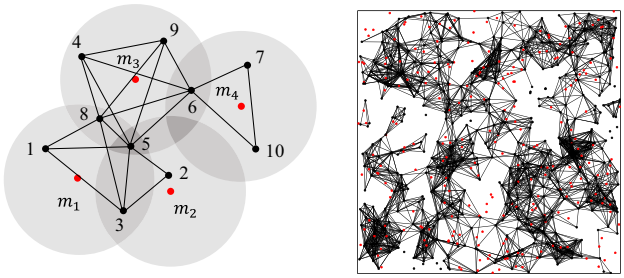
2. Model

2.1. Construction of two-layer network with umpire supervision

In this section, we first briefly introduce the construction method of BRGG. Then, based on the BRGG model and the BA scale-free network model,[26] we propose a construction algorithm for a two-layer network with umpire supervision, i.e., US-BRGBA.

2.1.1. Birandom geometric graph

In Ref. [31], the authors assumed that there are two types of nodes in the BRGG model, which are M public good nodes and N player nodes, and supposed that these M public good nodes are randomly assigned in the rectangle $[0,1]\times[0,1]$. In the BRGG model, each public good node influences the player nodes within a circular zone with radius d. Meanwhile, in the BRGG, a player node randomly emerges per unit time, and this player node is fully connected to all other player nodes within the same zone (i.e., the influence zone of the same public good node). The process stops until N player nodes are present in the network (see Fig. 1). Figure 1 shows schematic diagrams of the BRGG model with scales of $(M,N)=(4,10)$ and $(M,N)=(250,500)$, respectively.



(a) $(M,N)=(4,10)$ (b) $(M,N)=(250,500)$

Fig. 1. Schematic diagram of the BRGG model. (a) $m_1\sim m_4$ are four public good nodes: $1\sim 10$ are ten player nodes. (b) $d=0.07$.

2.1.2. Two-layer network with umpire supervision

It is evident that within the influence zone of each public good node in Fig. 1, players form a fully connected network. Real-world social networks are typically heterogeneous rather than homogeneous, such as the BA scale-free network structure. Considering the irrationality of player connections in the BRGG model above and the impact of corruption on social dilemmas, this section will connect players following the construction method of the BA scale-free network and propose a two-layer network with umpire supervision (US-BRGBA), as shown in Fig. 2.

In the two-layer network US-BRGBA, the lower layer is the game layer, which contains M public goods and $N=qM$ players, where $q>0$ is a constant. The upper layer, by contrast, is the umpire layer, which includes M umpires. These M umpires are not connected to each other, but they correspond

one-to-one with the M public goods in the game layer. According to the BRGG model in Subsection 2.1, the M public goods are generated. Meanwhile, a player is generated per unit time. Within the influence zone of each public good, the first three players that appear are connected in a specific manner, and subsequent players generated are connected to two other players in the same zone following the BA scale-free network model. It should be noted that according to the generation rules of the BA model, if the initial network of the BA model has isolated or disconnected nodes, the preferential attachment of new nodes cannot be carried out. Therefore, to ensure the normal generation of the BA network, it is crucial to set the initial network as connected. Therefore, the generation steps of the two-layer network US-BRGBA are as follows.

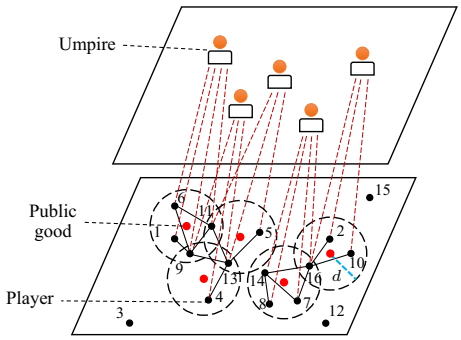


Fig. 2. Schematic diagram of a two-layer network with umpire supervision (US-BRGBA).

Step 1 Randomly generate M public goods within the rectangle $[0,1]\times[0,1]$.

Step 2 Randomly generate one player within the rectangle $[0,1]\times[0,1]$ per unit time, number them in the order of appearance, and determine which public good nodes' influence zones the player belongs to. If N players have been generated, proceed to Step 5.

Step 3 When three players have appeared within the influence zone of a public good node k $(k=1,2,\ldots,M)$, connect the last emerged player with the previous two players respectively. At this point, the first two players appearing within the influence zone of each public good are not connected.

Step 4 If more than three players have appeared within the influence zone of a public good node k, the last emerged player connects with two other players in the same zone according to the preferential attachment feature of the BA network, and then returns to Step 2. Conversely, directly return to Step 2.

Step 5 The game layer with N players has been constructed.

Step 6 Generate M umpires in the rectangle $[0,1]\times[0,1]$ of the umpire layer according to a specific spatial distribution pattern.

Step 7 Connect these M umpires with the players within the influence zone of the corresponding public goods in the game layer to construct the inter-coupling between the game layer and the umpire layer. Thus, the construction of the two-layer network US-BRGBA is completed.

The schematic diagram of the US-BRGBA construction algorithm is shown in Fig. 3.

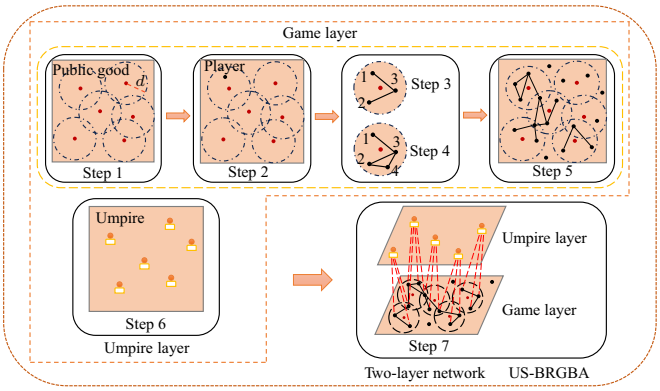


Fig. 3. The schematic diagram for constructing the US-BRGBA network.

In the gaming layer, let the matrix $A=(a_{st})_{N\times N}$, where a_{st} indicates whether there is a connection between the player s and the player t. If there is a connection between these two players, then $a_{st}=1$; conversely, $a_{st}=0$, where $s,t=1,2,\ldots,N$.

Remark 1 In this paper, isolated players are not considered in the two-layer network US-BRGBA.

2.2. US-BRGBA game model

In the two-layer network US-BRGBA, we assume that players have two states: cooperation and defection. Cooperators invest a certain amount of funds into the common pool, while defectors do not invest. The funds collected in the common pool are multiplied by a synergy factor r and then equally distributed to each player. It should be noted that only players within the same influence zone and being neighbors can jointly participate in the PGG, just as only residents within the same community can collectively enjoy the public resources of that community. Therefore, players within the influence zone of each public good engage in the PGG with their neighbors. Furthermore, we assume that umpires have two states: fairness and corruption. Fair umpires reward cooperators and punish defectors. Corrupt umpires only accept bribes from defectors and do not reward cooperators. Each umpire supervises the players within the influence zone of a public good to engage in games. The role of umpires is analogous to that of patrol officers with fixed patrol zones. Each patrol officer supervises the citizens in a fixed zone and rewards or punishes them according to their behaviors.

In each round of the game, a player needs to engage in as many games as the number of public goods influence zones they are in. Each game is participated in by the player together with all the neighbors within the same public goods

influence zone. Each player accumulates their own payoff after each round of the game. Umpires supervise players in the umpire layer. After each round of the game concludes and payoffs are accumulated, each player compares their payoff with those of their neighbors and then updates their strategy by imitating their neighbors in a synchronous manner. Specifically, each player i $(i = 1,2,\ldots,N)$ randomly selects one neighbor j from his/her neighbors to compare payoffs, and updates his/her strategy according to the Fermi function. To streamline the model, we posit that an umpire's spatial location and behavioral strategy remain fixed upon initialization.

Initially, each player i becomes a cooperator or a defector with equal probability. If the player i is a cooperator, the investment is $x_i\,(t) = 1$. Otherwise, the investment is $x_i\,(t) = 0$. When an umpire is fair, then let $y_k = 1$ $(k = 1,2,\ldots,M)$. Otherwise, let $y_k = 0$ $(k = 1,2,\ldots,M)$. Let G_m denote the set of players within the influence zone of the m-th public goods. Whether the player i is in G_m can be described by the following indicator function:

$$\delta_i\,(G_m) = \begin{cases} 0, & i \notin G_m, \\ 1, & i \in G_m, \end{cases} \quad i = 1,2,\ldots,N;\ m = 1,2,\ldots,M. \tag{1}$$

Based on the above analysis, the payoff of player i in the game within the group G_m is

$$P_i\,(G_m)\,(t) = \begin{cases} \dfrac{r\cdot\sum\limits_{l=1}^{N}\delta_l\,(G_m)\,a_{il}x_l\,(t)}{\sum\limits_{l=1}^{N}\delta_l\,(G_m)\,a_{il}} - x_i\,(t) - \gamma(1-x_i\,(t))\,, & y_k = 0, \\ \dfrac{r\cdot\sum\limits_{l=1}^{N}\delta_l\,(G_m)\,a_{il}x_l\,(t)}{\sum\limits_{l=1}^{N}\delta_l\,(G_m)\,a_{il}} - x_i\,(t) + \alpha x_i\,(t) - \beta\,(1-x_i\,(t))\,, & y_k = 1, \end{cases} \tag{2}$$

where r is the synergy factor in the PGG, $\alpha > 0$ represents the reward that cooperators receive when encountering fair umpires, $\beta > 0$ denotes the fine that fair umpires impose on defectors, and γ signifies the bribe that corrupt umpires collect from defectors.

Generally, the sum of the reward that fair umpires give to cooperators and the fine they impose on defectors should be greater than the cost of cooperation, while the bribe value should be kept below this cost. Therefore, in this paper, we assume that $(\alpha+\beta) \in (1,\infty)$ and $\gamma \in (0,1)$. The total payoff of the player i after one round of the game is

$$P_i\,(t) = \sum_{m=1}^{M} P_i\,(G_m)\,(t). \tag{3}$$

Given the total payoff from Eq. (3), each player i randomly selects a neighbor j and imitates his/her strategy using the following Fermi function:

$$W_{ij}\,(t) = \frac{1}{1+\exp\left[\dfrac{P_i\,(t)-P_j\,(t)}{k}\right]}. \tag{4}$$

The significance of the Fermi function lies in that when a game player i updates his/her strategy, if the selected neighbor player j has a higher payoff, then the strategy of the player j will be adopted by the player i with a higher probability. If the payoff of the player i is lower than that of the player j, the player i will still learn j with a lower probability (make an irrational choice). In Eq. (4), the parameter k denotes the amplitude of noise, which allows the players to irrationally learn their neighbors' strategies to a certain probability. In this paper, we set $k = 0.1$, which is a widely adopted value in related researches.

3. Results

To explore the cooperative behavior of players within the influence zone of public goods on the two-layer network US-BRGBA with umpire supervision, this section conducts MC simulations and evaluates the cooperation level after the PGG stabilizes in the US-BRGBA game model. In the numerical simulations of this section, we always use ρ_{c} to denote the cooperation level and ρ_{f} to represent the fraction of fair umpires in the umpire layer. Generally, the ρ_{c} is characterized by the fraction of cooperators, defined as the ratio of cooperators after the system reaches stable states. To ensure the stability of the game, each simulation runs for 10000 steps, and the value of ρ_{c} is obtained by averaging the results from the last 1000 steps following stabilization. Similar to the BRGG game model in Ref. [31], in this section, we still let $M = 250$, $q = 2$, and $d = 0.07$.

Specifically, in this section, we will first analyze and explore the feasibility and effectiveness of the two-layer network with umpire supervision and its game model proposed in Sections 2 and 3. Then, we will investigate the impacts of the spatial distribution of umpires, the fraction of fair umpires ρ_{f}, the reward value α, the fine value β, and the bribe value γ on players' cooperative behavior.

3.1. The feasibility and effectiveness of the US-BRGBA model

To explore the feasibility and effectiveness of the US-BRGBA model, this section will compare it with the BRGG game model. This section assumes that umpires are spatially distributed at random in the umpire layer and correspond one-to-one with public goods. Figure 4 shows the evolution process of player states in the BRGG model and the US-BRGBA model proposed in this paper with the same conditions.

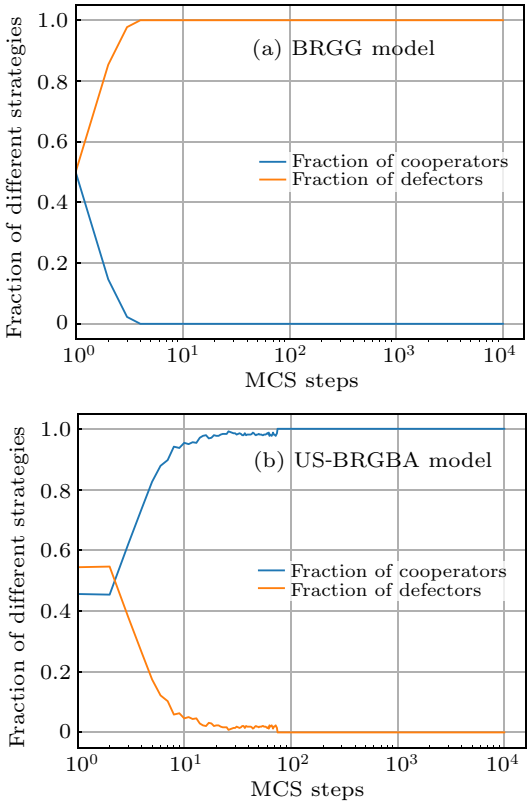


Fig. 4. The evolution process of player states. The parameters are $\rho_f = 0.5$, $\alpha = 0.7$, $\beta = 0.6$, $\gamma = 0.3$, and $r = 1.5$.

As shown in Fig. 4(a), from the beginning, the ρ_c of the BRGG model drops sharply and quickly falls to 0. Conversely, the fraction of defectors rises to 1 rapidly. However, Fig. 4(b) tells us that after introducing umpire supervision in the US-BRGBA model, the situation is completely opposite to that of the BRGG model. The fraction of cooperators rapidly rises and dominates the entire game layer after a short decline. Therefore, Fig. 4(b) demonstrates that the US-BRGBA game model with umpire supervision mechanism proposed in this paper is feasible and effective. The reasons for the phenomena in Fig. 4 are as follows: In the BRGG model, when the synergy factor r is small, the payoffs of cooperators are much lower than those of defectors. The cooperators are incapable of resisting the defectors. Consequently, even when cooperators form clusters, they cannot fend off the invasion of defectors. In the US-BRGBA model, the introduction of umpire supervision, whether fair or corrupt, reduces the relative payoff gap between the defectors and the cooperators. The players tend to become the cooperators, so the cooperators can completely dominate the game layer ultimately.

3.2. The influence of the umpires on the US-BRGBA game

3.2.1. The spatial distribution of umpires

In the previous section, we found that compared with the BRGG model, the introduction of umpire supervision fundamentally alters the cooperation level in the US-BRGBA model, demonstrating the critical importance of incorporating umpire supervision. In this section, we consistently assume that the fraction of fair umpires $\rho_f = 0.2$ or $\rho_f = 0.4$. In the following, from the perspective of umpires' spatial arrangement, we will continue to explore the evolutionary patterns of players' cooperative behaviors in the US-BRGBA model under two modes of umpires' spatial distribution, including random distribution and centralized distribution.

In Section 2, we mentioned that both umpires and public goods are distributed in $[0,1] \times [0,1]$ and maintain a one-to-one correspondence. For the random distribution of umpires, this section first randomly generates M public goods in the game layer, and then generates M umpires in the umpire layer to establish a one-to-one correspondence with the public goods. For the centralized distribution of umpires, when $\rho_f = 0.2$, this section first randomly generates $0.2M$ public goods within the rectangular zone $[0,1] \times [0,1/5]$, and then generates $0.2M$ fair umpires at the same rectangular positions in the umpire layer. Meanwhile, within the rectangular zone $[0,1] \times (1/5,1]$, $0.8M$ public goods are randomly generated, and then $0.8M$ corrupt umpires are generated at the same rectangular positions in the umpire layer. Similarly, for the centralized distribution of umpires, when $\rho_f = 0.4$, we first randomly generate $0.4M$ public goods within the rectangular zone $[0,1] \times [0,2/5]$, and then generate $0.4M$ fair umpires at the same rectangular positions in the umpire layer. Meanwhile, within the rectangular zone $[0,1] \times (2/5,1]$, $0.6M$ public goods are randomly generated, and then $0.6M$ corrupt umpires are generated at the same rectangular positions in the umpire layer.

When $\rho_f = 0.2$ and $\rho_f = 0.4$, with two types of umpire spatial distributions mentioned above, Fig. 5 illustrates the trajectories of the cooperation level ρ_c as a function of the synergy factor r when the US-BRGBA game reaches a stable state. In Section 3, we supposed that the reward value α, fine value β, and bribe value γ satisfy $(\alpha + \beta) \in (1, \infty)$ and $\gamma \in (0,1)$. In this section, we let $\alpha = 0.7$, $\beta = 0.6$, and $\gamma = 0.3$, respectively.

In Fig. 5(a), when the fraction of fair umpires $\rho_f = 0.2$ and the synergy factor r is small, the centralized distribution of umpires promotes cooperation more effectively than the random distribution. As r increases, the advantage of centralized distribution diminishes gradually. When $r > 1$, the situation reverses completely. At this point, the random distribution of

umpires plays a more positive role for the emergence of cooperative behaviors. When $r > 2.25$, there is virtually no difference between the random and centralized umpire distributions on cooperative behaviors. In Fig. 5(b), when $\rho_f = 0.4$ and $r < 2.5$, the random distribution of umpires consistently promotes cooperation more effectively than the centralized distribution. When $r > 2.5$, the random and centralized umpire distributions have essentially identical effects.

Therefore, it can be concluded that the promotional effect of umpires' spatial distribution on players' cooperative behaviors is not always the same. Rather, it depends on the fraction of fair umpires ρ_f and the value of the synergy factor r.

To further investigate the phenomena shown in Fig. 5, when all the players are in their stable state, Fig. 6 exhibits heatmaps of players' states and positions in the US-BRGBA game with different ρ_f and r. In this case, the spatial distributions of umpires are the same as in Fig. 5. Additionally, Fig. 6 mainly focuses on the transition phenomenon of the fraction of cooperative behaviors between random and centralized distributions near $r = 1$. Therefore, for $\rho_f = 0.2$ and $\rho_f = 0.4$, Fig. 6 takes $r = 0.8$ and $r = 1.7$.

To better observe the varieties of cooperators and defectors, in the initial stage, Fig. 6 sets the positions of players as identical for random distribution and centralized distribution. In Fig. 6(a), when $(\rho_f, r) = (0.2, 0.8)$, the fraction of fair umpires is low and r is also small. In this scenario, the cooperators are unable to resist the invasion of defectors because their payoffs are significantly smaller than those of defectors. As ρ_f or r increases, cooperators gradually form clusters, leading ρ_c to an increase. Specifically, when ρ_f increases, the fraction ρ_c only increases slightly, approaching a level comparable to that of defectors. Then, when r increases, cooperators significantly outcompete defectors. Further, when both ρ_f and r increase, cooperators gain an absolute advantage in the game layer.

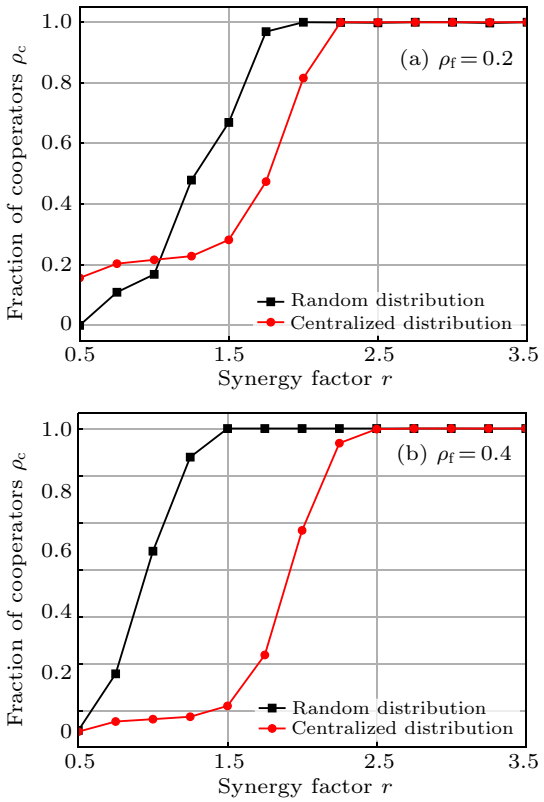


Fig. 5. The trajectories of the cooperation level ρ_c *versus* the synergy factor r in the US-BRGBA game model. The black line represents the random distribution of umpires, and the red line represents the centralized distribution of umpires. The parameters are $\alpha = 0.7$, $\beta = 0.6$, and $\gamma = 0.3$.

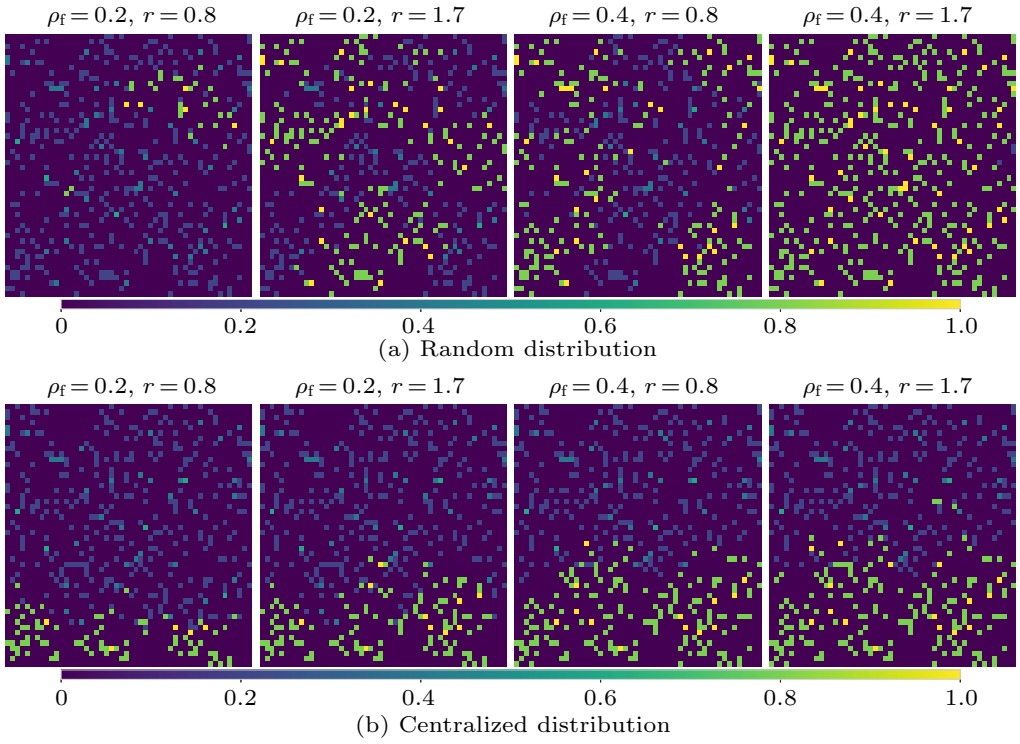


Fig. 6. The heatmaps of players' states and positions in the US-BRGBA game model.

In Fig. 6(b), it can be observed that cooperators form clusters to resist the invasion of defectors and distribute themselves at the positions corresponding to fair umpires. When $(\rho_f, r) = (0.2, 0.8)$, cooperators occupy approximately 20% of the positions in the game layer. When $(\rho_f, r) = (0.4, 0.8)$, the rate rises to approximately 40%. Increasing r makes cooperators expand from their original clusters and coexist with defectors in the game layer.

Comparing Figs. 6(a) and 6(b), it reveals that for smaller ρ_f and r, the centralized distribution enhances cooperation and performs better than the random distribution. However, for bigger ρ_f or r, the random distribution gradually becomes more effective in promoting cooperation than the centralized distribution. This is also consistent with the phenomenon observed in Fig. 5(a).

3.2.2. The fraction of fair umpires

It is evident that the fraction of fair umpires ρ_f plays a crucial role in the US-BRGBA game model. This section will explore the evolutionary patterns of the ρ_c when the ρ_f is altered in the US-BRGBA game model.

In this section, in the initial stage of the numerical simulations, it is assumed that fair umpires and corrupt umpires are randomly distributed in the umpire layer, while the cooperators and the defectors each occupy 50% of the game layer and are randomly distributed. With the increasing ρ_f, Fig. 7 illustrates the evolutionary trajectory of the ρ_c. Similarly, in this section, we also set $\alpha = 0.7$, $\beta = 0.6$, and $\gamma = 0.2$.

Figure 7 shows that regardless of the value r, the ρ_c generally exhibits an upward trend as the fraction ρ_f increases. In Fig. 7, we can also have three distinct findings. (i) When r is small, such as $r = 0.5$, $r = 1.0$, and $r = 1.5$, a low fraction ρ_f can lead the cooperators to be completely eliminated by the defectors. However, arising ρ_f brings the coexistence of cooperators and defectors. Further, increasing ρ_f continuously, the cooperators gradually gain dominance and eventually occupy the entire game layer. (ii) When r is moderate, such as $r = 2.0$, the coexistence of cooperators and defectors can be achieved as long as there is a smaller ρ_f. And, as ρ_f increases, the fraction ρ_c rises rapidly until no defectors remain in the game layer. (iii) When r is big, such as $r = 2.5$, $r = 3.0$, and $r = 3.5$, only the cooperators remain in the end regardless of how small ρ_f is.

Therefore, we come to the conclusion that in the relatively resource-abundant regions (i.e., r is relatively large), the corruption does not inhibit the emergence of cooperative behaviors. In this case, abundant resources can resist the invasion of corruption. Therefore, corruption mitigation in resource-poor regions warrants heightened attention. In resource-poor regions, interference by corrupt umpires may exacerbate wealth inequality. At this moment, the players refuse to cooperate, leading to further economic backwardness in this region.

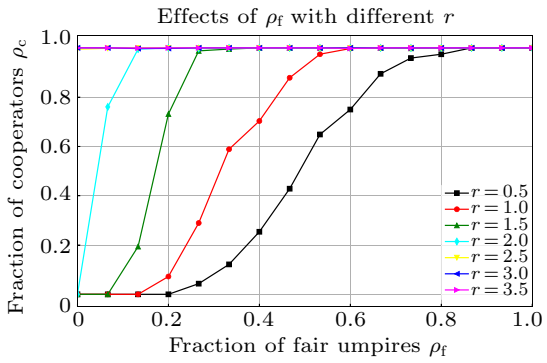


Fig. 7. The trajectory of the cooperation level ρ_c concerning the fraction of fair umpires ρ_f in the US-BRGBA game model. The parameters are $r = 0.5 : 0.5 : 3.5$, $\alpha = 0.7$, $\beta = 0.6$, and $\gamma = 0.2$.

To further discuss the characteristics of the cooperation level ρ_c when the US-BRGBA game model reaches a stable state with $\rho_f = 0$ in Fig. 7, Fig. 8 expresses the trajectory of the cooperation level ρ_c with respect to the synergy factor r. In this scenario, all the umpires in the umpire layer are corrupt.

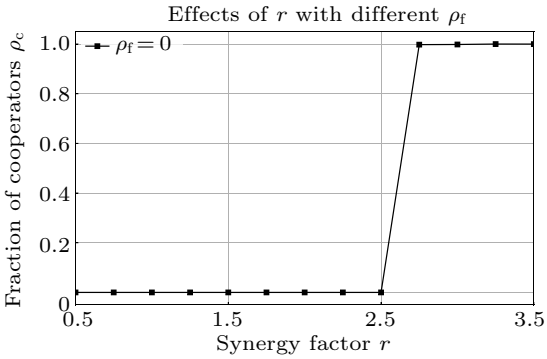


Fig. 8. The trajectory of cooperation level ρ_c with respect to the synergy factor r in the US-BRGBA game model. The parameters are $\rho_f = 0$, $\alpha = 0.7$, $\beta = 0.6$, and $\gamma = 0.2$.

Obviously, when all the umpires in the umpire layer are corrupt (i.e., $\rho_f = 0$), $r < 2.5$ leads to the entire game layer being dominated by defectors, $2.5 < r < 2.75$ allows for the coexistence of cooperators and defectors, and $r > 2.75$ makes the absence of defectors in the game layer. In Ref. [23], the PGG on square lattice appeared cooperative behavior only when $r > 3.64$. In the US-BRGBA model, the critical threshold can be reduced to $r = 2.5$, even in the absence of fair umpires, which demonstrates that compared to the PGG on square lattice, the presence of umpires consistently facilitates cooperation level even under conditions where all the umpires are corrupt.

3.3. Sensitivity analysis of the US-BRGBA game model

In the previous numerical simulations, the reward and punishment measures in the US-BRGBA game model, including reward value α, fine value β, and bribe value γ, are always

kept constant. If the values α, β, and γ are altered, what feature will emerge for the players' cooperative behavior? This section will analyze the sensitivity of the above three parameters in the proposed US-BRGBA game model. In Fig. 7, we know that larger r and ρ_f lead to an excessively high fraction of cooperators in the game layer, while smaller r and ρ_f result in an excessively low fraction of cooperators. Both scenarios are unfavorable for observing the sensitivity of these three parameters. Therefore, in this section, we assume that ρ_f and r are moderate, and set $\rho_f = 0.3$ and $r = 0.8$, respectively.

When the bribe value $\gamma = 0.3$ is fixed, Fig. 9 illustrates the effects of the α and the β on the ρ_c.

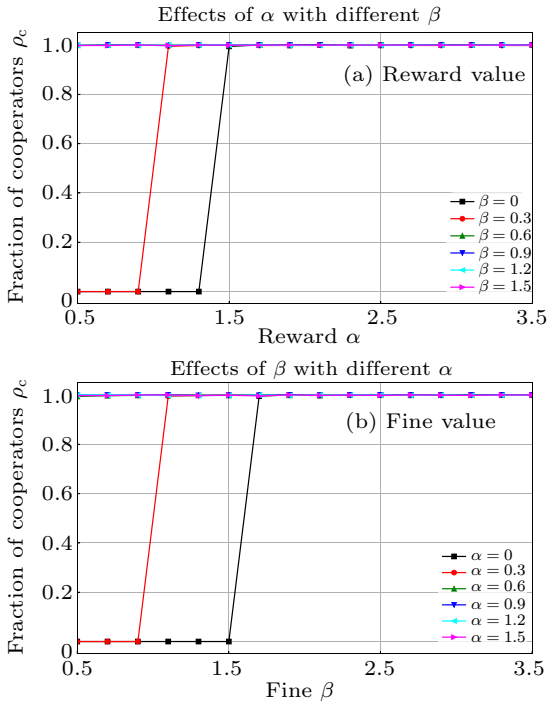


Fig. 9. ρ_c in the US-BRGBA game model with different α and β: (a) $\beta = 0 : 0.3 : 1.5$. (b) $\alpha = 0 : 0.3 : 1.5$. Other parameters are $\rho_f = 0.3$ and $r = 0.8$.

In Fig. 9(a), for $\beta = 0$, it can be seen that when $\alpha < 0.4$, both the punishment for defectors and the reward for cooperators are too small. The payoff gap between the cooperators and the defectors is excessively large, making it impossible for the cooperators to survive in the game layer. Then, when $0.4 < \alpha < 0.5$, the cooperators begin to coexist with the defectors. Further, when $\alpha \geq 0.5$, the cooperators can completely occupy the entire game layer. For $\beta = 0.3$, when the α is reduced to 0.2, the cooperators start to emerge. Then, when $\alpha > 0.3$, the cooperators completely dominate the game layer. Additionally, for $\beta > 0.6$, defectors eventually disappear regardless of the value α. In Fig. 9(b), when $\alpha = 0$, the cooperators start to emerge as the $\beta > 0.5$. Further, when $\beta > 0.6$, the cooperators eliminate the defectors. When $\alpha = 0.3$, the cooperators start to emerge as the value $\beta > 0.2$. At the same time, when $\beta > 0.3$, the cooperators completely occupy the entire game layer. When $\alpha > 0.6$, the ρ_c always reach 1 regardless of the value β.

Therefore, Fig. 9 tells us that when the bribe value γ is fixed, no matter what β is, there exists a threshold $\alpha = 0.5$, such that the cooperators dominate the entire game layer. Meanwhile, for different α, there is a threshold $\beta = 0.6$ such that the cooperators dominate the entire game layer.

Moreover, with the fixed fine value $\beta = 0.5$, Fig. 10 shows the trajectory of the ρ_c concerning α and γ.

In Fig. 10(a), with $\gamma = 0$, when $\alpha > 0.4$, the cooperators start to emerge. As α increases, the cooperators gradually eliminate the defectors. With $\gamma = 0.2$, the cooperators will emerge as long as $\alpha > 0.1$. And when the value $\alpha > 0.4$, the cooperators will completely occupy the entire game layer. In this case, the value α required for the cooperators to eliminate all the defectors becomes smaller. With $\gamma = 0.4$, the overall cooperation level remains high even if $\alpha = 0$. When $\alpha > 0.2$, the cooperators can completely dominate the game layer. With $\gamma > 0.6$, the defectors cannot survive regardless of the value α. In Fig. 10(b), increasing α makes ρ_c become larger. The value α required for the cooperators to start emerging and completely dominate the game layer becomes progressively smaller. When $\alpha > 1.2$, the cooperators can always occupy the game layer regardless of the value of γ.

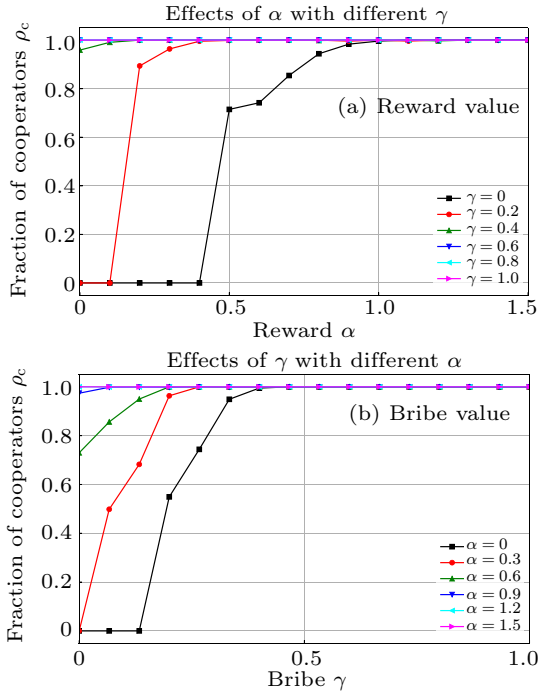


Fig. 10. The trajectory of ρ_c in the US-BRGBA game model: (a) $\gamma = 0 : 0.2 : 1$, (b) $\alpha = 0 : 0.3 : 1.5$. Other parameters are $\rho_f = 0.3$ and $r = 0.8$.

Therefore, Fig. 10 illustrates that for different values γ, $\alpha = 1.0$ is the threshold, which leads to the complete elimination of defectors. In addition, for different α, the threshold $\gamma = 0.47$ can also ensure $\rho_c = 1$.

When the reward value $\alpha = 0.5$ is fixed, Fig. 11 exhibits the trajectory of the ρ_c concerning the β and the γ.

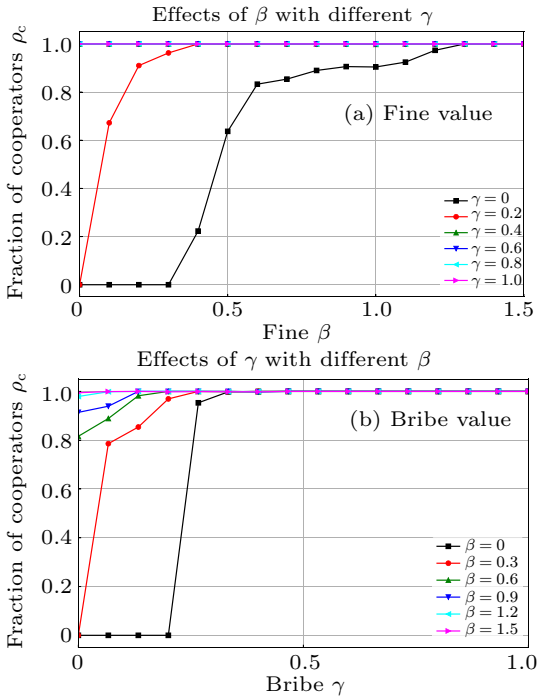


Fig. 11. The trajectory of cooperation level ρ_c in the US-BRGBA game model: (a) $\gamma = 0 : 0.2 : 1$, (b) $\beta = 0 : 0.3 : 1.5$. Other parameters are $\rho_f = 0.3$ and $r = 0.8$.

For $\gamma = 0$, Fig. 11(a) shows that when $\beta > 0.3$, the cooperators start to coexist with defectors. As the β continue to increase, the fraction of cooperators ρ_c first rises rapidly and then increases gently. When $\beta > 1.3$, the fraction $\rho_c = 100\%$. For $\gamma = 0.2$, the cooperators will emerge as long as $\beta > 0$, and their fraction rises rapidly with the increasing β. When $\beta > 0.4$, the fraction of cooperators ρ_c reaches 100%. For $\gamma > 0.4$, the cooperators completely eliminate the defectors independently of the value β. Similar to Fig. 10(b), in Fig. 11(b), as β increases, the ρ_c becomes larger. The value β required becomes progressively smaller to bring about the appearance of the cooperators and completely dominate the game layer. In this case, no matter what γ is, $\beta > 1.5$ can cause the occupation of cooperators in the game layer.

Therefore, Fig. 11 indicates that for different values of γ, $\beta = 1.3$ is the threshold, which guarantees $\rho_c = 100\%$. At the same time, for different values of β, the threshold $\gamma = 0.33$ brings $\rho_c = 100\%$ in the game layer.

As described above, when the value γ is fixed, α and β exhibit comparable influence capacities on the cooperative behavior. When fixing the value of β, the threshold of γ to achieve $\rho_c = 100\%$ in the game layer is smaller than that of α, indicating that the cooperative behavior of the US-BRGBA game model is very sensitive to the bribe value γ. When the value of α is unchanged, the bribe value γ also has a stronger influence on the cooperative behavior. Therefore, we conclude that the cooperative behavior of the US-BRGBA game model is more sensitive to the bribe value γ than to the reward value α and the fine value β.

4. Conclusions

"How human cooperative behavior evolves" has been listed as one of the most core issues driving fundamental scientific research proposed by *Science* magazine. Exploring the mechanisms underlying the emergence of cooperation holds significant scientific and theoretical value for correctly understanding the conditions and means of cooperation generation.

In this paper, we first propose a construction algorithm for a two-layer network with umpire supervision (US-BRGBA) by the BA scale-free network model and the BRGG model. Then, considering the fairness and corruption of umpires while introducing reward and punishment mechanisms exerted by umpires on players, a game model with umpire supervision on the US-BRGBA network is presented. Finally, based on MC simulations, this paper validates the feasibility and effectiveness of the proposed US-BRGBA game model, and deeply explores the impacts of umpires' spatial distribution, the fraction of fair umpires, reward values, fine values, and bribe values on players' cooperative behavior. The main conclusions are as follows. (i) Compared to the BRGG model, the proposed US-BRGBA game model in this paper can significantly promote cooperation, and even can completely occupy the entire game layer. (ii) For smaller fraction of fair umpires and synergy factor, concentrated distribution is more effective in promoting cooperation. Conversely, random distribution performs better. (iii) In the resource-abundant regions, corruption still brings about the emergence of cooperative behavior, which suggests that we should pay more attention to corruption issues in resource-poor regions. Further, even when all the umpires are corrupt, introducing umpires always promotes cooperation. (iv) Reward values, fine values, and bribe values all have certain influence on cooperative behavior. Moreover, the cooperative behavior of US-BRGBA game model is more sensitive to the bribe value than the reward value and the fine value.

This work provides a new framework for investigating PGG that better aligns with real-world scenarios and social dilemmas. However, the two-layer network constructed in this paper is static and only focuses on the relationships between two players, which may fail to fully capture the dynamic nature of PGG. Therefore, based on the results in this paper, we will continue to investigate the coevolution and cooperative behavior of PGG on temporal hypergraphs or high-order networks.

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