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New binary quantum stabilizer codes from classical quasi-cyclic codes of index two

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We study the algebraic structures of a generalized class of quasi-cyclic codes of index two, and prove that any quasi-cyclic code of index two can be transformed into our proposed quasi-cyclic codes. We determine the generator forms of the proposed quasi-cyclic codes and their Hermitian dual codes. We establish the necessary and sufficient conditions for Hermitian self-orthogonality and dual-containing property to construct quantum stabilizer codes via the Hermitian construction method. As an application, 10 record-breaking quantum stabilizer codes are constructed.

Keywords: quasi-cyclic codes, quantum stabilizer codes, Hermitian inner product, Hermitian construction

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1. Introduction

Quantum computers offer the ability to address problems such as integer factorization, unstructured dataset searches, and quantum system simulation-tasks that are intractable for classical computers within a practical timeframe.[1] However, quantum information is inherently fragile due to unavoidable decoherence and operational imperfections in physical quantum systems. To address this challenge, a framework of fault-tolerant quantum computation, which enables reliable quantum computations even in the presence of noise, was proposed. The realization of fault-tolerant quantum computation critically depends on robust quantum error correction (QEC) mechanisms that can preserve quantum coherence over long computational timescales. The fault tolerance theorem[2] has shown that the average error rate per quantum gate must fall below a certain threshold to enable arbitrarily long quantum computations.

Among the most promising frameworks for QEC are quantum stabilizer codes, which leverage the algebraic structure of classical codes to protect against quantum errors.[3] Quantum stabilizer codes allow for information to be redundantly encoded in entangled multi-qubit states, enabling the detection and correction of errors without measuring the logical information directly.[3] Recent studies have further advanced the theory and implementation of quantum stabilizer codes: Du *et al.*[4] analyzed the complete weight distributions of quantum stabilizer codes to better characterize code performance; Hong *et al.*[5] applied feedback control strategies and quantum stabilizer codes to improve the accuracy of quantum multi-parameter estimation; and Xiao *et al.*[6] proposed a new graph neural network based decoding method for XZZX code (a variant of the surface code) to address the limitations of traditional minimum weight perfect matching algorithms.

For example, surface code, as a two-dimensional topological stabilizer code, has become prominent in reaching fault-tolerant quantum computations because of its high error threshold and its compatibility with two-dimensional architectures.[7] The work of Krinner *et al.*[7] underscores the fundamental role of the code distance d: the logical error rate decreases exponentially with increasing d when the physical error rate is below the threshold. This finding highlights the importance of designing stabilizer codes with optimized parameters.

A common way to construct quantum stabilizer codes is to use classical codes.[8–11] Quasi-cyclic (QC) codes, which are extensions of cyclic codes and constant cyclic codes, provide a richer structure and better performance.[12–17] The QC codes used for constructing the quantum stabilizer codes are based primarily on the QC codes of index 2, including both one-generator and two-generator codes, following Ref. [18]. In Refs. [19,20], Lv *et al.* produced several record-breaking binary quantum stabilizer codes via two classes of QC codes of index 2: one-generator and two-generator codes. In 2022, Guan *et al.*[21,22] proposed other two-generator QC codes of index 2, investigated the Hermitian or symplectic self-orthogonality of these QC codes, and produced many record-breaking binary quantum stabilizer codes. In 2023, Abdukhalikov *et al.*[23] explored another class of one-generator QC codes of index 2 and obtained three record-breaking binary quantum stabilizer codes through the derived symplectic self-orthogonality conditions. In 2025, Guan *et al.*[24] extended the work of Ref. [23] to the QC codes of the index even and used the same one-generator QC codes of index 2 in Ref. [23]

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to construct quantum stabilizer codes, obtaining many record-breaking quantum stabilizer codes.

These studies demonstrate the potential of QC codes for constructing record-breaking quantum stabilizer codes. In this work, we study the algebraic structure of a generalized class of QC codes of index 2. We prove that any QC codes of index 2 can be transformed into our proposed QC codes with a standard generator form. We provide a systematic method for constructing quantum stabilizer codes via QC codes of index 2 under the Hermitian inner product and obtain two new classes of quantum stabilizer codes. We conclude that the quantum stabilizer codes constructed from Hermitian self-orthogonal (or dual-containing) QC codes of index 2 are all contained within our quantum stabilizer codes; for example, those in Refs. [18,19,21]. For practical application, we construct 10 binary record-breaking quantum stabilizer codes.

This paper is organized as follows. Section 2 provides essential concepts and notations for QC codes and quantum stabilizer codes. Section 3 presents the generator forms of our QC codes and their Hermitian dual codes, provides necessary and sufficient conditions for their Hermitian self-orthogonality and dual-containing property, and presents a systematic method for constructing quantum stabilizer codes from Hermitian self-orthogonal (or Hermitian dual-containing) QC codes of index 2. Section 4 provides several examples of the construction of quantum stabilizer code via our method. In Section 5, we present our results, including 10 record-breaking binary quantum stabilizer codes. Section 6 concludes this paper with a summary.

2. Preliminaries

2.1. Classical linear codes

Let $\mathbb{F}_q$ be a finite field with $q = p^r$, where p is a prime number and r is a positive integer. Recall that a classical linear code $\mathscr{C}$ of length n and dimension k over $\mathbb{F}_q$ is denoted by $[n,k]_q$. $\mathscr{C}$ is a k-dimensional linear subspace of $\mathbb{F}_q^n$. The Hamming weight of a vector v in $\mathbb{F}_q^n$, $\mathrm{wt}(v)$, is the number of nonzero entries in v. The minimum distance of $\mathscr{C}$ is equal to the minimum Hamming distance between any two distinct elements in $\mathscr{C}$ (i.e., two distinct codewords).

To define the Hermitian dual code, we first introduce the Hermitian inner product. The Hermitian inner products of two vectors $\boldsymbol{a} = (a_0,\ldots,a_{n-1})$ and $\boldsymbol{b} = (b_0,\ldots,b_{n-1})$ in $\mathbb{F}_{q^2}^n$ are defined as

$$\langle \boldsymbol{a},\boldsymbol{b}\rangle_h = \sum_{i=0}^{n-1} a_i^q b_i.$$

For a linear code $\mathscr{C} \subset \mathbb{F}_{q^2}^n$, its Hermitian dual code is defined as

$$\mathscr{C}^{\perp_h} = \left\{ \boldsymbol{a} \in \mathbb{F}_{q^2}^n \mid \langle \boldsymbol{a},\boldsymbol{b}\rangle_h = 0,\ \forall \boldsymbol{b} \in \mathscr{C} \right\}.$$

2.2. Quasi-cyclic codes

For a vector $\boldsymbol{c} = (c_0,\ldots,c_{m-1})$ in $\mathbb{F}_q^m$, the operation of the cyclic shift T is defined by $T(\boldsymbol{c}) = (c_{m-1},c_0,\ldots,c_{m-2})$. A linear code $\mathscr{C}$ in $\mathbb{F}_q^m$ is cyclic if $\mathscr{C} = T(\mathscr{C})$.

The codewords of cyclic codes can be represented in polynomial form. We denote the polynomial ring over $\mathbb{F}_{q^2}$ by $\mathbb{F}_{q^2}[x]$. Let $\langle x^m-1\rangle$ be the ideal of $\mathbb{F}_{q^2}[x]$ generated by the polynomial (x^m-1). Cyclic codes of length m over $\mathbb{F}_{q^2}$ can be considered as ideals in the quotient ring $\mathbb{R}_m = \mathbb{F}_{q^2}[x]/\langle x^m-1\rangle$. Let $\mathbb{R}_m^l$ be the set of l-tuples over $\mathbb{R}_m$, i.e., $\mathbb{R}_m^l = \{(a_0,a_1,\ldots,a_{l-1}) \mid a_i \in \mathbb{R}_m\}$, where $i = 0,1,\ldots,l-1$. Given a polynomial $g(x)$ in $\mathbb{F}_{q^2}[x]$, $[g(x)]$ denotes its residue class in $\mathbb{R}_m$. A vector $\boldsymbol{a} = (a_0,\ldots,a_{m-1})$ in $\mathbb{F}_q^m$ can be identified with the residue class

$$[a(x)] = \left[a_0 + a_1x + a_2x^2 + \cdots + a_{m-1}x^{m-1}\right]. \tag{1}$$

The vector $T(\boldsymbol{a})$ corresponds to the class $[xa(x)]$. Since $\mathbb{F}_{q^2}[x]$ is a principal ideal domain, any cyclic code can be generated by a single element $[g(x)] \in \mathbb{R}_m$. Let $g(x)$, $s(x) \in \mathbb{F}_{q^2}[x]$, the notation $g(x) \mid s(x)$ indicates that $g(x)$ divides $s(x)$. The greatest common divisor of $g(x)$ and $s(x)$ is denoted by $\gcd(g(x),s(x))$. Similarly, the least common multiple of $g(x)$ and $s(x)$ is denoted by $\mathrm{lcm}(g(x),s(x))$. Let $\partial g = \deg(g(x))$ be the degree of the polynomial $g(x)$. Let ∂g and ∂s be both less than m. If $g(x) \mid s(x)$, then there exists a polynomial $q(x)$ such that $s(x) = g(x)q(x)$. This implies that every codeword in the cyclic code generated by $[s(x)]$ is also a multiple of $g(x)$, hence the codewords of the cyclic code generated by $[s(x)]$ belong to that generated by $[g(x)]$. Therefore, the cyclic code generated by $[g(x)]$ contains that generated by $[s(x)]$.

Since QC codes are an extension of cyclic codes, they have definitions similar to those of cyclic codes. Let l and m be positive integers. For a vector $\boldsymbol{c} = \left(c_{00},c_{01},\ldots,c_{0,l-1},c_{10},\ldots,c_{1,l-1},\ldots c_{m-1,0},\ldots,c_{m-1,l-1}\right)$ in $\mathbb{F}_{q^2}^{lm}$, the operation of the cyclic shift T^l is defined as

$$T^l(\boldsymbol{c}) = \left(c_{m-1,0},c_{m-1,1},\ldots,c_{m-1,l-1},c_{00},c_{01},\ldots,c_{0,l-1},\ldots,c_{m-2,0},\ldots,c_{m-2,l-1}\right).$$

A linear code $\mathscr{C}$ in $\mathbb{F}_{q^2}^{lm}$ is considered a QC code of index l if $\mathscr{C} = T^l(\mathscr{C})$.

The codewords of the QC codes can also be represented in polynomial form. Map $\phi : \mathbb{F}_{q^2}^{lm} \to \mathbb{R}_m^l$ is defined by

$$\phi(\boldsymbol{c}) = ([c_0(x)],[c_1(x)],\ldots,\ [c_{l-1}(x)]) \in \mathbb{R}_m^l,$$

where

$$c_j(x) = \sum_{i=0}^{m-1} c_{ij}x^i \in \mathbb{F}_{q^2}[x],\ \ 0 \le j \le l-1.$$

The QC code of index l of length lm can be considered as an $\mathbb{R}_m$-submodule of $\mathbb{R}_m^l$. If a QC code $\mathscr{C}$ of index l of length

lm over $\mathbb{F}_{q^2}$ is generated by t generators $\boldsymbol{a}_1(x)$, $\boldsymbol{a}_2(x)$, …, and $\boldsymbol{a}_t(x)$ in $\mathbb{R}_m^l$, where

$$\begin{aligned}\boldsymbol{a}_1(x) &= \left([a_{1,0}(x)],[a_{1,1}(x)],\ldots,[a_{1,l-1}(x)]\right),\\ \boldsymbol{a}_2(x) &= \left([a_{2,0}(x)],[a_{2,1}(x)],\ldots,[a_{2,l-1}(x)]\right),\\ &\ldots,\\ \boldsymbol{a}_t(x) &= \left([a_{t,0}(x)],[a_{t,1}(x)],\ldots,[a_{t,l-1}(x)]\right),\end{aligned}$$

then the code $\mathscr{C}$ is called a t-generators QC code. The codewords of $\mathscr{C}$ can be identified via linear combinations of these generators

$$\sum_{i=1}^{t}[k_i(x)]\boldsymbol{a}_i(x) = \sum_{i=1}^{t}([k_i(x)a_{i,0}(x)],[k_i(x)a_{i,1}(x)],\ldots,[k_i(x)a_{i,l-1}(x)]),$$

where $k_i(x)\in\mathbb{F}_{q^2}[x]$. Let $\boldsymbol{g}(x)=(g_0(x),g_1(x),\ldots,g_{l-1}(x))\in\mathbb{F}_{q^2}[x]^l$. Map $\varphi:\mathbb{F}_{q^2}[x]^l\to\mathbb{R}_m^l$ is defined by

$$\varphi((\boldsymbol{g}(x)))=([g_0(x)],[g_1(x)],\ldots,[g_{l-1}(x)])\in\mathbb{R}_m^l.$$

For the QC codes $\mathscr{C}$ of index 2, it follows that:

Theorem 1 (Theorems 2.1 and 2.2 in Ref. [25]) For any QC code $\mathscr{C}$ of index 2, its preimage $\varphi^{-1}(\mathscr{C})$ of $\mathscr{C}$ in $\mathbb{F}_{q^2}[x]^2$ has an upper-triangular generator form

$$\begin{pmatrix} g_{10}(x) & g_{11}(x)\\ 0 & g_{21}(x)\end{pmatrix},$$

where $g_{10}(x),g_{21}(x)\mid(x^m-1)$ and $\partial g_{11}<\partial g_{21}$. There exists an upper-triangular matrix

$$\begin{pmatrix} m_{10}(x) & m_{11}(x)\\ 0 & m_{21}(x)\end{pmatrix}$$

such that

$$\begin{pmatrix} m_{10}(x) & m_{11}(x)\\ 0 & m_{21}(x)\end{pmatrix}\begin{pmatrix} g_{10}(x) & g_{11}(x)\\ 0 & g_{21}(x)\end{pmatrix} = \begin{pmatrix} x^m-1 & 0\\ 0 & x^m-1\end{pmatrix},$$

the dimension of the code $\mathscr{C}$ is $2m-\partial g_{10}-\partial g_{21}$.

We define some necessary polynomials at the end of this subsection. These polynomials help us define the supporting QC codes and prove the corresponding conclusions in the next section. Let $f(x)=f_0+f_1x+\cdots+f_{n-1}x^{n-1}$ be a polynomial in $\mathbb{F}_{q^2}[x]$ and $f_{n-1}\neq 0$. The conjugate polynomial $f^q(x)$ of $f(x)$ is defined

$$f^q(x)=f_0^q+f_1^qx+\cdots+f_{n-1}^qx^{n-1}.$$

If $\partial f<m$, the reciprocal polynomial $\bar{f}(x)$ of $f(x)$ in $\mathbb{F}_{q^2}[x]$ is defined

$$\bar{f}(x)=f(1/x)\mod(x^m-1).$$

When $f(x)\mid(x^m-1)$, the complementary factor $f'(x)$ of $f(x)$ and the dual polynomial $f^\perp(x)$ of $f(x)$ are defined as follows:

$$\begin{aligned}f'(x)&=(x^m-1)/f(x),\\ f^\perp(x)&=f'\left(\frac{1}{x}\right)x^{\partial f'}.\end{aligned}$$

2.3. Quantum stabilizer codes

Let p be a prime number, and let r be a positive integer. $\mathbb{C}^q$ is defined as a complex vector space of dimensions q, where $q=p^r$. Then, we let

$$V_n=(\mathbb{C}^q)^{\otimes n}=\underbrace{\mathbb{C}^q\otimes\cdots\otimes\mathbb{C}^q}_{n}.$$

An $((n,K,d))_q$ quantum code is a K-dimensional subspace of V_n. The minimum distance d allows it to detect all errors of error groups with weights less than d, but fails to detect some errors of weight d.[11] If $K=q^k$, the $((n,q^k,d))_q$ code can also be written as the $[[n,k,d]]_q$ code.

A quantum stabilizer code is the joint eigenvalue-1 eigenspace of a commutative subgroup of the error group generated by a nice error basis on V_n.[11] If the stabilizer group of a quantum stabilizer code does not contain non-scalar matrices of weight less than t, it is said to be pure to t. If a quantum stabilizer code is pure to its minimum distance, it is called pure.[11]

Quantum stabilizer codes can be obtained from Hermitian self-orthogonal classical linear codes.

Theorem 2 (Hermitian construction)[11] If there exists an $[n,k,d]_{q^2}$ classical linear code $\mathscr{C}$ that satisfies $\mathscr{C}\subseteq\mathscr{C}^{\perp_h}$, then there exists a quantum stabilizer code with parameter $[[n,n-2k,d_Q]]_q$, where $d_Q=\min\{\mathrm{wt}(\boldsymbol{x})\mid\boldsymbol{x}\in\mathscr{C}^{\perp_h}\backslash\mathscr{C}\}$.

In addition, quantum stabilizer codes can be obtained from existing codes via the following theorem.

Theorem 3 (Propagation rules)[10] Assume that a pure quantum stabilizer code $[[n,k,d]]_q$ exists. Then $[[n,k-1,d]]_q$ (for $k\geq 1$) and $[[n+1,k,d]]_q$ (for $k>0$) quantum stabilizer codes also exist.

3. Construction of quantum stabilizer codes

In this section, we construct quantum stabilizer codes via Hermitian self-orthogonal and Hermitian dual-containing QC codes. We define a generalized class of QC codes of index 2 and prove that any QC code of index 2 can be transformed into our QC codes with standard generator forms.

To establish the general form of generators for our QC codes of index 2, the following Theorem is essential.

Theorem 4 Any QC code $\mathscr{C}$ of index two of length $2m$ over $\mathbb{F}_{q^2}$ has a generator form

$$\begin{pmatrix}[g(x)] & [a(x)f(x)]\\ 0 & [s(x)]\end{pmatrix},$$

where $g(x)\mid(x^m-1), s(x)\mid(x^m-1)$, $a(x)$ satisfies $g(x)s(x)=a(x)d(x)$, where $d(x)=\gcd(g(x)s(x),x^m-1)$, and $f(x)$ is

some polynomial in $\mathbb{F}_{q^2}[x]$ with $\partial f < m$. The parameters of $\mathscr{C}$ are given by $[2m, 2m-\partial g-\partial s]_{q^2}$.

Proof According to Theorem 1, for any QC code $\mathscr{C}$ of index two, its preimage $\varphi^{-1}(\mathscr{C})$ in $\mathbb{F}_{q^2}[x]^2$ has an upper-triangular generator form

$$\begin{pmatrix} g(x) & g_{11}(x) \\ 0 & s(x) \end{pmatrix},$$

where $g(x), s(x) \mid (x^m-1)$ and $\partial g_{11} < \partial s$ and there exists an upper-triangular matrix

$$\begin{pmatrix} m_{10}(x) & m_{11}(x) \\ 0 & m_{21}(x) \end{pmatrix}$$

such that

$$\begin{pmatrix} m_{10}(x) & m_{11}(x) \\ 0 & m_{21}(x) \end{pmatrix} \begin{pmatrix} g(x) & g_{11}(x) \\ 0 & s(x) \end{pmatrix} = \begin{pmatrix} x^m-1 & 0 \\ 0 & x^m-1 \end{pmatrix},$$

we can obtain that $m_{10}(x)g(x) = (x^m-1)$, $m_{21}(x)s(x) = (x^m-1)$ and $m_{10}(x)g_{11}(x)+m_{11}(x)s(x) = 0$. According to $d(x) = \gcd(g(x)s(x), x^m-1)$, there exists a polynomial $b(x)$ such that $x^m-1 = d(x)b(x)$. Therefore,

$$\begin{aligned} &g(x)(m_{10}(x)g_{11}(x)+m_{11}(x)s(x)) \\ &= m_{10}(x)g(x)g_{11}(x)+m_{11}(x)g(x)s(x) \\ &= (x^m-1)g_{11}(x)+m_{11}(x)g(x)s(x) \\ &= d(x)b(x)g_{11}(x)+m_{11}(x)a(x)d(x) \\ &= d(x)(b(x)g_{11}(x)+m_{11}(x)a(x)) = 0. \end{aligned}$$

Since $d(x) \neq 0$, $b(x)g_{11}(x)+m_{11}(x)a(x) = 0$. We can conclude that $m_{11}(x)a(x) = -b(x)g_{11}(x)$. Therefore, $a(x) \mid b(x)g_{11}(x)$. Since $\gcd(a(x), b(x)) = 1$, we can obtain $a(x) \mid g_{11}(x)$, i.e., there exists a polynomial $f(x)$ such that $g_{11}(x) = a(x)f(x)$. According to Theorem 1, we can obtain $\partial g_{11} < \partial s$ and $\partial s \leqslant m$. Therefore, $\partial f \leqslant g_{11} < m$. QC code $\mathscr{C}$ has generators $([g(x)], [a(x)f(x)])$ and $(0, [s(x)])$.

To facilitate the presentation of QC codes used in this paper for constructing quantum stabilizer codes, we first give the definition of our QC codes.

Definition 1 Let $g(x)$, $s(x)$ be any polynomial in $\mathbb{F}_{q^2}[x]$ such that $g(x) \mid (x^m-1), s(x) \mid (x^m-1)$. Let $g(x)s(x) = a(x)d(x)$, where $d(x) = \gcd(g(x)s(x), x^m-1)$. Let $f(x)$ be any polynomial in $\mathbb{F}_{q^2}[x]$ such that $\partial f < m$. Define $\mathscr{C}_{q^2}(f,g,s)$ to be the QC code of index two of length $2m$ over $\mathbb{F}_{q^2}$ generated by $([g(x)], [a(x)f(x)])$ and $(0, [s(x)])$.

According to Theorem 4, we can conclude that the parameters of $\mathscr{C}_{q^2}(f,g,s)$ are $[2m, 2m-\partial g-\partial s]_{q^2}$, and any QC code of index 2 is some $\mathscr{C}_{q^2}(f,g,s)$ in Definition 1. Therefore, the QC codes $\mathscr{C}_{q^2}(f,g,s)$ in Definition 1 include the QC codes of index 2 presented in Refs. [18–21,23].

To obtain the generator form of the Hermitian dual code of $\mathscr{C}_{q^2}(f,g,s)$, the following lemma is needed.

Lemma 1[18] Let $n(x)$, $f(x)$ and $h(x)$ be monic polynomials in $\mathbb{F}_{q^2}[x]$ whose degrees are less than m, and consider the vectors determined by their residue class in $\mathbb{R}_m$ as described in Eq. (1). Then, the following equation for the Hermitian inner product of vectors in $\mathbb{F}_{q^2}^m$ holds:

$$\langle [n(x)f(x)], [h(x)] \rangle_h = \langle [n(x)], [\bar{f}^q(x)h(x)] \rangle_h.$$

On the basis of the results presented above, we can obtain the generator form of the Hermitian dual code of $\mathscr{C}_{q^2}(f,g,s)$.

Proposition 1 Given a QC code $\mathscr{C}_{q^2}(f,g,s)$ in Definition 1, let $x^m-1 = b(x)d(x)$, where $d(x) = \gcd(g(x)s(x), x^m-1)$. The Hermitian dual code $\mathscr{C}_{q^2}^{\perp_h}(f,g,s)$ of the QC code $\mathscr{C}_{q^2}(f,g,s)$ over $\mathbb{F}_{q^2}$ is a QC code generated by $([g^{\perp q}(x)], 0)$ and $([-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}], [s^{\perp q}(x)])$ with parameters $[2m, \partial g+\partial s]_{q^2}$.

Proof Let $\mathscr{C}_1$ be the QC code generated by $([g^{\perp q}(x)], 0)$ and $([-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}], [s^{\perp q}(x)])$ over $\mathbb{F}_{q^2}$. Recall that $[g^{\perp q}(x)]$ ($[s^{\perp q}(x)]$) generates the Hermitian dual code of the cyclic code generated by $[g(x)]$ ($[s(x)]$) and that the generator matrix G' of $\mathscr{C}_1$ is

$$G' = \begin{pmatrix} [g^{\perp q}(x)] & 0 \\ [xg^{\perp q}(x)] & 0 \\ \vdots & \vdots \\ [x^{\partial g-1}g^{\perp q}(x)] & 0 \\ [-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}] & [s^{\perp q}(x)] \\ [-x\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}] & [xs^{\perp q}(x)] \\ \vdots & \vdots \\ [-x^{\partial s-1}\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}] & [x^{\partial s-1}s^{\perp q}(x)] \end{pmatrix},$$

where the corresponding vectors, determined by their residue class in $\mathbb{R}_m$, are described in Eq. (1). Clearly, the generator matrix G' is full rank, the dimension of $\mathscr{C}_1$ is $2m-\partial g-\partial s$.

We next show that the Hermitian inner product between the codewords of $\mathscr{C}_1$ and $\mathscr{C}_{q^2}(f,g,s)$ is equal to zero. To simplify the expression, we let $a_{21}(x) = \bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}$, and let $a_1(x)$, $a_2(x)$, $b_1(x)$ and $b_2(x)$ be arbitrary polynomials in $\mathbb{F}_{q^2}[x]$. A codeword in $\mathscr{C}_{q^2}(f,g,s)$ can be written as

$$\begin{aligned} &[a_1(x)]([g(x)], [a(x)f(x)]) + [a_2(x)](0, [s(x)]) \\ &= ([a_1(x)g(x)], [a_1(x)a(x)f(x)]) + (0, [a_2(x)s(x)]) \\ &= ([a_1(x)g(x)], [a_1(x)a(x)f(x)+a_2(x)s(x)]). \end{aligned}$$

Similarly, a codeword in $\mathscr{C}_1$ can be written as

$$\begin{aligned} &[b_1(x)]([g^{\perp q}(x)], 0) + [b_2(x)]([-a_{21}(x)], [s^{\perp q}(x)]) \\ &= ([b_1(x)g^{\perp q}(x)], 0) + ([-b_2(x)a_{21}(x)], [b_2(x)s^{\perp q}(x)]) \\ &= ([b_1(x)g^{\perp q}(x) - b_2(x)a_{21}(x)]], [b_2(x)s^{\perp q}(x)]). \end{aligned}$$

The Hermitian inner product of the above two codewords is

$$\begin{aligned} &\langle [a_1(x)g(x)], \left[b_1(x)g^{\perp q}(x) - b_2(x)a_{21}(x)\right] \rangle_h \\ &+ \langle [a_1(x)a(x)f(x)+a_2(x)s(x)], [b_2(x)s^{\perp q}(x)] \rangle_h, \end{aligned} \quad (2)$$

and it is equal to

$$\begin{aligned}&\langle[a_1(x)g(x)],\left[b_1(x)g^{\perp q}(x)\right]\rangle_h\\&-\langle[a_1(x)g(x)],[b_2(x)a_{21}(x)]\rangle_h\\&+\langle[a_1(x)a(x)f(x)],[b_2(x)s^{\perp q}(x)]\rangle_h\\&+\langle[a_2(x)s(x)],[b_2(x)s^{\perp q}(x)]\rangle_h.\end{aligned}$$

According to $[g^{\perp q}(x)]$ ($[s^{\perp q}(x)]$) generates the Hermitian dual code of the cyclic code generated by $[g(x)]$ ($[s(x)]$), so

$$\begin{aligned}&\left\langle[a_1(x)g(x)],\left[b_1(x)g^{\perp q}(x)\right]\right\rangle_h=0,\\&\langle[a_2(x)s(x)],[b_2(x)s^{\perp q}(x)]\rangle_h=0.\end{aligned}$$

Since $g(x)\mid(x^m-1)$ and $s(x)\mid(x^m-1)$, $\mathrm{lcm}(g(x),s(x))\mid(x^m-1)$. Since $\mathrm{lcm}(g(x),s(x))\mid g(x)s(x)$, we can conclude that $\mathrm{lcm}(g(x),s(x))\mid d(x)$. Let $d_1(x)=d(x)/\mathrm{lcm}(g(x),s(x))$ and $t(x)=(x^m-1)/\mathrm{lcm}(g(x),s(x))$. According to $x^m-1=b(x)d(x)$, thus

$$\begin{aligned}t(x)&=(x^m-1)/\mathrm{lcm}(g(x),s(x))\\&=b(x)d(x)/\mathrm{lcm}(g(x),s(x))\\&=b(x)d_1(x).\end{aligned}$$

Let $d_2(x)=\gcd(g(x),s(x))$, $g(x)=d_2(x)m(x)$ and $s(x)=d_2(x)n(x)$. We can derive that $\mathrm{lcm}(g(x),s(x))=d_2(x)m(x)n(x)$. Since $x^m-1=\mathrm{lcm}(g(x),s(x))t(x)$, $x^m-1=d_2(x)m(x)n(x)t(x)$. Thus,

$$d(x)=\mathrm{lcm}(g(x),s(x))d_1(x)=d_2(x)m(x)n(x)d_1(x).$$

Based on $g(x)s(x)=a(x)d(x)$, thus

$$\begin{aligned}g(x)s(x)&=d_2^2(x)m(x)n(x)=a(x)d_2(x)m(x)n(x)d_1(x)\\&=a(x)d(x).\end{aligned}$$

Therefore,

$$d_2(x)=a(x)d_1(x).$$

Recall that $s'(x)=(x^m-1)/s(x)$, we can conclude that

$$\begin{aligned}s'(x)&=(d_2(x)m(x)n(x)t(x))/(d_2(x)n(x))=t(x)m(x)\\&=b(x)d_1(x)m(x).\end{aligned}$$

Therefore,

$$s^{\perp q}(x)=b^q(1/x)d_1^q(1/x)m^q(1/x)x^{\partial s'}.$$

Given that

$$\begin{aligned}&[b^q(1/x)d_1^q(1/x)m^q(1/x)x^{\partial s'}]\\&=[b^q(1/x)x^m d_1^q(1/x)x^m m^q(1/x)x^m x^{\partial s'}]\\&=[\bar{b}^q(x)\bar{d_1}^q(x)\bar{m}^q(x)x^{\partial s'}],\end{aligned}$$

we can conclude that $[s^{\perp q}(x)]=[\bar{b}^q(x)\bar{d_1}^q(x)\bar{m}^q(x)x^{\partial s'}]$. According to Lemma 1, the following holds:

$$\begin{aligned}&\langle[a_1(x)g(x)],[b_2(x)a_{12}(x)]\rangle_h\\&=\left\langle[a_1(x)d_2(x)m(x)],\left[b_2(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}\right]\right\rangle_h\\&=\left\langle[a_1(x)a(x)d_1(x)m(x)],\left[b_2(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}\right]\right\rangle_h\\&=\left\langle[a_1(x)a(x)f(x)],\left[b_2(x)\bar{b}^q(x)\bar{d_1}^q(x)\bar{m}^q(x)x^{\partial s'}\right]\right\rangle_h\\&=\langle[a_1(x)a(x)f(x)],[b_2(x)s^{\perp q}(x)]\rangle_h.\end{aligned}$$

Thus, the Hermitian inner product of the two codewords of Eq. (2) is equal to zero. The parameters of the QC code $\mathscr{C}_1$ are $[2m,\partial g+\partial s]_{q^2}$. Since the parameters of $\mathscr{C}_{q^2}(f,g,s)$ are $[2m,2m-\partial g-\partial s]_{q^2}$, we conclude that $\mathscr{C}_1$ is the Hermitian dual code of $\mathscr{C}_{q^2}(f,g,s)$ with parameters $[2m,\partial g+\partial s]_{q^2}$, which completes the proof.

According to Theorem 2, the construction of quantum stabilizer codes requires classical codes that are either Hermitian self-orthogonal or Hermitian dual-containing. Therefore, we first establish the necessary and sufficient conditions for the Hermitian self-orthogonality of $\mathscr{C}_{q^2}(f,g,s)$.

Proposition 2 The necessary and sufficient conditions for $\mathscr{C}_{q^2}(f,g,s)$ to be contained in its Hermitian dual code $\mathscr{C}_{q^2}^{\perp_h}(f,g,s)$ are as follows:

(1) there exists a polynomial $u(x)$ in $\mathbb{F}_{q^2}[x]$ such that $[u(x)s^{\perp q}(x)]=[s(x)]$,

(2) there exists a polynomial $v(x)$ in $\mathbb{F}_{q^2}[x]$ such that $[v(x)s^{\perp q}(x)]=[a(x)f(x)]$,

(3) $[g^{\perp q}(x)]\mid[u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]$,

(4) $[g^{\perp q}(x)]\mid[(g(x)+v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'})]$.

Proof Assume that the QC codes $\mathscr{C}_{q^2}(f,g,s)$ satisfy the conditions in Proposition 2. According to condition (1), $[u(x)s^{\perp q}(x)]=[s(x)]$. According to condition (2), $[v(x)s^{\perp q}(x)]=[a(x)f(x)]$. According to condition (3), there exists a polynomial $m(x)$ in $\mathbb{F}_{q^2}[x]$ such that $[m(x)g^{\perp q}(x)]=[u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]$. Therefore,

$$\begin{aligned}&[m(x)]([g^{\perp q}(x)],0)+[u(x)]([-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[s^{\perp q}(x)])\\&=([m(x)g^{\perp q}(x)],0)+([-u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[u(x)s^{\perp q}(x)])\\&=([m(x)g^{\perp q}(x)-u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[u(x)s^{\perp q}(x)])\\&=([u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}-u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[u(x)s^{\perp q}(x)])\\&=(0,[s(x)]).\end{aligned}$$

Similarly, according to condition (4), there exists a polynomial $n(x)$ in $\mathbb{F}_{q^2}[x]$ that satisfies the following equation:

$$[n(x)g^{\perp q}(x)]=[g(x)+v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}].$$

Therefore,

$$\begin{aligned}[n(x)]([g^{\perp q}(x)],0)+[v(x)]([-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[s^{\perp q}(x)]) &= ([n(x)g^{\perp q}(x)],0)+([-v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[v(x)s^{\perp q}(x)])\\ &= ([n(x)g^{\perp q}(x)-v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[v(x)s^{\perp q}(x)])\\ &= ([g(x)+v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}-v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[v(x)s^{\perp q}(x)])\\ &= ([g(x)],[a(x)f(x)]).\end{aligned}$$

Thus, $\mathscr{C}_{q^2}(f,g,s)$ is contained in its Hermitian dual code $\mathscr{C}_{q^2}^{\perp_h}(f,g,s)$.

Assume that $\mathscr{C}_{q^2}(f,g,s)$ is contained in its Hermitian dual code $\mathscr{C}_{q^2}^{\perp_h}(f,g,s)$. There exist polynomials $u(x),v(x),m(x),n(x)$ in $\mathbb{F}_{q^2}[x]$ such that the following equations hold:

$$\begin{aligned}([g(x)],[a(x)f(x)]) &= [n(x)]([g^{\perp q}(x)],0)+[v(x)]([-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[s^{\perp q}(x)])\\ &= ([n(x)g^{\perp q}(x)],0)+([v(x)(-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'})],[v(x)s^{\perp q}(x)])\\ &= ([n(x)g^{\perp q}(x)-v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[v(x)s^{\perp q}(x)]), \qquad (3)\\ (0,[s(x)]) &= [m(x)]([g^{\perp q}(x)],0)+[u(x)]([(-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'})],[s^{\perp q}(x)])\\ &= ([m(x)g^{\perp q}(x)],0)+([u(x)(-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'})],[u(x)s^{\perp q}(x)])\\ &= ([m(x)g^{\perp q}(x)-u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[u(x)s^{\perp q}(x)]). \qquad (4)\end{aligned}$$

According to Eq. (3), we can conclude that $[v(x)s^{\perp q}(x)]=[a(x)f(x)]$ and that $[n(x)g^{\perp q}(x)-v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]=[g(x)]$. And we can derive the following:

$$[n(x)g^{\perp q}(x)]=[g(x)+v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}].$$

Thus, $[g^{\perp q}(x)]\mid[g(x)+v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]$.

According to Eq. (4), we can conclude that $[u(x)s^{\perp q}(x)]=[s(x)]$ and that $[m(x)g^{\perp q}(x)-u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]=0$. And we can derive the following:

$$[m(x)g^{\perp q}(x)]=[u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}].$$

Thus, $[g^{\perp q}(x)]\mid[u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]$. The proof is concluded.

Now we establish the necessary and sufficient conditions for the Hermitian dual-containing property of $\mathscr{C}_{q^2}(f,g,s)$.

Proposition 3 The necessary and sufficient conditions for $\mathscr{C}_{q^2}(f,g,s)$ containing its Hermitian dual code $\mathscr{C}_{q^2}^{\perp_h}(f,g,s)$ are:

(1) there exists a polynomial $u_1(x)$ in $\mathbb{F}_{q^2}[x]$ such that $[u_1(x)g(x)]=[g^{\perp q}(x)]$,

(2) there exists a polynomial $v_1(x)$ in $\mathbb{F}_{q^2}[x]$ such that $[v_1(x)g(x)]=[\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]$,

(3) $[s(x)]\mid[u_1(x)a(x)f(x)]$,

(4) $[s(x)]\mid[(s^{\perp q}(x)+v_1(x)a(x)f(x))]$.

Proof Assume that the QC codes $\mathscr{C}_{q^2}(f,g,s)$ satisfy the conditions in Proposition 3. According to condition (1), $[u_1(x)g(x)]=[g^{\perp q}(x)]$. According to condition (2), $[v_1(x)g(x)]=[\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]$. According to condition (3), there exists a polynomial $m_1(x)$ in $\mathbb{F}_{q^2}[x]$ such that $[m_1(x)s(x)]=[u_1(x)a(x)f(x)]$. Therefore,

$$\begin{aligned}&[u_1(x)]([g(x)],[a(x)f(x)])-[m_1(x)](0,[s(x)])\\ &=([u_1(x)g(x)],[u_1(x)a(x)f(x)])-(0,[m_1(x)s(x)])\\ &=([u_1(x)g(x)],[u_1(x)a(x)f(x)-m_1(x)s(x)])\\ &=([u_1(x)g(x)],[u_1(x)a(x)f(x)-u_1(x)a(x)f(x)])\\ &=([g^{\perp q}(x)],0).\end{aligned}$$

Similarly, according to condition (4), there exists a polynomial $n_1(x)$ in $\mathbb{F}_{q^2}[x]$ that satisfies the following equation:

$$[n_1(x)s(x)]=[s^{\perp q}(x)+v_1(x)a(x)f(x)].$$

Therefore,

$$\begin{aligned}&[n_1(x)](0,[s(x)])-[v_1(x)]([g(x)],[a(x)f(x)])\\ &=(0,[n_1(x)s(x)])-([v_1(x)g(x)],[v_1(x)a(x)f(x)])\\ &=([-v_1(x)g(x)],[n_1(x)s(x)-v_1(x)a(x)f(x)])\\ &=([-v_1(x)g(x)],[s^{\perp q}(x)+v_1(x)a(x)f(x)-v_1(x)a(x)f(x)])\\ &=([-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[s^{\perp q}(x)]).\end{aligned}$$

Thus, $\mathscr{C}_{q^2}^{\perp_h}(f,g,s)$ is contained in $\mathscr{C}_{q^2}(f,g,s)$.

Assume that $\mathscr{C}_{q^2}^{\perp_h}(f,g,s)$ is contained in $\mathscr{C}_{q^2}(f,g,s)$. There exist polynomials $u_1(x),v_1(x),m_1(x),n_1(x)$ in $\mathbb{F}_{q^2}[x]$ such that the following equations hold:

$$\begin{aligned}&([g^{\perp q}(x)],0)\\ &= [u_1(x)]([g(x)],[a(x)f(x)])+[m_1(x)](0,[s(x)])\\ &= ([u_1(x)g(x)],[u_1(x)a(x)f(x)])+(0,[m_1(x)s(x)])\\ &= ([u_1(x)g(x)],[u_1(x)a(x)f(x)+m_1(x)s(x)]), \qquad (5)\\ &([-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}],[s^{\perp q}(x)])\\ &= [-v_1(x)]([g(x)],[a(x)f(x)])+[n_1(x)](0,[s(x)])\end{aligned}$$

$$= -([v_1(x)g(x)],[v_1(x)a(x)f(x)]) + (0,[n_1(x)s(x)])$$
$$= ([-v_1(x)g(x)],[n_1(x)s(x) - v_1(x)a(x)f(x)]). \quad (6)$$

According to Eq. (5), we can conclude that $[u_1(x)g(x)] = [g^{\perp_q}(x)]$ and $[u_1(x)a(x)f(x) + m_1(x)s(x)] = 0$. And we can derive the following:

$$[m_1(x)s(x)] = [-u_1(x)a(x)f(x)].$$

Thus, $[s(x)] \mid [u_1(x)a(x)f(x)]$.

According to Eq. (6), we can conclude that $[v_1(x)g(x)] = [\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]$ and that $[n_1(x)s(x) - v_1(x)a(x)f(x)] = [s^{\perp_q}(x)]$. And we can derive the following:

$$[n_1(x)s(x)] = [s^{\perp_q}(x) + v_1(x)a(x)f(x)].$$

Thus, $[s(x)] \mid [s^{\perp_q}(x) + v_1(x)a(x)f(x)]$. The proof is concluded.

By using the proposed generalized generator form of the QC codes of index 2 in Definition 1, we provide a systematic method for constructing quantum stabilizer codes via our QC codes. Additionally, we obtain two new classes of quantum stabilizer codes.

Theorem 5 Assume that $\mathscr{C}_{q^2}(f,g,s)$ is the QC code in Definition 1.

(1) If conditions (1)–(4) in Proposition 2 hold, the QC code $\mathscr{C}_{q^2}(f,g,s)$ is Hermitian self-orthogonal. Consequently, there exists a quantum stabilizer code $\mathscr{Q}$ with the following parameters:

$$\mathscr{Q} = [[2m, 2\partial g + 2\partial s - 2m, d_{\mathscr{Q}}]]_q,$$

where $d_{\mathscr{Q}} = \min\left\{\mathrm{wt}(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathscr{C}_{q^2}^{\perp_h}(f,g,s) \backslash \mathscr{C}_{q^2}(f,g,s)\right\}$.

(2) If conditions (1)–(4) in Proposition 3 hold, the QC code $\mathscr{C}_{q^2}(f,g,s)$ is Hermitian dual-containing. Consequently, there exists a quantum stabilizer code $\mathscr{Q}$ with the following parameters:

$$\mathscr{Q} = [[2m, 2m - 2\partial g - 2\partial s, d_{\mathscr{Q}}]]_q,$$

where $d_{\mathscr{Q}} = \min\left\{\mathrm{wt}(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathscr{C}_{q^2}(f,g,s) \backslash \mathscr{C}_{q^2}^{\perp_h}(f,g,s)\right\}$.

Since Propositions 2 and 3 provide necessary and sufficient conditions, rather than merely sufficient ones, for Hermitian self-orthogonality (or Hermitian dual-containing property) of QC codes of index 2, the conditions may appear relatively more complex. In fact, however, when constructing quantum stabilizer codes, it is often enough to consider simplified versions of these conditions (i.e., sufficient conditions) similar to the approach adopted in Refs. [18,19,21].

There are two main advantages to providing the necessary and sufficient conditions for this work: (1) All Hermitian self-orthogonal (or dual-containing) QC codes of index 2 have the generator form given in Definition 1 and satisfy the corresponding polynomial conditions in Proposition 2 or 3. Additionally, the quantum stabilizer codes constructed from Hermitian self-orthogonal (or dual-containing) QC codes of index 2 are contained within the class of quantum stabilizer codes derived via our systematic method presented in Theorem 5; therefore, the Hermitian self-orthogonal (or Hermitian dual-containing) QC codes presented in Refs. [18,19,21], along with the corresponding quantum stabilizer codes constructed from them, are all encompassed within the class of QC codes in Definition 1 and the quantum stabilizer codes obtained via our method. (2) The necessary and sufficient conditions we propose in Propositions 2 and 3 are less restrictive than any sufficient conditions that are, which allows for the identification of a broader class of Hermitian self-orthogonal (or Hermitian dual-containing) QC codes of index 2. This offers greater potential for constructing quantum stabilizer codes with improved parameters.

4. Examples

In this section, we present several simple examples to demonstrate the QC construction of quantum stabilizer codes. The algebraic software Magma[26] was used in the calculation of the code parameters.

Example 1 Let $q = 2$, $m = 10$ and w represent a primitive element in $\mathbb{F}_4$. Choose the polynomials in $\mathbb{F}_4[x]$ as follows:

$$g(x) = x^6 + w^2x^5 + wx^4 + wx^3 + wx^2 + w^2x + 1,$$
$$s(x) = x^8 + w^2x^7 + w^2x^6 + x^5 + x^3 + w^2x^2 + w^2x + 1,$$
$$f(x) = x^2 + w^2x + 1.$$

Due to

$$x^{10} - 1 = (x^6 + w^2x^5 + wx^4 + wx^3 + wx^2 + w^2x + 1) \times (x^4 + w^2x^3 + w^2x + 1),$$
$$x^{10} - 1 = (x^8 + w^2x^7 + w^2x^6 + x^5 + x^3 + w^2x^2 + w^2x + 1)(x^2 + w^2x + 1),$$

therefore,

$$x^{10} - 1 = g(x)(x^4 + w^2x^3 + w^2x + 1)$$
$$x^{10} - 1 = s(x)(x^2 + w^2x + 1).$$

We can conclude that $g(x) \mid (x^{10} - 1)$ and $s(x) \mid (x^{10} - 1)$. Thus, $g'(x) = x^4 + w^2x^3 + w^2x + 1$, $s'(x) = x^2 + w^2x + 1$. We can obtain that

$$g(x)s(x) = x^{14} + w^2x^{12} + x^{10} + x^4 + w^2x^2 + 1,$$
$$g^{\perp_q}(x) = x^4 + wx^3 + wx + 1, \quad s^{\perp_q}(x) = x^2 + wx + 1.$$

Due to

$$x^{14} + w^2x^{12} + x^{10} + x^4 + w^2x^2 + 1 = (x^{10} - 1)(x^4 + w^2x^2 + 1),$$

therefore, $d(x)=\gcd(g(x)s(x),x^{10}-1)=x^{10}-1$. We can obtain that

$$\begin{aligned}&a(x)=g(x)s(x)/d(x)=x^4+w^2x^2+1,\\&b(x)=(x^{10}-1)/d(x)=1,\\&-\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}=wx^{11}+x^{10}+x^2.\end{aligned}$$

Then, we verify the self-orthogonality of the QC code. Due to

$$\begin{aligned}&[x^8+w^2x^7+w^2x^6+x^5+x^3+w^2x^2+w^2x+1]\\=\ &[(x^2+wx+1)(x^6+x^5+x+1)],\end{aligned}$$

therefore,

$$[s(x)]=[s^{\perp q}(x)(x^6+x^5+x+1)].$$

Let $u(x)=x^6+x^5+x+1$, we can obtain $[s(x)]=[s^{\perp q}(x)u(x)]$, which satisfies condition (1) of Proposition 2. We can obtain that

$$\begin{aligned}u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}=\ &wx^{17}+w^2x^{16}+x^{15}+wx^{12}+w^2x^{11}\\&+x^{10}+x^8+x^7+x^3+x^2.\end{aligned}$$

Due to

$$\begin{aligned}&[wx^{17}+w^2x^{16}+x^{15}+wx^{12}+w^2x^{11}\\&+x^{10}+x^8+x^7+x^3+x^2]\\=\ &[(x^4+wx^3+wx+1)\\&\times(wx^{13}+x^{11}+x^{10}+x^4+w^2x^3+x^2)],\end{aligned}$$

therefore,

$$\begin{aligned}&[u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]\\=\ &[g^{\perp q}(x)(wx^{13}+x^{11}+x^{10}+x^4+w^2x^3+x^2)].\end{aligned}$$

We can obtain $[g^{\perp q}(x)]\mid[u(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]$, which satisfies condition (3) of Proposition 2. Due to

$$\begin{aligned}&[x^6+w^2x^5+wx^4+wx^3+wx^2+w^2x+1]\\=\ &[(x^2+wx+1)(x^4+x^3+x^2+x+1)],\end{aligned}$$

therefore,

$$[a(x)f(x)]=[s^{\perp q}(x)(x^4+x^3+x^2+x+1)].$$

Let $v(x)=x^4+x^3+x^2+x+1$, we can obtain $[a(x)f(x)]=[s^{\perp q}(x)v(x)]$, which satisfies condition (2) of Proposition 2. We can obtain that

$$\begin{aligned}&g(x)+v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}\\=\ &wx^{15}+w^2x^{14}+w^2x^{13}+w^2x^{12}+w^2x^{11}+x^{10}\\&+wx^5+w^2x^4+w^2x^3+w^2x^2+w^2x+1.\end{aligned}$$

Due to

$$\begin{aligned}&[wx^{15}+w^2x^{14}+w^2x^{13}+w^2x^{12}+w^2x^{11}\\&+x^{10}+wx^5+w^2x^4+w^2x^3+w^2x^2+w^2x+1]\\=\ &[(x^4+wx^3+wx+1)(wx^{11}+w^2x^9+x^8\\&+w^2x^7+x^6+w^2x^5+x^4+w^2x^3+x^2+x+1)],\end{aligned}$$

therefore,

$$\begin{aligned}&[g(x)+v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}]\\=\ &[g^{\perp q}(x)(wx^{11}+w^2x^9+x^8+w^2x^7+x^6+w^2x^5\\&+x^4+w^2x^3+x^2+x+1)].\end{aligned}$$

We can obtain $[g^{\perp q}(x)]\mid[(g(x)+v(x)\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'})]$, which satisfies condition (4) of Proposition 2.

Since the QC codes $\mathscr{C}_{q^2}(f,g,s)$ satisfy the conditions in Theorem 5, we can obtain a Hermitian self-orthogonal code $[20,6,8]_4$. Its Hermitian dual code is $[20,14,4]_4$. Then, a binary quantum stabilizer code with parameters $[[20,8,4]]_2$ can be obtained.

Example 2 Let $q=2$, $m=12$ and w represent a primitive element in $\mathbb{F}_4$. Choose the polynomials in $\mathbb{F}_4[x]$ as follows:

$$\begin{aligned}&g(x)=x+w^2,\\&s(x)=x^9+w^2x^8+w^2x^5+wx^4+wx+1,\\&f(x)=x^8+w^2x^7+w^2x^6+wx^4+x+1.\end{aligned}$$

Due to

$$\begin{aligned}x^{12}-1\ &=\ (x+w^2)(x^{11}+w^2x^{10}+wx^9+x^8+w^2x^7\\&\quad+wx^6+x^5+w^2x^4+wx^3+x^2+w^2x+w),\\x^{12}-1\ &=\ (x^9+w^2x^8+w^2x^5+wx^4+wx+1)\\&\quad\times(x^3+w^2x^2+wx+1),\end{aligned}$$

therefore,

$$\begin{aligned}x^{12}-1\ &=\ g(x)(x^{11}+w^2x^{10}+wx^9+x^8+w^2x^7\\&\quad+wx^6+x^5+w^2x^4+wx^3+x^2+w^2x+w),\\x^{12}-1\ &=\ s(x)(x^3+w^2x^2+wx+1).\end{aligned}$$

We can conclude that $g(x)\mid(x^{12}-1)$ and $s(x)\mid(x^{12}-1)$. Thus, $g'(x)=x^{11}+w^2x^{10}+wx^9+x^8+w^2x^7+wx^6+x^5+w^2x^4+wx^3+x^2+w^2x+w$, $s'(x)=x^3+w^2x^2+wx+1$. We can obtain that

$$\begin{aligned}g(x)s(x)\ &=\ x^{10}+wx^8+w^2x^6+x^4+wx^2+w^2,\\g^{\perp q}(x)\ &=\ w^2x^{11}+wx^{10}+x^9+w^2x^8+wx^7+x^6\\&\quad+w^2x^5+wx^4+x^3+w^2x^2+wx+1,\\s^{\perp q}(x)\ &=\ x^3+w^2x^2+wx+1.\end{aligned}$$

Due to

$$x^{12}-1=(x^{10}+wx^8+w^2x^6+x^4+wx^2+w^2)(x^2+w),$$

therefore, $d(x) = \gcd(g(x)s(x), x^{12}-1) = x^{10}+wx^8+w^2x^6+x^4+wx^2+w^2$. We can obtain that

$$\begin{aligned} a(x) &= g(x)s(x)/d(x) = 1, \\ b(x) &= (x^{12}-1)/d(x) = x^2+w, \\ -\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'} &= x^{24}+w^2x^{21}+wx^{19}+wx^{18}+x^{17}+w^2x^{14} \\ &\quad +x^{13}+wx^{11}+x^9+x^8+w^2x^7+w^2x^3. \end{aligned}$$

Then, we verify the self-orthogonality of the QC code. Due to

$$\begin{aligned} &[w^2x^{11}+wx^{10}+x^9+w^2x^8+wx^7+x^6 \\ &+w^2x^5+wx^4+x^3+w^2x^2] \\ &= [(x+w^2)(w^2x^{10}+x^8+wx^6+w^2x^4+x^2+w)], \end{aligned}$$

therefore,

$$[g^{\perp q}(x)] = [g(x)(w^2x^{10}+x^8+wx^6+w^2x^4+x^2+w)].$$

Let $u_1(x) = w^2x^{10}+x^8+wx^6+w^2x^4+x^2+w$, we can obtain $[g^{\perp q}(x)] = [g(x)u_1(x)]$, which satisfies condition (1) of Proposition 3. We can obtain that

$$\begin{aligned} u_1(x)a(x)f(x) &= w^2x^{18}+wx^{17}+w^2x^{16}+w^2x^{15}+x^{13} \\ &\quad +x^{11}+w^2x^{10}+wx^9+x^8+w^2x^7+x^6 \\ &\quad +w^2x^5+x^3+x^2+wx+w. \end{aligned}$$

Due to

$$\begin{aligned} &[w^2x^{18}+wx^{17}+w^2x^{16}+w^2x^{15}+x^{13}+x^{11} \\ &+w^2x^{10}+wx^9+x^8+w^2x^7+x^6 \\ &+w^2x^5+x^3+x^2+wx+w] \\ &= [(x^9+w^2x^8+w^2x^5+wx^4+wx+1)(w^2x^9 \\ &+w^2x^7+x^6+x^5+w^2x^4+w^2x^2+x+w)], \end{aligned}$$

therefore,

$$\begin{aligned} [u_1(x)a(x)f(x)] &= [s(x)(w^2x^9+w^2x^7+x^6+x^5 \\ &\quad +w^2x^4+w^2x^2+x+w)]. \end{aligned}$$

we can obtain $[s(x)] \mid [u_1(x)a(x)f(x)]$, which satisfies condition (3) of Proposition 3. Due to

$$\begin{aligned} &[x^{24}+w^2x^{21}+wx^{19}+wx^{18}+x^{17}+w^2x^{14} \\ &+x^{13}+wx^{11}+x^9+x^8+w^2x^7+w^2x^3] \\ &= [(x+w^2)(x^{23}+w^2x^{22}+wx^{21}+wx^{20}+x^{19}+x^{18} \\ &+x^{17}+wx^{16}+x^{15}+w^2x^{14}+x^{13}+wx^{12}+x^{11}+x^{10} \\ &+w^2x^9+w^2x^8+w^2x^7+x^6+w^2x^5+wx^4+x^3)], \end{aligned}$$

therefore,

$$\begin{aligned} &[\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}] \\ &= [g(x)(x^{23}+w^2x^{22}+wx^{21}+wx^{20}+x^{19}+x^{18} \\ &+x^{17}+wx^{16}+x^{15}+w^2x^{14}+x^{13}+wx^{12}+x^{11}+x^{10} \\ &+w^2x^9+w^2x^8+w^2x^7+x^6+w^2x^5+wx^4+x^3)]. \end{aligned}$$

Let $v_1(x) = x^{23}+w^2x^{22}+wx^{21}+wx^{20}+x^{19}+x^{18}+x^{17}+wx^{16}+x^{15}+w^2x^{14}+x^{13}+wx^{12}+x^{11}+x^{10}+w^2x^9+w^2x^8+w^2x^7+x^6+w^2x^5+wx^4+x^3$, we can obtain $[\bar{f}^q(x)\bar{b}^q(x)x^{\partial s'}] = [g(x)v_1(x)]$, which satisfies condition (1) of Proposition 3. We can obtain that

$$\begin{aligned} &s^{\perp q}(x)+v_1(x)a(x)f(x) \\ &= wx^{32}+x^{31}+x^{30}+wx^{29}+wx^{28}+wx^{27}+w^2x^{26} \\ &\quad +x^{25}+w^2x^{23}+w^2x^{22}+w^2x^{21}+w^2x^{20} \\ &\quad +wx^{18}+wx^{17}+x^{16}+wx^{14}+wx^{11} \\ &\quad +wx^{10}+wx^9+wx^6+x^5+wx^4+w^2x^3+1. \end{aligned}$$

Due to

$$\begin{aligned} &[wx^{32}+x^{31}+x^{30}+wx^{29}+wx^{28}+wx^{27} \\ &+w^2x^{26}+x^{25}+w^2x^{23}+w^2x^{22}+w^2x^{21}+w^2x^{20} \\ &+wx^{18}+wx^{17}+x^{16}+wx^{14}+wx^{11} \\ &+wx^{10}+wx^9+wx^6+x^5+wx^4+w^2x^3+1] \\ &= [(x^9+w^2x^8+w^2x^5+wx^4+wx+1) \\ &\times(x^{22}+w^2x^{21}+x^{20}+wx^{19}+x^{18}+x^{17}+w^2x^{16}+x^{15} \\ &+wx^{14}+wx^{13}+wx^{12}+wx^{11}+w^2x^{10}+x^9+w^2x^8 \\ &+w^2x^7+wx^6+w^2x^5+w^2x^4+x^3+w^2x^2+1)], \end{aligned}$$

therefore,

$$\begin{aligned} &[s^{\perp q}(x)+v_1(x)a(x)f(x)] \\ &= [s(x)(x^{22}+w^2x^{21}+x^{20}+wx^{19}+x^{18}+x^{17}+w^2x^{16} \\ &+x^{15}+wx^{14}+wx^{13}+wx^{12}+wx^{11} \\ &+w^2x^{10}+x^9+w^2x^8+w^2x^7+wx^6 \\ &+w^2x^5+w^2x^4+x^3+w^2x^2+1)]. \end{aligned}$$

We can obtain $[s(x)] \mid [s^{\perp q}(x)+v_1(x)a(x)f(x)]$, which satisfies conditions (4) of Proposition 3.

Since the QC codes $\mathscr{C}_{q^2}(f,g,s)$ satisfy the conditions in Theorem 5, we can obtain a Hermitian dual-containing code $[24,14,6]_4$. Its Hermitian dual code is $[24,10,8]_4$. Then, a binary quantum stabilizer code with parameters $[[12,4,6]]_2$ can be obtained.

5. New binary quantum stabilizer codes

In this section, we present record-breaking binary quantum stabilizer codes via our method. Record-breaking quantum stabilizer codes obtained via our method, with parameters $[[126,94,7]]_2$, $[[130,92,8]]_2$, and $[[136,96,8]]_2$ are presented. All the codes exhibit greater minimum distances, for the same length and dimension, than the binary quantum stabilizer codes listed in Ref. [27], and set new records. The construction details for these three quantum stabilizer codes

are provided below. Before we present the quantum stabilizer codes, we first provide some definitions and explanations that are used. We represent the elements $0,1,w,w^2$ as $0,1,2,3$ in $\mathbb{F}_4$ for brevity. Moreover, polynomials are simplified by representing them with their coefficient sequences in ascending order. The exponents of the coefficient sequences indicate the number of repeated coefficients. For instance, a polynomial $1+w^2x^3+wx^5$ over $\mathbb{F}_4$ can be written as 10^2302.

(1) $[[126,94,7]]_2$:

$$g(x) = 3120320312^212^332^2301301^221213^221213^221^430^321,$$

$s(x) = x^{63}-1$ and $f(x) = 0^21^20^231^20^33\ 203$. A Hermitian self-orthogonal code $[126,16,68]_4$ can be obtained. Its Hermitian dual code is $[126,110,7]_4$. The quantum code $[[126,94,7]]_2$ can be obtained from Theorem 5. Therefore, compared to the corresponding $[[126,94,6]]_2$ quantum stabilizer code in Ref. [27], this code is the best-known binary quantum stabilizer code with a length of 126 and dimension of 94, which improves the corresponding minimum distance record to 7.

(2) $[[130,92,8]]_2$:

$$g(x) = 103^20^22012^21^22^20^310^2202020^210^32^21^22^21020^23^201,$$

$s(x) = x^{65}-1$ and $f(x) = 0^21312102^2201^233^2231$. A Hermitian self-orthogonal code $[130,19,66]_4$ can be obtained. Its Hermitian dual code is $[130,111,8]_4$. The quantum code $[[130,92,8]]_2$ can be obtained from Theorem 5. Therefore, compared to the corresponding $[[130,92,7]]_2$ quantum stabilizer code in Ref. [27], this code is the best-known binary quantum stabilizer code with a length of 130 and dimension of 92, which improves the corresponding minimum distance record to 8.

(3) $[[136,96,8]]_2$:

$$g(x) = 1203^221^2032^3020210^23^423^40^2120202^3301^223^2021,$$

$s(x) = x^{68}-1$ and $f(x) = 301310^22^301^42312^2$. A Hermitian self-orthogonal code $[136,20,64]_4$ can be obtained. Its Hermitian dual code is $[136,116,8]_4$. The quantum code $[[136,96,8]]_2$ can be obtained from Theorem 5. Therefore, compared to the corresponding $[[136,96,7]]_2$ quantum stabilizer code in Ref. [27], this code is the best-known binary quantum stabilizer code with a length of 136 and dimension of 96, which improves the corresponding minimum distance record to 8.

Additionally, in Table 1, we present 10 record-breaking quantum stabilizer codes derived from the above three quantum stabilizer codes by Theorem 3, and compare them with the quantum stabilizer codes of the same length and dimension in Grassl's table.[27]

Table 1. New binary quantum stabilizer codes.

No.	Our quantum stabilizer codes	Quantum stabilizer codes[a)]
1	$[[126,94,7]]_2$	$[[126,94,6]]_2$
2	$[[126,93,7]]_2^*$	$[[126,93,6]]_2$
3	$[[127,94,7]]_2^*$	$[[127,94,6]]_2$
4	$[[130,92,8]]_2$	$[[130,92,7]]_2$
5	$[[131,92,8]]_2^*$	$[[131,92,7]]_2$
6	$[[132,92,8]]_2^*$	$[[132,92,7]]_2$
7	$[[133,92,8]]_2^*$	$[[133,92,7]]_2$
8	$[[136,96,8]]_2$	$[[136,96,7]]_2$
9	$[[136,95,8]]_2^*$	$[[136,95,7]]_2$
10	$[[137,96,8]]_2^*$	$[[137,96,7]]_2$

[a)] Ref. [27], *obtained from Theorem 3.

6. Conclusion

In this work, we study the algebraic structure of a generalized class of QC codes of index 2. We prove that any QC codes of index 2 can be transformed into our proposed QC codes with standard generator forms. We identify the generator forms of their Hermitian dual codes and establish the necessary and sufficient conditions for the Hermitian self-orthogonality and dual-containing property of our QC codes. We present a systematic method for constructing quantum stabilizer codes from the QC codes of index 2 via the Hermitian construction method. For the application, we construct 10 record-breaking quantum stabilizer codes. These results demonstrate that our method is superior and effective at constructing record-breaking quantum stabilizer codes. In subsequent research, we will construct record-breaking quantum stabilizer codes via self-orthogonal QC codes with symplectic inner products.

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