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Integrable decompositions and superposed nonlinear solutions of (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada equation

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The phenomenon of shallow water waves in nature attracts the attention of scholars and plays an important role in fields such as marine ecology, tidal waves, solitary waves, and offshore engineering. To better understand the phenomenon of shallow water waves, we investigate the $(2+1)$-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada (2D-CDGKS) equation from the perspective of integrable decomposition. By utilizing the Lax pairs and formal variable separation (FVS) method, the 2D-CDGKS equation can be decomposed into the Sharma–Tasso–Olver (STO) equation, the integrable Svinolupov–Sokolov (SS) equation, and the Sawada–Kotera (SK) equation. We construct some novel exact solutions by linear superposing the integrable decomposition relations. Additionally, the superposition of two-soliton solution with three-soliton solution, two-soliton solution propagating on periodic cnoidal background waves, and soliton–cnoidal wave interaction solutions with interesting dynamics, are explored. Our results have important significance for understanding of the physical events consolidate the complex system under consideration and offering vital insights into the intricate dynamics of its behavior. Furthermore, the present work will enrich the investigation of nonlinear dynamics in high-dimensional nonlinear system, and provide theoretical support for the related experimental phenomena.

Keywords: $(2+1)$-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada equation, integrable decomposition, linear superposed solutions, nonlinear wave interaction

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1. Introduction

Nonlinear partial differential equations (NPDEs) are indispensable mathematical tools for understanding the complex dynamic behavior of real natural phenomena. They have been extensively permeated in several disciplines, including quantum mechanics, fluid mechanics, nonlinear optics, water waves, biology.[1–4] Since the NPDEs are the most representative mathematical models of nonlinear phenomena indicated above, it has aroused the curiosity of scholars and carried out plenty of substantial explorations on NPDEs.[5,6] Therefore, NPDEs have emerged as a hot research topic in the current field of mathematical physics.[7,8]

In order to discover and understand the essence of complex nonlinear phenomena inherent in NPDEs, it is imperative to investigate methods for finding exact solutions of NPDEs. In contrast to nonlinear systems with finite freedoms described by ordinary differential equations (ODEs), there is currently no universal analytical theoretical method available to obtain exact solutions for systems with infinite freedoms described by NPDEs. In recent years, some effective method with various exact solutions for different types of NPDEs have been identified,[9,10] encompassing the inverse scattering transformation,[11] Hirota bilinear derivative method,[12–14] Painlevé analysis method,[15] Darboux transformation approach,[16,17] and improved physics-informed neural network method.[17,18] The widespread application of these NPDEs solving methods are significantly important to simulate complex nonlinear phenomena more comprehensively, which enabled researchers regarding the exact analytical solutions of NPDEs are hot topic.[19–21]

Integrable systems are important category of NPDEs, possessing a rich variety of exact analytical solutions, Lax pairs, recursive operators, and infinite conservation laws. Recently, Alexandrov imparted a provocative insight: integrable systems are the important part of modern mathematics and theoretical physics.[22] Establishing connections between different integrable systems can uncover unexpected relationships between different mathematical physics problems, thereby solving related problems. In certain physical scenarios, the originally integrable system cannot directly reflect the mechanism of manipulating nonlinear waves (specific types of exact analytical solutions). Fortunately, it becomes feasible within the subsystems obtained after integrable decomposition. Nevertheless, despite the reasonable improvement and expansion of the aforementioned methods, research on uncovering potential integrable decompositions of NPDEs, obtaining new types of exact analytical solutions, and clarifying the intrinsic connections among various integrable systems re-

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mains inadequate. If a complex equation or system can be decomposed into several relatively simple and solvable parts, then it is said to possess an integrable decomposition. Integrable decomposition offers a powerful tool for studying complex mathematical and physical systems. By breaking down complex problems into simpler, solvable parts, it enables researchers to gain a deeper understanding of and solve these issues.[23,24] To exemplify this concept, the integrable decomposition method is applied to the generalized coupled (2+1)-dimensional Sharma–Tasso–Olver–Burgers (STOB) equation with dissipative and dispersive effects, revealing that this dissipation-dispersion equation can be solved for exact solutions via a dispersionless system.[25] The method is further extended to investigate the multi-component decomposition of the BKP equation hierarchy and its linear superposition solutions; therefrom, a new integrable coupled KdV-type system with abundant high-order symmetries is derived.[26] Thus, the study of integrable decomposition is very important.[27,28]

Formal variable separation (FVS) method[29,30] is a fundamental tool in the study of integrable decomposition,[31–33] which can apply to non-Lax integrable systems due to this method do not rely on the Lax pairs of nonlinear systems. In terms of FVS method, its essence is to use a set of self-consistent over-determined and formally variable-separated low-dimensional equations to solve high-dimensional nonlinear equations, which provides a way to solve high-dimensional problems and reveals the inherent structure and properties of NPDEs. Based on the FVS method, we can also indicate the intrinsic relationship between classical integrable systems. Specifically, Hao and Lou constructed linear superposition solutions of nonlinear systems such as the BKP equation, Heavenly equation, and sine-Gordon equation based on the separation of variables method.[34–37] However, the existing improved FVS methods still cannot effectively solve the accurate description of wave formation, propagation, and interaction mechanisms in different nonlinear system environments (plasma physics, nonlinear optics). Thus, it cannot help researchers understand complex wave phenomena. This is primarily because the localized waves involved in various contexts in nonlinear systems ranging from fluid dynamics to nonlinear optics remain challenging to predict, control, and comprehend their transitions.

Considering the above issues, the main research objectives of this article are listed as follows.

(i) To investigate the integrable decomposition relationship of the 2D-CDGKS model in the sense of Lax integrability based on the FVS method, as well as the impact of its parameters on nonlinear wave dynamics.

(ii) To identify a simple yet effective mathematical tool for constructing complex waveforms holds significant importance in predicting ocean disasters and facilitating soliton fiber optic communication.

Thus, our research mainly focuses on the linear superpositions of the special decomposition for 2D-CDGKS equation, obtained by the new integrable decomposition method are still the solutions of the original equation under certain constraints, and study the salient dynamics characteristics of linear superpositions solutions achieved via simple systematic technique and graphical demonstrations.

To achieve above, in the forthcoming Sections 2 and 3 of this article, taking the 2D-CDGKS equation as simple example, the integrable decomposition relations are explicitly given via the formal variable separation (FVS) method, it is revealed that these integrable decomposition are all KdV decomposition and explored the intrinsic relationship between the Sharma–Tasso–Olver (STO) equation, the integrable Svinolupov–Sokolov (SS) equation, and the Sawada–Kotera (SK) equation. Advancing to Section 4, we delve into the linear combinations of these integrable decomposition and construct abundant linear superposed solutions of the 2D-CDGKS equation, and we also utilized the symbolic computations and categorical graphical illustrations to intuitively investigate the impact of parameters and constraints on the dynamic behavior of the solutions. The conclusion section gives a summary and discussion.

2. Derivation of integrable decomposition relations

Inspired by the existing interesting research results, we anchor our discussion in the higher-order generalization of the ubiquitous Korteweg–de Vries (KdV) equation, which leads to the 2D-CDGKS equation:

$$\begin{aligned}&36u_t+u_{x5}+15u_xu_{xx}+15uu_{x3}+45u^2u_x-5u_{xxy}\\&-15uu_y-15u_x\partial_x^{-1}u_y-5\partial_x^{-1}u_{yy}=0,\end{aligned}\tag{1}$$

where $u\equiv u(x,y,t), u_x\equiv\partial_x u, u_{xx}\equiv\partial_x^2 u$, and the operator ∂_x^{-1} denotes the inverse operator of ∂_x.

By a transformation, the 2D-CDGKS Eq. (1) is equivalent to the following system

$$\begin{aligned}&36u_t+u_{x5}+15u_xu_{xx}+15uu_{x3}+45u^2u_x\\&-5u_{x3}-15uv_x-15u_xv+5v_y=0,\\&u_y=v_x,\end{aligned}\tag{2}$$

which is one of the most mystery models having significant integrability and has rich applications in nonlinear dispersive physical phenomena. In order to obtain integrable decomposition relations of Eq. (1), we need to consider the potential form of Eq. (1) by using $u=w_x$,

$$\begin{aligned}36w_{xt}\ =\ &-w_{x6}-15w_{xx}w_{x3}-15w_xw_{x4}-45w_x^2\\&+5w_{x3y}+15w_xw_{xy}+15w_{xx}w_y+5w_{yy}.\end{aligned}\tag{3}$$

We all know that the symmetry of the field quantity w is the specific solution of the linearized equation of the potential CDGKS (pCDGKS) Eq. (3)

$$\begin{aligned} 36\sigma_{xt} &= (5\sigma_{x3}+15\sigma_x w_y+15w_x\sigma_y-\sigma_{x5}-15\sigma_x w_{x3} \\ &\quad -15w_x\sigma_{x3}-45\sigma_x w_x^2-5\partial^{-1}\sigma_{yy})_x, \end{aligned} \tag{4}$$

which means that equation (4) is invariant under the infinitesimal transformation $w \to w+\epsilon\sigma$ with infinitesimal parameter ϵ ($\epsilon \ll 1$). Generally, symmetry properties can provide clues and methods for finding exact or approximate solutions to integrable models, such as formally separating variables for a high-dimensional integrable model. Thus, we try to apply the symmetry constraints on its Lax pair to separate variables for the $(2+1)$-dimensional pCDGKS Eq. (3). After surveying relevant studies, it becomes evident that the $(2+1)$-dimensional pCDGKS equation possesses the Lax pair

$$\psi_y = 3w_x\psi_x+\psi_{x3}, \tag{5}$$

$$\begin{aligned} 12\psi_t &= 3\psi_{x5}+15w_x\psi_{x3}+15w_{x2}+15w_x^2\psi_x \\ &\quad +5w_y\psi_x+10w_{x3}\psi_x. \end{aligned} \tag{6}$$

By inserting $\sigma=\psi_x$ with Eqs. (5) and (6) into Eq. (4), one can ascertain that ψ_x is exactly one of the non-local symmetry of the pCDGKS equation (3). Substituting the symmetry constraints $\psi_x = w_x$, that is, $\psi = w$ into the Lax pairs (5) and (6), the resulting equations appear as

$$w_y = 3w_x^2+w_{x3}, \tag{7}$$

$$4w_t = w_{x5}+10w_xw_{x3}+5w_{x2}w_{x3}+10w_x^3. \tag{8}$$

It is well-suited that equation (7) is not explicitly t-dependent and equation (8) is not explicitly y-dependent, which means that the $(2+1)$-dimensional pCDGKS Eq. (3) has now been decomposed into a pair of $(1+1)$-dimensional equations, namely the KdV equation (7) and the fifth-order KdV equation (8), respectively. Then it is also a solution of the pCDGKS Eq. (3). The integrable decomposition obtained via the above variable separation method is also known as nonlinearity because w is still a function of variables y and t. This method is also named the formal variable separation method.

We assume that any $(2+1)$-dimensional nonlinear evolution equation possesses a solution in the decomposed form

$$w_y = F(w,w_x,w_{x2},\ldots,w_{xm}), \tag{9}$$

$$w_t = G(w,w_x,w_{x2},\ldots,w_{xn}), \tag{10}$$

and the compatibility constraint condition

$$w_{yt}-w_{ty}=[F,G]=0 \tag{11}$$

should be satisfied. Using symbolic calculation method and combining pCDGKS Eq. (3) and the compatibility conditions (11), we can obtain the exact forms of the functions F and G. In detail, the following decomposition proposition is proposed by us.

Proposition 1 If w_1, w_2, w_3, w_4, w_5, and w_6 are solutions of the following integrable decomposed systems

$$\begin{aligned} w_{1y} &= w_{1x3}+3w_{1x}^2+c_0w_{1x}+c_1, \\ w_{1t} &= \frac{1}{4}w_{1x5}+\frac{5}{2}w_{1x}w_{1x3}+\frac{5}{12}c_0w_{1x3}+\frac{5}{2}w_{1x}^3 \\ &\quad +\frac{5}{4}c_0w_{1x}^2+\frac{5}{36}c^2w_{1x}+\frac{5}{36}c_1w_{1x}+\frac{5}{4}w_{1x2}^2, \end{aligned} \tag{12}$$

$$\begin{aligned} w_{2y} &= w_{2x3}+\frac{3}{2}w_{2x}^2+c_1, \\ w_{2t} &= \frac{1}{4}w_{2x5}+\frac{5}{4}w_{2x}w_{2x3}+\frac{5}{8}w_{2x2}^2 \\ &\quad +\frac{5}{8}w_{2x}^3+\frac{5}{12}c_1w_{2x}+c_2, \end{aligned} \tag{13}$$

$$\begin{aligned} w_{3y} &= -\frac{1}{2}w_{3x3}-\frac{3}{2}w_{3x}^2+c_0w_{3x}+c_1, \\ w_{3t} &= -\frac{1}{16}w_{3x5}-\frac{5}{8}w_{3x3}w_{3x}-\frac{5}{8}w_{3x}^3 \\ &\quad +\frac{5c^2}{36}w_{3x}+\frac{5}{12}c_1w_{3x}-\frac{5}{16}w_{3x2}^2+c_3, \end{aligned} \tag{14}$$

$$\begin{aligned} w_{4y} &= w_{4x3}+\frac{3}{2}w_{4x}^2+\frac{3}{4}W^2w_{4x}+\frac{3}{2}Ww_{4x2}, \\ W &\equiv w_4+\frac{2}{3}c_0, \\ w_{4t} &= \frac{1}{4}w_{4x5}+\frac{5}{8}Ww_{4x4}+\left(\frac{5}{8}W^2+\frac{15}{8}w_{4x}\right)w_{4x3} \\ &\quad +\left(\frac{25}{8}Ww_{4x}+\frac{5}{16}W^3\right)w_{4x2} \\ &\quad +\frac{15}{16}W^2w_{4x}^2+\frac{5}{64}W^4w_{4x}+\frac{5}{4}w_{4x2}^2+\frac{15}{16}w_{4x}^3, \end{aligned} \tag{15}$$

$$\begin{aligned} w_{5y} &= w_{5x3}-\frac{3}{4}\frac{w_{5x2}^2}{W_1}+\frac{3}{2}w_{5x}^2+c_1, \\ W_1 &\equiv w_{5x}+c_0, \\ w_{5t} &= \frac{1}{4}w_{5x5}-\frac{5}{8}\frac{w_{5x2}w_{5x4}}{W_1}-\frac{5}{16}\frac{w_{5x3}^2}{W_1} \\ &\quad +\left(\frac{5}{4}w_{5x}+\frac{5}{4}\frac{w_{5x2}^2}{W_1^2}\right)w_{5x3}+\left(\frac{5}{8}-\frac{15}{16}\frac{w_{5x}}{W_1}\right)w_{5x2}^2 \\ &\quad -\frac{35}{64}\frac{w_{5x2}^4}{W_1^3}+\frac{5}{8}w_{5x}^3+\frac{5c_1}{12}w_{5x}+c_0, \end{aligned} \tag{16}$$

$$\begin{aligned} w_{6y} &= c_0w_{6x}+c_1, \\ w_{6t} &= -\frac{1}{36}w_{5x5}-\frac{5}{12}w_{6x}w_{6x3}+\frac{5c_0}{36}w_{6x3}-\frac{5}{12}w_{6x}^3 \\ &\quad +\frac{5c_0^2}{36}w_{6x}+\frac{5c}{12}w_{6x}^2+\frac{5c_1}{12}w_{6x}+c_2, \end{aligned} \tag{17}$$

with c_i ($i=0,1,2$) being arbitrary constants, then w_i ($i=1,2,\ldots,6$) are also solutions of the pCDGKS Eq. (3).

The detailed proof process of Proposition 1 is given in the subsequent appendix. According to the existing literature, the first three decomposition relations (12)–(14) belong to KdV-type integrable decompositions. When the constants c_0, c_1, and c_2 take the value 0, the integrable decomposition relation (12) is reduced to the decomposition rela-

tions (7) and (8) under symmetry constraint. The first equation of the decomposition relations (15) is a specific form of the Svinolupov–Sokolov (SS) equation which discussed in the related investigation. The decomposition (16) is related to the celebrated Sharma–Tasso–Olver (STO, a class of higher-order modified KdV equation).[38,39] The decomposition (17) is just a Sawada–Kotera-type decomposition. Based on the above integrable decomposition relationships, we can construct a new type of nonlinear local excitation solutions–linear superposed solutions of the (2+1)-dimensional pCDGKS equation.

3. Linear superposed solutions inherent in the 2D-CDGKS equation

Starting from a known solution to the NPDEs, another solution to the NPDEs is determined through a specific relationship. In this cycle, a plethora of exact solutions to the equation can be generated. The above mentioned specific relationship can be called Bäcklund transformation/Darboux transformation.[40–42] Actually, the above type of transformation reflects the common nonlinear superposition relationship that exists between the solutions of nonlinear systems. In nonlinear systems, linear superposition of solutions cannot hold as a general principle. However, the research findings obtained on the basis of the 2D-CDGKS equation indicate that the linear combination of specific integrable decompositions derived through the new integrable decomposition method remains the solution of the original equation under certain constraint, which are linear superposed solutions inherent in the NPDEs.

Proposition 2 If w_1, w_2, w_3, w_4, w_5, and w_6 are the solutions of the pCDGKS Eq. (3) with the conditions

$$
\begin{aligned}
w_{1y} &= w_{1x3}+3w_{1x}^2+c_0w_{1x},\\
w_{1t} &= \frac{1}{4}w_{1x5}+\left(\frac{5}{12}w_{1x3}+\frac{5}{4}w_{1x}^2\right)c_0\\
&\quad+\frac{5c_0^2}{36}w_{1x}+\frac{2}{5}w_{1x}w_{1x3}+\frac{5}{2}w_{1x}^3+\frac{5}{4}w_{1x2}^2+c_1, \qquad (18)\\
w_{2y} &= w_{2x3}+3w_{2x}^2-c_0w_{2x},\\
w_{2t} &= \frac{1}{4}w_{2x5}-\left(\frac{5}{12}w_{2x3}+\frac{5}{4}w_{2x}^2\right)c_0\\
&\quad+\frac{5c_0^2}{36}w_{2x}+\frac{2}{5}w_{2x}w_{2x3}+\frac{5}{2}w_{2x}^3+\frac{5}{4}w_{2x2}^2+c_1, \qquad (19)\\
w_{3y} &= w_{3x3}+3w_{3x}^2,\\
w_{3t} &= \frac{1}{4}w_{3x5}+\frac{5}{2}w_{3x}w_{3x3}+\frac{5}{2}w_{3x}^3+\frac{5}{4}w_{3x2}^2, \qquad (20)\\
w_{4y} &= w_{4x3}+\frac{3}{2}w_{4x}^2,\\
w_{4t} &= \frac{1}{4}w_{4x5}+\frac{5}{4}w_{4x}w_{3x3}+\frac{5}{8}w_{4x}^3+\frac{5}{8}w_{4x2}^2, \qquad (21)\\
w_{5y} &= w_{5x3}+\frac{3}{2}w_{5x}^2-\frac{c_2^2}{6},\\
w_{5t} &= \frac{1}{4}w_{5x5}+\frac{5}{4}w_{5x}w_{4x3}+\frac{5}{8}w_{5x}^3+\frac{5}{8}w_{5x2}^2-\frac{5c_2^2}{72}w_{5x}, \qquad (22)\\
w_{6y} &= w_{6x3}+\frac{3}{2}w_{6x}^2-\frac{c_2^2}{6},\\
w_{6t} &= \frac{1}{4}w_{6x5}+\frac{5}{4}w_{6x}w_{6x3}+\frac{5}{8}w_{6x}^3+\frac{5}{8}w_{6x2}^2-\frac{5c_2^2}{72}w_{6x}, \qquad (23)
\end{aligned}
$$

with c_i $(i=0,1,2)$ being arbitrary constants, then the following linear superpositions

$$w_7=w_1+w_2,\quad w_8=\frac{1}{2}(w_3+w_4),\quad w_9=w_5+\frac{1}{2}w_6 \qquad (24)$$

are also solutions of the pCDGKS Eq. (3).

Proof By inserting the above linear superposition relations as shown in Eq. (24) along with the integrable decomposition results from Eqs. (12)–(17) into pCDGKS Eq. (3), one is able to directly prove the proposition.

As can be seen from Proposition 1, the pCDGKS equation has six sets of possible integrable decomposition relations. However, upon careful verification, it was discovered that only the first two types of integrable decompositions, namely Eqs. (12) and (13), can still construct new types of nonlinear localized excitation solutions after appropriate linear superposition.

Next, we construct several typical linear superposition solutions and conduct a detailed study on their dynamical behaviors for the 2D-CDGKS Eq. (2). Furthermore, we investigate the propagation and superposition dynamics and provide graphical illustrations for a better understanding.

Case 1 Superposed phenomena between a two-soliton solution and a three-soliton solution

In this case, we construct a two-soliton solution and a three-soliton solution. For this purpose, we select the following exponential test function as an initial seed solution[43]

$$
\begin{aligned}
w_1 &= 2(\ln f_2^-)_x,\\
f_2^- &= 1+\exp(\xi_1)+\exp(\xi_2)+a_{12}^-\exp(\xi_1+\xi_2),\\
w_2 &= 2(\ln f_3^+)_x,\\
f_3^+ &= 1+\exp(\eta_1)+\exp(\eta_2)+\exp(\eta_3)+a_{12}^+\exp(\eta_1+\eta_2)\\
&\quad+a_{23}^+\exp(\eta_2+\eta_3)+a_{13}^+\exp(\eta_1+\eta_3)\\
&\quad+a_{12}^+a_{23}^+a_{13}^+\exp(\eta_1+\eta_2+\eta_3), \qquad (25)\\
\xi_{i-} &= k_{i-}+k_{i-}(k_{i-}^2-c_0)y\\
&\quad+\left(\frac{1}{4}k_{i-}^5-\frac{5}{12}c_0k_{i-}^3+\frac{5}{36}c_0^2k_{i-}\right)t,\\
\eta_{i+} &= k_{i+}+k_{i+}(k_{i+}^2+c_0)y\\
&\quad+\left(\frac{1}{4}k_{i+}^5+\frac{5}{12}c_0k_{i+}^3+\frac{5}{36}c_0^2k_{i+}\right)t,\\
a_{ij}^{\mp} &= \frac{(k_{i\mp}-k_{j\mp})^2}{(k_{i\mp}+k_{j\mp})^2}, \qquad (26)
\end{aligned}
$$

with $k_{i\mp}$, c being arbitrary constants. Combining expressions (25) and (26), we get the superposed phenomenon between a two-soliton solution and a three-soliton solution with

the constraint conditions $c_0 = 0.15$, $k_{1-} = 0.8$, $k_{2-} = -1.25$, $k_{1+} = -1$, $k_{2+} = -1.25$, $k_{3+} = 1.5$ at $y = 0$ plane as depicted in Fig. 1. The above solution (25) with expressions (26) is nothing but the 2D-CDGKS equation (1) with $u = w_x$ and expressed in Proposition 2 with Eqs. (18)–(23) and the linear superpositions solutions (24).

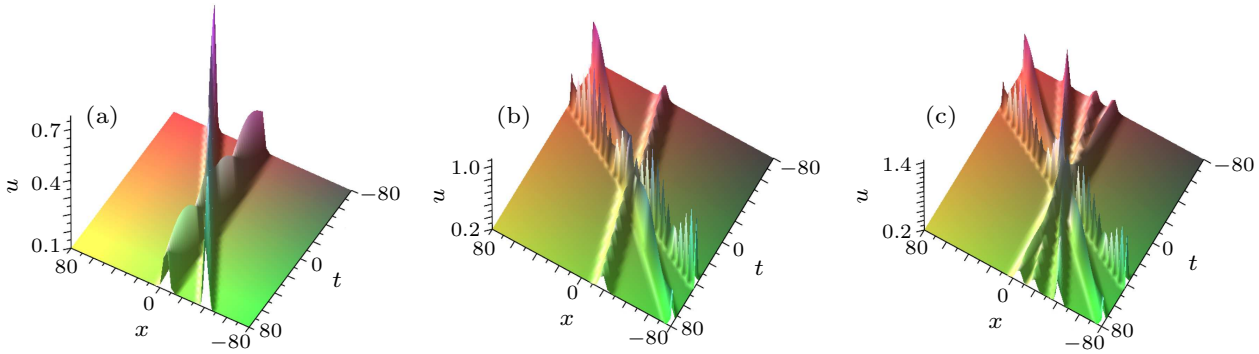


Fig. 1. The superposed phenomenon between a two-soliton and a three-soliton: (a) two-soliton solution $u = w_{1x}$, where w_1 is defined by Eq. (25) for the decomposition relation (18); (b) three-soliton solution $u = w_{2x}$ is defined by Eq. (25) for the decomposition relation (19); (c) the linear superposition of the two-soliton and the three-soliton yields $u = w_{1x} + w_{2x}$.

Case 2 Superposed phenomena between a two-soliton solution and a periodic cnoidal solution

Proceeding further, we discuss the superposed phenomena between a two-soliton solution and a periodic cnoidal solution. In case 1, if one of w_1 and w_2 is replaced by a periodic cnoidal solution, then the linear combination of the two-soliton and the periodic cnoidal structure will further enrich the solution patterns, making them more complex and intriguing. Thus, we select the traveling wave solutions of Eq. (18) in the form[44]

$$w_2 = W(k_0x + s_0y + \omega_0t) \equiv W(X). \tag{27}$$

Inserting Eq. (27) into Eq. (19), we have

$$\begin{aligned}
&-3W_X^2k_0^2 + (c_0k_0 + s_0)W_X - W_{XXX}k_0^3 = 0,\\
&\frac{5}{4}c_0k_0^2W_X^2 + \left(\omega_0 - \frac{5}{36}c_0^2k_0 - \frac{5}{2}k_0^4W_{XXX}\right)W_X\\
&+ \frac{5}{12}c_0k_0^3W_{XXX} - \frac{1}{4}k_0^5W_{XXXXX}\\
&- \frac{5}{2}W_X^3k_0^3 - \frac{5}{4}W_{XX}^2k_0^4 - c_1 = 0.
\end{aligned} \tag{28}$$

One of the essential aspects of soliton theory is the evolution of nonlinear waves on variable backgrounds. In order to search for the periodic cnoidal background waves of Jacobi elliptic functions, we assume that

$$\frac{\mathrm{d}W}{\mathrm{d}X} = \beta_0 + \beta_1 \mathrm{cn}(k_0x + s_0y + \omega_0t|m_0), \tag{29}$$

where cn are Jacobi elliptic function with elliptic modulus m_0. Among them, m_0 indicates not only the modulus of the Jacobi elliptic function, but also the amplitude of the Jacobi elliptic function. Substituting expression (29) into Eq. (19), we get the following parameters constrains

$$\begin{aligned}
\beta_0 &= k_0Z_0, \quad \beta_1 = 2k_0m_0^2,\\
s_0 &= 6Z_0k_0^3 + k_0^3(8m_0^2 - 4) - c_0k_0,\\
c_1 &= \left(\frac{4}{9}m_0^2 - \frac{4}{9}m_0^4 - \frac{4}{9}\right)k_0^6Z_0\\
&\quad - \frac{4}{9}k_0^6m_0^2(m_0^2 - 1)(2m_0^2 - 1),\\
\omega_0 &= -\frac{5}{2}Z_0c_0k_0^3 + (4m_0^4 - 4m_0^2 + 4)k_0^5\\
&\quad + \left(\frac{5}{3}c_0 - \frac{10}{3}m_0^2c_0\right)k_0^3 + \frac{5}{36}c_0^2k_0,\\
Z_0 &= -\frac{4}{3}m_0^2 + \frac{2}{3} \pm \sqrt{m_0^4 - m_0^2 + 1}.
\end{aligned} \tag{30}$$

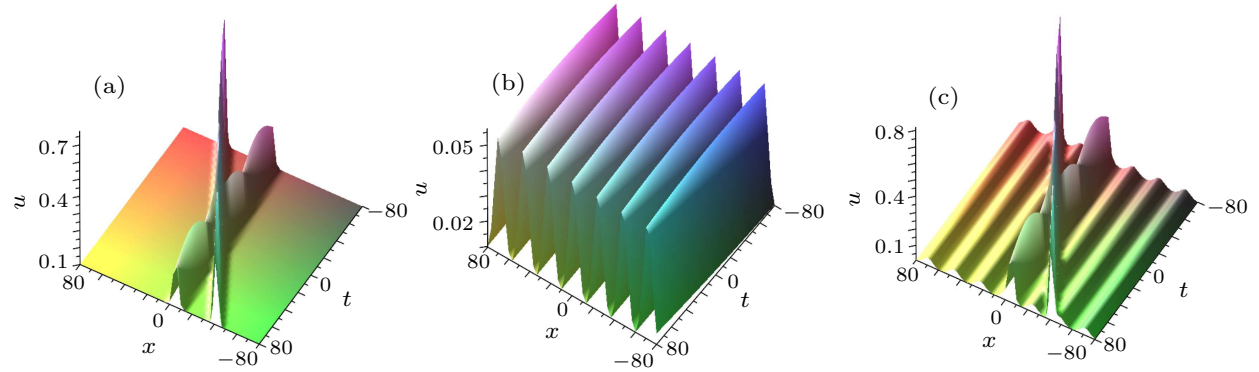


Fig. 2. The superposed phenomenon between a two-soliton and a periodic cnoidal solution: (a) two-soliton solution $u = w_{1x}$, where w_1 is defined by Eq. (25) for the decomposition relation (18); (b) periodic cnoidal solution $u = w_{2x}$ is defined by Eq. (25) for the decomposition relation (19); (c) the linear superposition of the two-soliton and the periodic cnoidal $u_7 = w_{1x} + w_{2x}$.

After integrating solution (29) over X once, we obtain the decomposed periodic wave solutions for Eq. (19)

$$\begin{aligned} w_2 &= 2k_0 \mathrm{E_e}(\mathrm{sn}(k_0x+s_0y+\omega_0t|m_0)|m_0) \\ &\quad +(k_0Z_0+2k_0m_0^2-2k_0)(k_0x+s_0y+\omega_0t), \end{aligned} \tag{31}$$

where E_e is the second incomplete elliptic integral, sn is Jacobi elliptic function with elliptic modulus m_0. Taking $c_0=-0.15$, $m_0=0.85$, $k_0=0.18$ in the decomposed periodic wave solutions (31) with parameter constraints (30), we derive the solutions $u=w_{7x}=w_1+w_2$ which can describe the superposed phenomenon between single two-soliton wave and one periodic cnoidal wave, as shown in Fig. 2. As can be seen from Fig. 2, the amplitude and shape of the two-soliton wave remain unchanged after superposition. Meanwhile, the amplitude and shape of the periodic cnoidal wave, which acts as the background wave, have also remain unchanged.

Case 3 Superposed phenomena between a two-soliton solution and a soliton–cnoidal interaction solution

In order to study the superposed phenomena between a two-soliton solution and a soliton–cnoidal interaction solution, we establish the connection between the decomposed solution (20) and the standard elliptic function. By utilizing a specific solution of the elliptic equation in the form of Jacobi elliptic function, the detailed expression for the interaction solution between a soliton and a cnoidal periodic wave is obtained.[45] If w_3 is fixed as

$$w_{3x}=f_x^2+\frac{1}{4}\frac{f_{x2}^2}{f_x^2}-\frac{1}{2}\frac{f_{x3}}{f_x}+2f_{x2}\tanh(f)-2f_x^2\tanh^2(f), \tag{32}$$

and combined with the above integrable decomposition relationship (20), the following compatibility conditions should be satisfied

$$f_{x3}-2f_x^3+f_y-\frac{3}{2}\frac{f_{x2}^2}{f_x}+\lambda f_x=0, \tag{33}$$

$$f_{y2}+\frac{1}{9}f_{xt}-\frac{f_{xy}f_y}{f_x}+\frac{2\lambda}{3}\frac{f_{x2}f_y}{f_x}-\frac{1}{9}\frac{f_{x2}f_t}{f_x}-\frac{2\lambda}{3}f_{xy}=0, \tag{34}$$

wherein λ is an arbitrary constant. It should be noted that equations (33) and (34) are compatible with each other. That is to say, $f_{x3y2}=f_{y2x3}$.

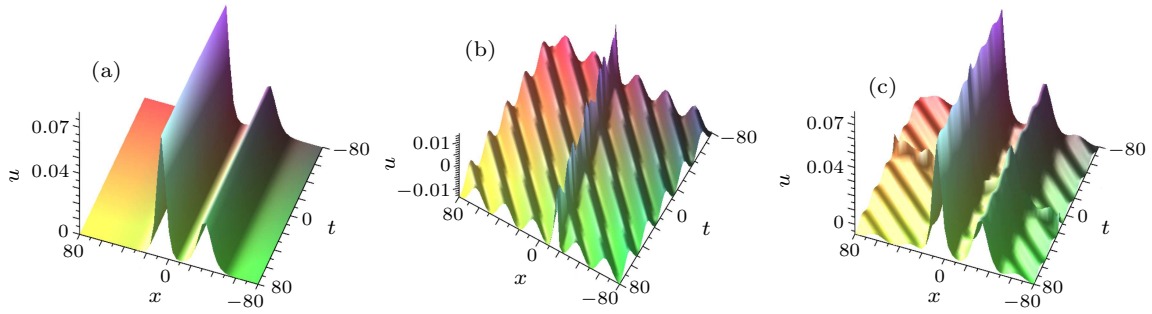


Fig. 3. The superposed phenomenon between a two-soliton and a periodic cnoidal–soliton interaction solution: (a) two-soliton solution $u=w_{4x}=w_2|_{c_0=0}$, where w_2 is defined by Eq. (26) with Eq. (25) for decomposition relation (18); (b) periodic soliton–cnoidal solution $u=w_{3x}$ is defined by Eq. (39) with Eqs. (36), (38), and (40) for decomposition relation (20); (c) the linear superposition of the two-soliton and the periodic cnoidal $u=\frac{1}{2}w_{3x}+w_{4x}$.

To obtain the soliton–cnoidal periodic interaction solution, we examine the solution formed by the combination of different functions

$$\begin{aligned} f &= k_1x+l_1y+\omega_1t+W(kx+ly+\omega t) \\ &\equiv k_1x+l_1y+\omega_1t+W(Z), \end{aligned} \tag{35}$$

where k, k_1, l, l_1, ω, ω_1 are real constants. Substituting ansatz (35) into Eq. (34), we naturally obtain the following expression

$$\lambda=\frac{k\omega_1-k_1\omega}{6(kl_1-k_1l)}+\frac{3l}{2}. \tag{36}$$

Next, inserting Eq. (35) into the condition equation (33) and letting $W_Z=W_1(Z)$, we find that equation (33) transforms to be a standard elliptic equation

$$W_{1Z}^2=\mu_4W_1^4+\mu_3W_1^3+\mu_2W_1^2+\mu_1W_1+\mu_0, \tag{37}$$

with coefficients μ_i $(i=0,1,2,3,4)$ taking the form

$$\begin{aligned} \mu_0 &= \frac{k_1^3\mu_3l_1}{k^2(kl_1-lk_1)}-\frac{((l\mu_3+12l_1)k_1^3-\frac{1}{3}k_1\omega_1-l_1^2)k_1}{k^3(kl_1-lk_1)} \\ &\quad +\frac{(36k_1^3-\omega k_1-9l_1l)k_1^2}{3k^4(kl_1-lk_1)}-\frac{4k_1^3l^2}{k^5(kl_1-lk_1)}, \\ \mu_1 &= \frac{3\mu_3l_1k_1^2}{k(kl_1-lk_1)}+\frac{2k_1\omega_1+3l_1^2-(9l\mu_3+96l_1)k_1^3}{3k^2(kl_1-lk_1)} \\ &\quad +\frac{2k_1(48lk_1^3-\omega k_1+12l_1l)}{3k^3(kl_1-lk_1)}-\frac{9l^2k_1^2}{k^4(kl_1-lk_1)}, \\ \mu_2 &= \frac{3k_1l_1\mu_3}{kl_1-lk_1}+\frac{\omega_1-(9l\mu_3+72l_1)k_1^2}{3k(kl_1-lk_1)} \\ &\quad +\frac{(72k_1^3+15l_1)l-\omega k_1}{3k^2(kl_1-lk_1)}-\frac{5l^2k_1^2}{k^3(kl_1-lk_1)}, \\ \mu_3 &= \mu_3, \quad \mu_4=4. \end{aligned} \tag{38}$$

We choose the solution of the elliptic equation as

$$W_1(Z)=\frac{c_1c_2\mathrm{cn}(Z|m)\mathrm{dn}(Z|m)}{1-c_2^2\mathrm{sn}^2(Z|m)}, \tag{39}$$

where c_1 and c_2 are two free constants, sn, cn, and dn are Jacobi elliptic functions with elliptic modulus m. Substituting solution (39) into elliptic equation (37) and the following parameters conditions are obtained

$$\begin{aligned} c_1 &= \frac{1}{2}, \quad c_2 = m, \quad k_1 = \frac{k}{2}, \\ l &= -\frac{(6m^4-36m^2+30)k^5-\omega+2\omega_1}{(30m^2-30)k^2}, \\ l_1 &= \frac{(54m^4-84m^2+30)k^5+\omega-2\omega_1}{(60m^2-60)k^2}. \end{aligned} \tag{40}$$

Figure 3 displays the plots of the solution $u = w_{8x} = \frac{1}{2}w_{3x} + w_{4x}$ with parameters $k_1 = 0.1$, $k = 0.2$, $m = 0.1$, $\omega = 4$, $\omega_1 = 0.5$, $k_3 = 0.25$, $k_4 = 0.4$. They demonstrate that the special superposed phenomenon between the two-soliton solution and the periodic cnoidal–soliton interaction solution on the $y = 0$ plane.

Case 4 Superposed phenomena between two different periodic cnoidal solutions

Similar to the specific calculation process of case 2, we obtain the decomposed periodic cnoidal wave solutions of Eqs. (22) and (23). It can be demonstrated that the decomposed equation (22) has periodic cnoidal solution

$$\begin{aligned} w_5 &= 4k_1\mathrm{E_e}(\mathrm{sn}(k_1x+s_1y+\omega_1t|m_1)|m_1) \\ &\quad +(k_1Z_1+2k_1m_1^2-2k_1)(k_1x+s_1y+\omega_1t), \end{aligned} \tag{41}$$

with the following constraints

$$\begin{aligned} \beta_0 &= Z_1k_1, \ \ \beta_1 = 4k_1m_1^2, \ \ s_1 = 8k_1^3m_1^2+3Z_1k_1^3-4k_1^3, \\ c_2^2 &= -9k_1^4Z_1+(24k_1^4-48k_1^4m_1^2)Z_1-48k_1^4m_1^4+48k_1^4m_1^2, \\ \omega_1 &= \frac{5}{2}Z_1^2k_1^5+\left(\frac{40}{3}k_1^5m_1^2-\frac{20}{3}k_1^5\right)Z_1+\frac{52}{3}k_1^5m_1^4-\frac{52}{3}k_1^5m_1^2+4k_1^5, \\ Z_1 &= \{80m_1^4-40m_1^2(30A+B)^{1/3}+2(30A+B)^{2/3}-80m_1^2+20(30A+B)^{1/3}+80\}/\{15(30A+B)^{1/3}\}, \\ A &= \sqrt{-60m_1^{12}-180m_1^{10}-435m_1^8+570m_1^6-435m_1^4+180m_1^2-60}, \\ B &= 100m_1^6-150m_1^4-150m_1^2+100, \end{aligned} \tag{42}$$

and further extending the result to Eq. (23)

$$w_6 = 4k_2\mathrm{E_e}(\mathrm{sn}(k_2x+s_2y+\omega_2t|m_2)|m_2)+(k_2Z_2+2k_2m_2^2-2k_2)(k_2x+s_2y+\omega_2t), \tag{43}$$

with

$$\begin{aligned} \beta_0 &= Z_2k_1, \ \ \beta_1 = 4k_2m_2^2, \ \ s_2 = 8k_2^3m_2^2+3Z_2k_2^3-4k_2^3, \\ c_2^2 &= -9k_2^4Z_2+(24k_2^4-48k_2^4m_2^2)Z_2-48k_2^4m_2^4+48k_2^4m_2^2, \\ \omega_2 &= \frac{5}{2}Z_2^2k_2^5+\left(\frac{40}{3}k_2^5m_2^2-\frac{20}{3}k_2^5\right)Z_2+\frac{52}{3}k_2^5m_2^4-\frac{52}{3}k_2^5m_2^2+4k_2^5, \\ Z_2 &= \{80m_2^4-40m_2^2(30C+D)^{1/3}+2(30C+D)^{2/3}-80m_2^2+20(30C+D)^{1/3}+80\}/\{15(30C+D)^{1/3}\}, \\ C &= \sqrt{-60m_2^{12}-180m_2^{10}-435m_2^8+570m_2^6-435m_2^4+180m_2^2-60}, \\ D &= 100m_2^6-150m_2^4-150m_2^2+100, \end{aligned} \tag{44}$$

where m_1, m_2, k_1, k_2 are arbitrary constants. Figure 4 displays the profiles of linear superposed interaction of two periodic cnoidal wave solutions $u = w_{9x} = \frac{1}{2}(w_{5x}+w_{6x})$ with wave parameters being fixed as $m_1 = m_2 = 0.7$, $k_1 = 0.12$, $k_2 = 0.15$.

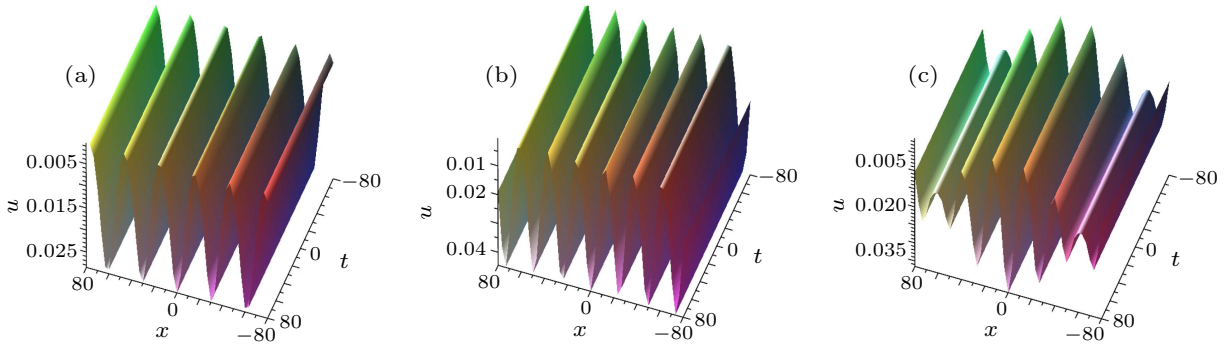


Fig. 4. The superposed phenomenon between two periodic cnoidal solutions: (a) the periodic cnoidal solution $u = w_{5x}$, where w_5 is defined by Eq. (41) with Eq. (42) for decomposition relation (22); (b) the periodic cnoidal solution $u = w_{6x}$ is defined by Eq. (43) with Eq. (44) for decomposition relation (23); (c) the linear superposition of the two-soliton and the periodic cnoidal $u = w_{9x} = \frac{1}{2}(w_{5x}+w_{6x})$.

4. Concluding remarks

In this paper, the integrable decomposition of 2D-CDGKS equation is studied by means of recursive relations and symmetry reductions, and it is found that the 2D-CDGKS equation possesses six decomposition relations, which provide the interrelations between different classic integrable systems involving ubiquitous KdV equation, SK equation, STO equation and integrable SS equation. The proposition in this paper intimates the tantalizing possibility of the existence of linear superposition in nonlinear systems. Some new linear superposition solutions of 2D-CDGKS equation are derived by linear combined of the above-mentioned integrable decomposition, including two-soliton, three-soliton, periodic cnoidal waves, interaction solutions of soliton and periodic cnoidal wave. We illustrate our results via categorical graphically with specific parameters that ensure better understanding. We delineates an innovation methodology for resolving new exact solutions, with a pronounced emphasis on linear superposition solutions. These solutions are inextricably linked to the integrable decomposition of NPDEs, this enables researchers to get better understanding of the connections between NPDEs under consideration and make more precise predictions regarding its behavior.

Appendix A: Proof of Proposition 1

This part is the proof of Proposition 1.

Proof Inserting Eqs. (9) and (10) into the PCDGKS equation (3) and balancing the highest nonlinearity and dispersive term, we obtain

$$n+1=6,\quad 2m=n+1. \tag{A1}$$

A simple computation of Eq. (A1) provides us

$$m=3,\quad n=5. \tag{A2}$$

Therefore, we assume that the decomposition relationships of Eq. (3) are in the following form

$$w_y = P(w,w_x,w_{x2},w_{x3}), \tag{A3}$$
$$w_t = Q(w,w_x,w_{x2},w_{x3},w_{x4},w_{x5}), \tag{A4}$$

and the corresponding compatibility condition is

$$w_{yt}-w_{ty}=[P,Q]=0. \tag{A5}$$

For the sake of brevity, we denote

$$\begin{aligned} P &= P(w,w_x,w_{x2},w_{x3}) \equiv P(x_0,x_1,x_2,x_3),\\ Q &= Q(w,w_x,w_{x2},w_{x3},w_{x4},w_{x5})\\ &\equiv Q(x_0,x_1,x_2,x_3,x_4,x_5). \end{aligned}$$

Substituting Eqs. (A3) and (A4) into the PCDGKS Eq. (3), we obtain

$$w_{x6}\left(Q_{x_5}-\frac{5}{36}P_{x_3}^2-\frac{5}{36}P_{x_3}+\frac{1}{36}\right) +W(x_0,x_1,\ldots,x_5)=0, \tag{A6}$$

where $W=W(x_0,x_1,\ldots,x_5)$ is a function of the $\{x_0,x_1,x_2,x_3,x_4,x_5\}$. Let the coefficient of w_{x6} equal zero, we can get

$$Q=\frac{1}{36}(5F_{x_3}^2+5F_{x_3}-1)x_5+G_1(x_0,x_1,x_2,x_3,x_4), \tag{A7}$$

where $G_1(x_0,x_1,x_2,x_3,x_4)$ is a function of $\{x_0,x_1,x_2,x_3,x_4\}$. Inserting Eq. (A7) into Eq. (A6), we obtain

$$\begin{aligned} &w_{x5}\Big[Q_{1x_4}-\frac{5}{36}(P_{x_3}+2)(x_1P_{x_0x_3}+x_2P_{x_1x_3}+x_3P_{x_2x_3}+x_4P_{x_3x_3})\\ &-\frac{5}{18}P_{x_2x_3-\frac{5}{36}F_{x_2}}\Big]+W_1=0, \end{aligned} \tag{A8}$$

where $W_1=W_1(x_0,x_1,x_2,x_3,x_4)$ is w_{x5}-independent. Eliminating the coefficient of w_{x5} yields

$$\begin{aligned} Q_1=&\frac{5}{36}(P_{x_3}+2)(x_1P_{x_0x_3}+x_2P_{x_1x_3}+x_3P_{x_2x_3}+x_4P_{x_3x_3})x_4\\ &+\frac{5}{18}x_4P_{x_2}\left(P_{x_3}+\frac{1}{2}P_{x_2}\right)+G_2, \end{aligned} \tag{A9}$$

with $G_2\equiv G_2(x_0,x_1,x_2,x_3)$ being a function of $\{x_0,x_1,x_2,x_3\}$.

Similarly, substituting the decompositions (A3) and (A4) with $m=3,n=5$ and the results (A7) and (A9) into the compatibility condition (A5), we have

$$\begin{aligned} &\frac{5}{36}w_{x7}(P_{x_3}-1)^2(x_1P_{x_0x_3}+x_2P_{x_1x_3}\\ &+x_3P_{x_2x_3}+x_4P_{x_3x_3})+\Omega=0, \end{aligned} \tag{A10}$$

where $\Omega=\Omega(x_0,x_1,\ldots,x_6)$ is a w_{x7}-independent function of the lower order differential form of w with respect to x.

Vanishing the coefficient of $w_{x7}w_{x4}$ in Eq. (A10), we get

$$P=P_1(x_0,x_1,x_2)x_3+H(x_0,x_1,x_2). \tag{A11}$$

Substituting Eq. (A11) into Eq. (A10) and requiring the coefficient of w_{x7} being zero results $P_1(x_0,x_1,x_2)=\alpha$, a constant. Thus,

$$P=\alpha x_3+H(x_0,x_1,x_2). \tag{A12}$$

Because of the expression (A12), equation (A8) is simplified to

$$\begin{aligned} &w_{x4}[Q_{2x_3}-\frac{5}{36}(\alpha+2)(x_1H_{x_0x_2}+x_2H_{x_1x_2}+x_3H_{x_2x_2}\\ &-\frac{5}{36}((2\alpha+1)H_{x_1}+3(\alpha-1)x_1+H_{x_2}^2)]+W_2=0, \end{aligned} \tag{A13}$$

with $W_2=W_2(x_0,x_1,x_2)$. Vanishing the coefficient of w_{x4} in Eq. (A13) leads to

$$\begin{aligned} Q_2=&\frac{5}{72}x_3^2H_{x_2x_2}+\frac{5}{36}((\alpha+2)x_1H_{x_0x_2}+(\alpha+2)x_2H_{x_1x_2}\\ &+(2\alpha+1)H_{x_1}+H_{x_2}^2+3\alpha x_1-3x_1)x_3+J=0, \end{aligned} \tag{A14}$$

where $J=J(x_0,x_1,x_2)$. After the above deduction process, the decomposition relationship is finally simplified to

$$w_y=\alpha w_{x3}+H, \tag{A15}$$

$$\begin{aligned}w_t=&\frac{5}{36}(\alpha^2+\alpha-1)w_{x5}+\frac{5}{36}(2\alpha+1)w_{x4}H_{x_2}\\&+\frac{5}{72}x_3^2H_{x_2x_2}+\frac{5}{36}((\alpha+2)x_1H_{x_0x_2}+(\alpha+2)x_2H_{x_1x_2}\\&+(2\alpha+1)H_{x_1}+H_{x_2}^2+3\alpha x_1-3x_1)x_3+J=0,\end{aligned} \tag{A16}$$

with single constant α and two undetermined functions $H=H(x_0,x_1,x_2)$ and $J=J(x_0,x_1,x_2)$.

Inserting Eqs. (A15) and (A16) into the CDGKS equation (2) and the consistent condition (A5) and then vanishing the coefficients of $w_{xi},i\geqslant 3$ leave the determining equations on $\{\alpha,H,J\}$, and they are solved in three different cases for $\alpha=0$, $\alpha=1$, and $\alpha\neq 0,1$, respectively. We can get the following results, because some expressions are relatively complex, we do not enumerate them one by one herein, but only present a portion of them

$$\begin{aligned}&\frac{5\alpha H_{x_2x_2x_2}}{72}=0,\quad\left(\frac{5}{18}-\frac{35\alpha^2}{72}\right)\partial_{x_2}^4H=0,\\&\left(\frac{1}{3}-\frac{5\alpha^2}{72}\right)\partial_{x_2}^5H=0,\quad\frac{(5\alpha-5)H_{x_2x_2}}{36}=0,\\&\frac{(5\alpha+10)(-1+\alpha)H_{x_2x_2x_2}}{36}=0,\\&\frac{5x_1(-1+\alpha)^2H_{x_0x_2}}{36}+\frac{5x_2(-1+\alpha)^2H_{x_1x_2}}{36}=0,\\&\frac{5\alpha(H_{x_2x_2x_2x_2}H_{x_2})}{72}+\left(\frac{5}{8}\alpha-\frac{5}{12}\right)H_{x_2x_2x_2}H_{x_2x_2}\\&+\left(\frac{5}{18}-\frac{35\alpha^2}{72}\right)H_{x_1x_2x_2x_2}\\&-\frac{5(5\alpha^2-2)\partial_{x_2}^4H}{72}(x_1H_{x_0}+x_2H_{x_1})=0,\end{aligned} \tag{A17}$$

$$\begin{aligned}&\left(\frac{5\alpha}{18}-\frac{5}{18}\right)H_{x_2x_2}H_{x_2}+\left(\frac{5}{12}-\frac{5\alpha}{12}\right)H_{x_1x_2}-\frac{5(\alpha+2)(\alpha-1)H_{x_2x_2}}{18}(x_1H_{x_1}+x_2H_{x_2})=0,\\&\left(-\frac{5\alpha}{18}-\frac{5}{36}\right)H_{x_0}-\frac{5H_{x_1}H_{x_2}}{18}-\frac{5x_1H_{x_2}}{12}+J_{x_2}-\frac{5x_1(\alpha+2)H_{x_0x_1}}{36}-\frac{5x_2(2\alpha+1)H_{x_0x_2}}{36}\\&-\frac{5x_2(\alpha+2)H_{x_1x_1}}{36}-\frac{5x_2^2(2\alpha+1)H_{x_0x_0x_2}}{36}-\frac{5x_2x_1(2\alpha+1)H_{x_0x_1x_2}}{18}-\frac{5x_2^2(2\alpha+1)H_{x_1x_1x_2}}{36}=0,\\&\left(\frac{5x_1(\alpha-1)H_{x_0x_2}}{18}+\frac{5x_2(\alpha-1)H_{x_1x_2}}{18}\right)H_{x_2}+\frac{5x_1(\alpha-1)^2H_{x_0x_1}}{36}-\frac{5x_2(\alpha+2)(\alpha-1)H_{x_0x_2}}{36}\\&+\frac{5x_2(\alpha-1)^2H_{x_1x_1}}{36}-\frac{5x_1^2(\alpha+2)(\alpha-1)H_{x_0x_0x_2}}{36}-\frac{5x_2x_1(\alpha+2)(\alpha-1)H_{x_0x_1x_2}}{18}\\&-\frac{5x_2^2(\alpha+2)(\alpha-1)H_{x_2x_1x_1}}{36}+\frac{5}{4}\alpha x_2(\alpha-1)=0,\end{aligned} \tag{A18}$$

$$\begin{aligned}&\left(-\frac{5x_2H_{x_2}}{36}-\frac{5x_1H_{x_1}}{36}-\frac{5H}{36}-\frac{5x_1^2}{12}\right)H_{x_0}-\left(\frac{5x_2^2H_{x_1x_1}}{36}+\frac{5x_1^2H_{x_0x_0}}{36}+\frac{5x_1x_2H_{x_0x_1}}{18}\right)H_{x_2}\\&-\frac{5x_2H_{x_1}^2}{36}-\frac{5x_1x_2x_1}{12}+x_1J_{x_0}+x_2J_{x_1}-\frac{5x_1x_2(\alpha+1)H_{x_0x_0}}{12}-\frac{5x_2^2(\alpha+1)H_{x_0x_1}}{12}\\&-\frac{5x_1^3(\alpha+1)H_{x_0x_0x_0}}{36}-\frac{5x_1^2x_2(\alpha+1)H_{x_0x_0x_1}}{12}-\frac{5x_2^2x_1(\alpha+1)H_{x_0x_1x_1}}{12}-\frac{5x_2^3(\alpha+1)H_{x_1x_1x_1}}{36}-\frac{5x_2(H-3x_1^2)}{12}=0,\end{aligned} \tag{A19}$$

$$\begin{aligned}&\left(-\frac{5x_2H_{x_2}}{36}-\frac{5x_1H_{x_1}}{36}-\frac{5H}{36}-\frac{5x_1^2}{12}\right)H_{x_0}+\left(-\frac{5x_2^2H_{x_1x_1}}{36}-\frac{5x_1^2H_{x_0x_0}}{36}-\frac{x_1x_2H_{x_0x_1}}{18}\right)H_{x_2}\\&-\frac{5x_1x_2(\alpha+1)H_{x_0x_0}}{12}-\frac{5x_2^2(\alpha+1)H_{x_0x_1}}{12}-\frac{5x_1^3(\alpha+1)H_{x_0x_0x_0}}{36}-\frac{5x_1^2x_2(\alpha+1)H_{x_0x_0x_1}}{12}\\&-\frac{5x_1x_2^2(\alpha+1)H_{x_0x_1x1}}{12}-\frac{5x_2^3(\alpha+1)H_{x_1x_1x1}}{36}-\frac{5x_2(-3x_1^2+H)}{12}-\frac{5x_2H_{x_1}^2}{36}-\frac{5x_1x_2H_{x_1}}{12}+x_1J_{x_0}+x_2J_{x_1}=0,\end{aligned} \tag{A20}$$

$$\begin{aligned}&\left(\left(-\frac{5}{9}-\frac{10\alpha}{9}\right)H_{x_1x_2}+\frac{5x_1(-1+\alpha)H_{x_0x_2x_2}}{18}+\frac{5x_2(-1+\alpha)H_{x_1x_2x_2}}{18}\right)H_{x_2}-\left(\frac{5}{12}+\frac{5}{6}\alpha\right)H_{x_2x_2}H_{x_1}\\&+\left(\frac{5x_1(4\alpha-7)H_{x_2x_2}}{18}-\frac{(5\alpha+5)(4\alpha-1)}{12}\right)H_{x_0x_2}+\frac{5H_{x_2}^2H_{x_2x_2}}{36}+\frac{5x_2(4\alpha-7)H_{x_2x_2}H_{x_1x_2}}{18}\\&-\frac{(5\alpha+10)(4\alpha-1)H_{x_1x_1}}{36}-\frac{5x_1(7\alpha^2+7\alpha-5)H_{x_0x_1x_2}}{18}-\frac{5x_2(4\alpha^2+\alpha-2)H_{x_0x_2x_2}}{12}\\&-\frac{5x_2(7\alpha^2+7\alpha-5)H_{x_1x_1x_2}}{18}-\frac{5x_1^2(4\alpha^2+\alpha-2)H_{x_0x_0x_2x_2}}{12}-\frac{5}{6}x_1x_2(4\alpha^2+\alpha-2)H_{x_0x_1x_2x_2}\\&-\frac{5x_2^2(4\alpha^2+\alpha-2)H_{x_1x_1x_2x_2}}{12}+3\alpha J_{x_2x_2}+\frac{5}{4}\alpha(\alpha-1)-\frac{5}{4}x_1(-1+\alpha)H_{x_2x_2}=0,\end{aligned} \tag{A21}$$

From the determining equation (A17), we know that the determining equations should be solved in three different cases for $\alpha=0$, $\alpha=1$, and $\alpha\neq 0,1$, respectively. By solving the related determining equations, we can get the following results.

Case 1 When $\alpha=0$, the final solutions of H and J read

$$\begin{aligned} H &= c_1x_1+c_2=c_1w_x+c_2,\\ J &= -\frac{5w_x^3}{12}+\frac{5c_1w_x^2}{12}+\frac{(5c_1^2+15c_2)w_x}{36}+c_3. \end{aligned} \tag{A22}$$

Case 2 When $\alpha=1$, we obtain four different solutions:

$$\begin{aligned} H &= \frac{3x_1^2}{2}+c_1,\\ J &= \frac{5}{8}w_{x2}^2+\frac{5}{8}w_x^3+\frac{5}{12}c_1w_x+c_2, \end{aligned} \tag{A23}$$

$$\begin{aligned} H &= c_1w_x+3w_x^2+c_2,\\ J &= \frac{5w_x^3}{2}+\frac{5c_1w_x^2}{4}+\frac{5(c_1^2+3c_2)w_x}{36}\\ &\quad+\frac{5w_{x2}^2}{4}+c_3, \end{aligned} \tag{A24}$$

$$\begin{aligned} H &= \frac{3w_x^2}{2}+\frac{3}{4}\left(w+\frac{2c_1}{3}\right)w_x+\frac{3}{2}\left(w+\frac{2c_1}{3}\right)w_{x2},\\ J &= \frac{15}{16}w_x^3+\frac{15}{16}\left(w+\frac{2}{3}c_1\right)^2w_x^2+\frac{5}{64}\left(w+\frac{10c_1}{3}\right)\\ &\quad\times\left(w^3+2c_1w^2+\frac{4}{3}c_1^2w+\frac{8c_1^3}{27}+40w_{x2}\right)w_x\\ &\quad+\frac{5}{16}\left(w^3+2c_1w^2+\frac{4}{3}c_1^2w+\frac{8c_1^3}{27}+4w_{x2}\right)w_{x2}, \end{aligned} \tag{A25}$$

$$\begin{aligned} H &= \frac{3w_{x2}^2}{3c_2-4w_x}+\frac{3w_x^2}{2}+c_1,\\ J &= \frac{5w_{x2}^2(27c_2^3-54c_2^2w_x+32w_x^3+56w_{x2})^2}{8(4w_x-3c_2)^3}\\ &\quad+\frac{5}{8}w_x^3+\frac{5c_1w_x}{12}+c_3. \end{aligned} \tag{A26}$$

Case 3 When $\alpha\neq 1$, we find that α should be fixed as $\alpha=-1/2$ and the functions H and J are also fixed as

$$\begin{aligned} H &= -\frac{3}{2}w_x^2+c_1w_x+c_2,\\ J &= -\frac{5w_x^3}{8}+\frac{5(c_1^2+3c_2)w_x}{36}-\frac{5w_{x2}^2}{16}+c_3, \end{aligned} \tag{A27}$$

where c_1, c_2, and c_3 in all cases are arbitrary constants. By substituting the above solutions (A22)–(A27) into the decomposition relations (A15) and (A16), Proposition 1 is proved.

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