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Asymmetric model of the dynamic quantum Cournot duopoly game with asymmetric information and heterogeneous players

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Building on the existing symmetric quantization model of the dynamic Cournot duopoly game (CDG) with asymmetric information, we extend it to an asymmetric quantization model and study the stability of the quantum Bayesian Nash equilibrium (QBNE) under heterogeneous expectations. We analyze the influence of various parameters on the stability of QBNE, with a particular focus on the impact of the parameter α on system stability. The results show that when $\alpha < 1$, under the same parameters, the quantum strategy of the asymmetric quantization model is more favorable for stabilizing the market. However, when $\alpha > 1$, the quantum strategy of the symmetric quantization model is more conducive to stabilizing the market.

Keywords: quantum Cournot games, Bayesian Nash equilibrium, asymmetric quantization scheme, asymmetric information

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1. Introduction

The integration of quantum theory and game theory has given rise to a new interdisciplinary field known as quantum game theory, which originated from Meyer's pioneering work on a quantum coin-flipping game.[1] Not long thereafter, Eisert *et al.*[2] applied the concept of quantum entanglement (QE) to the classic prisoners' dilemma game (PDG). The two-player quantum PDG was experimentally realized by Du *et al.* via NMR techniques.[3] Experimental realizations of the quantum PDG involving two to four players have subsequently been achieved via NMR[4] and linear optics techniques.[5,6]

Oligopoly is a common market structure in real-world economies. Quantum game theory has recently seen growing application in the analysis and modeling of behavior in oligopoly markets. A quantum approach based on continuous-variable strategies was proposed by Li *et al.*[7] Building on this, Du *et al.*[8] applied the method to quantize the CDG under asymmetric information. Chen *et al.*[8] further generalized Du *et al.*'s framework[9] to broader scenarios. Later studies employed the Li–Du–Massar scheme to investigate Cournot, Bertrand, and Stackelberg games.[10–15] These quantum models have uncovered outcomes and strategic dynamics that are fundamentally absent in classical game theory. More recently, the two-turn game[16] involving continuous-variable quantum strategies has been studied. In addition, the quantum frameworks proposed by Li–Du–Massar and Frackiewicz were independently generalized by Wang and Yang[17] to accommodate multiple players.

The above-mentioned research on quantum duopoly games has largely emphasized static analysis, which reveals outcomes but not the evolution of strategic interactions. Dynamic analysis is thus crucial for understanding firms' adjustments and changes in system stability. Rand[18] was the pioneer in introducing dynamical systems theory into duopoly game models. Since then, this approach has been adopted by many researchers in the study of duopoly games.[19–25] The findings underscore the significance of studying duopoly games dynamically under bounded rationality, yet the Nash equilibrium often remains locally unstable. A number of recent works have explored the local stability of equilibria by integrating quantum game theory and complex dynamics. The stability of a quantum CDG was investigated by Zhang *et al.*[26] A quantum CDG with isoelastic demand was developed by Shi and Xu[27] to examine how QE affects system stability. The stability of equilibria in quantum Stackelberg duopoly models was analyzed by Hu *et al.*[28] These findings indicate that QE can delay the emergence of bifurcations and chaotic behavior.

In addition, real-world markets rarely operate under conditions of complete information due to uncertainties arising from dynamic competition, evolving consumer preferences, and economic fluctuations. Consequently, exploring incomplete information scenarios becomes increasingly critical for practical market decision making. Harsanyi[29] proposed a Bayesian game to address the problem of incomplete information. Yu and Yu[30] constructed a dynamic CDG where firm 2 operates with two distinct marginal cost structures: one high and one low. Long and Zhao[31] explored equilibrium stability in dynamic Bertrand duopoly games influenced by

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cluster overflow and asymmetric information. Askar[21] studied a Cournot–Bertrand duopoly game involving firms with bounded rationality. Building on Yu's model,[30] Wang *et al.*[32] generalized it to an n-player setting and examined the equilibrium stability. In the public goods game, Kang *et al.*[33] studied the impact of the cost of information acquisition for discreet investors on the level of cooperation.

Recently, researchers have incorporated quantum theory into duopoly games,[34–37] prisoners' dilemma games,[38] and cooperation games[39] with asymmetric information. It is worth noting that although these quantum games involve asymmetric information, the quantization schemes applied are all symmetric, which creates a structural inconsistency. To address this, Qin *et al.*[40] proposed an asymmetric quantization scheme for the quantum Bertrand duopoly game and examined whether the asymmetric and symmetric quantum schemes can achieve optimal cooperative payoffs. Their results show that the asymmetric quantization scheme can achieve the optimal cooperative payoff, whereas the symmetric scheme fails to do so. Furthermore, Wang *et al.*[41,42] constructed a quantum Stackelberg duopoly game model using Qin *et al.*'s asymmetric quantization scheme[40] and studied the impact of changes in the asymmetry factor α and the QE γ on the Nash equilibrium profits. Similarly, Zhong *et al.*[43] used the asymmetric quantization scheme proposed by Qin *et al.*[40] to establish a quantum Stackelberg duopoly model under the condition of isoelastic demand and analyzed the impact of γ and α on the Nash equilibrium profits of the leader and the follower.

A natural question is: Does the stability of the asymmetric model of a dynamic quantum duopoly with asymmetric information perform better than that of the symmetric model? To address this question, we construct an asymmetric quantization model of a quantum CDG under asymmetric information and investigate the stability of its QBNE in comparison to the symmetric quantization model proposed by Zhou *et al.*[34] The findings show that the effectiveness of market stabilization depends on the value of α, with the asymmetric quantization model performing better when $\alpha < 1$, and the symmetric model being more effective when $\alpha > 1$.

The following sections are arranged as follows. An asymmetric quantization scheme for the quantum CDG under asymmetric information is detailed in Section 2. The stability of the QBNE in a dynamic model with heterogeneous expectations is investigated in Section 3. Theoretical results are confirmed by numerical simulations in Section 4. Concluding remarks are summarized succinctly in Section 5.

2. The model

We introduce a CDG under asymmetric information,[30] where both firms produce homogeneous products. The market price is

$$P = a - q_1 - q_2,$$

where q_i $(i = 1,2)$ is the output of firm i and $a > 0$. Let c_i represent the marginal cost for firm i. Firm 2 is assumed to have two possible marginal cost levels: a high cost $c_{\rm h}$ and a low cost $c_{\rm l}$. Let $c_{\rm h} = c_1 + \varepsilon$, $c_{\rm l} = c_1 - \varepsilon$ and the probability distribution $P(c_2 = c_{\rm h}) = \theta$, $P(c_2 = c_{\rm l}) = 1-\theta$, where $c_1 > 0$, $c_{\rm h} > 0$, $c_{\rm l} > 0, \varepsilon > 0$, and $\theta \in (0,1)$. Denote firm 2's output as $q_{2{\rm h}}$ and $q_{2{\rm l}}$ when its marginal cost is $c_{\rm h}$ and $c_{\rm l}$, respectively. The expected profit for Company 2 is

$$\begin{cases} \pi_{2{\rm h}} = (a - q_1 - q_{2{\rm h}} - c_{\rm h})q_{2{\rm h}}, \\ \pi_{2{\rm l}} = (a - q_1 - q_{2{\rm l}} - c_{\rm l})q_{2{\rm l}}. \end{cases} \tag{1}$$

The expected profit for Company 1 is

$$\begin{aligned} \pi_1 &= \theta(a - q_1 - q_{2{\rm h}} - c_1)q_1 + (1-\theta)(a - q_1 - q_{2{\rm l}} - c_1)q_1 \\ &= (a - q_1 - (\theta q_{2{\rm h}} + (1-\theta)q_{2{\rm l}}) - c_1)q_1. \end{aligned} \tag{2}$$

Next, utilizing Qin *et al.*'s asymmetric quantization approach,[40] we construct a quantum CDG with asymmetric information. Following the full implementation of strategic operations and measurement in the asymmetric quantization model, the obtained values, denoted by q_1 and q_2, reflect the respective outputs of the two competing firms[42]

$$\begin{cases} q_1 = y_1 \cosh\gamma + y_2 \alpha^{-1} \sinh\gamma, \\ q_2 = y_1 \alpha \sinh\gamma + y_2 \cosh\gamma, \end{cases} \tag{3}$$

where y_1 and y_2 denote the parameters through which the two firms carry out their strategic decisions; $\gamma = \sqrt{\gamma_1\gamma_2}$, $\alpha = \sqrt{\gamma_1/\gamma_2}$, γ_1 and γ_2 are two independent entanglement factors. If $\gamma_1 = \gamma_2$, our quantization model is equivalent to the minimal quantization model proposed by Zhou *et al.*[34]

Firm 1's expected quantum profit can be derived by substituting Eq. (3) into Eq. (2):

$$\begin{aligned} &\pi_1^{\rm Q}(y_1, y_{2{\rm h}}, y_{2{\rm l}}) \\ &= \theta(a - y_1(\cosh\gamma + \alpha\sinh\gamma) \\ &\quad - y_{2{\rm h}}(\alpha^{-1}\sinh\gamma + \cosh\gamma) - c_1)(y_1\cosh\gamma + y_{2{\rm h}}\alpha^{-1}\sinh\gamma) \\ &\quad + (1-\theta)(a - y_1(\cosh\gamma + \alpha\sinh\gamma) \\ &\quad - y_{2{\rm l}}(\alpha^{-1}\sinh\gamma + \cosh\gamma) - c_1)(y_1\cosh\gamma + y_{2{\rm l}}\alpha^{-1}\sinh\gamma). \end{aligned}$$

The quantum profits corresponding to firm 2's choices of $y_{2{\rm h}}$ and $y_{2{\rm l}}$ are derived by substituting Eq. (3) into Eq. (1)

$$\begin{aligned} \pi_{2{\rm h}}^{\rm Q}(y_1, y_{2{\rm h}}) &= (y_1\alpha\sinh\gamma + y_{2{\rm h}}\cosh\gamma) \\ &\quad \times (a - y_1(\cosh\gamma + \alpha\sinh\gamma) \\ &\quad - y_{2{\rm h}}(\alpha^{-1}\sinh\gamma + \cosh\gamma) - c_{\rm h}). \\ \pi_{2{\rm l}}^{\rm Q}(y_1, y_{2{\rm l}}) &= (y_1\alpha\sinh\gamma + y_{2{\rm l}}\cosh\gamma) \\ &\quad \times (a - y_1(\cosh\gamma + \alpha\sinh\gamma) \\ &\quad - y_{2{\rm l}}(\alpha^{-1}\sinh\gamma + \cosh\gamma) - c_{\rm l}). \end{aligned}$$

To simplify the calculation, we define $A = \cosh^2\gamma + \sinh^2\gamma + 2\alpha\sinh\gamma\cosh\gamma$ and $B = \cosh^2\gamma + \sinh^2\gamma + 2\alpha^{-1}\sinh\gamma\cosh\gamma$.

Below are the marginal profit functions for the two firms:

$$\begin{cases} \dfrac{\partial \pi_1^{\mathrm{Q}}}{\partial y_1} = (a-c_1)\cosh\gamma - y_1(A+1) \\ \qquad\qquad -B(\theta y_{2\mathrm{h}} + (1-\theta)y_{2\mathrm{l}}), \\ \dfrac{\partial \pi_{2\mathrm{h}}^{\mathrm{Q}}}{\partial y_{2\mathrm{h}}} = (a-c_1-\varepsilon)\cosh\gamma - y_{2\mathrm{h}}(B+1) - Ay_1, \\ \dfrac{\partial \pi_{2\mathrm{h}}^{\mathrm{Q}}}{\partial y_{2\mathrm{l}}} = (a-c_1+\varepsilon)\cosh\gamma - y_{2\mathrm{l}}(B+1) - Ay_1. \end{cases} \tag{4}$$

To ensure the economic validity of the model, we impose that all relevant variables must take strictly positive values. Therefore, the following assumption holds.

Assumption 1 $a - c_1 > \max\{A+1, B\}\varepsilon$.

Let $\frac{\partial \pi_{2\mathrm{h}}^{\mathrm{Q}}}{\partial y_{2\mathrm{h}}} = 0$ and $\frac{\partial \pi_{2\mathrm{l}}^{\mathrm{Q}}}{\partial y_{2\mathrm{l}}} = 0$, the following equations are derived:

$$y_{2\mathrm{h}}(y_1; c_1, \varepsilon) = \begin{cases} \frac{(a-c_1-\varepsilon)\cosh\gamma - Ay_1}{B+1}, & y_1 \le \frac{(a-c_1-\varepsilon)\cosh\gamma}{A}, \\ 0, & y_1 > \frac{(a-c_1-\varepsilon)\cosh\gamma}{A}, \end{cases} \tag{5}$$

$$y_{2\mathrm{l}}(y_1; c_1, \varepsilon) = \begin{cases} \frac{(a-c_1+\varepsilon)\cosh\gamma - Ay_1}{B+1}, & y_1 \le \frac{(a-c_1+\varepsilon)\cosh\gamma}{A}, \\ 0, & y_1 > \frac{(a-c_1+\varepsilon)\cosh\gamma}{A}. \end{cases} \tag{6}$$

The mathematical expectation of y_2 is $E_{y_2}(y_1; c_1, \varepsilon) = \theta y_{2\mathrm{h}}(y_1; c_1, \varepsilon) + (1-\theta)y_{2\mathrm{l}}(y_1; c_1, \varepsilon)$, then

$$E_{y_2} = \begin{cases} \dfrac{(a-c_1)\cosh\gamma - Ay_1 + (1-2\theta)\varepsilon\cosh\gamma}{B+1}, & y_1 \le \dfrac{(a-c_1-\varepsilon)\cosh\gamma}{A}, \\ (1-\theta)\dfrac{(a-c_1-\varepsilon)\cosh\gamma - Ay_1}{B+1}, & \dfrac{(a-c_1-\varepsilon)\cosh\gamma}{A} < y_1 \le \dfrac{(a-c_1+\varepsilon)\cosh\gamma}{A}, \\ 0, & y_1 > \dfrac{(a-c_1+\varepsilon)\cosh\gamma}{A}. \end{cases} \tag{7}$$

Let $\frac{\partial \pi_1^{\mathrm{Q}}}{\partial y_1} = 0$, we have the following equation:

$$y_1(E_{y_2}) = \begin{cases} \frac{(a-c_1)\cosh\gamma - BE_{y_2}}{A+1}, & E_{y_2} \le \frac{(a-c_1)\cosh\gamma}{B}, \\ 0, & E_{y_2} > \frac{(a-c_1)\cosh\gamma}{B}. \end{cases} \tag{8}$$

Theorem 1 If Assumption 1 holds, then

$$E_{y_2} = \frac{(a-c_1)\cosh\gamma - Ay_1 + (1-2\theta)\varepsilon\cosh\gamma}{B+1}. \tag{9}$$

Proof If $E_{y_2} = 0$, based on Eqs. (7) and (8), $y_1 \le \frac{(a-c_1-\varepsilon)\cosh\gamma}{A}$ and $y_1 = \frac{(a-c_1)\cosh\gamma}{A+1}$. However, since $\frac{(a-c_1)\cosh\gamma}{A+1} - \frac{(a-c_1-\varepsilon)\cosh\gamma}{A} > 0$, then $E_{y_2} = 0$ does not hold.

Then, if

$$E(y_2) = (1-\theta)\frac{(a-c_1-\varepsilon)\cosh\gamma - Ay_1}{B+1},$$

then $\frac{(a-c_1-\varepsilon)\cosh\gamma}{A} < y_1$ and $E_{y_2} \le \frac{(a-c_1)\cosh\gamma}{B}$. From Eqs. (7) and (8),

$$y_1 = \frac{(a-c_1)\cosh\gamma(1+B\theta) - B(1-\theta)\varepsilon\cosh\gamma}{(A+1)(B+1) - AB(1-\theta)}.$$

However, $y_1 > \frac{(a-c_1-\varepsilon)\cosh\gamma}{A}$ is equivalent to

$$\frac{(a-c_1)\cosh\gamma(1+B\theta) - B(1-\theta)\varepsilon\cosh\gamma}{(A+1)(B+1) - AB(1-\theta)} - \frac{(a-c_1-\varepsilon)\cosh\gamma}{A}$$

$$= \frac{\cosh\gamma(B+1)(\varepsilon(A+1) - (a-c_1))}{A((A+1)(B+1) - AB(1-\theta))} > 0,$$

namely $(a-c_1) \le \varepsilon(A+1)(B+1)$, which contradicts Assumption 1. So $E(y_2) \ne (1-\theta)\frac{(a-c_1-\varepsilon)\cosh\gamma - Ay_1}{B+1}$.

Finally, $E(y_2) = \frac{(a-c_1)\cosh\gamma - Ay_1 + (1-2\theta)\varepsilon\cosh\gamma}{B+1}$.

Theorem 2 If Assumption 1 is satisfied, then the equilibrium strategy profile in the quantum CDG with asymmetric information is given by $y^* = (y_1^*, y_{2\mathrm{h}}^*, y_{2\mathrm{l}}^*)$, where

$$\begin{cases} y_1^* = \dfrac{(a-c_1)\cosh\gamma - B(1-2\theta)\varepsilon\cosh\gamma}{A+B+1}, \\ y_{2\mathrm{h}}^* = \dfrac{(a-c_1-\varepsilon)\cosh\gamma - Ay_1^*}{B+1}, \\ y_{2\mathrm{l}}^* = \dfrac{(a-c_1+\varepsilon)\cosh\gamma - Ay_1^*}{B+1}. \end{cases} \tag{10}$$

Proof Based on Assumption 1 and Theorem 1, it follows that $E_{y_2} \le \frac{(a-c_1)\cosh\gamma}{B}$. Combining Eqs. (5), (6), and (8), the QBNE is determined to be $y^* = (y_1^*, y_{2\mathrm{h}}^*, y_{2\mathrm{l}}^*)$.

3. The stability of QBNE

Here, firm 2 is assumed to follow adaptive expectations, and firm 1 adjusts its decision via a gradient method based on marginal profit.[19]

Firms 1 and 2 adjust their strategies according to the following dynamic equations:

$$\begin{cases} y_1(t+1) = y_1(t) + vy_1(t)((a-c_1)\cosh\gamma - y_1(t)(A+1) - B(\theta y_{2\mathrm{h}}(t) + (1-\theta)y_{2\mathrm{l}}(t)), \\ y_{2\mathrm{h}}(t+1) = \beta y_{2\mathrm{h}}(t) + (1-\beta)\dfrac{(a-c_1-\varepsilon)\cosh\gamma - Ay_1(t)}{B+1}, \\ y_{2\mathrm{l}}(t+1) = \beta y_{2\mathrm{l}}(t) + (1-\beta)\dfrac{(a-c_1+\varepsilon)\cosh\gamma - Ay_1(t)}{B+1}, \end{cases} \tag{11}$$

where $v \in (0,+\infty)$ reflects firm 1's quantum strategy adjustment speed; β serves as firm 2's adaptive expectation.

Let $y_i(t+1) = y_i(t) = y_i$, $i = 1$, 2h, 2l, yields the following two quantum equilibrium points of system (11):

$$E^0 = (0, \frac{(a-c_1-\varepsilon)\cosh\gamma}{B+1}, \quad \frac{(a-c_1+\varepsilon)\cosh\gamma}{B+1}),$$
$$E^* = (y_1^*, y_{2\mathrm{h}}^*, y_{2\mathrm{l}}^*),$$

where $y_1^*, y_{2\mathrm{h}}^*, y_{2\mathrm{l}}^*$ are determined from Eq. (10). The Jacobian matrix can be expressed as

$$J = \begin{bmatrix} b_{11} & -B\theta y_1 v & -Bv(1-\theta)y_1 \\ -\frac{A(1-\beta)}{B+1} & \beta & 0 \\ -\frac{A(1-\beta)}{B+1} & 0 & \beta \end{bmatrix},$$

where $b_{11} = 1 + v((a-c_1)\cosh\gamma - 2y_1(A+1) - B(\theta y_{2\mathrm{h}} + (1-\theta)y_{2\mathrm{l}})$.

We present the theorem on the boundary equilibrium E^0 as follows.

Theorem 3 E^0 is a saddle point if Assumption 1 is satisfied.

Proof The Jacobian matrix $J(E^0)$ is

$$J(E^0) = \begin{bmatrix} 1 + \frac{v\cosh\gamma(a-c_1-(1-2\theta)B\varepsilon)}{B+1} & 0 & 0 \\ -\frac{A(1-\beta)}{B+1} & \beta & 0 \\ -\frac{A(1-\beta)}{B+1} & 0 & \beta \end{bmatrix}.$$

Since Assumption 1 holds, the eigenvalues are $\lambda_1 = 1 + \frac{v\cosh\gamma(a-c_1-(1-2\theta)B\varepsilon)}{B+1} > 1$, $\lambda_2 = \lambda_3 = \beta \in (0,1)$.

Regarding the QBNE E^*, the theorem below holds.

Theorem 4 If $vy_1^*(2AB + (1-\beta)(A+B+1)) - 2(1+\beta)(B+1) < 0$ and Assumption 1 holds, then QBNE E^* is locally asymptotically stable.

Proof At the QBNE, the Jacobian matrix is given by

$$J(E^*) = \begin{bmatrix} 1-(A+1)vy_1^* & -\theta Bvy_1^* & -(1-\theta)Bvy_1^* \\ -\frac{A(1-\beta)}{B+1} & \beta & 0 \\ -\frac{A(1-\beta)}{B+1} & 0 & \beta \end{bmatrix}.$$

Then the characteristic polynomial of $J(E^*)$ is

$$\begin{aligned} f(\lambda) &= |\lambda I - J(E^*)| \\ &= \begin{vmatrix} \lambda - (1-(A+1)vy_1^*) & \theta Bvy_1^* & (1-\theta)Bvy_1^* \\ \frac{A(1-\beta)}{B+1} & \lambda-\beta & 0 \\ \frac{A(1-\beta)}{B+1} & 0 & \lambda-\beta \end{vmatrix} \\ &= (\lambda-\beta)((\lambda-\beta)(\lambda-(1-(A+1)vy_1^*) \\ &\quad - \frac{ABv(1-\beta)}{B+1}y_1^*). \end{aligned}$$

Clearly, one of the eigenvalues is $\lambda_1 = \beta \in (0,1)$, and the two additional roots fulfill the equation below

$$\lambda^2 - \lambda(\beta + 1 - (A+1)vy_1^*) + \beta(1-(A+1)vy_1^*) - \frac{ABv(1-\beta)}{B+1}y_1^* = 0.$$

Thus, the two additional roots are

$$\lambda_2 = \frac{\beta+1-(A+1)vy_1^* + \sqrt{\Delta}}{2},$$
$$\lambda_3 = \frac{\beta+1-(A+1)vy_1^* - \sqrt{\Delta}}{2},$$

where

$$\begin{aligned} \Delta &= (\beta+1-(A+1)vy_1^*)^2 - 4(\beta(1-(A+1)vy_1^*) \\ &\quad - \frac{ABv(1-\beta)}{B+1}y_1^*). \end{aligned}$$

$\lambda_2 < 1$ is equivalent to

$$\frac{B}{B+1}Avy_1^* < (A+1)vy_1^*.$$

This is always holds.

$\lambda_3 > -1$ is equivalent to

$$vy_1^*(2AB + (1+\beta)(A+B+1)) < 2(1+\beta)(B+1).$$

Since $vy_1^*(2AB + (1+\beta)(A+B+1)) - 2(1+\beta)(B+1) < 0$, E^* is locally asymptotically stable, completing the proof.

The stability region S corresponding to the equilibrium E^* is

$$S = \{(\beta, v) \in R_+^2 : \beta \in (0,1), v \in (0, v_{\mathrm{G}}(\beta,\gamma))\},$$

where

$$\begin{cases} v_{\mathrm{G}}(\beta,\gamma) = \dfrac{2(1+\beta)(B+1)}{(2AB+(1+\beta)(A+B+1))y_1^*}, \\ y_1^* = \dfrac{(a-c_1)\cosh\gamma - B(1-2\theta)\varepsilon\cosh\gamma}{A+B+1}. \end{cases} \tag{12}$$

Then

$$\frac{\partial v_{\mathrm{G}}(\beta,\gamma)}{\partial\beta} = \frac{4AB(B+1)}{((2AB+(1+\beta)(A+B+1))y_1^*)^2} > 0,$$
$$\frac{\partial v_{\mathrm{G}}^2(\beta,\beta)}{\partial\beta^2} < 0.$$

Therefore, $v_{\mathrm{G}}(\beta,\gamma)$ is strictly increasing and concave. The QBNE is locally asymptotically stable when $(\beta, v) \in S$. Conversely, when $(\beta, v) \notin S$, the system may undergo bifurcation or even chaos. Based on Eq. (12), we derive the following proposition.

Proposition 3 The stability region of QBNE expands with increasing β and c_1, but contracts with increasing values of a and θ. Furthermore, if $\theta < 1/2$, as ε grows, the stability region becomes larger; conversely, if $\theta > 1/2$, it narrows with increasing ε. These findings are consistent with the results of Yu *et al.*,[30] Zhou *et al.*,[34] and Wang *et al.*[32]

4. Numerical simulations

This section investigates the dynamics of the quantum CDG using numerical simulations. We employ a set of bifurcation plots, chaos diagrams, Lyapunov exponents, and strange attractors to assess the stability differences between the symmetric and asymmetric quantum schemes of the dynamic CDG with informational asymmetry and heterogeneous expectations. In this Section, let $a = 6$, $c_1 = 3$, $\varepsilon = 1$, $\theta = 0.6$, $\gamma_1 = 0.2$, $\gamma_2 = 0.6$, then $y_1 = 0.712$, $y_{2h} = 0.26$, $y_{2l} = 0.86$. When $\gamma_1 = \gamma_2 = \sqrt{0.12}$ and the remaining parameters are kept unchanged, $y_1 = 0.72$, $y_{2h} = 0.23$, $y_{2l} = 0.93$. When $\gamma_1 = 0.6$, $\gamma_2 = 0.2$ and the remaining parameters are kept unchanged, $y_1 = 0.68$, $y_{2h} = 0.15$, $y_{2l} = 0.94$. When $\gamma_1 = 0.9$, $\gamma_2 = 0.3$ and the remaining parameters are kept unchanged, $y_1 = 0.56$, $y_{2h} = 0.06$, $y_{2l} = 0.75$.

Initially, the stability domain of the QBNE is investigated. Setting $\gamma = \sqrt{0.12}$, while keeping all other parameters constant. To investigate how parameters β and α influence the stability region, the stability region diagrams for $\alpha = 1/3$, $\alpha = 1$ and $\alpha = 3$ with respect to β are generated, as shown in Figs. 1(a), 1(b), and 1(c). Clearly, an increase in α leads to a contraction of the stability region, while it expands as β increases. Comparing Figs. 1(c) and 1(d), as the parameter γ increases, the stability region becomes smaller, which is consistent with the findings of Zhou *et al.*[34]

To explore how the parameters v and α influence QBNE stability, bifurcation diagrams are plotted for $\alpha = 1/3$, $\alpha = 1$, and $\alpha = 3$ with respect to v, as shown in Fig. 2(a). As v increases, the firm's output evolves from a stable equilibrium to bifurcation, and ultimately enters a chaotic regime. To more clearly illustrate how variations in α influence QBNE stability, the associated maximal Lyapunov exponent is plotted in Fig. 2(b). Obviously, a larger α causes bifurcation and chaos to occur earlier. In other words, when $\alpha < 1$, under the same parameters, the stability region of the asymmetric quantization model system is larger than that of the symmetric quantization model system. That is, the quantum strategy of the asymmetric quantization model is more favorable for stabilizing the market. However, when $\alpha > 1$, the quantum strategy of the symmetric quantization model is more conducive to stabilizing the market. Additionally, figures 2(c) and 2(d) further demonstrate that an increase in γ advances the emergence of bifurcations and chaos.

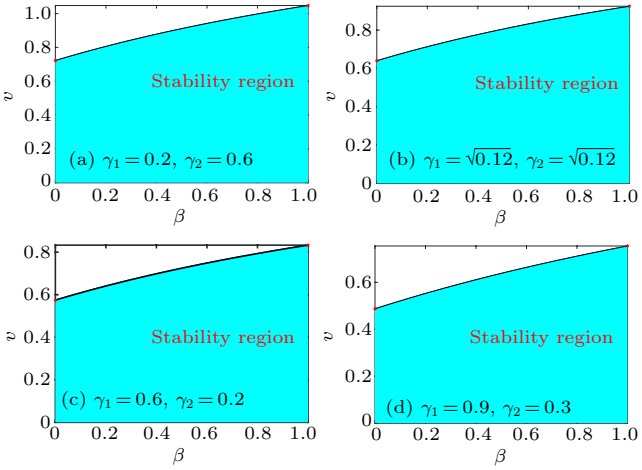


Fig. 1. Stability region of QBNE.

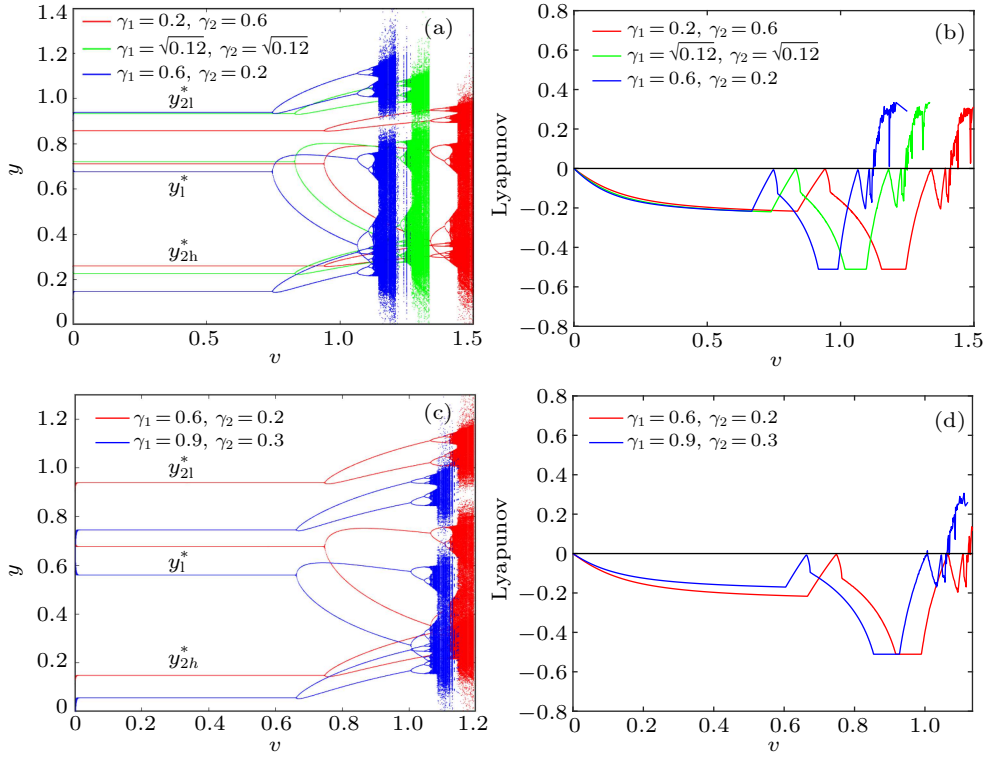


Fig. 2. Bifurcation plot and corresponding maximal Lyapunov exponent for v.

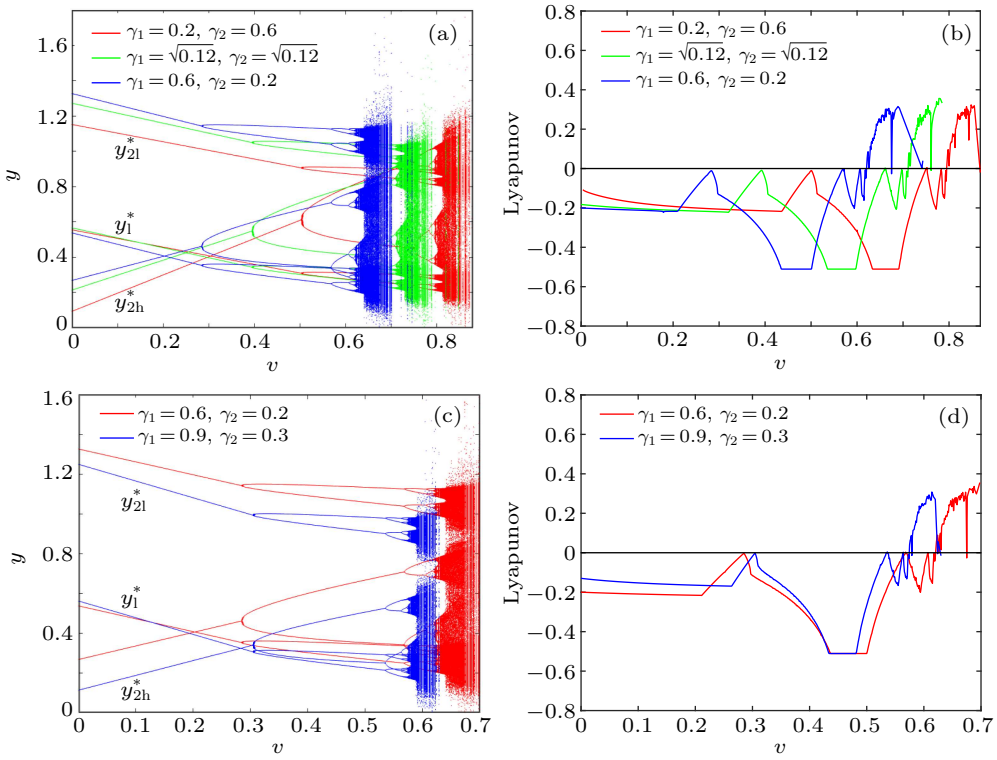


Fig. 3. Bifurcation plot and corresponding maximal Lyapunov exponent for θ.

To explore how the parameters θ and α influence QBNE stability, bifurcation diagrams are plotted for $\alpha = 1/3$, $\alpha = 1$ and $\alpha = 3$ with respect to θ, as shown in Fig. 3(a). As θ increases, the firm's output evolves from a stable equilibrium to bifurcation, and ultimately enters a chaotic regime. To more clearly illustrate how variations in α influence QBNE stability, the associated maximal Lyapunov exponent is plotted in Fig. 3(b). Obviously, a larger α causes bifurcation and chaos to occur earlier. Additionally, figures 3(c) and 3(d) indicate that a larger α delays bifurcation and leads to an earlier emergence of chaos.

Finally, we use strange attractors to further illustrate the impact of α on the system's stability. A hallmark of these strange attractors is the existence of at least one positive and one negative Lyapunov exponent, accompanied by a Lyapunov fractal dimension. The definition of the Lyapunov dimension is given by[44]

$$d_L = k + \sum_{i=1}^{k} \frac{L_i}{|L_{k+1}|},$$

where L_i $(i = 1, 2, \ldots, n)$ is the Lyapunov exponent, and $L_1 > L_2 > \cdots > L_n$; k is the largest positive integer such that $\sum_{i=1}^{k} L_i \geq 0$ and $\sum_{i=1}^{k+1} L_i < 0$.

Figure 4 is the attractor corresponding to Fig. 2(a) with $v = 1.32$. Using the same parameter settings as in Fig. 2(a), the Lyapunov exponents in Fig. 4(a) are $L_1 = -0.072$, $L_2 = -0.48$, and $L_3 = -0.5108$, indicating that the system reaches fixed points. The Lyapunov exponents in Fig. 4(b) are $L_1 = 0.3741$, $L_2 = -0.4073$, and $L_3 = -0.5108$. The Lyapunov dimension is $d(L) = 1 + \frac{L_1}{|L_2|} \approx 1.918$, indicating that the system exhibits a strange attractor. As α increases, the attractor transitions from fixed points to chaotic attractors.

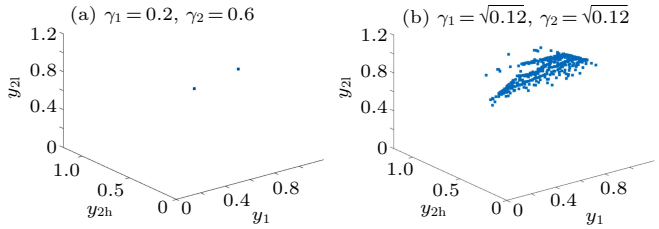


Fig. 4. The attractors corresponding to Fig. 2(a) with $v = 1.32$.

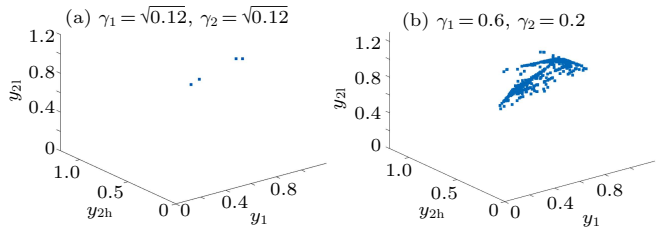


Fig. 5. The attractors corresponding to Fig. 2(a) with $v = 1.195$.

Figure 5 is the attractor corresponding to Fig. 2(a) with $v = 1.195$. Using the same parameter settings as in Fig. 2(a), the Lyapunov exponents in Fig. 5(a) are $L_1 = -0.8161$, $L_2 = -0.4173$, and $L_3 = -0.5108$, indicating that the system reaches fixed points. The Lyapunov exponents in Fig. 5(b) are $L_1 = 0.3742$, $L_2 = -0.4045$, and $L_3 = -0.5108$. The Lyapunov dimension is $d(L) = 1 + \frac{L_1}{|L_2|} \approx 1.925$, indicating that the system exhibits a strange attractor. As α increases, the attractor transitions from fixed points to chaotic attractors.

5. Conclusion

This paper proposes an asymmetric quantization model for dynamic quantum CDG with asymmetric information, mainly comparing the stability of the system between the quantum strategy of the asymmetric quantization model and that of the symmetric quantization model.[34] The results show that when $\alpha < 1$, under asymmetric information and heterogeneous expectations, the stability region of the asymmetric quantization model system for the dynamic quantum CDG is larger than that of the symmetric quantization model system. That is, the quantum strategy of the asymmetric quantization model is more favorable for stabilizing the market. However, when $\alpha > 1$, the system's stability region under the symmetric quantization model is larger than that under the asymmetric quantization model. That is, the quantum strategy of the symmetric quantization model is more conducive to stabilizing the market. Moreover, Proposition 3 indicates that the effects of the parameters β, θ, c_1, and ε on the stability of the system are consistent with the results of Zhou *et al*.[34]

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