

Rigorous verification of quantum contextuality from anomalous weak value

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Rigorous verification of quantum contextuality from anomalous weak value

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Weak measurement offers a powerful framework for probing nonclassical features of quantum mechanics, with anomalous weak values serving as operational signatures of contextuality. While the anomalous weak value verification of quantum contextuality has been predominantly investigated in the single-photon regime and analyzed under approximation condition of infinitesimally small perturbation strength. This study releases the approximation condition and takes into account the impact of perturbation strength on the rigor of the verification. And the investigation on the verification of contextuality is extended to the multi-photon scenarios for observing the influence of the correlation between photons on the verification. Without the limitation of infinitesimally small probability of disturbance, anomalous weak values are identified as necessary for contextuality to emerge, thereby refining the criterion proposed by Pusey [*Phys. Rev. Lett.* **113** 200401 (2014)]. In the multi-photon scenarios, the emergence of contextuality also depends strongly on both the photon number and the photon-number distribution state. In particular, contextuality is found to be maximized when the single-photon component dominates and the second-order correlation is lower. These results highlight the critical role of photon statistics in experimental tests of contextuality via anomalous weak values.

Keywords: quantum measurement, contextuality, weak measurement, weak values

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1. Introduction

The theory of weak measurements (WM) was first introduced by Aharonov, Albert, and Vaidman in 1988[1] as a novel quantum measurement paradigm for probing particle spin. Extending the standard von Neumann measurement model, WM enables partial information extraction through a weak interaction between the interested system and probe, without causing the collapse of the system into an eigenstate.[2–4] Today, WM stands as one of the most promising techniques for exploring fundamental aspects of quantum mechanics, with three major applications.[5–8] Among them, the real part of the weak-value (WV), representing the conditional mean of generalized observable eigenvalues, has proven to be a powerful tool for revealing anomalous quantum effects owing to its distinctive properties.[9–14] In particular, WM has emerged as a significant method for verifying quantum contextuality, offering a means to challenge noncontextual hidden-variable theories (NCHVTs).[15–21]

Previous studies have focused on the apparent disparity between anomalous WV and NCHVTs in quantum theory.[15,22] This contrast underscores a fundamental question: the supposed independence of measurement outcomes from simultaneous measurements of other observables in causally disconnected regions, a key assumption underlying NCHVTs.[23,24] Pusey's seminal work established that the observation of anomalous WVs can serve as a proof of contextuality,[15] a conclusion later experimentally verified by Piacentini *et al*.[25] Subsequent theoretical extensions, such as those by Kunjwal *et al*., have demonstrated that both the real and imaginary parts of anomalous WVs can signal incompatibility with NCHVTs.[26] Nevertheless, the genuine nonclassical nature of anomalous WVs remains a subject of debate.[27] As Pusey pointed out, it is essential to ask 'to what extent do they represent a genuinely nonclassical effect?',[15] motivating deeper investigations into the fundamental origins of contextuality manifested through anomalous WVs.

To further explore the nature of quantum contextuality revealed by anomalous WVs, we consider the influence of multiple photons and the correlation in it. Here the correlation in multiple photons is described by the second-order correlation function. Quantum entanglement, as another specific form of multi-photon correlation,[28] has been

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shown to enhance the effectiveness of weak-value amplification (WVA) techniques,[29,30] with its feasibility confirmed in experiments.[31] However, due to the difficulties in the preparation and the measurement of multi-photon entangled states, the entanglement between photons will not be considered in the work.

2. Verification with single photon

Theorem 1 Let us suppose that we have pre- and post-selection states $|i\rangle$ and $|f\rangle$ of the interested system, an initial probe state of $|\psi_i\rangle$, a nondestructive and WM process $\{\mathbb{M}_{\rm P}\}$ with unbiased noise $x\in\mathbb{R}$ dependent on the initial probe state, and a sharp measurement process $\{|f\rangle\langle f|\}$ on the interested system, such that:

(i) The pre- and post-selection are nonorthogonal, so the undisturbed probability of post-selection is given as

$$p_{\rm f}=|\langle f|i\rangle|^2>0. \tag{1}$$

(ii) If we ignore the sharp measurement process $\{|f\rangle\langle f|\}$, the nondestructive and WM process $\{\mathbb{M}_{\rm P}\}$ can be expanded by the projectors $\hat{\Pi}$, $\hat{\bar{\Pi}}=I-\hat{\Pi}$ weighed by the unbiased noise dependent on the initial probe state, *i.e.*,

$$\mathbb{P}_{\rm W}=\mathbb{M}_{\rm P}^{\dagger}\mathbb{M}_{\rm P}=p_n(x-g)\,\hat{\Pi}+p_n(x)\,\hat{\bar{\Pi}}, \tag{2}$$

where $p_n(x)=|\langle x|\psi_i\rangle|^2$ is initial probability distribution of the probe in position space with median $x=0$.

(iii) When the outcome of $\{\mathbb{M}_{\rm P}\}$ is ignored, the nondestructive and WM process $\{\mathbb{M}_{\rm P}\}$ affects sharp measurement process $\{|f\rangle\langle f|\}$ in the same way as mixing it with another POVM process $\{\mathbb{M}_{\rm d}\}$

$$\mathbb{P}=\int_{-\infty}^{\infty}\mathbb{M}_{\rm P}^{\dagger}|f\rangle\langle f|\mathbb{M}_{\rm P}\,{\rm d}x=(1-p_{\rm d})\,|f\rangle\langle f|+p_{\rm d}\mathbb{M}_{\rm d}, \tag{3}$$

with $p_{\rm d}$ defined as the 'probability of disturbance.

(iv) A contextuality inequality is satisfied if

$$p_-=\frac{1}{p_{\rm f}}\int_{-\infty}^{0}|\langle f|\mathbb{M}_{\rm P}|i\rangle|^2\,{\rm d}x>\frac{1}{2}+\frac{p_{\rm d}}{p_{\rm f}}, \tag{4}$$

where p_- is a negative bias of x values under the pre- and post-selection.

There is no noncontextual ontological model for the preselection of $|i\rangle$, measurement of $\{\mathbb{M}_{\rm P}\}$, and post-selection of $|f\rangle$ satisfying outcome determinism for sharp measurements.

Within the framework of WM with post-selection,[1] the WV of the projection operator $\Pi_{\rm w}$ is defined as

$$\Pi_{\rm w}=\frac{\langle f|\hat{\Pi}|i\rangle}{\langle f|i\rangle}. \tag{5}$$

The contextuality inequality can be rewritten as

$$p_-=A\,|\Pi_{\rm w}|^2+B\,\left|\tilde{\Pi}_{\rm w}\right|^2+2C\,{\rm Re}\left(\Pi_{\rm w}\tilde{\Pi}_{\rm w}^{*}\right)>\frac{1}{2}+\frac{p_{\rm d}}{p_{\rm f}}, \tag{6}$$

with $A=\int_{-\infty}^{0}p_n(x-g)\,{\rm d}x$, $B=\int_{-\infty}^{0}p_n(x)\,{\rm d}x$, and $C=\int_{-\infty}^{0}\sqrt{p_n(x-g)\,p_n(x)}\,{\rm d}x$. We can define a contextuality value as $I=p_--1/2-p_{\rm d}/p_{\rm f}>0$. From Eq. (6) Pusey has concluded that anomalous WV ${\rm Re}\,(\Pi_{\rm w})<0$ means contextuality, which has been extended and tightened in Ref. [26].

Experimental verification of this theoretical connection was first conducted using single-photon systems.[25] In these experiments, the photon's polarization served as the system of interest, with the pre-selected state $|i\rangle=(|H\rangle-|V\rangle)/\sqrt{2}$ and the post-selected state $|f\rangle=\cos\theta|H\rangle+\sin\theta|V\rangle$, where $|H\rangle$ and $|V\rangle$ represent the horizontal and vertical polarization states, respectively. The WM process was implemented through a spatial walk-off induced by birefringent crystals, described by the unitary operation $U={\rm e}^{-{\rm i}g\hat{\Pi}\otimes\hat{P}}$, where g is the weak interaction strength, $\hat{\Pi}=|V\rangle\langle V|$ is the observable, and $\hat{P}$ the transverse momentum operator.

The initial spatial distribution of the probe was a Gaussian beam:

$$|\psi_i\rangle=N\int_{-\infty}^{\infty}{\rm e}^{-x^2/2\sigma^2}|x\rangle\,{\rm d}x. \tag{7}$$

where $N=(\pi\sigma^2)^{-1/4}$ and σ is the width of the wave packet.

Under experimental conditions with a post-selection angle $\theta=0.18\pi$ and a disturbance probability $p_{\rm d}=1-{\rm e}^{-g^2/(4\sigma^2)}=0.0019\pm0.0002$, a contextuality value $I=0.063\pm0.011$ was observed.[25]

The previous research works are based on the assumption of WM with $p_{\rm d}\to0$, which is rigours for its experimental realization. Here we release slightly the requirement of $p_{\rm d}$, rewrite the negative bias as

$$\begin{aligned}p_-=\frac{1}{2}-\frac{g}{\sqrt{\pi}\sigma}{\rm Re}\,(\Pi_{\rm w})-\frac{1}{2}O\left(\frac{g}{\sigma}\right)|\Pi_{\rm w}|^2-\left[O\left(\frac{g}{2\sigma}\right)\right.\\+\left.\left(\frac{g}{\sqrt{\pi}\sigma}+O\left(\frac{g}{2\sigma}\right)-1\right)O\left(-\frac{g^2}{4\sigma^2}\right)\right]{\rm Re}\left(\Pi_{\rm w}\tilde{\Pi}_{\rm w}\right),\end{aligned} \tag{8}$$

where both $O(g/\sigma)$ and $O\left(-g^2/4\sigma^2\right)$ represent the first-order and second-order perturbative terms associated with the disturbance intensity g/σ, respectively.

Compared with previous approximations,[26] our corrections explicitly account for both the disturbance probability $p_{\rm d}$ and the WV $\Pi_{\rm w}$.

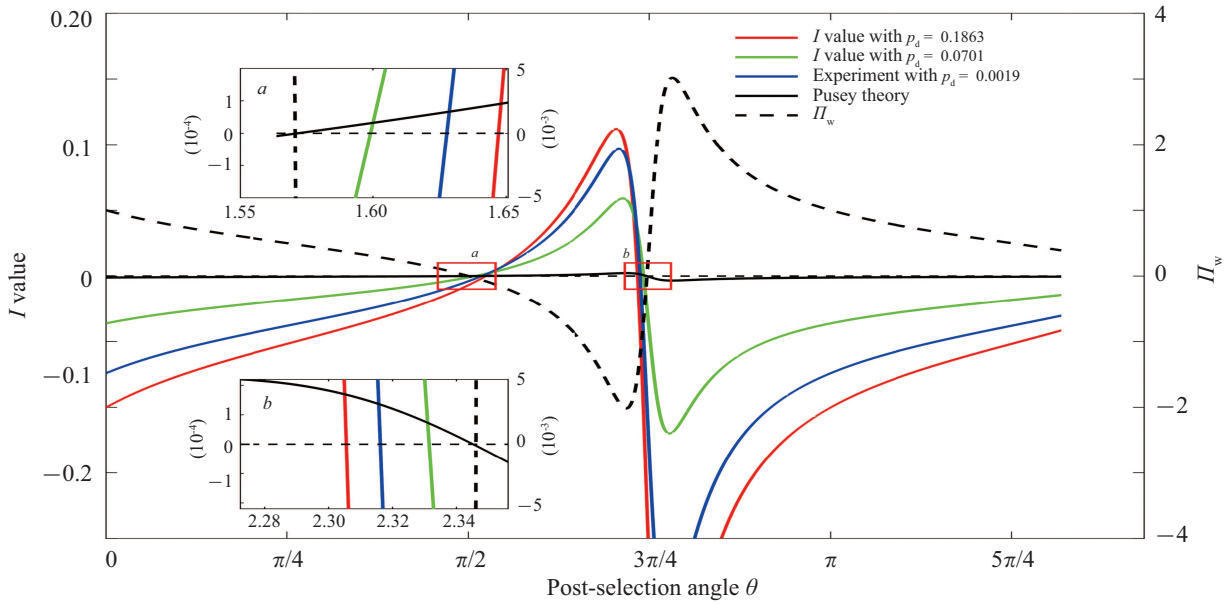


Fig. 1. The distribution of the contextuality quantities I (solid line) in different disturbance probability $p_{\rm d}$ and WV $\Pi_{\rm w}$ (dashed line) within a half period of the post-selection angle θ in the single photon case. Accompanying diagrams a and b depict detailed figures in the region where the WV $\Pi_{\rm w}$ and I value converge.

Furthermore, to probe slight deviations, we consider a small post-selection phase shift $\varepsilon = 0.1$ rad in the post-selection state

$$|f\rangle = \cos\theta\, {\rm e}^{{\rm i}\varepsilon}|H\rangle + \sin\theta\, {\rm e}^{-{\rm i}\varepsilon}|V\rangle. \tag{9}$$

We use the spatial walk-off experienced by the horizontally polarized photons, the distribution of the contextuality I value with respect to the post-selection angle θ in different disturbance intensity g/σ as shown in Fig. 1.

Without loss of generality, we use the anomalous WVs of the projection operator, ${\rm Re}(\Pi_{\rm w}) < 0$, to relate it to the anomalous WVs of the system's observable. Since the eigenvalues of the projection operator fall within the range $[0,1]$, ${\rm Re}(\Pi_{\rm w}) < 0$ is expected to indicate contextuality with $I > 0$. However, as shown in Figs. 1(a) and 1(b), this expectation is not realized. The manifestation of contextuality requires the anomalous WV to surpass a certain threshold. In Fig. 1, the black solid line represents Pusey's approximate theory, corresponding to ${\rm Re}(\Pi_{\rm w}) < 0$ and $I < 0$. In contrast, the other curves (red, blue, green) with different disturbance probability ($p_{\rm d} = 0.1863$, 0.0019, and 0.0704) show that a higher degree of WV anomaly is needed to observe contextuality. Furthermore, as $p_{\rm d}$ increases, a stronger anomaly in the WV is required for contextuality to emerge.

The above analysis shows that the verification between the quantum contextuality and anomalous WVs is in one-way model. Anomalous WVs are necessary but not sufficient for quantum contextuality. The answer to the question: 'To what extent do they represent genuinely nonclassical effects?' is that WVs must exceed a threshold depending on factors such as post-selection probability $p_{\rm f}$, disturbance probability $p_{\rm d}$, and the WV $\Pi_{\rm w}$ itself.

3. Verification with multi-photon

It is of practical significance to investigate the impact of multi-photon components in the light source on the verification of contextuality. These photons may contain some kinds of correlations,such as bunching effect.

To address the impact of bunching effect in multi-photon on the verification of contextuality, we start from a pre-selected state $|i\rangle^{\otimes n} = [(|H\rangle + |V\rangle)\sqrt{2}]^{\otimes n}$. The system evolves under Hamiltonian with an observable $\hat{\Pi}^{\otimes n}$, influenced by both the photon number n and the distribution $\bar{n}$. Post-selection onto $|f\rangle^{\otimes n} = (\cos\theta\, {\rm e}^{{\rm i}\varepsilon}|H\rangle + \sin\theta\, {\rm e}^{-{\rm i}\varepsilon}|V\rangle)^{\otimes n}$ yields a WV $\Pi_{\rm w}^{\otimes n}$, from which the contextuality indicator I is derived, with post-selection success probability

$$p_{\rm f}^{(\otimes n)} = \left[\frac{1}{2}(1+\cos 2\theta \sin 2\varepsilon)\right]^n.$$

In multi-photon system, the rank-1 projection operator for the system is expressed as

$$\hat{\Pi}^{(\otimes n)} = \hat{\Pi}_1 \otimes \cdots \otimes \hat{\Pi}_n, \tag{10}$$

where $\hat{\Pi}_n = |V\rangle_n\langle V|$ denotes the observable $\hat{\Pi}$ of the n-th single system to satisfy $p\left(\hat{\Pi}^{(\otimes n)}|\mathbb{M}_{\rm P}\right) \in \{0,1\}$, and $\hat{U}^{(\otimes n)} = {\rm e}^{-{\rm i}g\hat{\Pi}^{(\otimes n)}\otimes\hat{P}}$ represents the evolution operator for the interaction between the system and probe. We can obtain the WV with

$$\Pi_{\rm w}^{(\otimes n)} = \left(\frac{\cos\theta\, {\rm e}^{-{\rm i}\varepsilon}}{\cos\theta\, {\rm e}^{-{\rm i}\varepsilon} + \sin\theta\, {\rm e}^{{\rm i}\varepsilon}}\right)^n.$$

The unbiased noise applied to the nondestructive measurement $\{\mathbb{M}_{\rm P}\}$ described in Theorem 1,

$$\mathbb{P}_{\rm W}^{(\otimes n)} = (\mathbb{M}_{\rm P}^{(\otimes n)})^{\dagger}\mathbb{M}_{\rm P}^{(\otimes n)}$$

$$= p_n(x-g)\hat{\Pi}^{(\otimes n)} + p_n(x)\hat{\tilde{\Pi}}^{(\otimes n)}, \tag{11}$$

when a Gaussian probe is used as in the case of single photon, the expression for $\mathbb{M}_{\mathrm{P}}^{(\otimes n)}$ is given by

$$\mathbb{M}_{\mathrm{P}}^{(\otimes n)} = N\exp\left[-\frac{(x-g)^2}{2\sigma^2}\right]\hat{\Pi}^{(\otimes n)} + N\exp\left[-\frac{x^2}{2\sigma^2}\right]\hat{\tilde{\Pi}}^{(\otimes n)}, \tag{12}$$

and the 'probability of disturbance' $p_{\mathrm{d}}^{(\otimes n)}$ in multi-photon is $[1-\triangle^{(\otimes n)}]/2$, with $\triangle^{(\otimes n)} = \exp\left[-g^2/4\sigma^2\right]$.

The negative bias can be obtained,

$$\begin{aligned} p_{-}^{(\otimes n)} = {} & \frac{1}{2} - \frac{g}{\sqrt{\pi}\sigma}\mathrm{Re}\left(\Pi_{\mathrm{w}}^{(\otimes n)}\right) - \frac{1}{2}O\left(\frac{g}{\sigma}\right)\left|\Pi_{\mathrm{w}}^{(\otimes n)}\right|^2 \\ & - O\left(\frac{g}{2\sigma}\right)\mathrm{Re}\left[\Pi_{\mathrm{w}}^{(\otimes n)}\left(\tilde{\Pi}_{\mathrm{w}}^{(\otimes n)}\right)^*\right] \\ & + \left[1 - \frac{g}{\sqrt{\pi}\sigma} - O\left(\frac{g}{2\sigma}\right)\right]O\left(-\frac{g^2}{4\sigma^2}\right) \\ & \times \mathrm{Re}\left[\Pi_{\mathrm{w}}^{(\otimes n)}\left(\tilde{\Pi}_{\mathrm{w}}^{(\otimes n)}\right)^*\right], \end{aligned} \tag{13}$$

the meanings of $O(g/\sigma)$ and $O\left(-g^2/4\sigma^2\right)$ are consistent with those in Eq. (8).

Finally, multi-photon contextuality inequality can be obtained,

$$I^{(\otimes n)} = p_{-}^{(\otimes n)} - \frac{1}{2} - \frac{p_{\mathrm{d}}^{(\otimes n)}}{p_{\mathrm{f}}^{(\otimes n)}} > 0. \tag{14}$$

We visually depict the distributions of I value and Π_{w} concerning θ in a unified plot in Fig. 2 for different photon numbers.

It is obvious from Fig. 2 that the system fails to exhibit contextuality for the case with photon numbers greater than one. The reason is in this configuration of more than one photons, the contextuality of individual photon, manifesting in their spatial distribution, is overshadowed by the measurement outcomes of other photons.

As shown in Fig. 2, for contextuality to exist, the curves of I and Π_{w} *versus* θ must intersect the $I=0$ line at least twice. Figure 2 highlights the first intersection, with the anomalous WVs and I values magnified. In the red boxes of Fig. 2, one can identify regions where anomalous WVs appear without contextuality, similar to the single-photon regime, while the behavior changes with the number of photons. This observation suggests the need for further analysis of the WV anomalies and the influence of key parameters such as interaction strength and post-selection probability. Combined Eqs. (14) and (13) the boundary of anomalous WVs that giving rise to contextuality can be obtained. Under the approximation of $\mathrm{e}^{-g^2/4\sigma^2} \approx 1$, boundary value can be defined as $-\left(\sqrt{\pi\sigma^2}/g\right)(p_{\mathrm{d}}/p_{\mathrm{f}})$.

In Fig. 3, multi-photon scenarios of WV boundary values (solid lines) are presented, obtained without any approximation from the contextuality inequality. It can be found that if the anomalous WVs verifies contextuality, the WV (dashed lines) must break through the boundary values, as depicted in the accompanying figures for different photon numbers. The larger the area enclosed between the solid and dashed lines, the stronger the contextuality exhibited by the system.

The distribution of boundary values reveals that increasing the photon number significantly reduces the area between the anomalous WV and its threshold, indicating a suppression of contextuality. As equation (14) shows, the critical WV required to reveal contextuality depends not only on the probability of disturbance p_{d} and post-selection probability p_{f}, but also grows faster than the anomalous WV itself in multi-photon regimes, leading to the loss of contextuality.

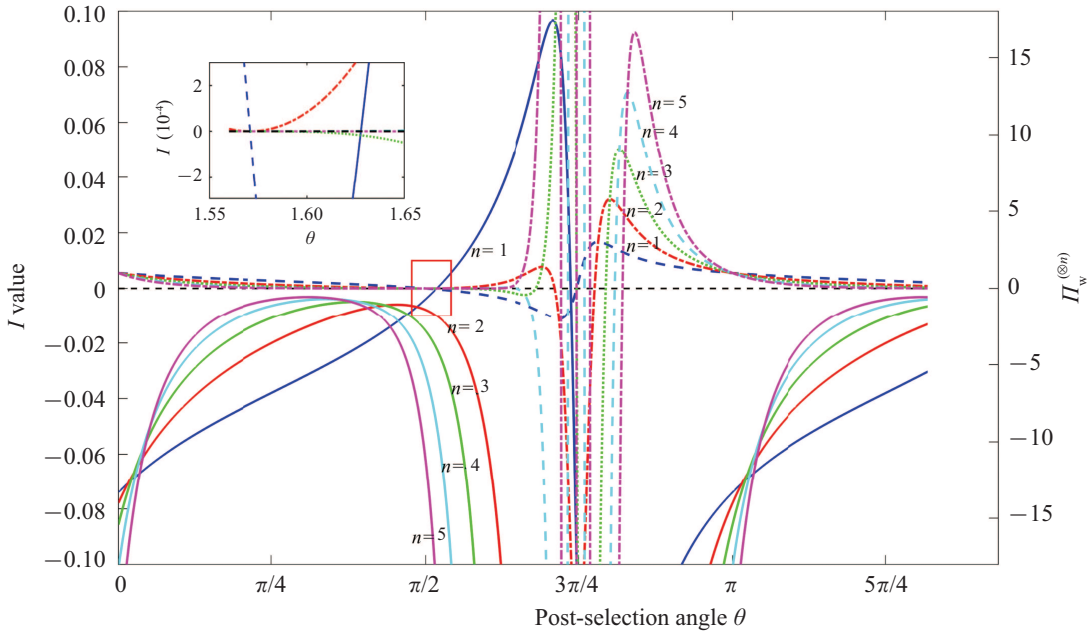


Fig. 2. The distributions of the I value (solid lines) and $\Pi_{\mathrm{w}}^{(\otimes n)}$ (dashed lines) as functions of θ in multi-photon scenarios with $p_{\mathrm{d}} = 0.0019$.

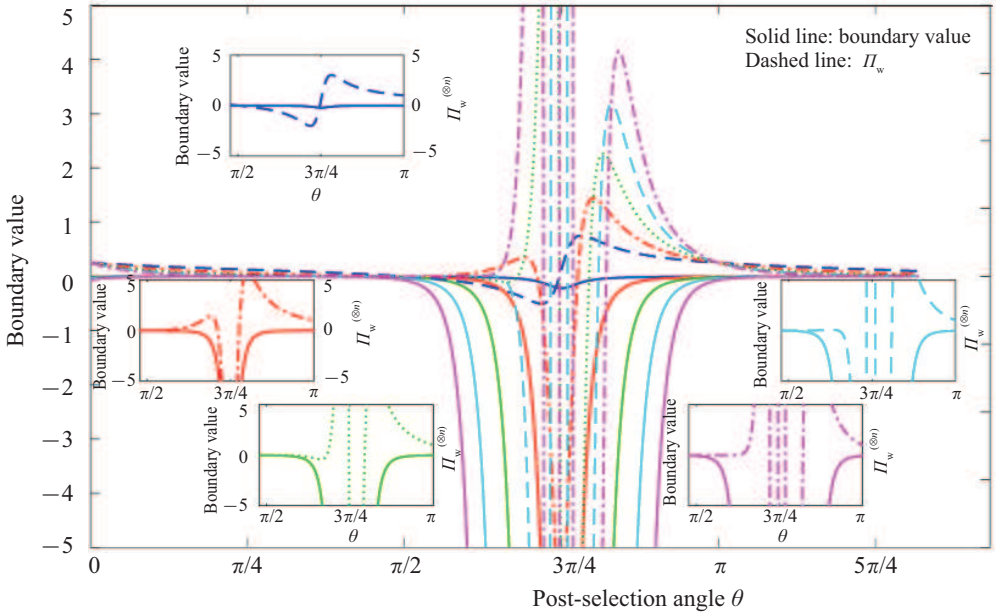


Fig. 3. The distributions of the boundary value as functions of θ in multi-photon scenarios with $p_{\rm d} = 0.0019$.

From another practical perspective, in real experimental scenarios, the emitted light from a source is not fixed at a particular photon number level but rather distributed among different photon numbers with a certain probability P_n.[32] In such cases, the measured I value should be described as the sum of I_n values generating by different numbers of photons weighted by their probabilities P_n, as expressed in the following equation:

$$\sum_n P_n I_n > 0. \tag{15}$$

Here, we represent the light source using the quantum states of single-mode fields in quantum optics, and discuss the contextuality for several common single-mode field quantum states: coherent state, squeezed state and thermal state.

The distribution of photon numbers in the coherent state $|\alpha\rangle$ obeys a Poisson distribution

$$P_n = \mathrm{e}^{-\bar{n}} \frac{\bar{n}^n}{n!}, \tag{16}$$

when the average photon number is $\bar{n}$, the probability of having n photons is denoted as P_n. The second-order correlation function at zero delay for a coherent state is given by $G_{\rm c}^{(2)}(0) = 1$, which means that the photons do not interact with each other, and randomly distribute in numbers. By limiting the maximum photon number to be 5, the distribution of contextual I values with respect to the post-selection angle θ and the average photon number $\bar{n}$ is depicted in Fig. 4(a). When $\bar{n} = 0.1$, it represents a weak coherent state. From Fig. 4(a), we observed that with the increase of $\bar{n}$, the corresponding positive I value gradually decreases until there is no region with $I > 0$ for the whole post-selection angle range.

For the single-mode squeezed vacuum state $|\xi\rangle$, the distribution of photon numbers is given by

$$P_{2m} = \frac{1}{\cosh r} \left(\frac{1}{2} \tanh r \right)^{2m} \frac{(2m)!}{(m!)^2}, \tag{17}$$

where r represents the squeezing strength describing the magnitude of squeezing. The photon number $n = 2m$ indicates that only an even number of photons can be detected in the squeezed state, and the relationship between the average photon number $\bar{n}$ and r satisfies: $\bar{n} = \sinh^2 r$. The second-order correlation function of the single-mode squeezed vacuum state is given by $G_{\rm s}^{(2)}(0) \geq 3$, which means that the photons interact with each other, and there is strong correlation betweens photons. By adjusting the average photon number $\bar{n}$ through r, the distribution of contextual I values with respect to the post-selection angle θ and the average photon number $\bar{n}$ is shown in Fig. 4(b). From Fig. 4(b), we observe that with the increase of $\bar{n}$, the variation of I with θ does not change significantly, and without exhibiting contextuality.

For quantum state in thermal radiation, known as the thermal state, its photon number distribution follows a super-Poissonian distribution. The probability distribution of photon numbers is expressed as

$$P_n = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}. \tag{18}$$

The second-order correlation function at zero delay for a thermal state is given by $G_t^{(2)}(0) = 2$, which means that photons in a thermal light field tend to occur in pairs and exhibit bunching effect. The distribution of I values with respect to the post-selection angle θ and the average photon number $\bar{n}$ is depicted in Fig. 4(c). Similar to the coherent state, the I value gradually decreases with $\bar{n}$ increasing. The maximum I value of the system approaches 0 for $\bar{n} = 5$, which indicates no contextuality.

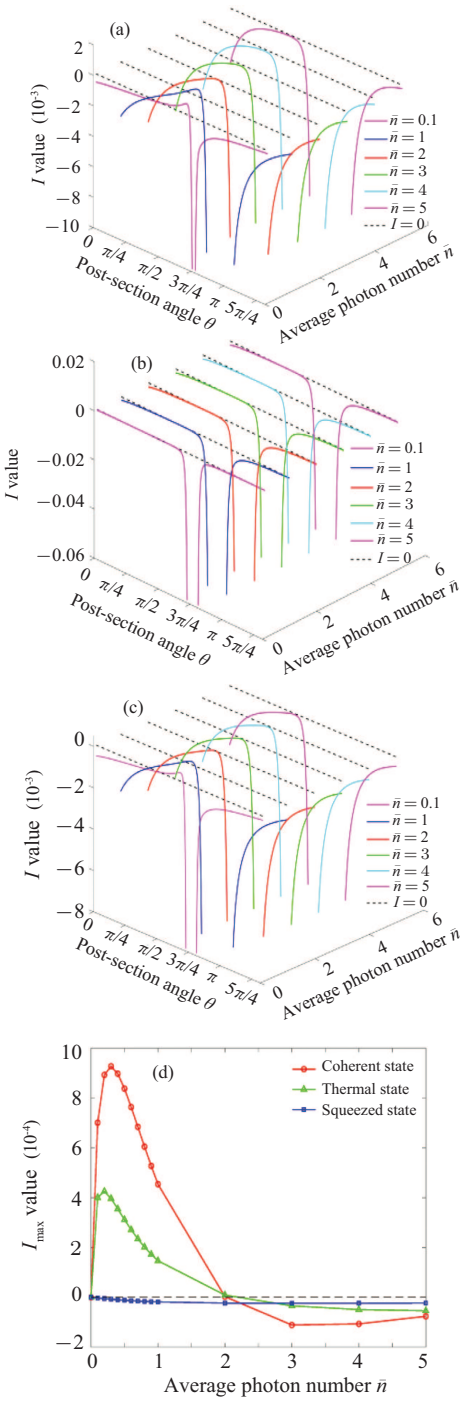


Fig. 4. Panels (a)–(c) depict the distributions of the I values for the coherent, squeezed, and thermal states at different average photon numbers $\bar{n}$ for post-selection angles with $p_{\rm d} = 0.0019$, respectively. Panel (d) shows the trend of the maximum value of I with the average photon number $\bar{n}$ for different states.

It can be summarized from the above analyses that in multi-photon scenarios, the contextuality is not enhanced as expected. Instead, it gradually diminishes as the average photon number increases. This kind of trend is the same for different photon-number distribution models. However, under coherent state, squeezed state, and thermal state of single-mode light field, the maximum of the contextuality value $I_{\max}$ exhibits distinct behavior as a function of the average photon number $\bar{n}$, as shown in Fig. 4(d). With comparison in the second-order correlation function of three kinds of quantum states, it can be concluded that the correlation in the light source does not make positive contribution to the presence of the contextuality. The same conclusion can be obtain from the situation where the photons are individually disturbed by unitary operation $\hat{U}$ with

$$\hat{\Pi}^{(\otimes n)} = \hat{\Pi}_1 \otimes \hat{I}_2 \cdots \otimes \hat{I}_n + \cdot + \hat{I}_1 \otimes \hat{I}_2 \cdots \otimes \hat{\Pi}_n, \tag{19}$$

as demonstrated in Fig. 5.

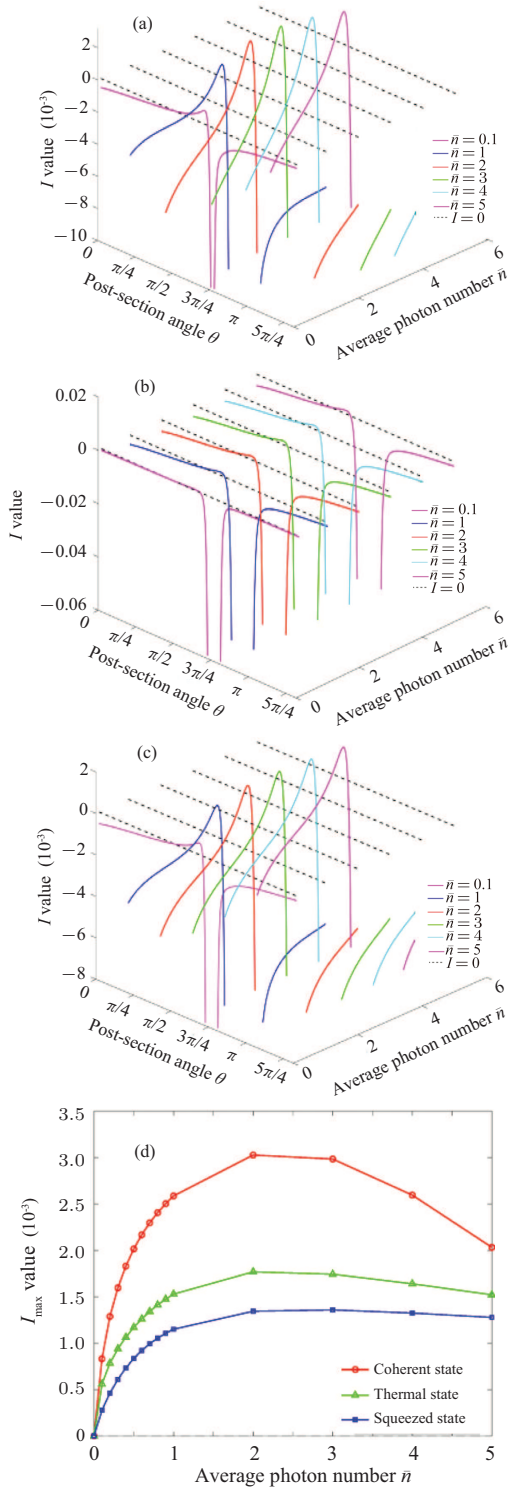


Fig. 5. The coherent, squeezed, and thermal states at different average photon numbers $\bar{n}$ for post-selection angles with $p_{\rm d} = 0.0019$ when the photons are individually disturbed, respectively.

4. Summary

Within the framework of Theorem 1, we investigated the verification of quantum contextuality via anomalous WVs, considering both single photon and multiple photons. Our analysis reveals that maximal contextuality emerges only in the single-photon regime and diminishes as the photon number increases. The contextuality occurs when the anomalous WVs exceed a certain threshold, which depends on factors such as the probability of disturbance p_{d}, the post-selection probability p_{f} and the numbers of photon numbers. The contextuality is discussed based on three kinds of common quantum states with differen correlation characteristic including coherent state, squeezed state, and thermal state. The correlation in the light source is not beneficial to the presence of the contextuality. In conclusion, strictly speaking, anomalous WVs just are necessary for contextuality to emerge.

Acknowledgments

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