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Degenerate solitons and asymptotic analysis for a three-coupled fourth-order nonlinear Schrödinger system in an alpha helical protein

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We investigate the alpha helical protein structure characterized by fourth-order interspine coupling, focusing on a three-coupled fourth-order nonlinear Schrödinger system. We introduce a generalized Darboux transformation, departing from the classical Darboux transformation. Based on this, we construct the two- and three-degenerate soliton solutions and four-degenerate asymptotic soliton solutions. Based on the asymptotic analysis, we find that the amplitudes of interacting solitons are retained upon the interactions. Elastic interactions between two degenerate solitons exhibiting four curve-type asymptotic solitons are depicted. When the lattice parameter β changes, the velocities of the two degenerate solitons also change. Elastic interaction among three degenerate solitons comprising four curve-type asymptotic solitons and two line-type solitons is presented. Interaction among one soliton and two degenerate solitons with different velocities is shown. Elastic interaction among four degenerate solitons comprising eight curve-type asymptotic solitons is also presented. Interaction among two two-degenerate solitons with two spectral parameters is shown. The relative distance between two asymptotic solitons exhibits logarithmic growth with $|t|$, where t represents the retarded time. Acceleration of soliton separation decays exponentially with relative distance, and eventually approaches zero. Phase shifts depend on t.

Keywords: three-coupled fourth-order nonlinear Schrödinger system, interactions of degenerate solitons, asymptotic analysis

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1. Introduction

Nonlinear properties have significantly influenced energy transfer and retention within physical, chemical and biological systems.[1,2] It has been widely recognized that many biological processes are linked to the conveyance of energy through protein molecules, with bio-energy being liberated via the hydrolysis of adenosine triphosphate.[1] A nonlinear dynamics has been induced via the interaction between vibrons and phonons, which achieves the dispersion balance of dipole–dipole coupling.[1] This has produced the Davydov soliton, which appears as a solution to the nonlinear Schrödinger equation (NLSE) in the continuum approximation.[3] Daniel and Deepamala[4] first studied the effects of higher-order molecular excitations and interactions on the soliton dynamics of alpha helical proteins. They found that the obtained results are favorable for soliton propagation through the chains. Studies have characterized the dynamic properties of the modified Davydov model for α-helical proteins at both discrete and continuous levels, and the dynamics controlled by some specific integrable three-coupled discrete NLSEs.[5] One soliton and the interactions of two and three solitons for the integrable three-coupled discrete NLSEs have been studied.[6]

As is well known, in the terminology of inverse scattering transformation the N-soliton solutions are related to some reflection coefficients allowing for N simple poles, where N is a positive integer.[7] When these simple poles merge, multi-pole solutions (also known as degenerate soliton solutions) have been obtained, where the interacting solitons form weakly bound states.[7–10] Contrary to conventional multi-soliton interactions, degenerate solitons display a strong interaction, with their relative distance exhibiting logarithmic growth with $|t|$.[7–11] There have been nondegenerate solutions of Manakov equations that have attracted much attention recently.[12–15] Liu *et al*. have developed an exact theory of the superregular breathers of Manakov equations.[12] The theory is based on eigenvalue analysis and on finding the exact links between the superregular breathers and modulation instability.[12] Non-degenerate complex solutions which admit the coexistence of multi-rogue waves with different amplitude profiles have been investigated.[13] Non-degenerate Akhmediev breathers that have unequal individual eigenvalues have been studied.[14] Non-degenerate Kuznetsov–Ma solitons formed by nonlinear superposition of two fundamental Kuznetsov–Ma solitons that have the same propagation period but different eigenvalues have been constructed.[15] The elastic and inelastic interactions of solitons have been studied.[16–18] These nonlinear waves are iteratively composed of multiple spectral parameters, but our degenerate solitons

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are composed of one spectral parameter that is iterated multiple times. Mathematically speaking, our degenerate soliton solutions are not simply composed of exponential or polynomial functions but are composed of exponential and polynomial functions. As a higher-order extension of the Manakov equations, a three-coupled fourth-order NLS system, accounting for quartic interspine coupling in an alpha helical protein, has been analyzed:[1,19–23]

$$\begin{aligned}&\mathrm{i}q_{\iota,t}+q_{\iota,xx}+\beta q_{\iota,xxxx}+2q_\iota\sum_{\rho=1}^{3}|q_\rho|^2\\&+\beta\Big[2q_\iota\sum_{\rho=1}^{3}|q_{\rho,x}|^2+2q_{\iota,x}\sum_{\rho=1}^{3}q_\rho q^*_{\rho,x}+6q_{\iota,x}\sum_{\rho=1}^{3}q^*_\rho q_{\rho,x}\\&+4q_{\iota,xx}\sum_{\rho=1}^{3}|q_\rho|^2+4q_\iota\sum_{\rho=1}^{3}q^*_\rho q_{\rho,xx}+2q_\iota\sum_{\rho=1}^{3}q_\rho q^*_{\rho,xx}\\&+6q_\iota\Big(\sum_{\rho=1}^{3}|q_\rho|^2\Big)^2\Big]=0,\quad \iota=1,2,3,\end{aligned}\tag{1}$$

where q_ι denote the three-component amplitudes of molecular excitation, β denotes lattice parameter, the variables t and x denote the retarded time and scaled distance, $\mathrm{i}^2=-1$ and the symbol $*$ indicates the operation of complex conjugation.

The Lax pair associated with system (1) has been reconstructed as[21]

$$\Phi_x=U\Phi,\quad \Phi_t=V\Phi,\tag{2}$$

where

$$\begin{aligned}U&=\mathrm{i}\lambda\sigma+P,\\V&=8\mathrm{i}\beta\sigma\lambda^4+8\beta P\lambda^3\\&\quad+\lambda^2[4\mathrm{i}\beta P_x\sigma+4\mathrm{i}\beta P^2\sigma+2\mathrm{i}(I_4-\sigma)]\\&\quad+\lambda(4\beta P^3-2\beta P_{xx}-2\beta PP_x\\&\quad+2\beta P_xP-2P)+\mathrm{i}\beta\sigma P_{xxx}-\mathrm{i}\beta P_{xx}P\sigma\\&\quad-\mathrm{i}\beta PP_{xx}\sigma+\mathrm{i}\beta P_x^2\sigma+3\mathrm{i}\beta P^2P_x\sigma+3\mathrm{i}\beta P_xP^2\sigma\\&\quad+\mathrm{i}\sigma P_x-\mathrm{i}\sigma P^2+3\mathrm{i}\beta\sigma P^4,\\\sigma&=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad I_4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\\P&=\begin{pmatrix}0&q_1^*&q_2^*&q_3^*\\-q_1&0&0&0\\-q_2&0&0&0\\-q_3&0&0&0\end{pmatrix},\end{aligned}$$

the spectral parameter λ is defined as a complex-valued constant, $\Phi=(\phi_1,\phi_2,\phi_3,\phi_4)^{\mathrm{T}}$ is a vector, T denotes the transpose operation applied to vectors or matrices and ϕ_1, ϕ_2, ϕ_3 and ϕ_4 represent complex-valued functions that maintain continuous differentiability under variations of x and t. Through application of the compatibility condition $U_t-V_x+UV-VU=0$, complete formulation of system (1) has been obtained.[21]

For system (1), through the application of the Darboux transformation (DT) not only the Lax integrability but also the one-, two- and three-soliton solutions have been systematically acquired.[1] Utilization of the Bell polynomials has enabled investigation of the bound states of two and three solitons.[19] Generalized DT (GDT) has been employed to yield semi-rational rogue waves.[20] Single vector rogue waves, vector rogue wave pairs, triple vector rogue waves and baseband modulation instability have been worked out[21] and three types of the Nth-order rogue wave solutions obtained.[22] Infinitely many conservation laws and Darboux-dressing transformations, breather-wave solutions and rogue wave solutions have been reported.[23]

In the following, we rewrite the GDT for system (1) in Ref. [20]. It has been assumed that Ω_1 is a solution for Lax pair (2) at $\lambda=\lambda_1+\varepsilon$ and $q_\iota=0$, where ε is a real constant. Ω_1 has been expanded at λ_1 as

$$\begin{aligned}&\Omega_1=\psi_0+\psi_1\varepsilon+\psi_2\varepsilon^2+\cdots+\psi_{N-1}\varepsilon^{N-1}+\cdots,\\&\psi_k=\frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\Omega_1(\lambda)|_{\lambda=\lambda_1}.\end{aligned}$$

Through the following GDT, Φ_N has been obtained as

$$\begin{aligned}\Phi_N&=\psi_0+\left[\sum_{s=1}^{N-1}(\lambda_1-\lambda_1^*)-(\lambda_1-\lambda_1^*)\frac{\Phi_s\Phi_s^\dagger}{\Phi_s^\dagger\Phi_s}\right]\psi_1+\cdots\\&\quad+\left\{\left[(\lambda_1-\lambda_1^*)-(\lambda_1-\lambda_1^*)\frac{\Phi_{N-1}\Phi_{N-1}^\dagger}{\Phi_{N-1}^\dagger\Phi_{N-1}}\right]\right.\\&\quad\left.\cdots\left[(\lambda_1-\lambda_1^*)-(\lambda_1-\lambda_1^*)\frac{\Phi_1\Phi_1^\dagger}{\Phi_1^\dagger\Phi_1}\right]\right\}\psi_{N-1},\end{aligned}\tag{3}$$

where $\Phi_1=\psi_0$, Φ_N represents the non-zero solution of Lax pair (2), $\dagger$ denotes the matrix transpose and complex conjugation, while $[N]$ stands for N iterations. Based on solutions (3), the Nth-order solutions $q_\iota[N]$ for system (1) are derived as

$$q_1[N]=2\mathrm{i}(\lambda_1^*-\lambda_1)\sum_{m=1}^{N}\frac{(\Phi_m\Phi_m^\dagger)_{21}}{\Phi_m^\dagger\Phi_m},\tag{4a}$$

$$q_2[N]=2\mathrm{i}(\lambda_1^*-\lambda_1)\sum_{m=1}^{N}\frac{(\Phi_m\Phi_m^\dagger)_{31}}{\Phi_m^\dagger\Phi_m},\tag{4b}$$

$$q_3[N]=2\mathrm{i}(\lambda_1^*-\lambda_1)\sum_{m=1}^{N}\frac{(\Phi_m\Phi_m^\dagger)_{41}}{\Phi_m^\dagger\Phi_m},\tag{4c}$$

where $(\Phi_m\Phi_m^\dagger)_{21}$, $(\Phi_m\Phi_m^\dagger)_{31}$ and $(\Phi_m\Phi_m^\dagger)_{41}$ denote the entries in the second, third and fourth rows and the first column of the matrix $\Phi_m\Phi_m^\dagger$, respectively.

However, to our knowledge, a GDT different from that presented in Ref. [20], and interactions of two, three and four degenerate solitons for system (1) have not been carried out. For system (1), Section 2 will construct a GDT at two spectral parameters. In Section 3, we will derive the solutions for two, three and four degenerate solitons and perform corresponding asymptotic analysis, and the interactions among three and four degenerate solitons with different velocities will be worked out. Section 4 will give our conclusions.

2. GDT for system (1)

To investigate the semirational solutions, motivated by Ref. [24], we find another GDT at two spectral parameters. To construct the degenerate soliton solutions for system (1), we begin with the seed solutions for system (1) by setting $q_1 = q_2 = q_3 = 0$. Supposing that Ω_1 and Ω_2 for Lax pair (2) at $\lambda = \lambda_1 + \varepsilon$ and $\lambda = \lambda_2 + \varepsilon$ respectively can be presented as

$$\Omega_1 = \begin{pmatrix} w_1 \mathrm{e}^{-\mathrm{i}(\lambda_1+\varepsilon)x+[4\mathrm{i}(\lambda_1+\varepsilon)^2-8\mathrm{i}\beta(\lambda_1+\varepsilon)^4]t} \\ w_2 \mathrm{e}^{\mathrm{i}(\lambda_1+\varepsilon)x+8\mathrm{i}\beta(\lambda_1+\varepsilon)^4 t} \\ w_3 \mathrm{e}^{\mathrm{i}(\lambda_1+\varepsilon)x+8\mathrm{i}\beta(\lambda_1+\varepsilon)^4 t} \\ w_4 \mathrm{e}^{\mathrm{i}(\lambda_1+\varepsilon)x+8\mathrm{i}\beta(\lambda_1+\varepsilon)^4 t} \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} w_1 \mathrm{e}^{-\mathrm{i}(\lambda_2+\varepsilon)x+[4\mathrm{i}(\lambda_2+\varepsilon)^2-8\mathrm{i}\beta(\lambda_2+\varepsilon)^4]t} \\ w_2 \mathrm{e}^{\mathrm{i}(\lambda_2+\varepsilon)x+8\mathrm{i}\beta(\lambda_2+\varepsilon)^4 t} \\ w_3 \mathrm{e}^{\mathrm{i}(\lambda_2+\varepsilon)x+8\mathrm{i}\beta(\lambda_2+\varepsilon)^4 t} \\ w_4 \mathrm{e}^{\mathrm{i}(\lambda_2+\varepsilon)x+8\mathrm{i}\beta(\lambda_2+\varepsilon)^4 t} \end{pmatrix}. \tag{5}$$

Next, we expand solutions Ω_1 at λ_1 as follows:

$$\Omega_1 = \begin{pmatrix} w_1 \mathrm{e}^{-\mathrm{i}\lambda_1 x+(4\mathrm{i}\lambda_1^2-8\mathrm{i}\beta\lambda_1^4)t} \\ w_2 \mathrm{e}^{\mathrm{i}\lambda_1 x+8\mathrm{i}\beta\lambda_1^4 t} \\ w_3 \mathrm{e}^{\mathrm{i}\lambda_1 x+8\mathrm{i}\beta\lambda_1^4 t} \\ w_4 \mathrm{e}^{\mathrm{i}\lambda_1 x+8\mathrm{i}\beta\lambda_1^4 t} \end{pmatrix} + \begin{pmatrix} -\mathrm{i}w_1 \mathrm{e}^{-\mathrm{i}\lambda_1 x+(4\mathrm{i}\lambda_1^2-8\mathrm{i}\beta\lambda_1^4)t}[x+8t\lambda_1(4\beta\lambda_1^2-1)] \\ \mathrm{i}w_2 \mathrm{e}^{\mathrm{i}\lambda_1 x+8\mathrm{i}\beta\lambda_1^4 t}(x+32t\beta\lambda_1^3) \\ \mathrm{i}w_3 \mathrm{e}^{\mathrm{i}\lambda_1 x+8\mathrm{i}\beta\lambda_1^4 t}(x+32t\beta\lambda_1^3) \\ \mathrm{i}w_4 \mathrm{e}^{\mathrm{i}\lambda_1 x+8\mathrm{i}\beta\lambda_1^4 t}(x+32t\beta\lambda_1^3) \end{pmatrix} \varepsilon + \cdots. \tag{6}$$

Substituting expression (6) into solutions (3) yields Φ_2 and Φ_1, and then we set

$$\begin{aligned} A = & \left[(\lambda-\lambda_1^*)-(\lambda_1-\lambda_1^*)\frac{\Phi_2\Phi_2^\dagger}{\Phi_2^\dagger\Phi_2}\right] \\ & \times\left[(\lambda-\lambda_1^*)-(\lambda_1-\lambda_1^*)\frac{\Phi_1\Phi_1^\dagger}{\Phi_1^\dagger\Phi_1}\right]\Omega_2|_{\lambda=\lambda_2+\varepsilon}. \end{aligned} \tag{7}$$

Expanding A at λ_2, we have

$$\begin{aligned} & A = \chi_0 + \chi_1\varepsilon + \chi_2\varepsilon^2 + \cdots + \chi_{N-1}\varepsilon^{N-1} + \cdots, \\ & \chi_k = \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}A(\lambda)|_{\lambda=\lambda_2}. \end{aligned}$$

We assume that $\hat{\Phi}_{N-2}$ $(N \geq 3)$ is another solution of Lax pair (2) at $\lambda = \lambda_2$, and is acquired through another GDT:

$$\begin{aligned} \hat{\Phi}_{N-2} = & \chi_0 + \left[\sum_{s=1}^{N-3}(\lambda_2-\lambda_2^*)-(\lambda_2-\lambda_2^*)\frac{\hat{\Phi}_s\hat{\Phi}_s^\dagger}{\hat{\Phi}_s^\dagger\hat{\Phi}_s}\right]\chi_1 + \cdots \\ & + \left\{\left[(\lambda_2-\lambda_2^*)-(\lambda_2-\lambda_2^*)\frac{\hat{\Phi}_{N-3}\hat{\Phi}_{N-3}^\dagger}{\hat{\Phi}_{N-3}^\dagger\hat{\Phi}_{N-3}}\right]\right. \\ & \left.\cdots\left[(\lambda_2-\lambda_2^*)-(\lambda_2-\lambda_2^*)\frac{\hat{\Phi}_1\hat{\Phi}_1^\dagger}{\hat{\Phi}_1^\dagger\hat{\Phi}_1}\right]\right\}\chi_{N-3}, \end{aligned} \tag{8}$$

where $\hat{\Phi}_1 = \chi_0$ and $\hat{\Phi}_{N-2}$ terms represent complex-valued functions. Based on expression (8), we derive the Nth-order solutions $q_\iota[N]$ for system (1) as

$$\begin{aligned} q_1[N] = & 2\mathrm{i}(\lambda_1^*-\lambda_1)\sum_{m=1}^{2}\frac{(\Phi_m\Phi_m^\dagger)_{21}}{\Phi_m^\dagger\Phi_m} \\ & +2\mathrm{i}(\lambda_2^*-\lambda_2)\sum_{n=1}^{N-2}\frac{(\hat{\Phi}_n\hat{\Phi}_n^\dagger)_{21}}{\hat{\Phi}_n^\dagger\hat{\Phi}_n}, \end{aligned} \tag{9a}$$

$$\begin{aligned} q_2[N] = & 2\mathrm{i}(\lambda_1^*-\lambda_1)\sum_{m=1}^{2}\frac{(\Phi_m\Phi_m^\dagger)_{31}}{\Phi_m^\dagger\Phi_m} \\ & +2\mathrm{i}(\lambda_2^*-\lambda_2)\sum_{n=1}^{N-2}\frac{(\hat{\Phi}_n\hat{\Phi}_n^\dagger)_{31}}{\hat{\Phi}_n^\dagger\hat{\Phi}_n}, \end{aligned} \tag{9b}$$

$$\begin{aligned} q_3[N] = & 2\mathrm{i}(\lambda_1^*-\lambda_1)\sum_{m=1}^{2}\frac{(\Phi_m\Phi_m^\dagger)_{41}}{\Phi_m^\dagger\Phi_m} \\ & +2\mathrm{i}(\lambda_2^*-\lambda_2)\sum_{n=1}^{N-2}\frac{(\hat{\Phi}_n\hat{\Phi}_n^\dagger)_{41}}{\hat{\Phi}_n^\dagger\hat{\Phi}_n}, \end{aligned} \tag{9c}$$

where $(\hat{\Phi}_n\hat{\Phi}_n^\dagger)_{21}$, $(\hat{\Phi}_n\hat{\Phi}_n^\dagger)_{31}$ and $(\hat{\Phi}_n\hat{\Phi}_n^\dagger)_{41}$ denote the entries in the second, third and fourth rows and the first column of the matrix $\hat{\Phi}_n\hat{\Phi}_n^\dagger$, respectively.

3. Degenerate soliton dynamics and asymptotic analysis for system (1)

3.1. Interactions between two degenerate solitons and asymptotic analysis for system (1)

We substitute Φ_1 and Φ_2 into solutions (4), where Φ_1 and Φ_2 are obtained by substituting expression (6) into solutions (3). As we set $N = 2$ in solutions (4), i.e., by applying two iterations of λ_1, solutions for two degenerate solitons with $\lambda_1 = \lambda_{1\mathrm{R}} + \mathrm{i}\lambda_{1\mathrm{I}}$ for system (1) can be worked out as

$$\begin{aligned} & q_\iota[2] = \frac{8\mathrm{e}^{\mathrm{i}\omega}w_1w_{\iota+1}\lambda_{1\mathrm{I}}[\mathrm{e}^{\zeta}(w_2^2+w_3^2+w_4^2)(1+2x\lambda_{1\mathrm{I}}+\eta+\rho)+\mathrm{e}^{-\zeta}w_1^2(1-2x\lambda_{1\mathrm{I}}-\eta+\rho)]}{\mathrm{e}^{-2\zeta}w_1^4+\mathrm{e}^{2\zeta}(w_2^2+w_3^2+w_4^2)^2+2w_1^2(w_2^2+w_3^2+w_4^2)[1+2(2x\lambda_{1\mathrm{I}}+\eta)^2-2\rho^2]}, \\ & \zeta = 2\lambda_{1\mathrm{I}}\{-x+4t\lambda_{1\mathrm{R}}[1+8\beta(\lambda_{1\mathrm{I}}^2-\lambda_{1\mathrm{R}}^2)]\}, \quad \eta = -8t\lambda_{1\mathrm{I}}\lambda_{1\mathrm{R}}(1+24\beta\lambda_{1\mathrm{I}}^2-8\beta\lambda_{1\mathrm{R}}^2), \\ & \rho = 8\mathrm{i}t\lambda_{1\mathrm{I}}^2[1+8\beta(\lambda_{1\mathrm{I}}^2-3\lambda_{1\mathrm{R}}^2)], \quad \omega = 2[x\lambda_{1\mathrm{R}}+2t(\lambda_{1\mathrm{I}}^2+4\beta\lambda_{1\mathrm{I}}^4-\lambda_{1\mathrm{R}}^2-24\beta\lambda_{1\mathrm{I}}^2\lambda_{1\mathrm{R}}^2+4\beta\lambda_{1\mathrm{R}}^4)], \end{aligned} \tag{10}$$

where $\lambda_{1\mathrm{I}}$ and $\lambda_{1\mathrm{R}}$ are the real constants. Figure 1 depicts the interactions between two degenerate solitons for two sets of parameter conditions. When β increases from 1 to $8/5$, the velocities of the two degenerate solitons change.

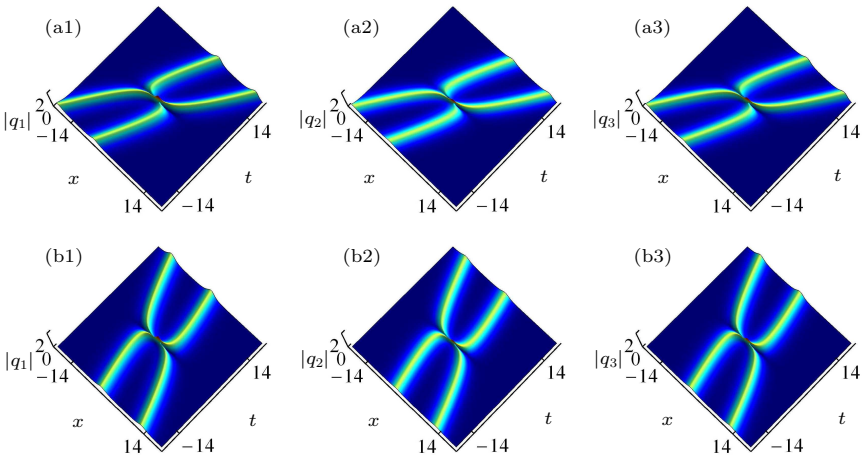


Fig. 1. Interactions between two degenerate solitons via solutions (10) with $\lambda_1 = \frac{1}{2} - \frac{2\mathrm{i}}{5}$, $w_1 = w_3 = 1$, $w_2 = 2$, $w_4 = \frac{3}{2}$: (a) $\beta = 1$ and (b) $\beta = \frac{8}{5}$.

Drawing inspiration from Refs. [25,26], we conduct an asymptotic analysis that depends on the balance between the exponential and algebraic terms to examine the asymptotic properties of solutions (10). Observing Fig. 1, it is evident that the trajectories of the two degenerate solitons depart from linearity, as we will verify later. t exhibits asymptotic behavior — it tends to $+\infty$ or $-\infty$, leading to the derivation of the following asymptotic soliton solutions:

$$q_\iota[2] \to q^+_{\iota,1}[2] = \frac{2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega} w_{\iota+1}\lambda_{1\mathrm{I}}[1+8\beta(\lambda_{1\mathrm{I}}+\mathrm{i}\lambda_{1\mathrm{R}})^2](\lambda_{1\mathrm{I}}+\mathrm{i}\lambda_{1\mathrm{R}})}{\sqrt{(w_2^2+w_3^2+w_4^2)\eta_0}}\operatorname{sech}\left[\zeta+\ln\left(\frac{\sqrt{(w_2^2+w_3^2+w_4^2)}}{16tw_1\lambda_{1\mathrm{I}}\sqrt{\eta_0}}\right)\right], \tag{11}$$

$$t\,\mathrm{e}^{-\zeta} = O(1), \quad \zeta \to +\infty, \quad t \to +\infty,$$

$$q_\iota[2] \to q^-_{\iota,1}[2] = \frac{-2\,\mathrm{e}^{\mathrm{i}\omega} w_{\iota+1}\lambda_{1\mathrm{I}}[1+8\beta(\lambda_{1\mathrm{I}}-\mathrm{i}\lambda_{1\mathrm{R}})^2](\mathrm{i}\lambda_{1\mathrm{I}}+\lambda_{1\mathrm{R}})}{\sqrt{(w_2^2+w_3^2+w_4^2)\eta_0}}\operatorname{sech}\left[\zeta+\ln\left(\frac{-16t\lambda_{1\mathrm{I}}\sqrt{(w_2^2+w_3^2+w_4^2)\eta_0}}{w_1}\right)\right], \tag{12}$$

$$t\,\mathrm{e}^{\zeta} = O(1), \quad \zeta \to -\infty, \quad t \to -\infty,$$

$$q_\iota[2] \to q^+_{\iota,2}[2] = \frac{2\,\mathrm{e}^{\mathrm{i}\omega} w_{\iota+1}\lambda_{1\mathrm{I}}[1+8\beta(\lambda_{1\mathrm{I}}-\mathrm{i}\lambda_{1\mathrm{R}})^2](\mathrm{i}\lambda_{1\mathrm{I}}+\lambda_{1\mathrm{R}})}{\sqrt{(w_2^2+w_3^2+w_4^2)\eta_0}}\operatorname{sech}\left[\zeta+\ln\left(\frac{16t\lambda_{1\mathrm{I}}\sqrt{(w_2^2+w_3^2+w_4^2)\eta_0}}{w_1}\right)\right], \tag{13}$$

$$t\,\mathrm{e}^{\zeta} = O(1), \quad \zeta \to -\infty, \quad t \to +\infty,$$

$$q_\iota[2] \to q^-_{\iota,2}[2] = \frac{-2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega} w_{\iota+1}\lambda_{1\mathrm{I}}[1+8\beta(\lambda_{1\mathrm{I}}+\mathrm{i}\lambda_{1\mathrm{R}})^2](\lambda_{1\mathrm{I}}+\mathrm{i}\lambda_{1\mathrm{R}})}{\sqrt{(w_2^2+w_3^2+w_4^2)\eta_0}}\operatorname{sech}\left[\zeta+\ln\left(-\frac{\sqrt{(w_2^2+w_3^2+w_4^2)}}{16tw_1\lambda_{1\mathrm{I}}\sqrt{\eta_0}}\right)\right], \tag{14}$$

$$t\,\mathrm{e}^{-\zeta} = O(1), \quad \zeta \to +\infty, \quad t \to -\infty,$$

$$\eta_0 = (\lambda_{1\mathrm{I}}^2+\lambda_{1\mathrm{R}}^2)[1+16\beta(\lambda_{1\mathrm{I}}^2-\lambda_{1\mathrm{R}}^2)+64\beta^2(\lambda_{1\mathrm{I}}^2+\lambda_{1\mathrm{R}}^2)^2].$$

From asymptotic solitons (11)–(14), it can be found that the bright solitons $q^\pm_{\iota,1}[2]$ and $q^\pm_{\iota,2}[2]$ exhibit some asymptotic properties.

(a) Amplitudes

$$A(q^\pm_{\iota,1}[2]) = A(q^\pm_{\iota,2}[2]) = \left|\frac{2w_{\iota+1}\lambda_{1\mathrm{I}}[1+8\beta(\lambda_{1\mathrm{I}}+\mathrm{i}\lambda_{1\mathrm{R}})^2](\lambda_{1\mathrm{I}}+\mathrm{i}\lambda_{1\mathrm{R}})}{\sqrt{(w_2^2+w_3^2+w_4^2)\eta_0}}\right|.$$

(b) Characteristic lines

$$q^+_{\iota,1}[2]: \ \frac{\sqrt{w_2^2+w_3^2+w_4^2}}{16w_1\lambda_{1\mathrm{I}}\sqrt{\eta_0}}\,\mathrm{e}^{\zeta} - t = 0, \quad q^-_{\iota,1}[2]: w_1\,\mathrm{e}^{-\zeta} + 16t\lambda_{1\mathrm{I}}\sqrt{(w_2^2+w_3^2+w_4^2)\eta_0} = 0,$$

$$q^+_{\iota,2}[2]: \ w_1\,\mathrm{e}^{-\zeta} - 16t\lambda_{1\mathrm{I}}\sqrt{(w_2^2+w_3^2+w_4^2)\eta_0} = 0, \quad q^-_{\iota,2}[2]: \frac{\sqrt{w_2^2+w_3^2+w_4^2}}{16w_1\lambda_{1\mathrm{I}}\sqrt{\eta_0}}\,\mathrm{e}^{\zeta} + t = 0.$$

(c) Phase shift $P(1)$ between $q^{+}_{\iota,1}[2]$ and $q^{-}_{\iota,1}[2]$ (or $P(2)$ between $q^{+}_{\iota,2}[2]$ and $q^{-}_{\iota,2}[2]$)

$$P(1)=2\ln\left[\frac{1}{16|\lambda_{1\mathrm{I}}t|\sqrt{\eta_0}}\right],\quad P(2)=2\ln[16|\lambda_{1\mathrm{I}}t|\sqrt{\eta_0}].$$

(d) Relative distance between $q^{+}_{\iota,1}[2]$ and $q^{+}_{\iota,2}[2]$ (or $q^{-}_{\iota,1}[2]$ and $q^{-}_{\iota,2}[2]$)

$$d_{12}=\frac{\ln(16|\lambda_{1\mathrm{I}}t|\sqrt{\eta_0})}{|\lambda_{1\mathrm{I}}|},\quad |t|\geq\frac{1}{16|\lambda_{1\mathrm{I}}\sqrt{\eta_0}|}.$$

We find that the amplitudes of the two solitons in each component remain unchanged before and after the interaction. It can be deduced that the interaction between $q_{\iota,1}[2]$ and $q_{\iota,2}[2]$ is elastic. Amplitude variation between each component depends only on w_2, w_3, w_4, β and λ_1. The phase shifts of $P(1)$ and $P(2)$ exhibit inversely opposite values. We find that the relative distance between $q^{+}_{\iota,1}[2]$ and $q^{+}_{\iota,2}[2]$ exhibits logarithmic growth with $|t|$. Furthermore, we compute that $\partial^2 d_{12}/\partial|t|^2=-256|\lambda_{1\mathrm{I}}|(\lambda_{1\mathrm{R}}^2+\lambda_{1\mathrm{I}}^2)[1+16\beta(\lambda_{1\mathrm{I}}^2-\lambda_{1\mathrm{R}}^2)+64\beta^2(\lambda_{1\mathrm{R}}^2+\lambda_{1\mathrm{I}}^2)^2]\,\mathrm{e}^{-2d_{12}|\lambda_{1\mathrm{I}}|}$, revealing that the acceleration of soliton separation decays exponentially with relative distance and eventually tends to zero. Unlike the interaction of two solitons in a straight line shape, the solitons of the above-mentioned interactions exhibit a curved shape. When performing the DT, spectral parameter also plays a key role in wave formation. This has been shown clearly in Ref. [27], where different types of beating solitons were studied. Based on above asymptotic analysis, we find that the amplitude, velocity, relative distance and phase shift are all affected by the spectral parameter.

3.2. Interactions among three degenerate solitons and asymptotic analysis for system (1)

We substitute Φ_1, Φ_2 and Φ_3 into solutions (4), where Φ_1, Φ_2 and Φ_3 are obtained by substituting expression (6) into solutions (3). Since many parameters are very difficult to calculate, for the sake of simplicity in the calculation, we set $w_1=w_2=w_3=w_4=1$. When we assign $N=3$ in solutions (4), i.e., by applying three iterations of λ_1, we obtain the three-degenerate soliton solutions for system (1) with $\lambda_1=\mathrm{i}\lambda_{1\mathrm{I}}$ as

$$\begin{aligned}
q_\iota[3] &= -4\mathrm{e}^{\mathrm{i}\omega_1}\lambda_{1\mathrm{I}}\frac{\Pi_1\mathrm{e}^{2\zeta_1}+\Pi_2\mathrm{e}^{-2\zeta_1}+\Pi_3}{27\mathrm{e}^{3\zeta_1}+\mathrm{e}^{-3\zeta_1}+\Omega_1\mathrm{e}^{\zeta_1}+\Omega_2\mathrm{e}^{-\zeta_1}},\quad \zeta_1=-2\lambda_{1\mathrm{I}}x,\quad \omega_1=4t\lambda_{1\mathrm{I}}^2(1+4\beta\lambda_{1\mathrm{I}}^4),\\
\Pi_1 &= 9\{-3-8x^2\lambda_{1\mathrm{I}}^2+128\lambda_{1\mathrm{I}}^4(t+8t\beta\lambda_{1\mathrm{I}}^2)^2-64\mathrm{i}t(\lambda_{1\mathrm{I}}^2+12\beta\lambda_{1\mathrm{I}}^4)-4\mathrm{i}x\lambda_{1\mathrm{I}}[-3\mathrm{i}+16t(\lambda_{1\mathrm{I}}^2+8\beta\lambda_{1\mathrm{I}}^4)]\},\\
\Pi_2 &= -3-8x^2\lambda_{1\mathrm{I}}^2+128\lambda_{1\mathrm{I}}^4(t+8t\beta\lambda_{1\mathrm{I}}^2)^2-64\mathrm{i}t(\lambda_{1\mathrm{I}}^2+12\beta\lambda_{1\mathrm{I}}^4)+4x\lambda_{1\mathrm{I}}[3+16\mathrm{i}t(\lambda_{1\mathrm{I}}^2+8\beta\lambda_{1\mathrm{I}}^4)],\\
\Pi_3 &= 6\{-3+16x^2\lambda_{1\mathrm{I}}^2+32x^4\lambda_{1\mathrm{I}}^4+32768\mathrm{i}t^3\beta\lambda_{1\mathrm{I}}^8(1+8\beta\lambda_{1\mathrm{I}}^2)^2+8192\lambda_{1\mathrm{I}}^8(t+8t\beta\lambda_{1\mathrm{I}}^2)^4-64\mathrm{i}t[\lambda_{1\mathrm{I}}^2\\
&\quad +4(x^2+3\beta)\lambda_{1\mathrm{I}}^4+64x^2\beta\lambda_{1\mathrm{I}}^6]+128t^2[\lambda_{1\mathrm{I}}^4+8(x^2+2\beta)\lambda_{1\mathrm{I}}^6+64(2x^2-3\beta)\beta\lambda_{1\mathrm{I}}^8+512x^2\beta^2\lambda_{1\mathrm{I}}^{10}]\},\\
\Omega_1 &= 9\{3+64x^3\lambda_{1\mathrm{I}}^3+64x^4\lambda_{1\mathrm{I}}^4+16384\lambda_{1\mathrm{I}}^8(t+8t\beta\lambda_{1\mathrm{I}}^2)^4+1024t^2x\lambda_{1\mathrm{I}}^5(3+80\beta\lambda_{1\mathrm{I}}^2+448\beta^2\lambda_{1\mathrm{I}}^4)\\
&\quad +512t^2\lambda_{1\mathrm{I}}^4(3+80\beta\lambda_{1\mathrm{I}}^2+576\beta^2\lambda_{1\mathrm{I}}^4)+16x^2\lambda_{1\mathrm{I}}^2[3+128\lambda_{1\mathrm{I}}^4(t+8t\beta\lambda_{1\mathrm{I}}^2)^2]\},\\
\Omega_2 &= 3\{3-64x^3\lambda_{1\mathrm{I}}^3+64x^4\lambda_{1\mathrm{I}}^4+16384\lambda_{1\mathrm{I}}^8(t+8t\beta\lambda_{1\mathrm{I}}^2)^4-1024t^2x\lambda_{1\mathrm{I}}^5(3+80\beta\lambda_{1\mathrm{I}}^2+448\beta^2\lambda_{1\mathrm{I}}^4)\\
&\quad +512t^2\lambda_{1\mathrm{I}}^4(3+80\beta\lambda_{1\mathrm{I}}^2+576\beta^2\lambda_{1\mathrm{I}}^4)+16x^2\lambda_{1\mathrm{I}}^2[3+128\lambda_{1\mathrm{I}}^4(t+8t\beta\lambda_{1\mathrm{I}}^2)^2]\}.
\end{aligned}\tag{15}$$

Solutions (15) comprise three line-type asymptotic solitons and six curve-type ones. We determine the three line-type solitons, which share an identical form as

$$q_\iota[3]\to q^{\pm}_{\iota,1}[3]=-\frac{2\mathrm{e}^{\mathrm{i}\omega_1}\lambda_{1\mathrm{I}}}{\sqrt{3}}\mathrm{sech}\left(\zeta_1+\ln\sqrt{3}\right),\ t\to\pm\infty.\tag{16}$$

Following the method of asymptotic analysis in Subsection 3.1, under the scaling relations $t\sim\mathrm{e}^{\frac{\zeta_1}{2}}$ or $t\sim\mathrm{e}^{-\frac{\zeta_1}{2}}$, we proceed to compute

$$\begin{aligned}
q_\iota[3]\to\ & q^{+}_{\iota,2}[3]=q^{-}_{\iota,3}[3]=-\frac{2\mathrm{e}^{\mathrm{i}\omega_1}\lambda_{1\mathrm{I}}}{\sqrt{3}}\mathrm{sech}\left[\zeta_1+\ln\left(\frac{\sqrt{3}}{128\lambda_{1\mathrm{I}}^4(t+8t\beta\lambda_{1\mathrm{I}}^2)^2}\right)\right],\\
& t\mathrm{e}^{-\frac{\zeta_1}{2}}=O(1),\ \zeta_1\to+\infty,\ t\to\pm\infty,
\end{aligned}\tag{17}$$

$$\begin{aligned}
q_\iota[3]\to\ & q^{-}_{\iota,2}[3]=q^{+}_{\iota,3}[3]=-\frac{2\mathrm{e}^{\mathrm{i}\omega_1}\lambda_{1\mathrm{I}}}{\sqrt{3}}\mathrm{sech}\left[\zeta_1+\ln\left(128\sqrt{3}\lambda_{1\mathrm{I}}^4(t+8t\beta\lambda_{1\mathrm{I}}^2)^2\right)\right],\\
& t\mathrm{e}^{\frac{\zeta_1}{2}}=O(1),\ \zeta_1\to-\infty,\ t\to\pm\infty.
\end{aligned}\tag{18}$$

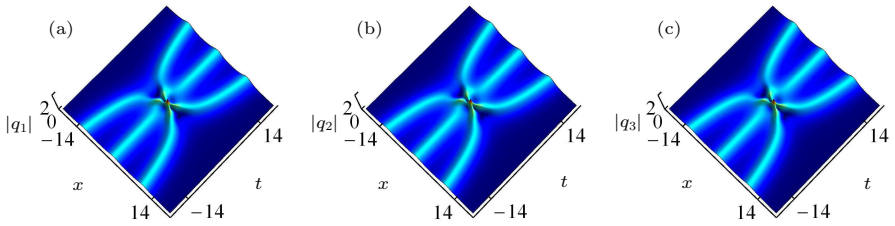


Fig. 2. Interaction of three degenerate solitons via solutions (15) with $\lambda_1 = -\frac{\mathrm{i}}{3}$, $w_1 = w_2 = w_3 = w_4 = 1$ and $\beta = \frac{1}{2}$.

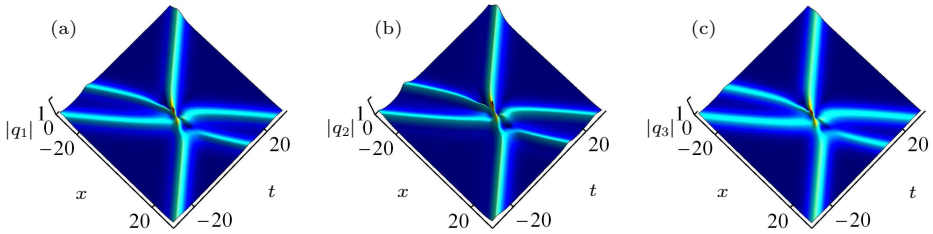


Fig. 3. Interaction among one soliton and two degenerate solitons via solutions (9) with $\lambda_1 = -\frac{1}{2} + \frac{\mathrm{i}}{5}$, $\lambda_2 = -\frac{1}{3} + \frac{\mathrm{i}}{4}$, $w_1 = w_4 = \frac{1}{2}$, $w_2 = \beta = 1$ and $w_3 = 2$.

We find that the amplitudes of these three degenerate solitons are $2|\lambda_{1\mathrm{I}}|/\sqrt{3}$ before and after the interaction, without any change. This indicates that the interaction is elastic, and the amplitude is only related to $\lambda_{1\mathrm{I}}$. When $\lambda_{1\mathrm{I}}$ increases, the amplitude of the solitons will increase. Center trajectories of the four asymptotic solitons are (1) $q^+_{t,2}[3], q^-_{t,3}[3]$, $\frac{\sqrt{3}}{128\lambda_{1\mathrm{I}}^4(1+8\beta\lambda_{1\mathrm{I}}^2)^2}\mathrm{e}^{\zeta_1} - t^2 = 0$; (2) $q^-_{t,2}[3], q^+_{t,3}[3]$, $\frac{1}{128\sqrt{3}\lambda_{1\mathrm{I}}^4(1+8\beta\lambda_{1\mathrm{I}}^2)^2}\mathrm{e}^{-\zeta_1} - t^2 = 0$. Respective phase shifts between $q^+_{t,1}[3]$ and $q^-_{t,1}[3]$, $q^+_{t,2}[3]$ and $q^-_{t,2}[3]$, $q^+_{t,3}[3]$ and $q^-_{t,3}[3]$ are $P(1) = 0$, $P(2) = -4\ln[8\sqrt{2}\lambda_{1\mathrm{I}}^2|(1+8\beta\lambda_{1\mathrm{I}}^2)t|]$ and $P(3) = 4\ln[8\sqrt{2}\lambda_{1\mathrm{I}}^2|(1+8\beta\lambda_{1\mathrm{I}}^2)t|]$. No phase shift is observed between $q^+_{t,1}[3]$ and $q^-_{t,1}[3]$, as both are characterized by the same asymptotic expression. Concurrently, $P(2)$ and $P(3)$ demonstrate opposite characteristics for matching $|t|$. Because the amplitude is unchanged, solutions (15) characterize the elastic interactions between six curve-type solitons, with three line-type solitons remaining unaffected, as shown in Fig. 2. Relative distances between the adjacent solitons are that $d_{12} = d_{13} = \ln[8\sqrt{2}\lambda_{1\mathrm{I}}^2|(1+8\beta\lambda_{1\mathrm{I}}^2)t|]/|\lambda_{1\mathrm{I}}|$. We find that the relative distance grows logarithmically with $|t|$. Furthermore, we compute that $a_{12} = -128|\lambda_{1\mathrm{I}}|^3(1+8\beta\lambda_{1\mathrm{I}}^2)^2\mathrm{e}^{-2d_{12}|\lambda_{1\mathrm{I}}|}$, revealing that the acceleration of soliton separation decays exponentially with relative distance, eventually tending to zero. When we set $N = 3$ in solutions (9), e.g., by applying two iterations of λ_1 and applying one iteration of λ_2, another three degenerate soliton solution for system (1) can be derived. Unlike the interaction of the three degenerate solitons with the same velocity in Fig. 2, we see the interaction among one soliton and two degenerate solitons with different velocities in Fig. 3. Therefore, the reason why multi-degenerate solitons are separated in Figs. 3 and 5 is the iteration with multiple different spectral parameters.

3.3. Interactions among four degenerate solitons and asymptotic analysis for system (1)

We substitute Φ_1, Φ_2, Φ_3 and Φ_4 into solutions (4), where Φ_1, Φ_2, Φ_3 and Φ_4 are obtained by substituting expression (6) into solutions (3). When we assign that $N = 4$ in solutions (4), i.e., by applying four iterations of λ_1, the four-degenerate soliton solutions for system (1) can be obtained. Due to their intricate nature, the complete expressions of these solutions are not provided here. Since many parameters are very difficult to calculate, for the sake of simplicity in the calculation, we set $w_1 = w_2 = w_3 = w_4 = 1$ and $\lambda_1 = \mathrm{i}\lambda_{1\mathrm{I}}$. Based on the asymptotic analysis, the derivation simplifies to retaining only the dominant highest-order terms within the four-degenerate soliton expressions as

$$q_t[4] \to \frac{8192\lambda_{1\mathrm{I}}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega_2}[27\mathrm{e}^{-3\zeta_2}\lambda_{1\mathrm{I}}^6(1+8\beta\lambda_{1\mathrm{I}}^2)^3t^3 + \mathrm{e}^{3\zeta_2}\lambda_{1\mathrm{I}}^6(1+8\beta\lambda_{1\mathrm{I}}^2)^3t^3 + 49152\mathrm{e}^{-\zeta_2}\lambda_{1\mathrm{I}}^{14}(1+8\beta\lambda_{1\mathrm{I}}^2)^7t^7 + 16384\mathrm{e}^{\zeta_2}\lambda_{1\mathrm{I}}^{14}(1+8\beta\lambda_{1\mathrm{I}}^2)^7t^7]}{243\mathrm{e}^{-4\zeta_2} + 3\mathrm{e}^{4\zeta_2} + 4194304\mathrm{e}^{2\zeta_2}\lambda_{1\mathrm{I}}^{12}(1+8\beta\lambda_{1\mathrm{I}}^2)^6t^6 + 37748736\mathrm{e}^{-2\zeta_2}\lambda_{1\mathrm{I}}^{12}(1+8\beta\lambda_{1\mathrm{I}}^2)^6t^6 + 805306368\lambda_{1\mathrm{I}}^{16}(1+8\beta\lambda_{1\mathrm{I}}^2)^8t^8}, \tag{19}$$

where $\omega_2 = 4t\lambda_{1\mathrm{I}}^2(1+4\beta\lambda_{1\mathrm{I}}^4)$ and $\zeta_2 = -2\lambda_{1\mathrm{I}}x$. Analysis reveals that the four degenerate solitons (19) exhibit four distinct asymptotic soliton balances as $t\mathrm{e}^{-\frac{\zeta_2}{3}} = O(1)$, $t\mathrm{e}^{\frac{\zeta_2}{3}} = O(1)$, $t\mathrm{e}^{-\zeta_2} = O(1)$ and $t\mathrm{e}^{\zeta_2} = O(1)$. When $t\mathrm{e}^{\pm\frac{\zeta_2}{3}} = O(1)$, we obtain

$$\begin{aligned} q_t[4] \to \quad & q^+_{t,1}[4] = -\frac{2\lambda_{1\mathrm{I}}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega_2}}{\sqrt{3}}\mathrm{sech}\left\{\zeta_2 + \ln\left[\frac{\sqrt{3}}{2048\lambda_{1\mathrm{I}}^6(1+8\beta\lambda_{1\mathrm{I}}^2)^3t^3}\right]\right\}, \\ & t\mathrm{e}^{-\frac{\zeta_2}{3}} = O(1), \quad \zeta_2 \to +\infty, \quad t \to +\infty, \end{aligned} \tag{20}$$

$$q_l[4]\to\ q^{-}_{l,1}[4]=\frac{2\lambda_{1I}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega_2}}{\sqrt{3}}\mathrm{sech}\left\{\zeta_2+\ln\left[-\frac{2048}{3\sqrt{3}}\lambda_{1I}^6(1+8\beta\lambda_{1I}^2)^3t^3\right]\right\},$$
$$t\,\mathrm{e}^{\frac{\zeta_2}{3}}=O(1),\ \ \zeta_2\to-\infty,\ \ t\to-\infty, \tag{21}$$
$$q_l[4]\to\ q^{+}_{l,4}[4]=-\frac{2\lambda_{1I}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega_2}}{\sqrt{3}}\mathrm{sech}\left\{\zeta_2+\ln\left[\frac{2048}{3\sqrt{3}}\lambda_{1I}^6(1+8\beta\lambda_{1I}^2)^3t^3\right]\right\},$$
$$t\,\mathrm{e}^{\frac{\zeta_2}{3}}=O(1),\ \ \zeta_2\to-\infty,\ \ t\to+\infty, \tag{22}$$
$$q_l[4]\to\ q^{-}_{l,4}[4]=\frac{2\lambda_{1I}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega_2}}{\sqrt{3}}\mathrm{sech}\left\{\zeta_2+\ln\left[-\frac{\sqrt{3}}{2048\lambda_{1I}^6(1+8\beta\lambda_{1I}^2)^3t^3}\right]\right\},$$
$$t\,\mathrm{e}^{-\frac{\zeta_2}{3}}=O(1),\ \ \zeta_2\to+\infty,\ \ t\to-\infty. \tag{23}$$

When $t\,\mathrm{e}^{\pm\zeta_2}=O(1)$, we obtain

$$q_l[4]\to\ q^{+}_{l,2}[4]=-\frac{2\lambda_{1I}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega_2}}{\sqrt{3}}\mathrm{sech}\left\{\zeta_2+\ln\left[\frac{1}{8\sqrt{3}\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)t}\right]\right\},$$
$$t\,\mathrm{e}^{-\frac{\zeta_2}{3}}=O(1),\ \ \zeta_2\to+\infty,\ \ t\to+\infty, \tag{24}$$
$$q_l[4]\to\ q^{-}_{l,2}[4]=\frac{2\lambda_{1I}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega_2}}{\sqrt{3}}\mathrm{sech}\left\{\zeta_2+\ln\left[-\frac{8}{\sqrt{3}}\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)t\right]\right\},$$
$$t\,\mathrm{e}^{\frac{\zeta_2}{3}}=O(1),\ \ \zeta_2\to-\infty,\ \ t\to-\infty, \tag{25}$$
$$q_l[4]\to\ q^{+}_{l,3}[4]=-\frac{2\lambda_{1I}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega_2}}{\sqrt{3}}\mathrm{sech}\left\{\zeta_2+\ln\left[\frac{8}{\sqrt{3}}\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)t\right]\right\},$$
$$t\,\mathrm{e}^{\frac{\zeta_2}{3}}=O(1),\ \ \zeta_2\to-\infty,\ \ t\to+\infty, \tag{26}$$
$$q_l[4]\to\ q^{-}_{l,3}[4]=\frac{2\lambda_{1I}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega_2}}{\sqrt{3}}\mathrm{sech}\left\{\zeta_2+\ln\left[-\frac{1}{8\sqrt{3}\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)t}\right]\right\},$$
$$t\,\mathrm{e}^{-\frac{\zeta_2}{3}}=O(1),\ \ \zeta_2\to+\infty,\ \ t\to-\infty. \tag{27}$$

Center trajectories are as follows:

$$q^{+}_{l,1}[4]:\frac{\sqrt{3}}{2048\lambda_{1I}^6(1+8\beta\lambda_{1I}^2)^3}\,\mathrm{e}^{\zeta_2}-t^3=0,$$
$$q^{-}_{l,1}[4]:3\sqrt{3}\,\mathrm{e}^{-\zeta_2}+2048\lambda_{1I}^6(1+8\beta\lambda_{1I}^2)^3t^3=0, \tag{28}$$
$$q^{+}_{l,2}[4]:\frac{1}{8\sqrt{3}\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)}\,\mathrm{e}^{\zeta_2}-t=0,$$
$$q^{-}_{l,2}[4]:\sqrt{3}\,\mathrm{e}^{-\zeta_2}+8\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)t=0, \tag{29}$$
$$q^{+}_{l,3}[4]:\sqrt{3}\,\mathrm{e}^{-\zeta_2}-8\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)t=0,$$
$$q^{-}_{l,3}[4]:\frac{1}{8\sqrt{3}\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)}\,\mathrm{e}^{\zeta_2}+t=0, \tag{30}$$
$$q^{+}_{l,4}[4]:3\sqrt{3}\,\mathrm{e}^{-\zeta_2}-2048\lambda_{1I}^6(1+8\beta\lambda_{1I}^2)^3t^3=0,$$
$$q^{-}_{l,4}[4]:\frac{\sqrt{3}}{2048\lambda_{1I}^6(1+8\beta\lambda_{1I}^2)^3}\,\mathrm{e}^{\zeta_2}+t^3=0. \tag{31}$$

Phase shifts between $q^{+}_{l,1}[4]$ and $q^{-}_{l,1}[4]$, $q^{+}_{l,2}[4]$ and $q^{-}_{l,2}[4]$, $q^{+}_{l,3}[4]$ and $q^{-}_{l,3}[4]$, $q^{+}_{l,4}[4]$ and $q^{-}_{l,4}[4]$ are

$$P(1)=2\ln\left[\frac{3}{2048\lambda_{1I}^6(1+8\beta\lambda_{1I}^2)^2t^2}\right],$$
$$P(2)=2\ln\left[\frac{1}{8\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)|t|}\right], \tag{32}$$
$$P(3)=2\ln\left[8\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)|t|\right],$$
$$P(4)=2\ln\left[\frac{2048\lambda_{1I}^6(1+8\beta\lambda_{1I}^2)^2t^2}{3}\right]. \tag{33}$$

Based on the above analysis, it is established that $q^{\pm}_{l,1}[4]$ and $q^{\pm}_{l,4}[4]$ comprise the two-degenerate soliton pair corresponding to $t\,\mathrm{e}^{\pm\frac{\zeta_2}{3}}=O(1)$, while $q^{\pm}_{l,2}[4]$ and $q^{\pm}_{l,3}[4]$ comprise the second pair associated with $t\,\mathrm{e}^{\pm\zeta_2}=O(1)$. Additionally, the eight bright asymptotic solitons are characterized by uniform amplitude $2|\lambda_{1I}|/\sqrt{3}$. Based on phase shifts (32) and (33), $P(1)$ and $P(4)$ demonstrate opposite characteristics for matching $|t|$, and $P(2)$ and $P(3)$ are opposite under the same $|t|$. Relative distances between the adjacent solitons are

$$d_{12}=d_{34}=\frac{1}{|\lambda_{1I}|}\ln\left[\frac{16}{\sqrt{3}}\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)|t|\right],$$
$$d_{23}=\frac{1}{|\lambda_{1I}|}\ln\left[8\lambda_{1I}^2(1+8\beta\lambda_{1I}^2)|t|\right].$$

The corresponding accelerations are

$$a_{12}=a_{34}=-\frac{256}{3}|\lambda_{1I}|^3(1+8\beta\lambda_{1I}^2)^2\,\mathrm{e}^{-2|\lambda_{1I}|d_{12}},$$

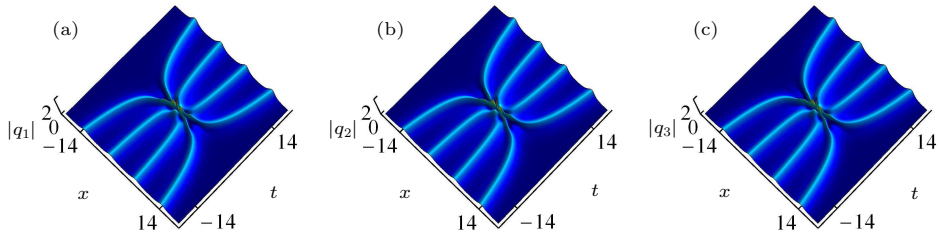


Fig. 4. Interaction of four degenerate solitons via solutions (4) with $\lambda_1 = \frac{\mathrm{i}}{2}$, $w_1 = w_2 = w_3 = w_4 = 1$ and $\beta = \frac{1}{2}$.

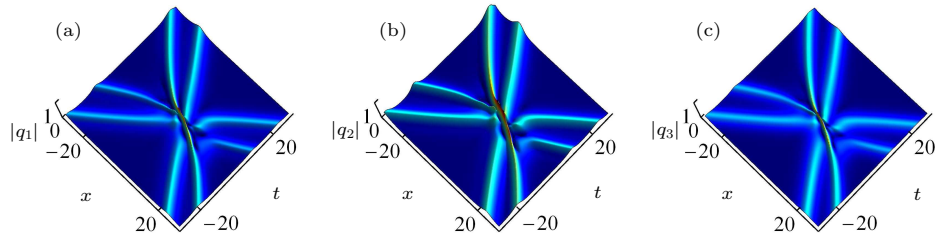


Fig. 5. Interaction of four degenerate solitons via solutions (9) with $\lambda_1 = -\frac{1}{2} + \frac{\mathrm{i}}{5}$, $\lambda_2 = -\frac{1}{3} + \frac{\mathrm{i}}{4}$, $w_3 = 2$, $w_1 = w_4 = \frac{1}{2}$ and $w_2 = \beta = 1$.

and $a_{23} = -64|\lambda_{1\mathrm{I}}|^3(1 + 8\beta\lambda_{1\mathrm{I}}^2)^2\,\mathrm{e}^{-2|\lambda_{1\mathrm{I}}|d_{23}}$. Soliton separation acceleration decays exponentially with relative distance, and eventually tends to zero. Because the amplitudes of four solitons are unchanged after the interaction, elastic interaction among the four degenerate solitons is shown in Fig. 4. When we set $N = 4$ in solutions (9), e.g., by applying two iterations of λ_1 and two iterations of λ_2, we construct the other four-degenerate soliton solutions for system (1). Different from the interaction of the four degenerate solitons in Fig. 4, we see the interaction between two two-degenerate solitons in Fig. 5.

4. Conclusions

A three-coupled fourth-order NLS system, accounting for quartic interspine coupling in an alpha helix protein, has been investigated, i.e., system (1). We have constructed solutions (9), which is different from that in the previous reference. With solutions (4) and (9), we have constructed the two-degenerate soliton solutions and three-degenerate soliton solutions, i.e., solutions (10), solutions (15), and four-degenerate asymptotic soliton solutions, i.e., four-degenerate solitons (19).

Based on the asymptotic analysis, we find that amplitudes of the interacting solitons are retained upon the interactions. Figure 1 depicts the elastic interactions between two degenerate solitons exhibiting the curve-type asymptotic solitons. When lattice parameter β increases from 1 to $8/5$, the velocities of the two degenerate solitons have changed. Elastic interactions of three degenerate solitons comprising six curve-type asymptotic solitons and three line-type solitons have been presented in Fig. 2. Interaction among one soliton and two degenerate solitons has been shown in Fig. 3. Elastic interactions of four degenerate solitons comprising twelve curve-type asymptotic solitons have been presented in Fig. 4. Interaction among two two-degenerate solitons with two spectral parameters has been presented in Fig. 5. Relative distance between multi-degenerate solitons has been found to grow logarithmically with $|t|$. Soliton separation acceleration has decayed exponentially with relative distance, and eventually tends to zero.

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