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Adaptive distributed optimization of high-order nonlinear multi-agent systems with predefined accuracy under state observer

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This paper discusses adaptive distributed optimization with predefined accuracy for high-order nonlinear multi-agent systems (MASs) that are subject to disturbances and nonlinear uncertainties. To estimate the global optimal solution in real-time, a distributed proportional-integral optimization technique is used to generate a virtual system for each agent. For the unknown control gain of the controller, the Nussbaum function is employed. Then, a fuzzy adaptive observer is designed to estimate the unmeasured state by leveraging the general approximation capabilities of fuzzy logic systems. Using the Lyapunov stability method and backstepping technique, we develop the adaptive law and a new distributed controller. This ensures that the outputs of multi-agent systems converge to optimal values. Finally, a simulation example is used to confirm the viability of the presented control mechanism.

Keywords: distributed optimization, nonlinear multi-agent systems (MASs), fuzzy state observer, predefined accuracy control

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1. Introduction

In recent years, there have been significant advancements in the domain of distributed optimization, which continues to evolve toward greater intelligence, efficiency, and distribution. This evolution has been effectively applied in areas such as resource allocation,[1,2] smart home technologies,[3,4] and distributed computing.[5,6] Every agent in a distributed optimization task has a local cost function that only it can access, and these local functions combine to form a global cost function. The purpose of distributed optimization is to minimize the global cost function. In Ref. [7], an integral sliding mode control framework was proposed to solve the robust distributed optimization problem for first-order MASs with time-varying costs. The distributed optimal cooperation problem for multiple systems was addressed in Ref. [8] by proposing a state-dependent distributed optimal cooperative algorithm. The integrator algorithm was used in Ref. [9] to tackle the optimal value problem for MASs with uncertain control gain. It is worth noting that current research on MASs has mainly focused on optimization problems,[10,11] consensus control,[12,13] and resilient control.[13–15] Therefore, this paper focuses on studying the optimization problem of high-order MASs with predefined accuracy.

The convergence performance of MASs is an important evaluation criterion. It has been investigated in numerous distinct aspects, such as predefined performance consensus control,[16,17] adaptive event-triggered consensus control,[18] finite time consensus control,[19,20] fixed time consensus control,[21,22] and so on. In contrast to the consensus controls mentioned above, predefined accuracy control indicates that the system's error concerning the target converges to an a priori-specified accuracy. For example, an adaptive tracking control with predefined accuracy is proposed for high-order nonlinear uncertain MASs in Ref. [23]. It does not consider the possibility of unknown states and high-order disturbances. For high-order nonlinear MASs with input saturation and unknown disturbances, a disturbance observer was created in Ref. [24] in order to accomplish adaptive consensus control with a predefined accuracy. In Ref. [25], the adaptive dynamic surface problem for nonlinear MASs with predefined accuracy, input quantization, and unmodeled dynamics was studied. In Ref. [26], neural network-based predetermined time convergence was reached using filters to solve distributed optimization problems. Research on predefined accuracy has not extended to analyzing optimization problems, and there are relatively few research results in this field. Furthermore, the existing research has not addressed unknown states, making it necessary to apply these results to the analysis of optimization problems.

In addition to this, the states of high-order nonlinear MASs are sometimes not fully available. Currently, for the

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states of MASs, fuzzy logic systems or neural network algorithms are generally used to design state observers to estimate the unmeasurable states and obtain the available observations. For example, in Ref. [27], an event-triggered adaptive fuzzy fractional-order control scheme based on an observer was proposed to address the cooperative control problem of uncertain integer-order MASs. Similarly, in Ref. [28], an adaptive neural control strategy was developed for nonlinear MASs with actuator failures, which estimates unmeasurable system states by integrating a state observer. The state unmeasurability problem can be effectively solved using fuzzy state observers.[29–31]

Motivated by the above analyses, this paper focuses on achieving adaptive distributed optimization with predefined accuracy for high-order nonlinear MASs under state observer. This article designs a new adaptive control protocol that introduces predefined accuracy into distributed optimization problems. Considering the incomplete availability of unmeasurable states in the high-order MASs, this paper proposes a fuzzy state observer to achieve effective estimation of the unmeasurable states. In order to address the issue of uncertain control direction, which frequently arises in generic systems, the Nussbaum function is presented. At the same time, the control parameters are dynamically adjusted according to the real-time changes of the system by combining with the adaptive control strategy to effectively adapt to the uncertainty of the system. In addition, the distributed optimization algorithm promotes the cooperation of agent to optimize the global performance of the system based on a predefined accuracy. The primary contributions of this article are listed below:

(i) A distributed optimal control strategy with predefined accuracy is proposed for high-order nonlinear uncertain MASs with bounded perturbations. The problem of predefined accuracy for the system has already had some relevant results. However, they have been targeted at the consensus tracking problem, with no relevant results for the optimization problem. This paper extends the investigation of the distributed optimization problem for high-order nonlinear MASs and more general system.

(ii) Compared to the results of recent studies, the distributed optimization problem is solved in this study using the PI algorithm and prefilter. This paper eliminates the need for high-order derivatives of the objective function. The complexity is reduced by a filter-based distributed optimization algorithm using a two-layer design framework, a virtual system layer to estimate the optimal signal, and a physical system layer to transform it into a tracking problem for distributed optimization.

(iii) The approximation properties of fuzzy logic systems are used to design distributed fuzzy state observers to approximate instead of neural network analysis. The fuzzy state observer can reconstruct the unmeasurable state of a nonlinear MAS and ensure that the observer error is bounded.

2. Preliminaries and problem statement

2.1. Graph theory and convex analysis

The communication interaction between the agents is shown as an undirected graph $\mathcal{G} = (\mathcal{V},\ \mathcal{E},\ \mathcal{A})$, where $\mathcal{V} = \{1, \ldots,\ N\}$ indicates the node-set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the edge-set, and $\mathcal{A} \subseteq \mathbb{R}^{N\times N}$ represents the adjacency matrix. It is implied from the edge $(\vartheta,\ \tau) \in \mathcal{E}$ that agent τ can get information from agent ϑ. The set of neighbours of an agent τ is denoted by $\mathcal{N}_\tau = \{\vartheta \in \mathcal{V} \mid (\vartheta,\ \tau) \in \mathcal{E}\}$. The adjacency matrix $\mathcal{A} = [a_{\tau\vartheta}] \in \mathbb{R}^{N\times N}$ satisfies that $a_{\tau\tau} = 0$, $a_{\tau\vartheta} = 1$ if $(\vartheta,\ \tau) \in \mathcal{E}$; otherwise, $a_{\tau\vartheta} = 0$. A path is an ordered sequence of different vertexes in set $\mathcal{V}$. If there is a path connecting any two different vertexes, $\mathcal{G}$ is considered strongly connected. A strongly connected and undirected graph is called a connected graph. $\mathcal{L} = [l_{\tau\vartheta}]$ describes the Laplacian matrix, where $l_{\tau\tau} = \sum_{\vartheta\in\mathcal{N}_\tau} a_{\tau\vartheta}$ and $l_{\tau\vartheta} = -a_{\tau\vartheta}$ $(\tau \neq \vartheta)$.

This article introduces the concept of convex functions. If the differentiable function f is convex in a convex set $C \subseteq \mathbb{R}^n$, then the inequality $f(x) - f(y) \geqslant \nabla f(y)^{\mathrm{T}}(x-y)$ must be satisfied, for $\forall x,\ y \in C$. The function f is strictly convex if the inequality above is strict. Moreover, if f is m-strongly $(m > 0)$ convex then it needs to satisfy $(\nabla f(x) - \nabla f(y))(x-y) \geq m|x-y|^2$ for $\forall x,\ y \in C$.

Table 1. Explanation of mathematical symbols.

Symbol	Explanation
$\Phi(X)$	fuzzy basis function vector
W	ideal weight vector
$\delta_{\tau,\vartheta}(X)$	fuzzy error term
Θ	ideal weight vector
Υ	auxiliary function
$\mathcal{H}$	auxiliary function
ψ	a designable constant

2.2. Model

Consider MASs made up of N followers with disturbance and an unknown control direction. The dynamics of each agent can be explained as follows:

$$\begin{cases} \dot{x}_{\tau,\vartheta} = x_{\tau,\vartheta+1} + f_{\tau,\vartheta}\left(\bar{x}_{\tau,\vartheta}\right) + d_{\tau,\vartheta}, \\ \dot{x}_{\tau,n_i} = \ell_{\tau,n_i} u_\tau + f_{\tau,n_i}\left(\bar{x}_{\tau,n_i}\right) + d_{\tau,n_i}, \\ y_\tau = x_{\tau,1}, \end{cases} \tag{1}$$

where $\bar{x}_{\tau,\vartheta} = \left[x_{\tau,1}, x_{\tau,2}, \ldots, x_{\tau,\vartheta}\right]^{\mathrm{T}}$ and y_τ represent the state vector and output vector of the τ-th agent, respectively, and u_τ is the control input, $\tau = 1, \ldots, N$, $\vartheta = 1, \ldots, n_i - 1$. The unknown bounded disturbances $d_{\tau,\vartheta}$ are not dependent on the state variables. The function $f_{\tau,\vartheta} \in \mathbb{R}$ is an unknown smoothing function and the constant $\ell_{\tau,n}$ is an unknown bounded constant. In this setting, only the output signal y_τ can be acquired.

Remark 1 The subject of this study is high-order nonlinear MASs. To simplify the following discussion, all systems discussed hereafter refer to high-order nonlinear MASs.

The purpose of this short is to create a distributed control law u_τ that will enable MASs (1) to address the optimization issue that follows:

$$\begin{aligned} &\min_{y\in\mathbb{R}^N} f(y)=\sum_{\tau=1}^{N} f_\tau(y_\tau),\\ &\text{subject to } \ Ly=0, \end{aligned} \tag{2}$$

where $y=[y_1,y_2,\ldots,y_N]^{\mathrm{T}}$ and $f_\tau(y_\tau)$ is the defined local cost function. Equivalently, the optimization problem can be expressed as follows:

$$\min_{Y\in\mathbb{R}} \bar{f}(Y)=\sum_{\tau=1}^{N} f_\tau(Y),$$

where the global cost function is indicated by $\bar{f}(Y)$. The objective is to discover an optimal solution to Eq. (2), which enables each agent to accomplish the minimiser y^* of the global cost function within a predefined time. Here, $y^*=\arg\min_{Y\in\mathbb{R}}\bar{f}(Y)$.

Assumption 1[32] Each local cost function f_τ is differentiable and convex, while the global cost function $\bar{f}$ is strictly convex.

In order to determine the estimate of the global optimal consensus, a virtual system is first constructed using the distributed PI optimization approach. This estimate is obtained as follows:

$$\begin{cases} \dot{\breve{y}}_\tau=-\nu\nabla f_\tau(\breve{y}_\tau)-\omega\sum\limits_{\vartheta=1}^{N} a_{\tau\vartheta}(\breve{y}_\tau-\breve{y}_\vartheta)-\breve{\chi}_\tau,\\ \dot{\breve{\chi}}_\tau=\nu\omega\sum\limits_{\vartheta=1}^{N} a_{\tau\vartheta}(\breve{y}_\tau-\breve{y}_\vartheta), \quad \tau\in\mathcal{V}. \end{cases} \tag{3}$$

Lemma 1[26] If Assumption 1 is satisfied under the connected topology $\mathcal{G}$, then it follows that $\breve{y}_\tau(t)\to y^*$ for the virtual system (1) with the following conditions: $\nu>0$, $\omega>0$, initial states $\breve{\chi}_\tau(t_0)$, $\breve{y}_\tau(t_0)\in R$ and $\sum_{i=1}^{N}\breve{\chi}_\tau(t_0)=0$.

Remark 2 Distributed PI optimization methods can estimate the optimal value of the objective function for high-order MASs, but this necessitates high-order differentiation of $\breve{y}_\tau(t)$. However, for compute-intensive systems, high-order differentiation of the goal function is sometimes unavailable. Consequently, a pre-filtering technique is proposed.

In order to obtain the optimal reference signal for reconstruction, the state $\breve{y}_\tau(t)$ of each agent is input to the prefilter, represented as follows:

$$\zeta_\tau^{(n_\tau)}=-q_{\tau,0}(\zeta_\tau-\breve{y}_\tau)-\rho_{n_\tau}\sum_{k=1}^{n_\tau-1} q_{\tau,k}\zeta_\tau^{(k)}, \tag{4}$$

where n_τ is the highest order number of the reference signal's high-order derivatives required for later tracking design stages, and $q_{\tau,o}$, ρ_{n_τ}, $q_{\tau,k}$ are constants such that $\sum_{k=0}^{n_\tau-1} q_{\tau,k}s^k+s^{n_\tau}$ is Hurwitz.

Lemma 2[33] If Assumption 1 is valid under the connectivity topology $\mathcal{G}$, then the following result is true in the context of the virtual system (3) and the prefilter (4): $\zeta_\tau\to y^*$.

We make the following assumption in order to make the analysis that follows easier.

Assumption 2 For ℓ_{τ,n_i} exists a positive constant $\bar{\ell}_{\tau,n_i}$ satisfying $|\ell_{\tau,n_i}|\le\bar{\ell}_{\tau,n_i}$.

Remark 3 The problem of predefined accuracy control for nonlinear MASs with input saturation was studied in Ref. [24]. In Ref. [32], the adaptive fuzzy distributed optimization problem of uncertain nonlinear MASs is studied, which can ensure that the state asymptotically converges to the optimal value, but cannot guarantee the accuracy problem. Compared with the above research, we have solved the predefined accuracy problem of the optimization problem for the first time, which can set the accuracy range that needs to be converged, and considers the problem of unknown direction. To handle the unknown state, we introduce a fuzzy logic system.

2.3. Design of fuzzy state observer

In this subsection, a distributed fuzzy state observer is established to acquire the unmeasurable state of the system (1).

The fuzzy logic system's powerful approximation ability is employed to derive the approximation function $\check{f}$ from the unknown function f. The formula for the fuzzy approximation function $\check{f}$ is represented as follows:

$$\check{f}(X)=\frac{\sum_{\tau=1}^{M} f^\gamma(\prod_{\gamma=1}^{n_i}\mu_{R_\gamma^\tau}(X_\gamma))}{\sum_{\tau=1}^{M}(\prod_{\gamma=1}^{n_i}\mu_{R_\gamma^\tau}(X_\gamma))}=W^{*\mathrm{T}}\Phi(X),$$

where M represents the number of rules, input signal $X=[X_1,X_2,\ldots,X_{n_i}]^{\mathrm{T}}$, $\mu_{R_\gamma^\tau}(X_\gamma)$ is the membership degree of X_γ to R_γ^τ, $W^{*\mathrm{T}}=\left[W_1^{*\mathrm{T}},\ldots,W_M^{*\mathrm{T}}\right]$ is an ideal weight vector and $\Phi^{\mathrm{T}}=\left[\Phi^1,\ldots,\Phi^M\right]$ is a fuzzy basis function vector. Then, the following result is obtained.

Lemma 3[30] If $f(X)$ is a continuous function established on a compact set Ω, and if $\varepsilon>0$, then there is an FLS that will satisfy

$$\sup\left|f(X)-W^{\mathrm{T}}\Phi(X)\right|\le\varepsilon,$$

where the nonlinear function $f_{\tau,\vartheta}(\bar{x}_{\tau,\vartheta})$, $(1\le\vartheta\le n)$ in the considered system (1) is approximated by $\hat{f}_{\tau,\vartheta}(\hat{\bar{x}}_{\tau,\vartheta})=\hat{W}_{\tau,\vartheta}^{\mathrm{T}}\Phi(\hat{\bar{x}}_{\tau,\vartheta})+\varepsilon$.

Remark 4 In Ref. [27], the state unknown problem was resolved using the fuzzy state estimator. In Ref. [28], the state unknown problem was resolved using the neural network observer. The design of state observers utilizing neural network methods or fuzzy logic systems is comparable for state estimation of MASs. However, fuzzy rules can be used to directly

build fuzzy observers without the need for a training procedure. Consequently, a fuzzy state observer is used in this paper.

By using the approximation property of fuzzy logic system, we create the distributed fuzzy state observer as follows:

$$\begin{cases} \dot{\hat{x}}_{\tau,\vartheta} = \hat{x}_{\tau,\vartheta+1} + \widehat{W}_{\tau,\vartheta}^{\mathrm{T}} \Phi_{\tau,\vartheta}\left(\hat{\bar{x}}_{\tau,\vartheta}\right) \\ \qquad\qquad + g_{\tau,\vartheta}\left(x_{\tau,1} - \hat{x}_{\tau,1}\right), \\ \dot{\hat{x}}_{\tau,n_i} = \ell_{\tau,n_i} u_\tau + \widehat{W}_{\tau,n_i}^{\mathrm{T}} \Phi_{\tau,n_i}\left(\hat{\bar{x}}_{\tau,n_i}\right) \\ \qquad\qquad + g_{\tau,n_i}\left(x_{\tau,1} - \hat{x}_{\tau,1}\right), \\ \hat{y}_\tau = \hat{x}_{\tau,1}, \end{cases} \tag{5}$$

where $G_\tau = [g_{\tau,1},\ldots,g_{\tau,n_i}]^{\mathrm{T}}$ and $\hat{\bar{x}}_{\tau,\vartheta} = [\hat{x}_{\tau 1},\hat{x}_{\tau 2},\ldots,\hat{x}_{\tau\vartheta}]^{\mathrm{T}}$ is the estimation of the state vector $\bar{x}_{\tau,\vartheta} = [x_{\tau 1},x_{\tau 2},\ldots,x_{\tau\vartheta}]^{\mathrm{T}}$. From Eqs. (1) and (5), the observer error dynamics are rewritten as

$$\begin{cases} \dot{\hat{e}}_\tau = A_\tau \hat{e}_\tau + \Gamma_\tau u_\tau + G_\tau x_{\tau,1} + \sum_{\gamma=1}^{n_i} B_{\tau,\gamma} \widehat{W}_{\tau,\gamma}^{\mathrm{T}} \Phi_{\tau,\gamma}\left(\hat{\bar{x}}_{\tau,\gamma}\right), \\ \hat{y}_\tau = C_\tau^{\mathrm{T}} \hat{e}_\tau, \end{cases} \tag{6}$$

where $A_\tau = \begin{bmatrix} -g_{\tau,1} & 1 & 0 & \cdots & 0 \\ -g_{\tau,2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -g_{\tau,n_i} & 0 & 0 & \cdots & 0 \end{bmatrix}$, $B_{\tau,\gamma} = \left[\underbrace{0,\ldots,1}_{\gamma},\ldots,0\right]^{\mathrm{T}}$, $\Gamma_\tau = [0,\ \ldots,\ 0,\ r_{\tau,n_i}]^{\mathrm{T}}$, $C_\tau = [1,0,\cdots,0]^{\mathrm{T}}$, $\Pi_\tau = [d_{\tau,1},\ d_{\tau,2},\ \ldots,\ d_{\tau,n_i}]^{\mathrm{T}}$, $\hat{e}_\tau = [\hat{x}_{\tau,1},\ \ldots,\ \hat{x}_{\tau,n_i}]^{\mathrm{T}}$, and $e_{\tau,1} = x_{\tau,1} - \hat{x}_{\tau,1}$.

Then, the error system simplified as

$$\begin{aligned} \dot{\tilde{e}}_\tau &= A_\tau \tilde{e}_\tau + \sum_{\gamma=1}^{n_i} B_{\tau,\gamma}\left(f_{\tau,\gamma}\left(\bar{x}_{\tau,\gamma}\right) - \widehat{W}_{\tau,\gamma}^{\mathrm{T}} \Phi_{\tau,\gamma}\left(\hat{\bar{x}}_{\tau,\gamma}\right)\right) + \Pi_\tau \\ &= A_\tau \tilde{e}_\tau + \Pi_\tau + \delta_\tau, \end{aligned} \tag{7}$$

where $\delta_\tau = [\delta_{\tau,1},\ldots,\delta_{\tau,n_i}]^{\mathrm{T}}$, $\Pi_\tau = [d_{\tau,1},d_{\tau,2},\ldots,d_{\tau,n_i}]^{\mathrm{T}}$, $\tilde{W}_{\tau,\gamma} = W_{\tau,\gamma}^* - \hat{W}_{\tau,\gamma}$, $\varepsilon_\tau = [\varepsilon_{\tau,1},\ldots,\varepsilon_{\tau,n_i}]^{\mathrm{T}}$. Then, we can obtain

$$\dot{\tilde{e}}_\tau = A_\tau \tilde{e}_\tau + \sum_{\gamma=1}^{n_i} B_{\tau,\gamma}\left(\tilde{W}_{\tau,\gamma}^{\mathrm{T}} \Phi_{\tau,\gamma}\left(\hat{\bar{x}}_{\tau,\gamma}\right)\right) + \Pi_\tau + \varepsilon_\tau.$$

The vector G_τ must be chosen so that A_τ is a Hurwitz matrix. Consequently, for every arbitrary matrix $Q_\tau = Q_\tau^{\mathrm{T}} > 0$, there is a positive symmetric matrix P_τ which means

$$A_\tau^{\mathrm{T}} P_\tau + P_\tau A_\tau = -Q_\tau.$$

The selection of the Lyapunov function as $V_{\tau,0} = \tilde{e}_\tau^{\mathrm{T}} P_\tau \tilde{e}_\tau$ results in the following expression for $\dot{V}_{\tau,0}$:

$$\begin{aligned} \dot{V}_{\tau,0} \le\ & -\lambda_{\min}(Q_\tau)\|\tilde{e}_\tau\|^2 + 2\tilde{e}_\tau^{\mathrm{T}} P_\tau \\ & \times \left(\Pi_\tau + \varepsilon_\tau + \sum_{\gamma=1}^{n_i} B_{\tau,\gamma} \tilde{W}_{\tau,\gamma}^{\mathrm{T}} \Phi_{\tau,\gamma}\left(\hat{\bar{x}}_{\tau,\gamma}\right)\right), \end{aligned} \tag{8}$$

where $-\lambda_{\min}(Q_\tau)$ is defined as the smallest eigenvalue of the matrix Q_τ. Using Young's inequality and the property $0 < \Phi_{\tau,0}^{\mathrm{T}}(.)\,\Phi_{\tau,0}(.) \le 1$, we can obtain the following inequality:

$$\begin{aligned} & 2\tilde{e}_\tau^{\mathrm{T}} P_\tau (\Pi_\tau + \varepsilon_\tau) \\ \le\ & \frac{2\|\tilde{e}_\tau\|^2}{\psi} + \psi\|P_\tau\|^2\|\varepsilon_\tau^*\|^2 + \psi\|P_\tau\|^2\|\bar{d}_\tau\|^2, \\ & 2\tilde{e}_\tau^{\mathrm{T}} P_\tau \left(\sum_{\gamma=1}^{n_i} B_{\tau,\gamma} \tilde{W}_{\tau,\gamma}^{\mathrm{T}} \Phi_{\tau,\gamma}\left(\hat{\bar{x}}_{\tau,\gamma}\right)\right) \\ \le\ & \frac{n_i\|\tilde{e}_\tau\|^2}{\psi} + \psi\|P_\tau\|^2 \sum_{\gamma=1}^{n_i} \tilde{W}_{\tau,\gamma}^{\mathrm{T}} \tilde{W}_{\tau,\gamma}, \end{aligned} \tag{9}$$

where $\varepsilon_\tau^* = \left[\varepsilon_{\tau,1}^*,\ldots,\varepsilon_{\tau,n_i}^*\right]^{\mathrm{T}}$ and ψ is a constant that can be designed. Substituting Eq. (9) into Eq. (8) yields

$$\begin{cases} \dot{V}_{\tau,0} \le -\lambda_{\tau,o}\|\tilde{e}_\tau\|^2 + \psi\|P_\tau\|^2 \sum_{\gamma=1}^{n_i} \tilde{W}_{\tau,\gamma}^{\mathrm{T}} \tilde{W}_{\tau,\gamma} \\ \qquad\qquad + \psi\|P_\tau\|^2\|\varepsilon_\tau^*\|^2 + \psi\|P_\tau\|^2\|\bar{d}_\tau\|^2, \\ -\lambda_{\tau,o} = \lambda_{\min}(Q_\tau) - \dfrac{n_i}{\psi} - \dfrac{2}{\psi}. \end{cases} \tag{10}$$

The Lyapunov function is chosen as $V_1 = \sum_{\tau=0}^{N} V_{\tau,0} = \sum_{\tau=0}^{N} \tilde{e}_\tau^{\mathrm{T}} P_\tau \tilde{e}_\tau$.

$$\begin{aligned} \dot{V}_1 \le\ & \sum_{\tau=1}^{N} \dot{V}_{\tau,0} \le -P_0\|\tilde{e}_\tau\|^2 + \psi \sum_{\tau=1}^{N} \|P_\tau\|^2\|\varepsilon_\tau^*\|^2 \\ & + \psi \sum_{\tau=1}^{N} \|P_\tau\|^2\|\bar{d}_\tau\|^2 + \sum_{\tau=1}^{N}\sum_{\gamma=1}^{n_i} \|P_\tau\|^2 \tilde{W}_{\tau,\gamma}^{\mathrm{T}} \tilde{W}_{\tau,\gamma}, \end{aligned} \tag{11}$$

where $P_0 = \min\{\lambda_{\min}(Q_\tau) - n_i/\psi - 2/\psi\} > 0$.

2.4. Nussbaum function

In order to deal with the problem of unknown control direction of the system, we introduce functions with the following property to solve the problem.

Definition 1[9] Even funtion $N(\sigma)$ named Nussbaum-type has some properties:

$$\lim_{\sigma\to\infty} \sup \frac{1}{\sigma}\int_0^\sigma N(\sigma)\,\mathrm{d}\sigma = +\infty,$$
$$\lim_{\sigma\to\infty} \inf \frac{1}{\sigma}\int_0^\sigma N(\sigma)\,\mathrm{d}\sigma = -\infty.$$

Since the definition of Nussbaum-type functions allows for many such functions, here we choose $\sigma^2\cos(\sigma)$.

Lemma 4[25] A smooth continuous function $V(t) \ge 0$, a function $\sigma(t)$ defined on $[0,T_{\mathrm{h}}]$ and the Nussbaum-type function $N(\sigma(t))$ fulfil the following inequality:

$$V(t) \le \int_0^t \left[\ell N(\sigma(z))\,\dot{\sigma}(z) + \dot{\sigma}(z)\right] \mathrm{e}^{c_1(z-t)}\,\mathrm{d}z + c_2,$$

where c_1, $c_2 \ge 0$ are constants, then $V(t)$, $\sigma(t)$, and $\int_0^t N(\sigma(z))\,\dot{\sigma}(z)\,\mathrm{d}z$ must be bounded on $[0,T_{\mathrm{h}}]$.

Lemma 5[25] Consider the following system:

$$\dot{\hat{\rho}}(t) = -p\hat{\rho}(t) + ql(t),$$

where $p>0$ and $q>0$ are constants, $l(t)>0$ is a variable. For $t\geq 0$, if the initial value of $\hat{\rho}(t)$ is bounded and positive, then $\rho(t)\geq 0$.

2.5. Two key helper functions

To design the desired controller, we present the following two functions in this subsection:

$$\Upsilon_l^{\varsigma}(\hbar)=\begin{cases}\dfrac{(|\hbar|-\varsigma)^l}{l!}, & |\hbar|>\varsigma,\\ 0, & |\hbar|\leq\varsigma,\end{cases}$$

$$\mathcal{H}_l^{\varsigma}(\hbar)=\begin{cases}\mathrm{sgn}(\hbar), & |\hbar|>\varsigma,\\ 1-2\cos^l\left(\dfrac{\pi}{2}\sin^l\left(\dfrac{\pi}{4\varsigma}(\hbar+\varsigma)\right)\right), & |\hbar|\leq\varsigma,\end{cases}$$

where l and ς are the positive integer and constant, respectively. The following lemma holds for the two functions mentioned above.

Lemma 6[25] For functions $\Upsilon_l^{\varsigma}(\hbar)$ and $\mathcal{H}_l^{\varsigma}(\hbar)$, the following properties hold.

(i) Both $\Upsilon_l^{\varsigma}(\hbar)$ and $\mathcal{H}_l^{\varsigma}(\hbar)$ are the n-th-order continuously differentiable functions. In addition, one has

$$\frac{\mathrm{d}\Upsilon_{l+1}^{\varsigma}(\hbar)}{\mathrm{d}\hbar^r}=\Upsilon_{l-r+1}^{\varsigma}(\hbar)\left[\mathcal{H}_{l-r+1}^{\varsigma}(\hbar)\right]^r,\ r=1,\ldots,l, \tag{12}$$

and $\left(\frac{1}{l-i+1}\right)\Upsilon_{l-i+2}^{\varsigma}(\hbar)\Upsilon_1^{\varsigma}(\hbar)=\Upsilon_{l-i+1}^{\varsigma}(\hbar)$.

(ii) The function $\Upsilon_{l+1}^{\varsigma}(\hbar)$ is non-negative, and $\Upsilon_{l+1}^{\varsigma}(\hbar)=0$ only if $|\hbar|\leq\varsigma$. So that $\Upsilon_r^{\varsigma}(\hbar)\left[\mathcal{H}_r^{\varsigma}(\hbar)\right]^2=\Upsilon_r^{\varsigma}(\hbar)$.

3. Main results

In this section, we use the backsteping technique to design the controller. At first, we design the coordinate transformation as follows:

$$\begin{cases}\hbar_{\tau,1}=x_{\tau,1}-\alpha_{\tau,0},\\ \hbar_{\tau,\vartheta}=\hat{x}_{\tau,\vartheta}-\alpha_{\tau,\vartheta-1},\vartheta=2,\ldots,n_i,\end{cases} \tag{13}$$

where $\hbar_{\tau,1}$ is consensus error, $\alpha_{\tau,0}=\zeta_\tau$.

This section provides the derivation methods for virtual control laws and adaptive laws in each design step. The Appendix A contains the derivation information.

Step 1 The following Lyapunov function has been chosen

$$\begin{aligned}V_{\tau,1}&=n_i!\Upsilon_{\tau,n_i+1}^{\varsigma_\tau}(\hbar_{\tau,1})+\frac{1}{2\varphi_{\tau,1}}\tilde{\theta}_{\tau,1}^2\\&\quad+\frac{1}{2\mu_{\tau,1}}\tilde{\phi}_{\tau,1}^2+\frac{1}{2\beta_{\tau,1}}\tilde{W}_{\tau,1}^{\mathrm{T}}\tilde{W}_{\tau,1},\end{aligned} \tag{14}$$

where $\beta_{\tau,1}$, $\varphi_{\tau,1}$, $\mu_{\tau,1}$, $\xi_{\tau,1}$ and $\bar{\delta}_{\tau,1}$ are designable positive constants. The variable $\phi_{\tau,1}$ is designed as $\phi_{\tau,1}=\frac{1}{2}\xi_{\tau,1}^2+\bar{\delta}_{\tau,1}$.

The virtual controller $\alpha_{\tau,1}$, the update laws $\hat{\theta}_{\tau,1}$ and $\hat{W}_{\tau,1}$, and the other variables are designed as follows:

$$\alpha_{\tau,1}=N(\sigma_{\tau,1})\,\bar{\alpha}_{\tau,1}, \tag{15a}$$

$$\begin{aligned}\bar{\alpha}_{\tau,1}&=\frac{s_{\tau,1}}{n_i+1}\Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,1})\mathcal{H}_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1})+\frac{\hat{\theta}_{\tau,1}}{2\xi_{\tau,1}^2}\mathcal{H}_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1})\\&\quad+\hat{\phi}_{\tau,1}\mathcal{H}_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1})+(\varsigma_\tau+1)\,\mathcal{H}_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1})\\&\quad-\frac{1}{4}n_i!\Upsilon_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1})\mathcal{H}_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1}),\end{aligned} \tag{15b}$$

$$\dot{\sigma}_{\tau,1}=n_i!\Upsilon_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1})\mathcal{H}_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1})\bar{\alpha}_{\tau,1}, \tag{15c}$$

$$\dot{\hat{\theta}}_{\tau,1}=\frac{\varphi_{\tau,1}}{2\xi_{\tau,1}^2}n_i!\Upsilon_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1})-a_{\tau,1}\hat{\theta}_{\tau,1}, \tag{15d}$$

$$\dot{\hat{W}}_{\tau,1}=t_{\tau,1}\beta_{\tau,1}\hbar_{\tau,1}\Phi_{\tau,1}(\hat{x}_{\tau,1})-b_{\tau,1}\hat{W}_{\tau,1}, \tag{15e}$$

$$\dot{\hat{\phi}}_{\tau,1}=\mu_{\tau,1}n_i!\Upsilon_{\tau,n_i}^{\varsigma_\tau}(\hbar_{\tau,1})-c_{\tau,1}\hat{\phi}_{\tau,1}, \tag{15f}$$

where $t_{\tau,1}$, $s_{\tau,1}$, $a_{\tau,1}$, $b_{\tau,1}$, and $c_{\tau,1}$ are positive-design parameters.

Step ϑ $(\vartheta=2,\ldots,n_i-1)$ The Lyapunov function that has been chosen is as follows:

$$\begin{aligned}V_{\tau,\vartheta}&=V_{\tau,\vartheta-1}+m_i!\Upsilon_{\tau,m_i+1}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})+\frac{1}{2\varphi_{\tau,\vartheta}}\tilde{\theta}_{\tau,\vartheta}^2\\&\quad+\frac{1}{2\mu_{\tau,\vartheta}}\tilde{\phi}_{\tau,k}^2+\frac{1}{2\beta_{\tau,\vartheta}}\tilde{W}_{\tau,\vartheta}^{\mathrm{T}}\tilde{W}_{\tau,\vartheta},\end{aligned} \tag{16}$$

where $\beta_{\tau,\vartheta}$, $\varphi_{\tau,\vartheta}$, $\mu_{\tau,\vartheta}$, $\xi_{\tau,\vartheta}$, and $\bar{\delta}_{\tau,\vartheta}$ are five positive constants and the adaptive variable $\phi_{\tau,\vartheta}=\frac{1}{2}\xi_{\tau,\vartheta}^2+\bar{\delta}_{\tau,\vartheta}$.

Design virtual controller $\alpha_{\tau,\vartheta}$, the adaptation laws $\hat{\theta}_{\tau,\vartheta}$ and $\hat{\phi}_{\tau,\vartheta}$, and the subsequent variables as follows:

$$\alpha_{\tau,\vartheta}=N(\sigma_{\tau,\vartheta})\,\bar{\alpha}_{\tau,\vartheta}, \tag{17a}$$

$$\begin{aligned}\bar{\alpha}_{\tau,\vartheta}&=\frac{s_{\tau,\vartheta}}{m_i+1}\Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})\mathcal{H}_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})\\&\quad+\frac{\hat{\theta}_{\tau,\vartheta}}{2\xi_{\tau,\vartheta}^2}\mathcal{H}_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})+\hat{\phi}_{\tau,\vartheta}\mathcal{H}_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})\\&\quad+(\varsigma_\tau+1)\,\mathcal{H}_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})\\&\quad+\frac{5}{4}m_i!\Upsilon_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})\mathcal{H}_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta}),\end{aligned}$$

$$\dot{\sigma}_{\tau,\vartheta}=m_i!\Upsilon_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})\mathcal{H}_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})\bar{\alpha}_{\tau,\vartheta}, \tag{17b}$$

$$\dot{\hat{\theta}}_{\tau,\vartheta}=\frac{\varphi_{\tau,\vartheta}}{2\xi_{\tau,\vartheta}^2}m_i!\Upsilon_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})-a_{\tau,\vartheta}\hat{\theta}_{\tau,\vartheta}, \tag{17c}$$

$$\dot{\hat{W}}_{\tau,\vartheta}=t_{\tau,\vartheta}\beta_{\tau,\vartheta}\hbar_{\tau,\vartheta}\Phi_{\tau,\vartheta}\left(\hat{\bar{x}}_{\tau,\vartheta}\right)-b_{\tau,\vartheta}\hat{W}_{\tau,\vartheta}, \tag{17d}$$

$$\dot{\hat{\phi}}_{\tau,\vartheta}=\mu_{\tau,\vartheta}m_i!\Upsilon_{\tau,m_i}^{\varsigma_\tau}(\hbar_{\tau,\vartheta})-c_{\tau,\vartheta}\hat{\phi}_{\tau,\vartheta}, \tag{17e}$$

where $m_i=n_i-\vartheta+1$, $t_{\tau,\vartheta}$, $s_{\tau,\vartheta}$, $a_{\tau,\vartheta}$, $b_{\tau,\vartheta}$, and $c_{\tau,\vartheta}$ are positive-design parameters.

Step n_i Choosing Lyapunov funtion

$$\begin{aligned}V_{\tau,n_i}&=V_{\tau,n_i-1}+\Upsilon_{\tau,2}^{\varsigma_\tau}(\hbar_{\tau,n_i})+\frac{1}{2\varphi_{\tau,n_i}}\tilde{\theta}_{\tau,n_i}^2\\&\quad+\frac{1}{2\mu_{\tau,n_i}}\tilde{\phi}_{\tau,n_i}^2+\frac{1}{2\beta_{\tau,n_i}}\tilde{W}_{\tau,n_i}^{\mathrm{T}}\tilde{W}_{\tau,n_i},\end{aligned} \tag{18}$$

where $\bar{\delta}_{\tau,n_i}$, β_{τ,n_i}, φ_{τ,n_i}, ξ_{τ,n_i}, and μ_{τ,n_i} are five designable constants and the adaptive variable $\phi_{\tau,n_i}=\frac{1}{2}\xi_{\tau,n_i}^2+\bar{\delta}_{\tau,n_i}$.

We can select the controller u_τ and adaptive variables $\hat{\theta}_{\tau,n_i}$, $\hat{W}_{\tau,n_i}$, and $\hat{\phi}_{\tau,n_i}$, which are denoted by

$$u_\tau=N(\sigma_{\tau,n_i})\,\bar{\alpha}_{\tau,n_i}, \tag{19a}$$

$$\begin{aligned}\bar{\alpha}_{\tau,n_i} =& \frac{s_{\tau,n_i}}{2}\Upsilon^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})\mathcal{H}^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})\\ &+\frac{\hat{\theta}_{\tau,n_i}}{2\xi^2_{\tau,n_i}}\mathcal{H}^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})+\hat{\phi}_{\tau,n_i}\mathcal{H}^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})\\ &+(\varsigma_\tau+1)\bar{\ell}_{\tau,n_i}\mathcal{H}^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})\\ &+\left[\frac{\bar{\ell}_{\tau,n_i}}{4}+1\right]\Upsilon^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})\mathcal{H}^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i}), \qquad (19b)\\ \dot{\sigma}_{\tau,n_i} =& \Upsilon^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})\mathcal{H}^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})\bar{\alpha}_{\tau,n_i}, \qquad (19c)\\ \dot{\hat{\theta}}_{\tau,n_i} =& \frac{\varphi_{\tau,n_i}}{2\xi^2_{\tau,n_i}}\Upsilon^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})-a_{\tau,n_i}\hat{\theta}_{\tau,n_i}, \qquad (19d)\\ \dot{\hat{W}}_{\tau,n_i} =& t_{\tau,n_i}\beta_{\tau,n_i}\hbar_{\tau,n_i}\Phi_{\tau,n_i}\left(\hat{\bar{x}}_{\tau,n_i}\right)-b_{\tau,n_i}\hat{W}_{\tau,n_i}, \qquad (19e)\\ \dot{\hat{\phi}}_{\tau,n_i} =& \mu_{\tau,n_i}\Upsilon^{\varsigma_\tau}_{\tau,1}(\hbar_{\tau,n_i})-c_{\tau,n_i}\hat{\phi}_{\tau,n_i}, \qquad (19f)\end{aligned}$$

where t_{τ,n_i}, s_{τ,n_i}, a_{τ,n_i}, b_{τ,n_i}, and c_{τ,n_i} are positive-design constants.

Theorem 1 For the considered nonlinear MASs (1) with unknown control gains, under the conditions of Assumptions 1 and 2, a distributed fuzzy state observer (5) is created. The adaptive laws (19d)–(19f), virtual controllers (17a), and distributed controller (19a), all system signals are bounded and the desired consensus optimization is achieved. The errors can converge to a predefined accuracy.

Proof The Lyapunov function is as follows:

$$V(t)=V_1+\sum_{\tau=1}^{N}V_{\tau,n_i}. \qquad (20)$$

We apply Young's inequality to obtain $\tilde{\theta}_{\tau,\gamma}\hat{\theta}_{\tau,\gamma}\le\frac{1}{2}\theta^2_{\tau,\gamma}-\frac{1}{2}\tilde{\theta}^2_{\tau,\gamma}$, $\tilde{W}^{\mathrm{T}}_{\tau,\gamma}\hat{W}_{\tau,\gamma}\le\frac{1}{2}W^{\mathrm{T}}_{\tau,\gamma}W_{\tau,\gamma}-\frac{1}{2}\tilde{W}^{\mathrm{T}}_{\tau,\gamma}\tilde{W}_{\tau,\gamma}$, $\tilde{\phi}_{\tau,\gamma}\hat{\phi}_{\tau,\gamma}\le\frac{1}{2}\phi^2_{\tau,\gamma}-\frac{1}{2}\tilde{\phi}^2_{\tau,\gamma}$. Substituting this into the above equation we get

$$\begin{aligned}\dot{V}(t)\le&\sum_{\tau=1}^{N}\sum_{\gamma=1}^{n_i}\left[-s_{\tau,\gamma}(n_i-\gamma+1)!\Upsilon^{\varsigma_\tau}_{\tau,n_i-\gamma+2}(\hbar_{\tau,\gamma})-\frac{a_{\tau,\gamma}}{2\varphi_{\tau,\gamma}}\tilde{\theta}^2_{\tau,\gamma}-\frac{b_{\tau,\gamma}}{2\beta_{\tau,\gamma}}\tilde{W}^{\mathrm{T}}_{\tau,\gamma}\tilde{W}_{\tau,\gamma}-\frac{c_{\tau,\gamma}}{2\mu_{\tau,\gamma}}\tilde{\phi}^2_{\tau,\gamma}+Z_{\tau,\gamma}\right]\\ &+\sum_{\tau=1}^{N}\left[\sum_{\gamma=1}^{n_i-1}\left[\dot{\sigma}_{\tau,\gamma}\left(N\left(\sigma_{\tau,\gamma}\right)+1\right)\right]+\dot{\sigma}_{\tau,n_i}\left(\ell_{\tau,n_i}N\left(\sigma_{\tau,n_i}\right)+1\right)\right]+\sum_{\tau=1}^{N}\sum_{\gamma=2}^{n_i}g_{\tau,\gamma}\frac{\|\tilde{e}_\tau\|^2}{2\psi}+\sum_{\tau=1}^{N}\frac{\|\tilde{e}_\tau\|^2}{2}\\ &+\sum_{\tau=1}^{N}\frac{\left\|\bar{d}_{\tau,1}\right\|^2}{2}+\sum_{\tau=1}^{N}\sum_{\gamma=1}^{n_i}\psi\|P_\tau\|^2\tilde{W}^{\mathrm{T}}_{\tau,\gamma}\tilde{W}_{\tau,\gamma}+\sum_{\tau=1}^{N}\psi\|P_\tau\|^2\|\varepsilon^*_\tau\|^2+\sum_{\tau=1}^{N}\psi\|P_\tau\|^2\left\|\bar{d}_\tau\right\|^2-P_0\|\tilde{e}_\tau\|^2. \qquad (21)\end{aligned}$$

Let $Z_{\tau,\gamma}=\frac{a_{\tau,\gamma}}{2\varphi_{\tau,\gamma}}\theta^2_{\tau,\gamma}+\frac{b_{\tau,\gamma}}{2\beta_{\tau,\gamma}}W^{\mathrm{T}}_{\tau,\gamma}W_{\tau,\gamma}+\frac{c_{\tau,\gamma}}{2\mu_{\tau,\gamma}}\phi^2_{\tau,\gamma}$, $b_{\tau,\gamma}-2\psi\|P_\tau\|^2>0$, $P_0-\frac{N}{2}-\frac{N\bar{g}_{\tau,\gamma}}{2}>0$, and $\varpi=\min\left\{m_{\tau,\gamma},a_{\tau,\gamma},b_{\tau,\gamma}-2\psi\|P_\tau\|^2,c_{\tau,\gamma},P_0-\frac{N}{2}-\frac{N\bar{g}_{\tau,\gamma}}{2}\right\}$. Consequently , when $\ell_{\tau,\gamma}=1$, $\tau=1,\ldots,n_i-1$, we have

$$\dot{V}(t)\le\sum_{\tau=1}^{N}\sum_{\gamma=1}^{n_i}\dot{\sigma}_{\tau,\gamma}\left(\ell_{\tau,\gamma}N\left(\sigma_{\tau,\gamma}\right)+1\right)-\varpi V(t)+\rho, \qquad (22)$$

where $\rho=\sum_{\tau=1}^{N}\sum_{\gamma=1}^{n_i}Z_{\tau,\gamma}+\sum_{\tau=1}^{N}\frac{\left\|\bar{d}_{\tau,1}\right\|^2}{2}+\sum_{\tau=1}^{N}\psi\|P_\tau\|^2\|\varepsilon^*_\tau\|^2+\sum_{\tau=1}^{N}\psi\|P_\tau\|^2\left\|\bar{d}_\tau\right\|^2$. The multiplication of both sides of Eq. (22) by $\mathrm{e}^{\varpi t}$ results in the following equation:

$$\begin{aligned}\mathrm{e}^{\varpi t}\dot{V}(t)\le&\sum_{\tau=1}^{N}\sum_{\gamma=1}^{n_i}\dot{\sigma}_{\tau,\gamma}\left(\ell_{\tau,\gamma}N\left(\sigma_{\tau,\gamma}\right)+1\right)\mathrm{e}^{\varpi t}\\ &-\varpi V(t)\,\mathrm{e}^{\varpi t}+\rho\,\mathrm{e}^{\varpi t},\end{aligned}$$

then, we get

$$\begin{aligned}\dot{V}(t)\le&\left(V(t)(0)-\frac{\rho}{\varpi}\right)\mathrm{e}^{-\varpi t}+\frac{\rho}{\varpi}\\ &+\mathrm{e}^{-\varpi t}\sum_{\tau=1}^{N}\sum_{\gamma=1}^{n_i}\int_o^t\dot{\sigma}_{\tau,\gamma}\left(\ell_{\tau,\gamma}N\left(\sigma_{\tau,\gamma}\right)+1\right)\mathrm{e}^{\varpi t}. \qquad (23)\end{aligned}$$

First, by Theorem 1, we can obtain the conclusion of all constrained signals of MASs. Then, according to Lemma 5, we can obtain that $\hat{\theta}_{\tau,\gamma}\ge0$ and $\hat{\omega}_{\tau,\gamma}\ge0$ are true if their initial conditions are given by non-negative values. In this way, we can derive the following inequality:

$$\begin{aligned}&\frac{\hat{\theta}_{\tau,\gamma}}{2\xi^2_{\tau,\gamma}}\mathcal{H}^{\varsigma_\tau}_{\tau,n_i-\gamma+1}(\hbar_{\tau,\gamma})\Phi^{\mathrm{T}}_{\tau,\gamma}\left(X_{\tau,\gamma}\right)\Phi_{\tau,\gamma}\left(X_{\tau,\gamma}\right)\ge0,\\ &\hat{\phi}_{\tau,\gamma}\mathcal{H}^{\varsigma_\tau}_{\tau,n_i-\gamma+1}(\hbar_{\tau,\gamma})\ge0, \qquad (24)\end{aligned}$$

where $\tau=1,\ldots,N$, $\gamma=1,\ldots,n_i$. Then, we can obtain $\dot{\sigma}_{\tau,\gamma}\ge s_{\tau,\gamma}(n_i-\gamma+1)!\Upsilon^{\varsigma_\tau}_{\tau,n_i-\gamma+2}(\hbar_{\tau,\gamma})$, so we can see that

$$\begin{aligned}&\int_0^t s_{\tau,\gamma}(n_i-\gamma+1)!\Upsilon^{\varsigma_\tau}_{\tau,n_i-\gamma+2}(\hbar_{\tau,\gamma})(t)(s)\,\mathrm{d}s\\ &\le u_{\tau,\gamma}(t)-u_{\tau,\gamma}(0). \qquad (25)\end{aligned}$$

The inequality is bounded because $u_{\tau,\gamma}(t)-u_{\tau,\gamma}(0)$ is bounded and $\Upsilon^{\varsigma}_{l}(\hbar)$ is the n-th-order continuously differentiable functions. According to Barbalat's Lemma, we get that $\left|\hbar_{\tau,\gamma}(t)\right|\le\varsigma_\tau$ holds as $t\to\infty$. The proof is completed.

4. Simulation results

In this section, to illustrate the efficacy of the proposed control method for fuzzy adaptive distributed optimisation with predefined accuracy, we conduct a simulation study using a set of MASs consisting of four followers as an example.

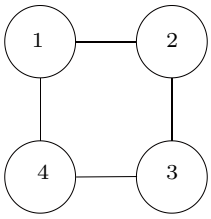


Fig. 1. Communication topology a.

The communication topology is shown in Fig. 1. Each follower is modeled as

$$\begin{cases} \dot{x}_{\tau,1} = x_{\tau,2} + x_{\tau,1}^2 + d_{\tau,1}, \\ \dot{x}_{\tau,2} = \ell_{\tau,2} u_\tau + x_{\tau,2}\sin(x_{\tau,1}) + d_{\tau,2}, \\ y_\tau = x_{\tau,1}, \quad \tau = 1,\ldots,4, \end{cases}$$

where $\ell_{\tau,2}$ is selected as 1. Furthermore, we choose $d_{\tau,1} = 0.2\sin(x_{\tau,1})\cos(t)$ and $d_{\tau,2} = 0,01\cos(t)$. The Laplacian matrix of the considered topological graph is $L = [2,-1,0,-1;-1,2,-1,0;0,-1,2,-1;-1,0,-1,2]$. The local cost function that satisfies Assumption 1 is chosen as $f_1(y_1) = (y_1+3)^{4/3}$, $f_2(y_2) = 0.7y_2^2 + \sin(y_2)$, $f_3(y_3) = \ln(\mathrm{e}^{-0.05y_3} + \mathrm{e}^{0.05y_3} + 0.6y_3^2$, $f_4(y_4) = \ln(2+0.1y_4^2) + y_4^2$. The optimization generator can be seen through Fig. 2 to obtain the optimized value $y^* = 0.56$.

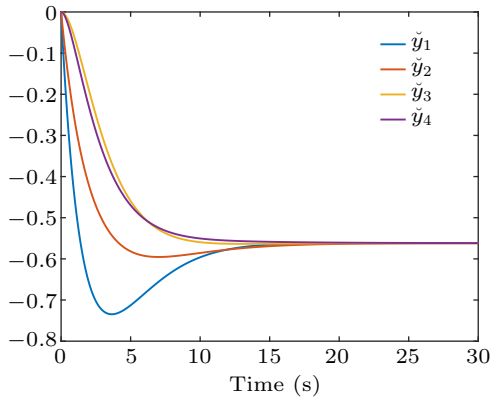


Fig. 2. Optimized estimate $\breve{y}_\tau$.

In this paper, the fuzzy membership functions we have chosen are $\mu_{R^l_{\tau,\vartheta}}(X_{\tau,\vartheta}) = \mathrm{e}^{-(X_{\tau,\vartheta}-4+l)^2/4}$, $\vartheta = 1,2$, $l = 1,\ldots,7$. The initial states are set as $x_1 = [-4,5,0.1]^{\mathrm{T}}$, $x_2 = [-4.3,0.1]^{\mathrm{T}}$, $x_3 = x_4 = [-4.1,0.1]^{\mathrm{T}}$, $\hat{x}_1 = [3,0.1]^{\mathrm{T}}$, $\hat{x}_2 = [1.7,0.1]^{\mathrm{T}}$, $\hat{x}_3 = [1.4,0.1]^{\mathrm{T}}$, $\hat{x}_4 = [2,0.1]$, $[\theta_{\tau,1},\theta_{\tau,2}] = [0.1,0.1]^{\mathrm{T}}$, $[\phi_{\tau,1},\phi_{\tau,2}] = [0.1,0.1]^{\mathrm{T}}$, and the rest initial values are set to 0. The design parameters are selected as $\nu = 3.5$, $\omega = 5$, $s_{\tau,1} = s_{\tau,2} = 0.12$, $\xi_{\tau,1} = 6$, $\xi_{\tau,1} = 8$, $\varphi_{\tau,1} = \varphi_{\tau,2} = 8$, $a_{\tau,1} = 4$, $a_{\tau,2} = 8$, $c_{\tau,1} = c_{\tau,2} = 0.01$, $\mu_{\tau,1} = 0.01$, $\beta_{\tau,1} = \beta_{\tau,2} = 8$, $\beta_{\tau,2} = 1$, $b_{1,1} = b_{4,1} = 0.25$, $b_{2,1} = b_{3,1} = 0.5$, $q_{\tau,0} = 1$, $t_{\tau,1} = t_{\tau,2} = 1$, and $b_{\tau,2} = 1$.

The simulation results are shown in Figs. 3–10. Figure 3 indicates that the output of the MAS eventually converges to the optimal value. Figures 4 and 5 show the approximation effect of the fuzzy state observer, indicating that it can accurately observe the system state. The predefined accuracy tracking performance is described in Fig. 6 and $\varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4 = 0.01$. Figure 6 shows that the four followers work well to achieve the optimal values within the predefined accuracy. In addition to this, figure 7 represents the optimization error map with different accuracies and $\varsigma_1 = 0.01, \varsigma_2 = 0.02, \varsigma_3 = 0.03, \varsigma_4 = 0.005$. Figure 8 depicts the evolution curves of the fuzzy adaptive parameters $W_{\tau,1}$ and $W_{\tau,2}$, demonstrating that $W_{\tau,1}$ and $W_{\tau,2}$ remain bounded and meet the necessary requirements. Figures 9 and 10 are adaptive laws, which show that

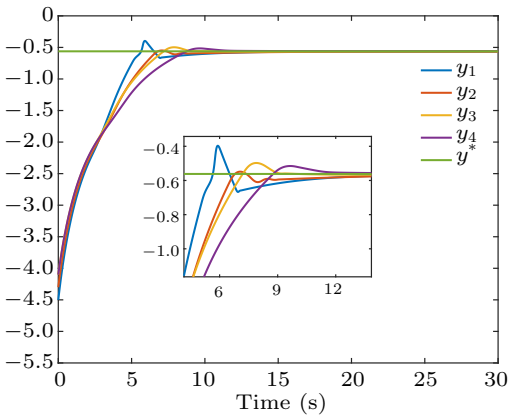


Fig. 3. Time responses of y_τ and $y^* = -0.56$ ($\tau = 1,2,3,4$).

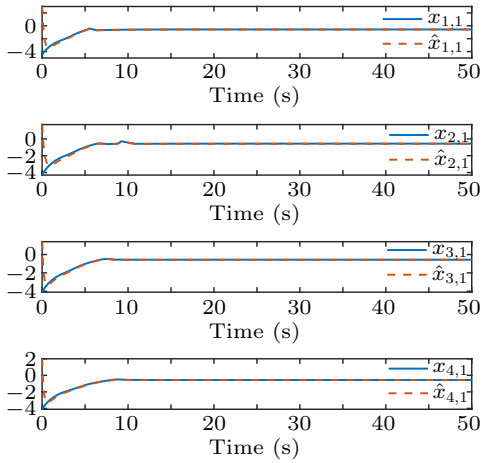


Fig. 4. State estimation of $x_{\tau,1}$.

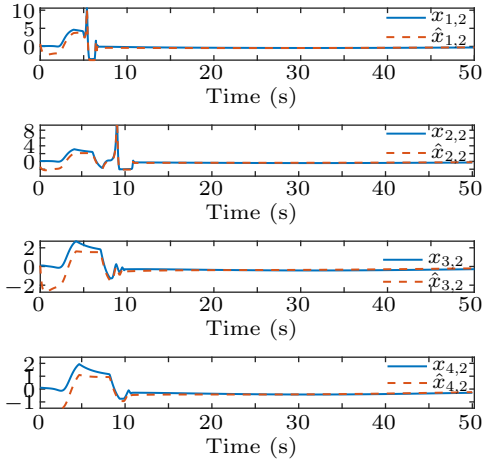


Fig. 5. State estimation of $x_{\tau,2}$.

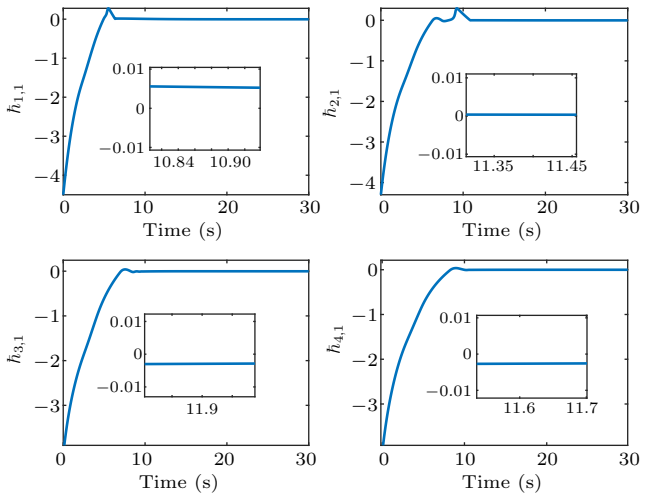


Fig. 6. Given the tracking error $\hbar_{\tau,1}$ with the same accuracy $\varsigma_\tau = 0.01$.

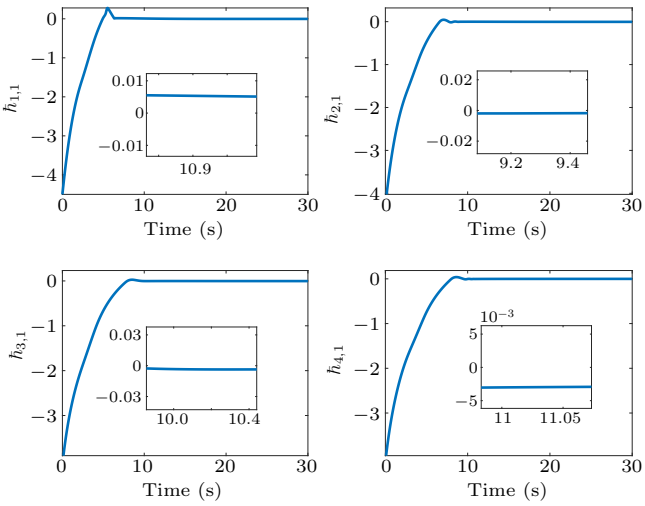


Fig. 7. Given the tracking error $\hbar_{\tau,1}$ with different accuracies.

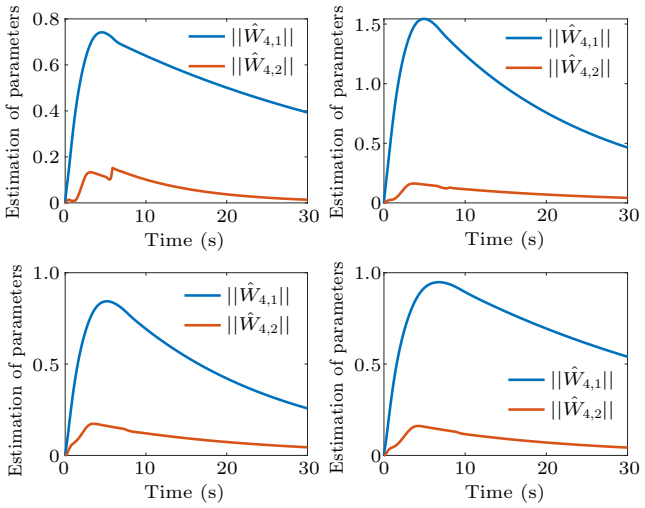


Fig. 8. Time response of adaptive parameters $W_{\tau,1}$ and $W_{\tau,2}$.

all signals are bounded. In addition, figure 11 presents the dynamic evolution of errors $\hbar_{\tau,1}$ in the absence of predefined accuracy constraints. A comparative analysis of the simulation results in Figs. 6 and 11 reveals that the optimization strategy incorporating a predefined accuracy mechanism effectively confines the error within the predefined accuracy range. This finding confirms the controllability of convergence accuracy in the proposed method.

Remark 5 From the design of the controller, each of the predefined accuracy can be set to a different accuracy value, which is validated by the simulation results in Section 4.

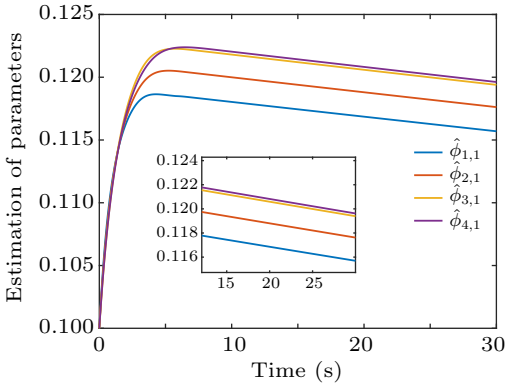


Fig. 9. Adaptive parameter $\phi_{\tau,1}$.

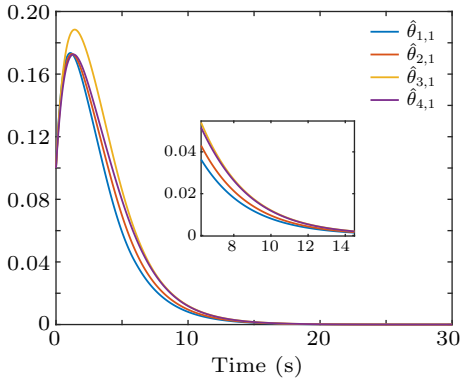


Fig. 10. Adaptive parameter $\theta_{\tau,1}$.

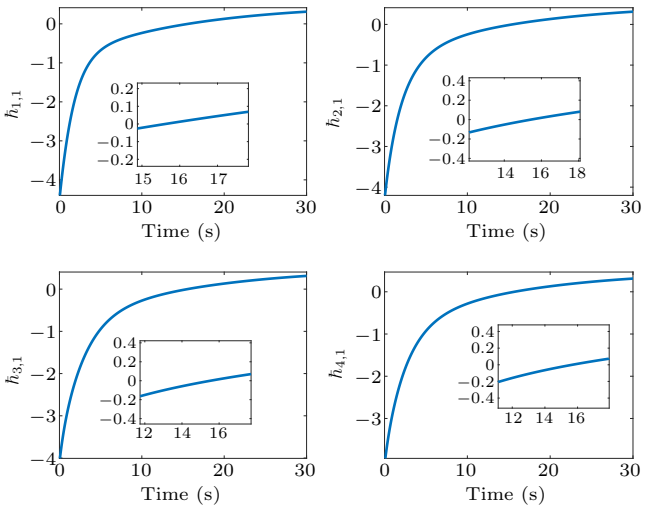


Fig. 11. Tracking error $\hbar_{\tau,1}$ without predefined accuracy.

To verify the robustness of the MASs, we introduced small amounts of noise Δ_τ and Δ_ϑ into the communication topology, where $\Delta_\tau = 0.001\sin(t)$ and $\Delta_\vartheta = 0.001\cos(t)$. The communication between agents is $a_{\tau\vartheta}(\breve{y}_\tau + \Delta_\tau - \breve{y}_\vartheta - \Delta_\vartheta)$.

As shown in Fig. 12, under noisy conditions, the optimizer still converges to the target value, with an error of less than 0.01. This result fully demonstrates the system's robustness to noise disturbance. Additionally, we constructed topology b, and the system switched between the communication topologies in Figs. 1 and 13, with the results shown in Fig. 14. Figure 14 indicates that the system can still converge stably to the optimal value $y^* = 0.56$ under topology change conditions. The results of the above two experiments demonstrate the system's robustness, and the simulation images for other parameters exhibit consistency, with the results remaining valid. Comparing it with the study in Ref. [24], we are not able to achieve the optimization of the predefined accuracy using the data from that study, whereas this article solves the optimization problem and can satisfy the smaller same predefined accuracy, giving better results.

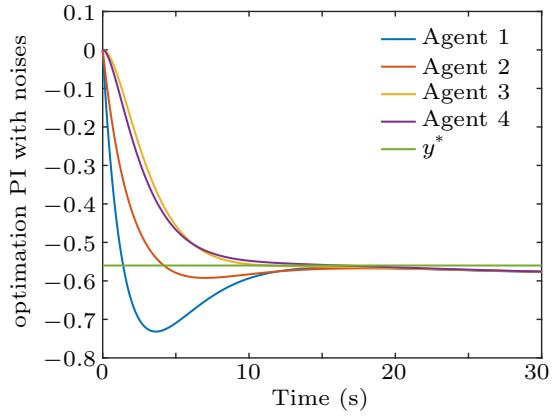


Fig. 12. Optimized estimate $\check{y}_\tau$ with noises.

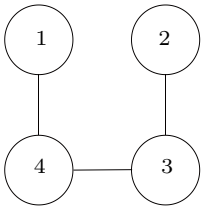


Fig. 13. Communication topology *b*.

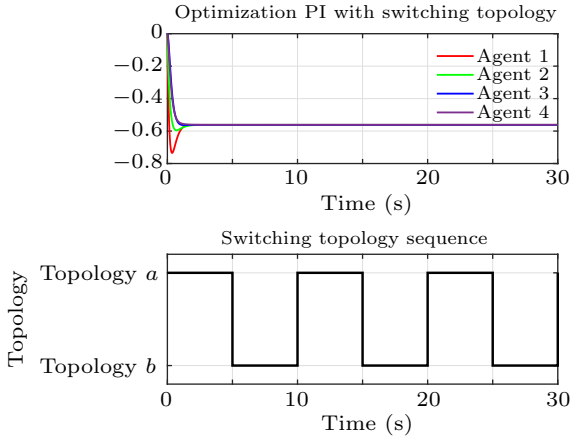


Fig. 14. Optimized estimation of $\check{y}_\tau$ under switching topology and switching topology sequence.

5. Conclusions

In this paper, we proposed a control strategy for the predefined accuracy adaptive distributed optimization problem of higher-order nonlinear multi-agent systems (MASs) with disturbances and nonlinear uncertainties. The strategy obtained a globally optimal consensus solution in a distributed manner by applying the PI algorithm with a prefiltering framework. To address the issue of unknown control gains, Nussbaum functions were introduced, and a fuzzy state observer was employed to estimate unmeasurable states. Finally, the Lyapunov stability method and backstepping technique were used to demonstrate that the distributed controller allowed the MASs to be controlled with predefined accuracy.

Appendix A: Back-stepping derivation steps

The n_i-step derivation process based on the back-stepping technique is shown below.

Step 1 Combining Eqs. (1), (13), and $e_{\tau,1} = x_{\tau,1} - \hat{x}_{\tau,1}$, the derivative of $\hbar_{\tau,1}$ is

$$\begin{aligned}\dot{\hbar}_{\tau,1} = \dot{x}_{\tau,1} - \dot{\alpha}_{\tau,0} = \hbar_{\tau,2} + \alpha_{\tau,1} + e_{\tau,2} \\ + f_{\tau,1}(x_{\tau,1}) + d_{\tau,1} - \dot{\alpha}_{\tau,0}.\end{aligned} \tag{A1}$$

The Lyapunov function is $V_{\tau,1}$. Combining Lemma 6, we have

$$\dot{V}_{\tau,1} = n_i! \Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1}) \mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1}) \dot{\hbar}_{\tau,1} - \Lambda_{\tau,1}, \tag{A2}$$

where

$$\begin{aligned}\tilde{f}_{\tau,1}(X_{\tau,1}) = f_{\tau,1}(x_{\tau,1}) - \dot{\alpha}_{\tau,0} + n_i! \Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1}) \mathcal{H}^{\xi}_{\tau,n_i}(\hbar_{\tau,1}) \\ - \frac{t_{\tau,1}\hbar_{\tau,1}}{n_i! \Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1}) \mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})} \tilde{W}^{\mathrm{T}}_{\tau,1} \Phi_{\tau,1}(\hat{x}_{\tau,1}),\end{aligned}$$

with

$$t_{\tau,1} = h_{\tau,1} \tan^2 \left[\frac{\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1}) \mathcal{H}^{\xi}_{\tau,n_i}(\hbar_{\tau,1})}{\varepsilon_{\tau,1}} \right], \quad \varepsilon_{\tau,1} > 0,$$

$$\Lambda_{\tau,1} = \frac{1}{\varphi_{\tau,1}} \tilde{\theta}_{\tau,1} \dot{\hat{\theta}}_{\tau,1} + \frac{1}{\mu_{\tau,1}} \tilde{\phi}_{\tau,1} \dot{\hat{\phi}}_{\tau,1} + \frac{1}{\beta_{\tau,1}} \tilde{W}^{\mathrm{T}}_{\tau,1} \dot{\hat{W}}_{\tau,1},$$

and use $\Theta^{*\mathrm{T}}_{\tau,1} \Phi_{\tau,1}(X_{\tau,1})$ to estimate unknown funtion $\tilde{f}_{\tau,1}(X_{\tau,1})$, which is

$$\tilde{f}_{\tau,1}(X_{\tau,1}) = \Theta^{*\mathrm{T}}_{\tau,1} \Phi_{\tau,1}(X_{\tau,1}) + \delta_{\tau,1}(X_{\tau,1}), \tag{A3}$$

where $\delta_{\tau,1}(X_{\tau,1})$ satisfies the condition that $|\delta_{\tau,1}(X_{\tau,1})| \le \bar{\delta}_{\tau,1}$, and $\bar{\delta}_{\tau,1} > 0$ is a constant.

Applying Lemma 3 and $0 \le \Phi^{\mathrm{T}}_{\tau,1}(.) \Phi_{\tau,1}(.) \le 1$, we have

$$\begin{aligned}&n_i! \Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1}) \mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1}) \Theta^{*\mathrm{T}}_{\tau,1} \Phi_{\tau,1}(X_{\tau,1}) \\ &\le n_i! \Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1}) \left[\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\right]^2 \left(\frac{\theta_{\tau,1}}{2\xi^2_{\tau,1}} + \frac{1}{2}\xi^2_{\tau,1} \right),\end{aligned} \tag{A4}$$

where $\theta_{\tau,1} = \left\| \Theta^{*\mathrm{T}}_{\tau,1} \right\|^2$.

Furthermore, we obtain the following inequality:

$$\begin{aligned} & n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\delta_{\tau,1}\left(X_{\tau,1}\right) \\ \le & n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\left[\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\right]^2\bar{\delta}_{\tau,1}, \\ & n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})d_{\tau,1} \\ \le & n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\left[\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\right]^2\bar{d}_{\tau,1}, \end{aligned} \tag{A5}$$

and set $\phi_{\tau,1}=\frac{1}{2}\xi^2_{\tau,1}+\bar{\delta}_{\tau,1}$ yields

$$\begin{aligned} & n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})e_{\tau,2} \le \frac{\|\tilde{e}_\tau\|^2}{2}+\frac{\left[n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\right]^2}{2}, \\ & n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})d_{\tau,1} \le \frac{\left\|\bar{d}_{\tau,1}\right\|^2}{2}+\frac{\left[n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\right]^2}{2}. \end{aligned} \tag{A6}$$

Therefore, combining the above inequalities yields

$$\begin{aligned} \dot{V}_{\tau,1} \le & n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1}) \\ & \times\left[\hbar_{\tau,2}+\alpha_{\tau,1}+\frac{\theta_{\tau,1}}{2\xi^2_{\tau,1}}\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})+\phi_{\tau,1}\mathcal{H}^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})\right] \\ & -t_{\tau,1}\hbar_{\tau,1}\tilde{W}^{\mathrm{T}}_{\tau,1}\Phi_{\tau,1}\left(\hat{x}_{\tau,1}\right)-\Lambda_{\tau,1}+\frac{\|\tilde{e}_\tau\|^2}{2}+\frac{\left\|\bar{d}_{\tau,1}\right\|^2}{2}. \end{aligned} \tag{A7}$$

Substituting Eq. (15) into Eq. (A7) yields

$$\begin{aligned} \dot{V}_{\tau,1} \le & \dot{\sigma}_{\tau,1}\left[N\left(\sigma_{\tau,1}\right)+1\right]-s_{\tau,1}n_i!\Upsilon^{\varsigma_\tau}_{\tau,n_i+1}(\hbar_{\tau,1})+\frac{a_{\tau,1}}{\varphi_{\tau,1}}\tilde{\theta}_{\tau,1}\hat{\theta}_{\tau,1} \\ & +\frac{b_{\tau,1}}{\beta_{\tau,1}}\tilde{W}^{\mathrm{T}}_{\tau,1}\hat{W}_{\tau,1}+\frac{c_{\tau,1}}{\mu_{\tau,1}}\tilde{\phi}_{\tau,1}\hat{\phi}_{\tau,1}-\frac{1}{4}\left[\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})n_i!\right]^2 \\ & +\Upsilon^{\varsigma_\tau}_{\tau,n_i}(\hbar_{\tau,1})n_i!\left[|\hbar_{\tau,2}|-(\varsigma_\tau+1)\right]+\frac{\|\tilde{e}_\tau\|^2}{2}+\frac{\left\|\bar{d}_{\tau,1}\right\|^2}{2}. \end{aligned} \tag{A8}$$

Step ϑ $(\vartheta=2,\ldots,n_i-1)$ From the definition of $\hbar_{\tau,\vartheta}=\hat{x}_{\tau,\vartheta}-\alpha_{\tau,\vartheta-1}$ and $m_i=n_i-\vartheta+1$, we know the derivative of $\hbar_{\tau,\vartheta}$ is

$$\dot{\hbar}_{\tau,\vartheta}=\hbar_{\tau,\vartheta+1}+\alpha_{\tau,\vartheta}+\widehat{W}^{\mathrm{T}}_{\tau,\vartheta}\Phi_{\tau,\vartheta}\left(\hat{\bar{x}}_{\tau,\vartheta}\right)+g_{\tau,\vartheta}e_{\tau,1}-\dot{\alpha}_{\tau,\vartheta-1}. \tag{A9}$$

Subsequently, we can differentiate $V_{\tau,\vartheta}$ to obtain

$$\dot{V}_{\tau,\vartheta}=\dot{V}_{\tau,\vartheta-1}+m_i!\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\mathcal{H}^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\dot{\hbar}_{\tau,\vartheta}-\Lambda_{\tau,\vartheta}, \tag{A10}$$

where

$$\begin{aligned} \tilde{f}_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right) = & \widehat{W}^{\mathrm{T}}_{\tau,\vartheta}\Phi_{\tau,\vartheta}\left(\hat{\bar{x}}_{\tau,\vartheta}\right)+g_{\tau,\vartheta}\frac{\psi\left[m_i!\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\right]^2}{2} \\ & -\frac{t_{\tau,\vartheta}\hbar_{\tau,\vartheta}}{m_i!\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\mathcal{H}^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})} \\ & \times\tilde{W}^{\mathrm{T}}_{\tau,\vartheta}\Phi_{\tau,\vartheta}\left(\hat{\bar{x}}_{\tau,\vartheta}\right)-\dot{\alpha}_{\tau,\vartheta-1}, \end{aligned}$$

with

$$t_{\tau,\vartheta}=h_{\tau,\vartheta}\tan^2\left[\frac{\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\mathcal{H}^{\xi}_{\tau,m_i}(\hbar_{\tau,\vartheta})}{\varepsilon_{\tau,\vartheta}}\right],\quad \varepsilon_{\tau,\vartheta}>0,$$

and the estimated value $\Theta^{*\mathrm{T}}_{\tau,\vartheta}\Phi_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right)$ can be used to estimated unknown funtion $\tilde{f}_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right)$ such that

$$\tilde{f}_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right)=\Theta^{*\mathrm{T}}_{\tau,\vartheta}\Phi_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right)+\delta_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right), \tag{A11}$$

where $\delta_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right)$ is the estimation error satisfying $\left|\delta_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right)\right|\le\bar{\delta}_{\tau,\vartheta}$ and $\bar{\delta}_{\tau,\vartheta}>0$ is set to a constant.

Similarly, the following inequality is true depending on Young's inequality:

$$\begin{aligned} & m_i!\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\mathcal{H}^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\left(\Theta^{*\mathrm{T}}_{\tau,\vartheta}\Phi_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right)+\delta_{\tau,\vartheta}\left(X_{\tau,\vartheta}\right)\right) \\ \le & m_i!\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\left[\mathcal{H}^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\right]^2\left(\frac{\theta_{\tau,\vartheta}}{2\xi^2_{\tau,\vartheta}}+\frac{1}{2}\xi^2_{\tau,\vartheta}+\bar{\delta}_{\tau,\vartheta}\right), \end{aligned}$$

where $\theta_{\tau,\vartheta}=\left\|\Theta^{*\mathrm{T}}_{\tau,\vartheta}\right\|^2$. At the same time, we set $\phi_{\tau,\vartheta}=\frac{1}{2}\xi^2_{\tau,\vartheta}+\bar{\delta}_{\tau,\vartheta}$ to obtain

$$\begin{aligned} & m_i!\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\mathcal{H}^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})e_{\tau,1} \\ \le & \frac{\|e_{\tau,1}\|^2}{2\psi}+\frac{\psi\left[m_i!\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})\right]^2}{2}. \end{aligned} \tag{A12}$$

Consequently, we have

$$\begin{aligned} \dot{V}_{\tau,\vartheta} \le & \sum_{\gamma=1}^{\vartheta}\left[\dot{\sigma}_{\tau,\gamma}\left(N\left(\sigma_{\tau,\gamma}\right)+1\right)\right. \\ & -s_{\tau,\gamma}(n_i-\gamma+1)!\Upsilon^{\varsigma_\tau}_{\tau,n_i-\gamma+2}(\hbar_{\tau,\gamma}) \\ & \left.+\frac{a_{\tau,\gamma}}{\varphi_{\tau,\gamma}}\tilde{\theta}_{\tau,\gamma}\hat{\theta}_{\tau,\gamma}+\frac{b_{\tau,\gamma}}{\beta_{\tau,\gamma}}\tilde{W}^{\mathrm{T}}_{\tau,\gamma}\hat{W}_{\tau,\gamma}+\frac{c_{\tau,\gamma}}{\mu_{\tau,\gamma}}\tilde{\phi}_{\tau,\gamma}\hat{\phi}_{\tau,\gamma}\right] \\ & +\kappa_{\tau,\vartheta}-\frac{1}{4}\left[\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})m_i!\right]^2 \\ & +\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})m_i!\left[\left|\hbar_{\tau,\vartheta+1}\right|-(\varsigma_\tau+1)\right] \\ & +\sum_{\gamma=2}^{\vartheta}g_{\tau,\gamma}\frac{\|\tilde{e}_\tau\|^2}{2l}+\frac{\|\tilde{e}_\tau\|^2}{2}+\frac{\left\|\bar{d}_{\tau,1}\right\|^2}{2}, \end{aligned} \tag{A13}$$

where

$$\begin{aligned} \kappa_{\tau,\vartheta} = & -\frac{1}{4}\left[\Upsilon^{\varsigma_\tau}_{\tau,m_i+1}(\hbar_{\tau,\vartheta-1})(m_i+1)!\right]^2 \\ & +\Upsilon^{\varsigma_\tau}_{\tau,m_i+1}(\hbar_{\tau,\vartheta-1})(m_i+1)!\left[\left|\hbar_{\tau,\vartheta}\right|-(\varsigma_\tau+1)\right] \\ & -\left[\Upsilon^{\varsigma_\tau}_{\tau,m_i}(\hbar_{\tau,\vartheta})m_i!\right]^2 \end{aligned}$$

and $\kappa_{\tau,\vartheta}\le 0$.

Consequently, by combining Eqs. (17) and (A13), it can be deduced that

$$\begin{aligned} \dot{V}_{\tau,\vartheta} \le & \sum_{\gamma=1}^{\vartheta}\left[\dot{\sigma}_{\tau,\gamma}\left(N\left(\sigma_{\tau,\gamma}\right)+1\right)\right. \\ & -s_{\tau,\gamma}(n_i-\gamma+1)!\Upsilon^{\varsigma_\tau}_{\tau,n_i-\gamma+2}(s_{\tau,\gamma}) \\ & \left.+\frac{a_{\tau,\gamma}}{\varphi_{\tau,\gamma}}\tilde{\theta}_{\tau,\gamma}\hat{\theta}_{\tau,\gamma}+\frac{b_{\tau,\gamma}}{\beta_{\tau,\gamma}}\tilde{W}^{\mathrm{T}}_{\tau,\gamma}\hat{W}_{\tau,\gamma}+\frac{c_{\tau,\gamma}}{\mu_{\tau,\gamma}}\tilde{\phi}_{\tau,\gamma}\hat{\phi}_{\tau,\gamma}\right] \\ & -\frac{1}{4}[\Upsilon^{\varsigma_\tau}_{\tau,m_i}(s_{\tau,\vartheta})m_i!]^2+\Upsilon^{\varsigma_\tau}_{\tau,m_i}(s_{\tau,\vartheta})m_i! \\ & \times\left[\left|s_{\tau,\vartheta+1}\right|-(\varsigma_\tau+1)\right]+\frac{\|\tilde{e}_\tau\|^2}{2}+\frac{\left\|\bar{d}_{\tau,1}\right\|^2}{2}. \end{aligned} \tag{A14}$$

Step n_i By defining $\hbar_{\tau,n_i} = \hat{x}_{\tau,n_i} - \alpha_{\tau,n_i-1}$, we can derive the formula for the derivative of $\hbar_{\tau,n_i}$ as follows:

$$\begin{aligned}\dot{\hbar}_{\tau,n_i} &= \dot{\hat{x}}_{\tau,n_i} - \dot{\alpha}_{\tau,n_i-1} \\ &= \ell_{\tau,n_i} u_\tau + \widehat{W}_{\tau,n_i}^{\mathrm{T}} \Phi_{\tau,n_i}\left(\hat{\bar{x}}_{\tau,n_i}\right) + g_{\tau,n_i} e_{\tau,1} - \dot{\alpha}_{\tau,n_i-1}.\end{aligned} \tag{A15}$$

We can get

$$\dot{V}_{\tau,n_i} = \dot{V}_{\tau,n_i-1} + \Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\mathcal{H}_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\dot{\hbar}_{\tau,n_i} - \Lambda_{\tau,n_i}, \tag{A16}$$

where

$$\begin{aligned}\tilde{f}_{\tau,n_i}(X_{\tau,n_i}) \;=\; & \widehat{W}_{\tau,n_i}^{\mathrm{T}} \Phi_{\tau,n_i}\left(\hat{\bar{x}}_{\tau,n_i}\right) \\ & + g_{\tau,n_i} \frac{\psi\left[\Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\right]^2}{2} - \dot{\alpha}_{\tau,n_i-1} \\ & - \frac{t_{\tau,n_i}\hbar_{\tau,n_i}}{\Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\mathcal{H}_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})} \tilde{W}_{\tau,n_i}^{\mathrm{T}} \Phi_{\tau,n_i}\left(\hat{\bar{x}}_{\tau,n_i}\right),\end{aligned}$$

with

$$t_{\tau,n_i} = h_{\tau,n_i} \tan^2\left[\frac{\Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\mathcal{H}_{\tau,1}^{\varsigma}(\Phi_{\tau,n_i})}{\varepsilon_{\tau,n_i}}\right], \quad \varepsilon_{\tau,n_i} > 0,$$

and we use $\Theta_{\tau,n_i}^{*\mathrm{T}} \Phi_{\tau,n_i}(X_{\tau,n_i})$ to estimated unknown function $\tilde{f}_{\tau,n_i}(X_{\tau,n_i})$,

$$\tilde{f}_{\tau,n_i}(X_{\tau,n_i}) = \Theta_{\tau,n_i}^{*\mathrm{T}} \Phi_{\tau,n_i}(X_{\tau,n_i}) + \delta_{\tau,n_i}(X_{\tau,n_i}).$$

By applying Young's inequality, we obtain

$$\begin{aligned}&\Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\mathcal{H}_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\Theta_{\tau,n_i}^{*\mathrm{T}} \Phi_{\tau,n_i}(X_{\tau,n_i}) \\ &\le \Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\left[\mathcal{H}_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\right]^2 \left(\frac{\theta_{\tau,n_i}}{2\xi_{\tau,n_i}^2} + \frac{1}{2}\xi_{\tau,n_i}^2\right),\end{aligned} \tag{A17}$$

$$\begin{aligned}&\Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\mathcal{H}_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\delta_{\tau,n_i}(X_{\tau,n_i}) \\ &\le \Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\left[\mathcal{H}_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\right]^2 \bar{\delta}_{\tau,n_i},\end{aligned} \tag{A18}$$

where $\theta_{\tau,n_i} = \left\|\Theta_{\tau,n_i}^{*\mathrm{T}}\right\|^2$. Also, let $\phi_{\tau,n_i} = \frac{1}{2}\xi_{\tau,n_i}^2 + \bar{\delta}_{\tau,n_i}$, the following inequalities can be obtained

$$\begin{aligned}&\Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\mathcal{H}_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})e_{\tau,1} \\ &\le \frac{\psi\left[\Upsilon_{\tau,1}^{\varsigma_\tau}(\hbar_{\tau,n_i})\right]^2}{2} + \frac{\|e_{\tau,1}\|^2}{2\psi}.\end{aligned} \tag{A19}$$

Combining the above variables gives

$$\begin{aligned}\dot{V}_{\tau,n_i} \le & \sum_{\gamma=1}^{n_i-1}\left[\dot{\sigma}_{\tau,\gamma}\left(N\left(\sigma_{\tau,\gamma}\right)+1\right)\right] + \dot{\sigma}_{\tau,n_i}\left(\ell_{\tau,n_i}N\left(\sigma_{\tau,n_i}\right)+1\right) \\ & + \kappa_{\tau,n_i} + \sum_{\gamma=2}^{n_i} g_{\tau,\gamma}\frac{\|\tilde{e}_\tau\|^2}{2\psi} + \frac{\|\tilde{e}_\tau\|^2}{2} + \frac{\left\|\bar{d}_{\tau,1}\right\|^2}{2} \\ & + \sum_{\gamma=1}^{n_i}\Bigg[-s_{\tau,\gamma}(n_i-\gamma+1)!\Upsilon_{\tau,n_i-\gamma+2}^{\varsigma_\tau}(\hbar_{\tau,\gamma}) \\ & + \frac{a_{\tau,\gamma}}{\varphi_{\tau,\gamma}}\tilde{\theta}_{\tau,\gamma}\hat{\theta}_{\tau,\gamma} + \frac{b_{\tau,\gamma}}{\beta_{\tau,\gamma}}\tilde{W}_{\tau,\gamma}^{\mathrm{T}}\hat{W}_{\tau,\gamma} \\ & + \frac{c_{\tau,\gamma}}{\mu_{\tau,\gamma}}\tilde{\phi}_{\tau,\gamma}\hat{\phi}_{\tau,\gamma}\Bigg].\end{aligned} \tag{A20}$$

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