

Evolutionary analysis of multi-player snowdrift game with cost sharing under Aspiration-Fermi hybrid rule

Bolin Yanga(杨柏林) and Guanghui Yang(杨光惠)
Citation: Chin. Phys. B, 2026, 35 (**3**): 030203. DOI: 10.1088/1674-1056/adf31a
http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Strategy persistence-consistency and reputation promote cooperation in dual-layer networks for prisoner's dilemma games

Qianwei Zhang(张倩伟), Jiaqi Liu(刘佳琪), and Leiman Fu(付蕾蔓)
Chin. Phys. B, 2025, 34 (**12**): 120201. DOI: 10.1088/1674-1056/ade665

Impact of peer pressure on cooperation evolution in the networked prisoner's dilemma game with migration mechanisms

Xianjia Wang(王先甲), Yanan Li(李亚楠), and Zhipeng Yang(杨志鹏)
Chin. Phys. B, 2025, 34 (**11**): 110201. DOI: 10.1088/1674-1056/ade425

Generic stability of cooperative equilibria for multi-leader-follower-population mixed games

Wenjun Wu(武文俊), Hui Yang(杨辉), and Guanghui Yang(杨光惠)
Chin. Phys. B, 2025, 34 (**10**): 100202. DOI: 10.1088/1674-1056/ade666

Influence of conformity on the evolution of cooperation in games with sampling rules on networks

Xianjia Wang(王先甲) and Qiaoyu He(何翘愚)
Chin. Phys. B, 2025, 34 (**3**): 030202. DOI: 10.1088/1674-1056/ada42e

Non-monotonic behavior of jam probability and stretched exponential distribution in pedestrian counterflow

Ze-Hao Chen(陈泽昊), Zhi-Xi Wu(吴枝喜), and Jian-Yue Guan(关剑月)
Chin. Phys. B, 2024, 33 (**9**): 090206. DOI: 10.1088/1674-1056/ad57ab

---

Evolutionary analysis of multi-player snowdrift game with cost sharing under Aspiration–Fermi hybrid rule

Bolin Yang(杨柏林)[1,2] and Guanghui Yang(杨光惠)[1,2,†]

[1] *School of Mathematics and Statistics, Guizhou University, Guiyang* 550025, *China*
[2] *Guizhou Provincial Key Laboratory for Games Decision-Making and Control Systems, Guiyang* 550025, *China*

(Received 29 May 2025; revised manuscript received 16 July 2025; accepted manuscript online 23 July 2025)

This paper firstly constructs a multi-player snowdrift game with all players' cost sharing once the snowdrift is removed for a fairness. Secondly, an Aspiration–Fermi hybrid rule is proposed to derive an extended average abundance function via Markov chain evolutionary processes. Besides, extensive numerical simulations well verify theoretical results. Our findings show that both cost sharing and the Aspiration–Fermi hybrid rule significantly promote the cooperation by enhancing the average abundance in multi-person snowdrift game. By comparison with the existing woks, our proposed model and hybrid strategy-update rule provide novel insights into the evolution of cooperation in multi-person games.

Keywords: multi-player snowdrift evolutionary game, cost sharing, Aspiration–Fermi hybrid rule, average abundance function

PACS: 02.50.Le, 02.50.Ga, 89.75.–k

DOI: 10.1088/1674-1056/adf31a **CSTR:** 32038.14.CPB.adf31a

1. Introduction

The evolution of cooperation pervades natural systems[1–5] and human societies,[6–9] and evolutionary game theory has served as a robust framework for cooperative dynamics.[10–18]

Given the prevalence of group interactions, multiplayer games[15–30] extend classical two-player models. For infinitely large well-mixed populations, replicator dynamics describe strategy propagation.[21–30] However, finite populations require stochastic modeling due to inherent randomness. For mutation-free scenarios, absorbing Markov chains characterize evolutionary processes, where fixation probabilities are key metrics.[21,31–34] Conversely, mutation-inclusive processes utilize ergodic Markov chains, with average abundance functions quantifying cooperative states.[35–38] Specifically, Wang and Xia[35] established an extended average abundance function for N-person snowdrift games (NSG) under aspiration-driven dynamics, demonstrating positive correlations between aspiration levels and cooperation. Building on this, Xia[36,37] emphasized average abundance as a critical cooperation indicator that elucidates cooperative diffusion mechanisms. Furthermore, Kroumi and Lessard[38] revealed enhanced cooperation under reduced benefit-cost variance or increased covariance in stag hunt and snowdrift games. Collectively, these studies establish average abundance as a pivotal metric for quantifying and analyzing cooperative evolution.

As a generalization of the two-player snowdrift game, NSG has garnered significant attention in collective behavior studies.[21] Recent extensions include threshold-dependent cooperation: Souza *et al.*[22] introduced minimum cooperator thresholds, while Zheng and Zhang[25] incorporated time costs into threshold-based NSG. Subsequent work by Pi *et al.*[28] examined stochastic equilibrium stability, and Sui *et al.*[24] and Pi *et al.*[30] explored dual-threshold models with cost boundaries. However, existing models neglect fairness[39] considerations, as snow removal costs are disproportionately borne by cooperators, while defectors free-ride. To address this equity gap, we propose cost-sharing mechanisms where all beneficiaries contribute equally — a novel approach not previously documented.

Strategy update rules critically influence cooperative outcomes. Predominant paradigms include Moran processes,[16,40,41] PW–Fermi rule,[42–45] and Aspiration rule.[36,46–49] Hybrid rules have emerged as promising tools: Liu *et al.*[50] combined aspiration and imitation rules, finding imitation-driven cooperation increases with aspiration-user density. Liu *et al.*[51] integrated Moran and Fermi processes, observing accelerated fixation *versus* pure Fermi dynamics. Wang *et al.*[32] designed a Moran-imitation hybrid, showing prolonged cooperator fixation under imitation-biased updates. These studies collectively demonstrate hybrid rules' capacity to enhance cooperation.

To bridge existing gaps, we develop an NSG model with cost-sharing and an Aspiration–Fermi hybrid rule. Key innovations include: (i) cost is equally shared among all players once the snowdrift is removed; (ii) an Aspiration–Fermi hybrid rule is proposed as the strategy update mechanism. These are exactly our two principal contributions.

[†]Corresponding author. E-mail: ghuiyang@126.com

© 2026 Chinese Physical Society and IOP Publishing Ltd. All rights, including for text and data mining, AI training, and similar technologies, are reserved.

The paper is structured as follows: in Section 2, we mainly propose a model of multi-person snowdrift game with cost sharing and an Aspiration–Fermi hybrid rule. Furthermore, the extended average abundance function of our model under the hybrid rule is precisely derived. Section 3 devotes to the simulations of the evolution of cooperation. Finally, conclusions is made for the paper in Section 4.

2. Model

2.1. A multi-person evolutionary game model

The multi-person evolutionary game model involves a uniformly mixed population including N agents with A and B strategies. The number of players playing A is j (defined as an evolutionary state) evolves over time and that of players with B is $N-j$. Therefore, multi-person evolutionary game is essentially a dynamic process in which any agent in a population continuously repeats the game by a certain strategy updating rule.

The interaction between players are described by the following three steps:

Step 1 a focal agent X, is randomly chosen from the above population. And $N_{\rm d}-1$ ($N_{\rm d}<N$) agents are selected from the remaining $N-1$ agents to constitute a group with i agents playing A and $N_{\rm d}-1-i$ agents playing B; the focal agent X interacts with the agents in the group.

Step 2 The payoffs for the focal agent X are followed the payoff matrix:

$$\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{ccccc} N_{\rm d}-1 & N_{\rm d}-2 & \cdots & 1 & 0 \end{array} \\ \begin{pmatrix} a_{N_{\rm d}-1} & a_{N_{\rm d}-2} & \cdots & a_1 & a_0 \\ b_{N_{\rm d}-1} & b_{N_{\rm d}-2} & \cdots & b_1 & b_0 \end{pmatrix} \end{array}.$$

This is the payoff matrix of the multi-person game.

Step 3 The agent X updates his strategies between A and B by a resulting strategy-switching rule based on the evaluation of the payoffs for a round of games.

2.2. Expected payoffs

According Step 2 of Subsection 2.1, the probability of the focal agent X playing strategy A or B follows a hypergeometric distribution, thus the expected payoffs of the focal agent X in a round of game is respectively as follows:

$$\Pi_A(j)=\sum_{i=0}^{N_{\rm d}-1}\frac{C_{j-1}^{i}C_{N-j}^{N_{\rm d}-1-i}}{C_{N-1}^{N_{\rm d}-1}}a_i, \tag{1}$$

$$\Pi_B(j)=\sum_{i=0}^{N_{\rm d}-1}\frac{C_{j}^{i}C_{N-j-1}^{N_{\rm d}-1-i}}{C_{N-1}^{N_{\rm d}-1}}b_i. \tag{2}$$

2.3. Strategy-updating mechanism

2.3.1. Aspiration rule

The Aspiration rule[46] involves a comparison between the expected payoffs and aspiration payoffs α for an agent, hence this rule is subjectively dominated by the agent. Driven by the Aspiration rule, the probability for the X agent switching strategies from A to B is

$$P_{A\to B}=\frac{1}{1+{\rm e}^{-\lambda(\alpha-\Pi_A(j))}}, \tag{3}$$

similarly, the probability of strategy-switching from B to A is

$$P_{B\to A}=\frac{1}{1+{\rm e}^{-\lambda(\alpha-\Pi_B(j))}}, \tag{4}$$

where the parameter λ indicates the intensity of selection.

2.3.2. PW–Fermi rule

Unlike the Aspiration rule, the PW–Fermi rule[42–45] compares an agent's own expected payoff with that of its neighbors, this rule thus embodies the objectivity of agents' consideration of the game environment. Driven by the PW–Fermi rule, the probability for the X agent switching strategies from A to B is

$$P_{A\to B}=\frac{1}{1+\exp\left[-\frac{1}{\kappa}(\Pi_B(j)-\Pi_A(j))\right]}, \tag{5}$$

and that of from B to A is

$$P_{B\to A}=\frac{1}{1+\exp\left[-\frac{1}{\kappa}(\Pi_A(j)-\Pi_B(j))\right]}, \tag{6}$$

where $\kappa>0$ denotes the environmental noise level of the game, it depicts the uncertainty in an agent's strategy selection.[42–45] If $\kappa\to 0$, the strategy with higher payoffs is imitated with a probability of 1, if $\kappa\to\infty$, players update their strategies randomly.

2.3.3. Aspiration–Fermi hybrid rule

In consideration of the subjectivity of an agents' aspiration payoff and the objectivity of game environment, we establish the following Aspiration–Fermi hybrid rule. Driven by Aspiration–Fermi hybrid rule, the probability of the focal agent switching strategies from A to B is

$$\begin{aligned} P_{A\to B} =& \frac{\theta}{1+\exp\left[-\lambda(\alpha-\Pi_A(j))\right]} \\ &+\frac{1-\theta}{1+\exp\left[-\frac{1}{\kappa}(\Pi_B(j)-\Pi_A(j))\right]}, \end{aligned} \tag{7}$$

and that of from B to A is

$$\begin{aligned} P_{B\to A} =& \frac{\theta}{1+\left[\exp-\lambda(\alpha-\Pi_B(j))\right]} \\ &+\frac{1-\theta}{1+\exp\left[-\frac{1}{\kappa}(\Pi_A(j)-\Pi_B(j))\right]}, \end{aligned} \tag{8}$$

where $\theta\in[0,1]$. In particular, it is completely the Fermi rule for $\theta=0$ and it is exactly the aspiration rule for $\theta=1$.

In a multi-person evolutionary game based on Aspiration–Fermi hybrid rule, the number j of A-strategy agents in the population is evolutionary over time. Here, we

mainly discuss the following three cases: (i) $j \to j-1$, (ii) $j \to j+1$, (iii) $j \to j$, and assume that the probability of other situations is all zero.

The evolution process of the above three cases is often described by a Markov chain without absorption states

$$\begin{aligned} &P(j \to j+1) = T_j^+ \\ &= \frac{N-j}{N}\left(\frac{\theta}{1+\exp\left[-\lambda(\alpha-\Pi_B(j))\right]}\right. \\ &\left.+\frac{1-\theta}{1+\exp\left[-\frac{1}{\kappa}(\Pi_A(j)-\Pi_B(j))\right]}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} &P(j \to j-1) = T_j^- \\ &= \frac{j}{N}\left(\frac{\theta}{1+\exp\left[-\lambda(\alpha-\Pi_A(j))\right]}\right. \\ &\left.+\frac{1-\theta}{1+\exp\left[-\frac{1}{\kappa}(\Pi_B(j)-\Pi_A(j))\right]}\right), \end{aligned} \tag{10}$$

$$\begin{aligned} &P(j \to j) = 1 - T_j^+ - T_j^- \\ &= 1 - P(j \to j+1) - P(j \to j-1). \end{aligned} \tag{11}$$

2.4. An extended average abundance function

The proportion k/N for A-strategy players in an evolutionary population is a random variable, and let $\vartheta_k(\theta,\lambda,\kappa)$ be the probability distribution for the corresponding variable k ($k = 0,1,2,\ldots,N$) under Aspiration–Fermi hybrid rule. The expected limit of the proportion of A strategy players in well-mixed population called the extended average abundance function (EAAF) is denoted by $\langle X_A(\theta,\lambda,\kappa)\rangle$:[35]

$$\langle X_A(\theta,\lambda,\kappa)\rangle = \sum_{k=0}^{N} \frac{k}{N}\vartheta_k(\theta,\lambda,\kappa). \tag{12}$$

Generally $\langle X_A(\theta,\lambda,\kappa)\rangle$ is available for analyzing the evolutionarily stable state for a Markov chain without absorbing states. Here ϑ_k satisfies the detailed balance formula:[54] $\vartheta_k T_k^+ = \vartheta_{k+1} T_{k+1}^-$.

Hence, the EAAF of A-strategy players under Aspiration–Fermi hybrid rule is

$$\langle X_A(\theta,\lambda,\kappa)\rangle = \frac{1}{N}\frac{\sum_{k=1}^{N} k \prod_{j=0}^{k-1} h(j)}{1+\sum_{k=1}^{N}\prod_{j=0}^{k-1} h(j)}, \tag{13}$$

where

$$h(j) = \frac{T_j^+}{T_{j+1}^-} = \frac{(N-j)\left(\frac{\theta}{1+\exp\left[-\lambda(\alpha-\Pi_B(j))\right]}+\frac{1-\theta}{1+\exp\left[-\frac{1}{\kappa}(\Pi_A(j)-\Pi_B(j))\right]}\right)}{(j+1)\left(\frac{\theta}{1+\exp\left[-\lambda(\alpha-\Pi_A(j+1))\right]}+\frac{1-\theta}{1+\exp\left[-\frac{1}{\kappa}(\Pi_B(j+1)-\Pi_A(j+1))\right]}\right)}. \tag{14}$$

2.5. Multi-person snowdrift game model with cost sharing

We further consider NSG[25] with cost sharing: $N_{\rm d}$ players are randomly sampled from a well-mixed population with size N and they are blocked by a huge snowdrift on the way to their destination. Cooperation (C) and defection (D) are two available strategies for each player. Within the $N_{\rm d}$-person group, at least M cooperators ($1 \le M \le N_{\rm d}$) are desirable to clear the snowdrift and then each player obtains a benefit b given a total cost c of endeavor. Precisely, if the number of cooperators is below M, these cooperators are too few to fulfill the snowdrift task so that each player additionally incurs a time cost $\delta(b)$ dependent on the benefit b, and meanwhile each cooperator bear the maximum cost c/M of shoveling cost. If the number of cooperators is no less than M, these cooperators are enough to complete the snowdrift, all $N_{\rm d}$ players equally share the cost c of shoveling snow for a fairness,[39] and each of them pay an extra time cost to a certain extent. Hence, the payoffs of the focal agent with cooperation and defection are given by

$$P_C(i) = \begin{cases} -\frac{c}{M} - \delta(b), & i \in [0, M-2], \\ b - \frac{c}{N_{\rm d}} - \frac{l\delta(b)}{i+1}, & i \in [M-1, N_{\rm d}-1], \end{cases} \tag{15}$$

$$P_D(i) = \begin{cases} -\delta(b), & i \in [0, M-1], \\ b - \frac{c}{N_{\rm d}} - \frac{l\delta(b)}{i+1}, & i \in [M, N_{\rm d}-1], \end{cases} \tag{16}$$

where $l \in [0,1]$.

Simply, GNSGC denotes our proposed above model. By comparisons with the existing models, in the rest of this paper, GNSG denotes the reduction model of GNSGC without cost sharing in Ref. [25], GSH represents the model[28] with time preferences of defectors, and TNSG signifies another reduction of GNSGC without cost sharing and time cost in Ref. [22].

Suppose $H_{\rm s}(x)$ represents a Heaviside step function, with the property that when $x < 0$, $H_{\rm s}(x) = 0$, and when $x \ge 0$, $H_{\rm s}(x) = 1$. Thus, the payoffs $P_C(i)$ and $P_D(i)$ are normalized as

$$\begin{aligned} P_C(i) &= H_{\rm s}(i+1-M)\left[b + \delta(b) + \frac{c}{M} - \frac{c}{N_{\rm d}} - \frac{l\delta(b)}{i+1}\right] \\ &\quad - \frac{c}{M} - \delta(b), \end{aligned} \tag{17}$$

$$P_D(i) = H_{\rm s}(i-M)\left[b + \delta(b) - \frac{c}{N_{\rm d}} - \frac{l\delta(b)}{i+1}\right] - \delta(b). \tag{18}$$

Consequently, the expected payoffs $\Pi_D(j)$ and $\Pi_C(j)$ are as follows:

$$\begin{aligned}\Pi_D(j) &= \sum_{i=0}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}P_D(i) = \sum_{i=0}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(H_s(i-M)\left[b+\delta(b)-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right]-\delta(b)\right)\\ &= -\delta(b)\sum_{i=0}^{M-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}+\sum_{i=M}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right),\end{aligned} \tag{19}$$

$$\begin{aligned}\Pi_C(j) &= \sum_{i=0}^{N_d-1}\frac{C_{j-1}^i C_{N-j}^{N_d-1-i}}{C_{N-1}^{N_d-1}}P_C(i) = \sum_{i=0}^{N_d-1}\frac{C_{j-1}^i C_{N-j}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(H_s(i+1-M)\left[b+\delta(b)+\frac{c}{M}-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right]-\frac{c}{M}-\delta(b)\right)\\ &= \begin{cases}\sum\limits_{i=0}^{N_d-1}\frac{C_{j-1}^i C_{N-j}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right), & M=1,\\ -\sum\limits_{i=0}^{M-2}\frac{C_{j-1}^i C_{N-j}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(\frac{c}{M}+\delta(b)\right)+\sum\limits_{i=M-1}^{N_d-1}\frac{C_{j-1}^i C_{N-j}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{ik+1}\right), & M\geq 2,\end{cases}\end{aligned} \tag{20}$$

Accordingly, the EAAF for cooperation in GNSGC is

$$\langle X_C(\theta,\lambda,\kappa)\rangle = \frac{1}{N}\frac{\sum\limits_{k=1}^{N}k\prod\limits_{j=0}^{k-1}h(j)}{1+\sum\limits_{k=1}^{N}\prod\limits_{j=0}^{k-1}h(j)} = \begin{cases}\frac{1}{N}\frac{\sum\limits_{k=0}^{N}k\prod\limits_{j=0}^{k-1}\frac{(N-j)(y(\theta,\lambda)+z(\theta,\kappa))}{(j+1)Y(\theta,\lambda)+Z(\theta,\kappa))}}{1+\sum\limits_{k=0}^{N}k\prod\limits_{j=0}^{k-1}\frac{(N-j)(y(\theta,\lambda)+z(\theta,\kappa))}{(j+1)(Y(\theta,\lambda)+Z(\theta,\kappa))}}, & M=1,\\ \frac{1}{N}\frac{\sum\limits_{k=0}^{N}k\prod\limits_{j=0}^{k-1}\frac{(N-j)(f(\theta,\lambda)+g(\theta,\kappa))}{(j+1)(F(\theta,\lambda)+G(\theta,\kappa))}}{1+\sum\limits_{k=0}^{N}k\prod\limits_{j=0}^{k-1}\frac{(N-j)(f(\theta,\lambda)+g(\theta,\kappa))}{(j+1)(F(\theta,\lambda)+G(\theta,\kappa))}}, & M\geq 2,\end{cases} \tag{21}$$

where

$$y(\theta,\lambda)=\frac{\theta}{1+\exp\left(-\lambda\left[\alpha+\delta(b)\frac{C_{N-j-1}^{d-1}}{C_{N-1}^{N_d-1}}-\sum\limits_{i=1}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)\right]\right)},$$

$$Y(\theta,\lambda)=\frac{\theta}{1+\exp\left(-\lambda\left[\alpha-\sum\limits_{i=0}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)\right]\right)},$$

$$z(\theta,\kappa)=\frac{1-\theta}{1+\exp\left(-\frac{1}{\kappa}\left[\sum\limits_{i=0}^{N_d-1}\frac{C_{j-1}^i C_{N-j}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)+\delta(b)\frac{C_{N-j-1}^{N_d-1}}{C_{N-1}^{N_d-1}}-\sum\limits_{i=1}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)\right]\right)},$$

$$Z(\theta,\kappa)=\frac{1-\theta}{1+\exp\left(-\frac{1}{\kappa}\left[-\delta(b)\frac{C_{N-j-2}^{N_d-1}}{C_{N-1}^{d-1}}+\sum\limits_{i=1}^{N_d-1}\frac{C_{j+1}^i C_{N-j-2}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)-\sum\limits_{i=0}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)\right]\right)},$$

$$f(\theta,\lambda)=\frac{\theta}{1+\exp\left(-\lambda\left[\alpha+\delta(b)\sum\limits_{i=0}^{M-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}-\sum\limits_{i=M}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)\right]\right)},$$

$$F(\theta,\lambda)=\frac{\theta}{1+\exp\left(-\lambda\left[\alpha+\sum\limits_{i=0}^{M-2}\frac{C_{N_t}^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(\frac{c}{M}+\delta(b)\right)-\sum\limits_{i=M-1}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)\right]\right)},$$

$$g(\theta,\kappa)=\frac{1-\theta}{1+\exp\left[-\frac{1}{\kappa}\left[-\sum\limits_{i=0}^{M-2}\frac{C_{j-1}^i C_{N-j}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(\frac{c}{M}+\delta(b)\right)+\sum\limits_{i=M-1}^{N_d-1}\frac{C_{j-1}^i C_{N-j}^{d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)+\delta(b)\sum\limits_{i=0}^{M-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}-\sum\limits_{i=M}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)\right]\right]},$$

$$G(\theta,\kappa)=\frac{1-\theta}{1+\exp\left[-\frac{1}{\kappa}\left[-\delta(b)\sum\limits_{i=0}^{M-1}\frac{C_{j+1}^i C_{N-j-1}^{d-1-i}}{C_{N-1}^{N_d-1}}+\sum\limits_{i=M}^{N_d-1}\frac{C_{j+1}^i C_{N-j-2}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)+\sum\limits_{i=0}^{M-2}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(\frac{c}{M}+\delta(b)\right)-\sum\limits_{i=M-1}^{N_d-1}\frac{C_j^i C_{N-j-1}^{N_d-1-i}}{C_{N-1}^{N_d-1}}\left(b-\frac{c}{N_d}-\frac{l\delta(b)}{i+1}\right)\right]\right]}.$$

3. Simulation results

3.1. Evolutionary analysis of equal benefits both cooperators and defectors

In this section, we investigate the effects of all parameters on the EAAF for GNSGC under the equal payoffs both cooperators and defectors. For the convenience of research, in this article, it is assumed that the time cost $\delta(b)=\mu b$, where $\mu\geq 0$ is the time cost factor. Select the basic parameters $N_d=15$, $\alpha=1$, $b=10$, $c=5$, $\mu=4$, $M=10$, $\lambda=4$, $\kappa=0.7$, $\theta=0.99$, $l=0.3$, $N=20$.

Firstly, in Fig. 1, we study the effect of Aspiration–Fermi

hybrid rule on the EAAFs for GNSGC, GNSG[25] (without considering cost sharing), GSH[28] (without considering the cost sharing, but taking into account the time preference of defectors) and TNSG[22] (without considering both cost sharing and time cost). Obviously, for $\theta \in [0,1]$, the EAAF of GNSGC is greater than that of GNSG, GSH, and TNSG as shown in Fig. 1. This indicates cost sharing helpfully enhance the cooperation. Additionally, at $\theta = 0.99$ the EAAFs for GNSGC, GNSG, and GSH attains its maximum which is conspicuously higher than the EAAFs at $\theta = 0$ (PW–Fermi rule) and $\theta = 1$ (Aspiration rule).This demonstrates that a higher proportion of the Aspiration rule in the hybrid framework more effectively facilitates cooperation, indicating that individual adherence to subjective aspirations enhances collaborative behavior. Meanwhile, for $\theta \in [0, 0.99]$ the EAAFs for GNSGC, GNSG, and GSH and for $\theta \in [0,1]$ the EAAF for TNSG gradually increase as θ. This suggests that the Aspiration–Fermi hybrid rule proposed by us exerts a positive effect on enhancing the EAAF and can more effectively facilitate the generation of cooperation.

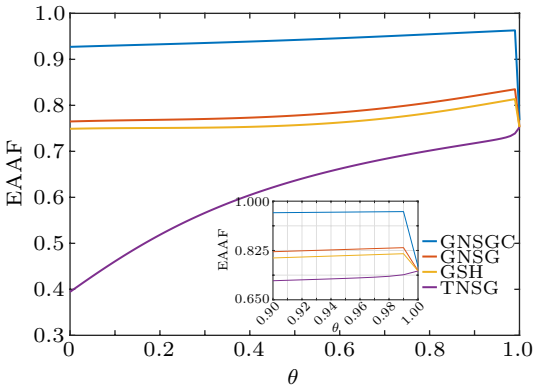


Fig. 1. The effect of Aspiration–Fermi hybrid rule on the EAAFs for GNSGC and other models.

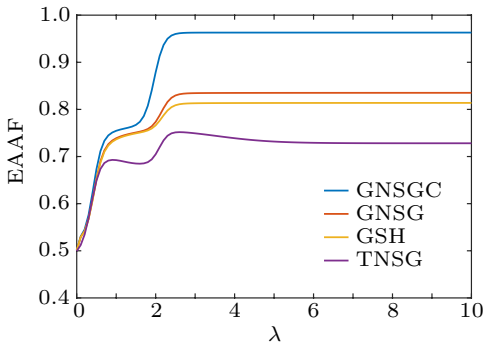


Fig. 2. The effect of selection intensity on the EAAFs for GNSGC and other models.

In order to further investigate the impact of parameters on the EAAF for GNSGC under the equal payoffs both cooperators and defectors, we conduct simulation analyses on the selection intensity λ within the Aspiration–Fermi hybrid rule. Evidently, in Fig. 2, the EAAFs for GNSGC, GNSG, GSH, and TNSG gradually increase and eventually stabilize at a certain fixed value as the selection intensity continuously increases, this indicates that appropriately increasing the selection intensity is helpful for cooperation. Furthermore, for $\lambda \in [0,10]$, the EAAF of GNSGC is greater than that of GNSG, GSH, and TNSG; this once again indicates that cost sharing helpfully enhances cooperation.

In Fig. 3, we probe into the impact of noise parameters κ on the EAAF for GNSGC. For $\kappa \in (0,1]$, the EAAF of GNSGC is greater than that of GNSG, GSH, and TNSG, this once again indicates that cost sharing helpfully enhances cooperation. Additionally, the EAAFs of GNSGC, GNSG, GSH, and TNSG first increase to the maximum value and then gradually decrease. This indicates that the selection of noise cannot be too large. Only an appropriate noise level can better promote cooperation.

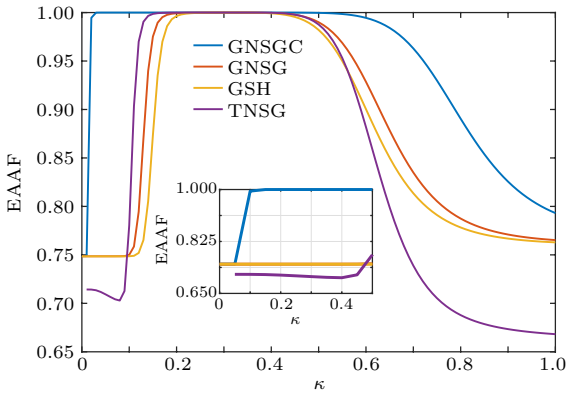


Fig. 3. The effect of environmental noise level on the EAAFs for GNSGC and other models.

In Fig. 4, we examine the influence of threshold M on the EAAF for GNSGC. As shown in the figure, as the threshold increases, the EAAF for GNSGC, GNSG, and GSH gradually increases, for $M \in [1,14]$. This indicates that the threshold helpfully enhances cooperation. Additionally, for $M \in [1,12]$ the EAAF for GNSGC is greater than that for GNSG, GSH, and TNSG, but the EAAF for TNSG is greater than that for GNSGC when $M \geq 12$, this indicates that when the threshold is below a certain value, cost sharing is beneficial for enhancing cooperation. Whereas when the threshold is close to the number of participants, TNSG can better promote cooperation.

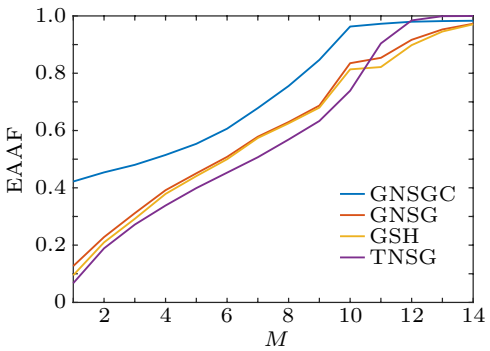


Fig. 4. The effect of threshold on the EAAFs for GNSGC and other models.

To explore the impact of the number of participants N_d on the EAAF, in Fig. 5, we investigate the influence of N_d on

the EAAF for GNSGC under Aspiration–Fermi hybrid rule. As can be seen from the figure, as N_d increases, the EAAF for GNSGC is increases first and then decreases, the EAAFs for both GNSG and GSH are first decreases, then increases, and finally decreases again but the EAAF for TNSG is gradually decreases. This indicates that the impact of the number of participants on the EAAF fluctuates greatly. Therefore, choosing an appropriate number of participants can better promote cooperation. Moreover, for $N_d \in [13, 20]$, the EAAF for GNSGC is greater than for GNSG, GSH, and TNSG, but the EAAF for TNSG is greater than GNSGC, GNSG, and GSH, for $N_d \in [11, 13]$, this indicates that when N_d is relatively close to the threshold M, TNSG is more capable of promoting cooperation. Conversely, when parameter N_d is significantly distant from the threshold, GNSGC is better at promoting cooperation. This also indicates that to a certain degree, cost sharing is beneficial for the increase in cooperation.

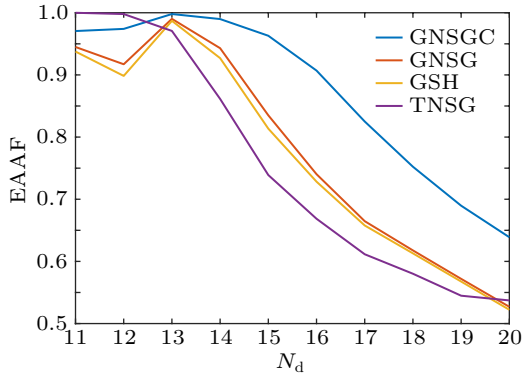


Fig. 5. The effect of participants on the EAAFs for GNSGC and other models.

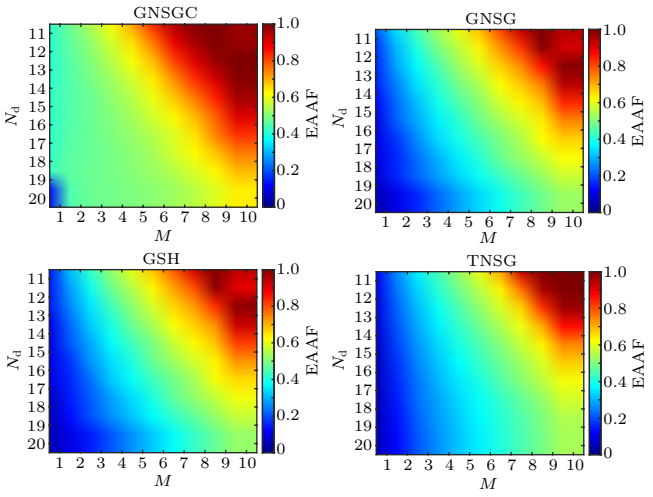


Fig. 6. The effects of both threshold and participants on the EAAFs for GNSGC and other models.

Furthermore, we draw in Fig. 6 to further explore the impacts of threshold M and participants N_d on the EAAF. The graphical results demonstrate that the closer the number of N_d is to the M, the higher the EAAF will be. This suggests that when the threshold is close to the number of participants, cooperation can be better promoted. Additionally, the regions with higher EAAF in GNSGC are larger than those in GNSG, GSH, and TNSG. This once again indicates that cost sharing is beneficial for cooperation.

In Fig. 7, we investigate the impact of the benefit b on the EAAF for GNSGC under Aspiration–Fermi hybrid rule. Just as we expect, as the benefit increases, the EAAFs for GNSGC, GNSG, GSH, and TNSG continuously rises until they increases to 1, this indicates that when the cost c is fixed, increasing benefit is beneficial to improving cooperation. In addition, the EAAF for GNSGC is greater than that of GNSG, GSH, and TNSG. This once again indicates that cost sharing helpfully enhances cooperation.

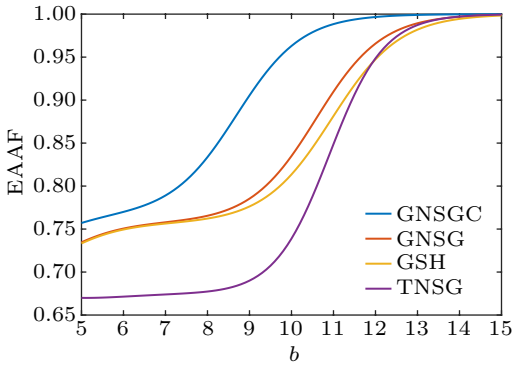


Fig. 7. The effect of benefit on the EAAFs for GNSGC and other models.

We depict in Fig. 8 to investigate the influence of cost c on the EAAF for GNSGC under Aspiration–Fermi hybrid rule. The graph indicates that the EAAF for GNSGC, GNSG, GSH, and TNSG gradually decline as the cost increases, this indicates that when the benefit b is fixed, the lower the cost, the more helpfully promote the cooperation. Additionally, the EAAF for GNSGC is greater than that of GNSG, GSH, and TNSG. This once again indicates that cost sharing helpfully enhances cooperation.

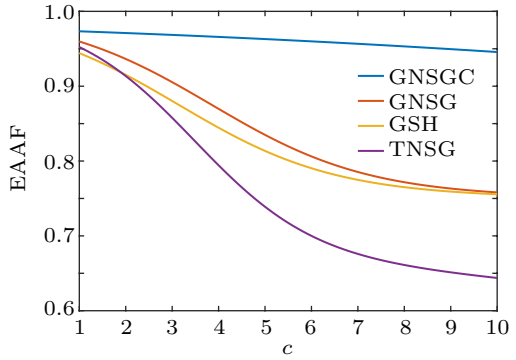


Fig. 8. The effect of cost on the EAAFs for GNSGC and other models.

In Fig. 9, we examine the influence of the time cost factor μ on the EAAF for GNSGC under Aspiration–Fermi hybrid rule. Obviously, as time cost factors increases, the EAAF for GNSGC, GNSG, and GSH generally shows a gradually decreasing trend. And when it exceeds a certain value, their value will be lower than the EAAF for TNSG. This indicates that time cost can promote cooperation to a certain extent.

However, when time cost is excessive, it will instead inhibit cooperation. Additionally, the EAAF for GNSGC is greater than that of GNSG, GSH. This once again indicates that cost sharing helpfully enhances cooperation.

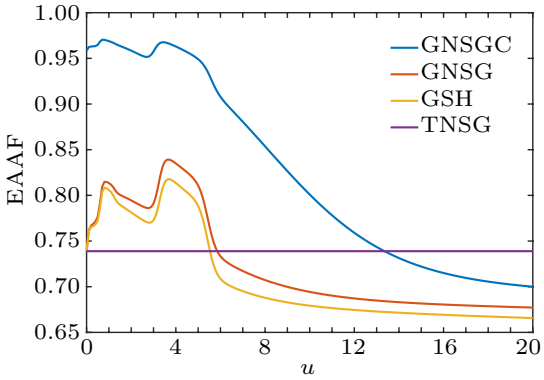


Fig. 9. The effect of time cost factors on the EAAFs for GNSGC and other models.

In Fig. 10, we investigate the impact of aspiration level α on the EAAF for GNSGC under the Aspiration–Fermi hybrid rule. As the aspiration level continues to increase, the EAAFs for GNSGC, GNSG, and GSH generally shows a gradually decreasing trend, but the EAAF for TNSG first increase to their maxima, then gradually decline, and finally tend towards a stable value. This shows that, just as in real life, neither an extremely high nor an extremely low aspiration level is appropriate. An appropriate aspiration level is more conducive to promoting cooperation. At the same time, generally speaking, the EAAF for GNSGC is greater than that for GNSG, GSH, and TNSG. To a certain extent, this once again indicates that cost sharing is beneficial for cooperation.

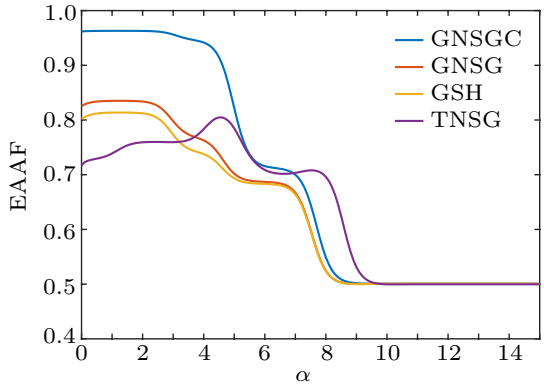


Fig. 10. The effect of aspiration level on the EAAFs for GNSGC and other models.

In Fig. 11, we investigate the impact of population size N on the EAAF for GNSGC under the Aspiration–Fermi hybrid rule. As the population size continues to increase, the EAAF for GNSGC, GNSG, GSH, and TNSG generally shows a trend of first decreasing and then slowly increasing, but the EAAF for GNSGC is greater than that of GNSG, GSH. This once again indicates that cost sharing helpfully enhances cooperation.

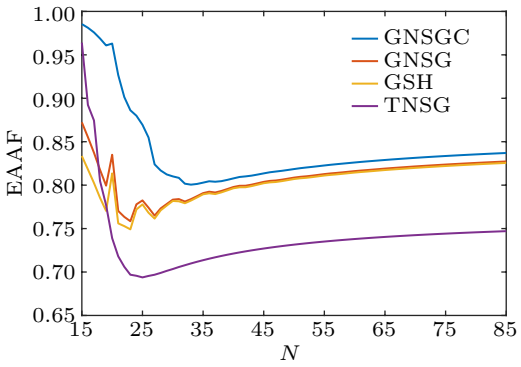


Fig. 11. The effect of population size on the EAAFs for GNSGC and other models.

3.2. Evolutionary analysis of unequal benefits of both cooperators and defectors

In this subsection, we study the impact of unequal benefits of both cooperators and defectors on the EAAF for GNSGC under Aspiration–Fermi hybrid rule in order to encourage more players to participate in cooperation. Unlike the Subsection 3.1, we increase the benefit $b_C = 11$ of cooperators without changing the benefit $b = 10$ of defectors. Other parameters are the same as in Subsection 3.1.

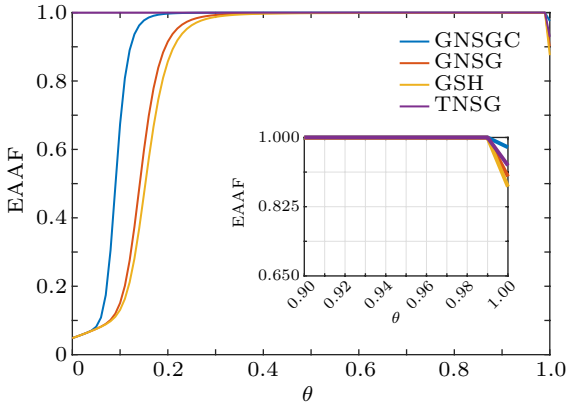


Fig. 12. The effect of Aspiration–Fermi hybrid rule on the EAAFs for GNSGC and other models under the benefit $b_C = 11$ of cooperators.

Firstly, in Fig. 12, we study the effect of Aspiration–Fermi hybrid rule on the EAAFs for GNSGC, GNSG, GSH, and TNSG under the benefit $b_C = 11$ of cooperators. By comparison with Fig. 1, it can be observed that increasing the benefit of cooperators while keeping the benefit of defectors unchanged will lead to extremely high the EAAFs for GNSGC, GNSG, GSH, and TNSG, thereby making cooperation more emergent. Obviously, for $\theta \in [0,1]$, the EAAF of GNSGC is greater than that of GNSG, GSH, and TNSG as shown in Fig. 1. This indicates that cost sharing helpfully enhances cooperation. Additionally, within the ranges of $\theta \in [0.2, 0.99]$ for GNSGC and $\theta \in [0.4, 0.99]$ for GNSG and GSH, the EAAF values reach their maximum which are conspicuously higher than the EAAFS at $\theta = 0$ (Fermi rule) and $\theta = 1$ (Aspiration rule). This once again suggests that the Aspiration–Fermi hybrid rule proposed by us exerts a positive effect on enhancing

the EAAF and can more effectively facilitate the generation of cooperation.

Secondly, to further explore the influence of cooperator benefits and defector benefits on EAAF, we conduct a sensitivity analysis on b_C and b. As shown in Fig. 13, for $b_C > b$, the EAAFs for GNSGC, GNSG, GSH, and TNSG are almost close to 1, are greater than the EAAFs for $b_C \leq b$. This once again indicates that only increasing the benefits of cooperators can better improve cooperation. In addition, the EAAF for GNSGC is greater than that of GNSG, GSH, and TNSG. This once again indicates that cost sharing helpfully enhances cooperation.

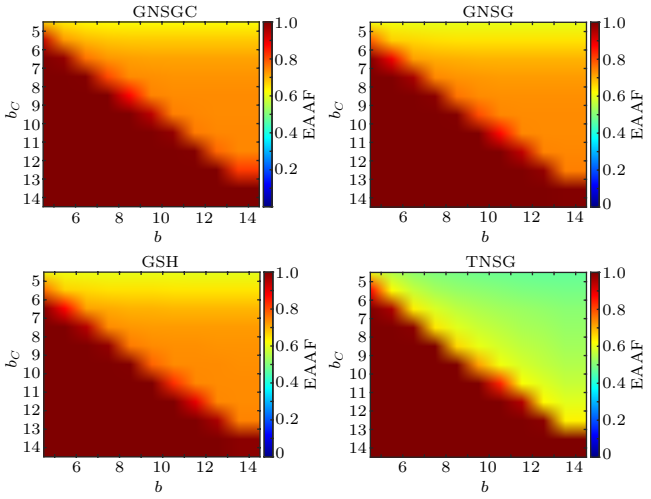


Fig. 13. The effects of both cooperator benefits and defector benefits on the EAAFs for GNSGC and other models.

4. Conclusion

In this paper, we initially construct a multi-player snowdrift evolutionary game model based on cost sharing. Building on GNSG, we stipulate that everyone must bear the cost of snow shoveling. Simultaneously, we propose an Aspiration–Fermi hybrid rule. Through the evolutionary process of Markov chains, we establish the EAAF for GNSGC under the Aspiration–Fermi hybrid rule. Subsequently, through numerical simulations, we first conduct numerical simulation analysis and comparison of different parameters in GNSGC, GNSG, GSH, and TNSG under the condition where the benefits of cooperators and defectors are the same. We deeply explore the influence of parameters such as update rules, selection intensity, noise parameters, thresholds, the number of participants, benefits, costs, time cost factors, aspiration levels, and population sizes on the EAAF. Finally, under the condition of unequal benefits of cooperators and defectors, we study the influence of update rules, cooperator benefits and defector benefits on the EAAFs for GNSGC, GNSG, GSH, and TNSG. The results are in line with the theory: The Aspiration–Fermi hybrid rule and cost sharing can increase the EAAF and thereby promote the improvement of cooperation levels. To a certain extent, selection intensity, noise parameters, thresholds, the number of participants, benefits, costs, time cost factors, and aspiration levels can also enhance cooperation. Moreover, only increasing the benefits of cooperators while keeping the benefits of defectors unchanged will directly make the average abundance reach 1, which can better promote cooperation. In short, the Aspiration–Fermi hybrid rule and cost sharing can facilitate the improvement of cooperation, and only increasing cooperation benefits will lead to a higher cooperation level. In conclusion, this paper introduces innovative Aspiration–Fermi hybrid rule and cost sharing into the multi-person snowdrift evolutionary game model. Through comprehensive parameter analysis and comparative analysis, our findings yield significant implications for understanding the evolution of cooperation and offer innovative approaches for enhancing cooperative behavior.

Appendix A: The proof for formula (13)

Proof The following equation can be obtained from the detailed balance formula:[54]

$$\begin{aligned}\vartheta_1 &= \frac{T_0^+}{T_1^-}\vartheta_0, \quad \vartheta_2 = \frac{T_1^+}{T_2^-}\vartheta_1 = \frac{T_0^+T_1^+}{T_1^-T_2^-}\vartheta_0,\\ \vartheta_3 &= \frac{T_2^+}{T_3^-}\vartheta_2 = \frac{T_0^+T_1^+T_2^+}{T_1^-T_2^-T_3^-}\vartheta_0, \quad \cdots.\end{aligned} \tag{A1}$$

Carry out an inductive analysis on formula (A1). When $N_j > 0$, there is

$$\vartheta_{N_j} = \frac{\prod_{N_i=0}^{N_j-1} T_{N_i}^+}{\prod_{N_i=0}^{N_j} T_{N_i}^-}\vartheta_0 = \prod_{N_i=0}^{N_j-1} h(N_i)\vartheta_0, \tag{A2}$$

where $h(N_i) = \frac{T_{N_i}^+}{T_{N_i+1}^-}$, $h(N_i)$ is a type-dominant function. When $h(N_i) > 1$, it indicates that type A is dominant; otherwise, type B is dominant. Therefore, $h(N_i)$ reflects the situation where type A is dominant after each game. To simplify, let $F(N_j) = \prod_{N_i=0}^{N_j-1} h(N_i)$. From the constraint condition $\sum_{N_j=0}^{N} \vartheta_{N_j} = 1$, it follows that

$$\sum_{N_j=0}^{N} \vartheta_{N_j} = \vartheta_0 + \sum_{N_j=0}^{N} F(N_j)\vartheta_0 = 1. \tag{A3}$$

It is solved from formula (A3) that $\vartheta_0 = \frac{1}{1+\sum_{N_j=0}^{N} F(N_j)}$, substitute it into formula (A2) to obtain that when $j > 1$

$$\vartheta_{N_j} = \frac{F(N_j)}{1+\sum_{N_j=0}^{N} F(N_j)}. \tag{A4}$$

By inserting formula (A4) into Eq. (12), it can be deduced that

$$\langle X_A(\theta,\lambda,\kappa)\rangle = \frac{1}{N}\sum_{N_j=1}^{N} N_j \frac{F(N_j)}{1+\sum_{N_j=0}^{N} F(N_j)}. \tag{A5}$$

Furthermore, it can be obtained that

$$\langle X_A(\theta,\lambda,\kappa)\rangle = \frac{1}{N}\frac{\sum_{N_j=1}^{N} N_j F(N_j)}{1+\sum_{N_j=0}^{N} F(N_j)} = \frac{1}{N}\frac{\sum_{N_j=1}^{N} N_j \prod_{N_i=0}^{N_j-1} h(N_i)}{1+\sum_{N_j=0}^{N} \prod_{N_i=0}^{N_j-1} h(N_i)}. \tag{A6}$$

References

[1] Milinski M 1987 *Nature* **325** 433
[2] Sinervo B and Clobert J 2003 *Science* **300** 1949
[3] Bshary R and Grutter A S 2006 *Nature* **441** 975
[4] Tilman A R, Plotkin J B and Akçay E 2020 *Nat. Commun.* **11** 915
[5] Wang C, Feng T J, Li C, *et al*. 2023 *Chaos, Solitons & Fractals* **168** 113141
[6] Fehr E and Fischbacher U 2003 *Nature* **425** 785
[7] Nowak M A 2006 *Science* **314** 1560
[8] Rand D G and Nowak M A 2013 *Trends Cogn. Sci.* **17** 413
[9] Apicella C L and Silk J B 2019 *Curr. Biol.* **29** R447
[10] Maynard Smith J 1982 *Evolution and the Theory of Games* (Cambridge: Cambridge University Press)
[11] Weibull J W 1995 *Evolutionary Game Theory* (Cambridge: MIT Press)
[12] Hofbauer J and Sigmund K 1998 *Evolutionary Games and Population Dynamics* (Cambridge: Cambridge University Press)
[13] Nowak M A 2006 *Evolutionary Dynamics* (Cambridge: Harvard University Press)
[14] Li A, Liu Y Y, Wang L, *et al*. 2020 *Nat. Commun.* **11** 2259
[15] Pi J X, Yang H and Yang G H 2022 *Physica A* **592** 126839
[16] Zhang Q C X, Shu L and Jiang B C 2023 *Appl. Math. Comput.* **446** 127875
[17] Tang W, Wang C, Pi J X, *et al*. 2024 *New J. Phys.* **26** 013017
[18] Wang C, Tang W and Pi J X 2024 *Chaos, Solitons & Fractals* **179** 114409
[19] Xu Y, Feng M L, Xia C Y, *et al*. 2022 *Physica A* **604** 127698
[20] Xu Y, Wang J, Xia C Y, *et al*. 2023 *Science China Information Sciences* **66** 222208
[21] Zheng D F, Yin H P, *et al*. 2007 *Europhys. Lett.* **80** 18002
[22] Souza M O, Pacheco J M and Santos F C 2009 *J. Theor. Biol.* **260** 581
[23] Xu M, Zheng D F, Xu C, *et al*. 2015 *Physica A* **424** 322
[24] Sui X, Cong R, Li K and Wang L 2015 *Phys. Lett. A* **379** 2922
[25] Zheng Y L and Zhang W G 2016 *J. Syst. Sci. Math. Sci.* **36** 1933 (in Chinese)
[26] Fang X and Chen X J 2021 *Eur. Phys. J. B* **94** 176
[27] Li M Y, Kang H W, Sun X P, *et al*. 2022 *Chaos, Solitons & Fractals* **164** 112747
[28] Pi J X, Yang G H, Yang H, *et al*. 2022 *Physica A* **604** 127927
[29] Ma X J, Quan J and Wang X J 2024 *Nonlinear Dyn.* **112** 709
[30] Pi J X, Yang G H, Wang C, *et al*. 2024 *Chaos, Solitons & Fractals* **180** 114593
[31] Ghang W and Nowak M A 2014 *J. Theor. Biol.* **360** 129
[32] Wang X J, Gu C L, Quan J, *et al*. 2019 *Chin. Phys. B* **28** 020203
[33] Tkadlec J, Pavlogiannis A, Nowak M A, *et al*. 2021 *Nat. Commun.* **12** 4009
[34] Gu C L, Wang X J, Ding R, *et al*. 2022 *Chaos, Solitons & Fractals* **164** 112658
[35] Wang X J and Xia K 2019 *Syst. Eng. Theory Pract.* **39** 1128 (in Chinese)
[36] Xia K 2021 *Chaos, Solitons & Fractals* **142** 110490
[37] Xia K 2021 *Appl. Math. Comput.* **392** 125733
[38] Kroumi D and Lessard S 2022 *J. Math. Biol.* **85** 27
[39] Debove S, Baumard N and André J B 2016 *Evol. Hum. Behav.* **37** 245
[40] Nowak M A, Sasaki A, Taylor C, *et al*. 2004 *Nature* **428** 646
[41] Lieberman E, Hauert C and Nowak M A 2005 *Nature* **433** 312
[42] Liu R R, Rong Z H, *et al*. 2010 *EPL* **91** 20002
[43] Nagashima K and Tanimoto J 2019 *Appl. Math. Comput.* **361** 661
[44] György S and Csaba T 1998 *Phys. Rev. E* **58** 69
[45] Traulsen A, Pacheco J M and Nowak M A 2007 *J. Theor. Biol.* **246** 522
[46] Liu Y, Chen X J, Wang L, *et al*. 2011 *Europhys. Lett.* **94** 60002
[47] Chen Y S, Yang H X and Guo W Z 2017 *Physica A* **469** 625
[48] Płatkowski T 2015 *Appl. Math. Comput.* **251** 46
[49] Wang X J and Liu J 2018 *Syst. Eng. Theory Pract.* **38** 1173 (in Chinese)
[50] Liu X S, He M F, Pan Q H, *et al*. 2016 *Phys. Rev. E* **94** 012124
[51] Liu X S, He M F, Pan Q H, *et al*. 2017 *Physica A* **484** 336
[52] Hong L J, Geng Y N, Shi L, *et al*. 2021 *Appl. Math. Comput.* **394** 125784
[53] Gao H Y, Wang J, Xia C Y, *et al*. 2021 *Appl. Math. Comput.* **410** 126433
[54] Pan F, Xi B and Wang L 2015 *Syst. Eng. Theory Pract.* **35** 1393 (in Chinese)