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Citation: Chin. Phys. B, 2026, 35 (**2**): 024204. DOI: 10.1088/1674-1056/adeb5f
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Decoherence and evolution of a general quadratic state for amplitude decay

Zhi-Long Wan(万志龙)[1,†], Hong-Chun Yuan(袁洪春)[1,2], Xiao-Lei Yin(尹晓蕾)[1], and Chang-Ying Wang(王昌英)[1]

[1] *School of Sciences, Changzhou Institute of Technology, Changzhou* 213032, *China*

[2] *School of Electrical and Information Engineering, Changzhou Institute of Technology, Changzhou* 213032, *China*

(Received 3 May 2025; revised manuscript received 24 June 2025; accepted manuscript online 3 July 2025)

Making full use of the operator ordering method and the integration within ordered products, we obtain the analytical evolution law of a general quadratic state in the amplitude decay channel, and find that it is determined not only by the decay rate of the amplitude decay channel but also by the coefficients of the initial quadratic state. Further, the quantum statistical properties of the initial quadratic state for amplitude decay are investigated via its average photon number and photon-counting distribution, and its Wigner distribution function evolution is discussed in detail.

Keywords: general quadratic state, amplitude decay channel, quantum statistical property, operator ordering, integration within ordered products

PACS: 03.65.Yz, 05.30.–d, 42.50.Dv

DOI: 10.1088/1674-1056/adeb5f **CSTR:** 32038.14.CPB.adeb5f

1. Introduction

Quantum decoherence is the consequence of quantum entanglement between a quantum system and its environment. It can also be understood as information leakage or quantum state collapse caused by the interaction between the quantum system and the external environment. This interaction may cause quantum systems that were originally in a superposition state to transform into classical mixed states, thereby losing the advantage of quantum coherence.

Recently, the decoherence effects in open-interaction systems have attracted significant attention. It is known that the evolution due to the interaction between a system and its environment can be described as the evolution from the initial density operator ρ_0 to the time-dependent density operator ρ_t. In other words, the decoherence evolution of any open system can be characterized by the master equation governing the operator ρ_t of the system. Usually, instead of solving these master equations, the evolution laws of the systems are investigated by solving the equation for the P-representation of the density operator or the Fock–Planck equation for the Wigner function.[1–3] Here, we can use entangled state representation and the operator-ordering method to directly solve the master equation describing the amplitude decay channel, which is of the form[4–6]

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = \kappa\left(2a\rho_t a^\dagger - a^\dagger a\rho_t - \rho_t a^\dagger a\right), \tag{1}$$

where κ represents the cavity decay rate, and obtain the analytical Kraus operator-sum solution of Eq. (1) as[7–9]

$$\rho_t = \sum_{l=0}^{\infty} \mathcal{M}_l \rho_0 \mathcal{M}_l^\dagger, \tag{2}$$

where M_l is the Kraus operator, i.e.,

$$\mathcal{M}_l = \sqrt{\frac{\mathcal{T}^l}{l!}}\mathrm{e}^{-\kappa t a^\dagger a} a^l \tag{3}$$

with $\mathcal{T} = 1 - \mathrm{e}^{-2\kappa t}$. One can check $\sum_{l=0}^{\infty}\mathcal{M}_l^\dagger\mathcal{M}_l = 1$, thus $\mathrm{tr}\,\rho_t = \mathrm{tr}\left(\sum_{l=0}^{\infty}\mathcal{M}_l^\dagger\mathcal{M}_l\rho_0\right) = \mathrm{tr}\,\rho_0 = 1$.

In this work, we focus on the decoherence and evolution of a system with a general quadratic normally ordered density operator in an amplitude decay channel. The system is described by the following density operator:[10]

$$\rho_0 = \mathrm{e}^{\delta} :\, \exp(\alpha a^{\dagger 2} + \beta a^\dagger a + \alpha^* a^2 + \gamma a^\dagger + \gamma^* a) :\,, \tag{4}$$

where the symbol : : denotes normal ordering, meaning that all creation operators are located to the left of all the annihilation operators in ρ_0. δ is determined by the other three parameters α, β and γ since we should let $\mathrm{tr}\,\rho_0 = 1$, and $a^\dagger$ and a are the photon creation and annihilation operators, respectively. It is worth noting that the density operator ρ_0 is general, since it can involve squeezing, displacement, degenerate parametric amplification, etc. For example, $\alpha = 0$, ρ_0 becomes $\mathrm{e}^{\delta}\mathrm{e}^{\gamma a^\dagger}\mathrm{e}^{a^\dagger a\ln(\beta+1)}\mathrm{e}^{\gamma^* a} \equiv \rho_{\mathrm{th}}$, corresponding to a mixture of thermal states, where we have used the relation $\exp(\upsilon a^\dagger a) =: \exp\left[(\mathrm{e}^{\upsilon} - 1)a^\dagger a\right] :$. For $\alpha = 0$ and $\beta = -1$, ρ_0 reduces to the coherent state $|\gamma\rangle\langle\gamma| \equiv \rho_{\mathrm{coh}}$. In addition, for $\gamma = 0$ and $\beta = -1$, ρ_0 becomes the squeezed state

†Corresponding author. E-mail: wanzl@czu.cn

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$\mathrm{e}^{\delta}\,\mathrm{e}^{\alpha a^{\dagger 2}}\,|0\rangle\langle 0|\,\mathrm{e}^{\alpha^* a^2} \equiv \rho_{\rm squ}$. On the other hand, the operator-ordering method offers a new way to investigate the decoherence evolution of the state ρ_0 for amplitude decay, which is also suitable for solving quantum decoherence problems in other systems. Since it is necessary to maintain good quantum coherence and control quantum decoherence to perform quantum operations, the research on decoherence evolution of a general quadratic state in the amplitude decay channel is more universal and meaningful for understanding decoherence theory in implementing quantum computers.

The paper is arranged as follows. In Section 2, we prove the normalization of a general quadratic state ρ_0. In Section 3, we obtain the analytical evolution law of the initial state ρ_0 in the amplitude decay channel. In Sections 4 and 5, we examine the evolution of quantum statistical properties of ρ_0 via its average photon number and photon counting distribution, and then discuss the Wigner distribution function evolution for amplitude decay. Our main results are presented in Section 6.

2. Normalization

Normalization ensures the correct description and calculation of quantum states, and thereby the correctness and reliability of quantum computing. The normalization of the general quadratic state ρ_0 is represented as $\mathrm{tr}\,\rho_0 = 1$, thus using Eq. (4) and the completeness relation of coherent states $|z\rangle$,[12–16] i.e.,

$$\int \frac{\mathrm{d}^2 z}{\pi}\,|z\rangle\langle z| = 1, \quad |z\rangle = \mathrm{e}^{-|z|^2/2+za^{\dagger}}\,|0\rangle, \tag{5}$$

as well as the integration within ordered products,[18–20] we obtain

$$\begin{aligned}\mathrm{tr}\rho_0 &= \mathrm{e}^{\delta}\int \frac{\mathrm{d}^2 z}{\pi}\exp(\beta\,|z|^2+\alpha z^{*2}+\alpha^* z^2+\gamma z^*+\gamma^* z)\\ &= \frac{\mathrm{e}^{\delta}}{\sqrt{\beta^2-4\,|\alpha|^2}}\mathrm{e}^{\frac{\alpha\gamma^{*2}-\beta|\gamma|^2+\alpha^*\gamma^2}{\beta^2-4|\alpha|^2}} = 1,\end{aligned} \tag{6}$$

where we have used the integral formula

$$\begin{aligned}&\int \frac{\mathrm{d}^2\alpha}{\pi}\mathrm{e}^{h|\alpha|^2+s\alpha+\eta\alpha^*+f\alpha^2+g\alpha^{*2}}\\ &= \frac{1}{\sqrt{h^2-4fg}}\exp\left(\frac{-hs\eta+s^2 g+\eta^2 f}{h^2-4fg}\right),\end{aligned} \tag{7}$$

which holds for the convergence condition $\mathrm{Re}(h\pm f\pm g)<0$. Therefore, the uniqueness of δ is

$$\delta = \frac{1}{2}\ln\left(\beta^2-4\,|\alpha|^2\right)-\frac{\alpha\gamma^{*2}-\beta\,|\gamma|^2+\alpha^*\gamma^2}{\beta^2-4\,|\alpha|^2}. \tag{8}$$

To ensure that this state ρ_0 is physically valid, the parameters α, β must meet the following conditions:

$$\beta<0, \quad \beta^2-4\,|\alpha|^2>0, \tag{9}$$

or equivalently,

$$\beta<0, \quad \beta+2\,|\alpha|<0. \tag{10}$$

3. Evolution of ρ_0

In this section, we investigate how the general quadratic state ρ_0 evolves when it undergoes an amplitude decay channel. According to the Kraus operator-sum form of ρ_t in Eq. (2), we find that the antinormal ordering of ρ_0 is convenient for obtaining its evolution for amplitude decay. So, using the formula[21]

$$\rho = \int \frac{\mathrm{d}^2 z}{\pi}\,\langle -z|\,\rho\,|z\rangle\,\vdots\,\mathrm{e}^{|z|^2+z^* a-za^{\dagger}+a^{\dagger}a}\,\vdots \tag{11}$$

that converts any boson operator ρ into its antinormal ordering to derive the antinormal-ordering product of ρ_0, that is

$$\begin{aligned}\rho &= \mathrm{e}^{\delta}\int \frac{\mathrm{d}^2 z}{\pi}\vdots\exp\Big[-(\beta+1)\,|z|^2+\left(\gamma^*-a^{\dagger}\right)z\\ &\quad -(\gamma-a)z^*+\alpha^* z^2+\alpha z^{*2}+a^{\dagger}a\Big]\vdots\\ &= \mathrm{e}^{\delta}\mathcal{G}^{1/2}\vdots\exp\left[-\mathcal{G}\,(\beta+1)\left(\gamma^*-a^{\dagger}\right)(\gamma-a)\right.\\ &\quad \left.+\alpha\mathcal{G}\left(\gamma^*-a^{\dagger}\right)^2+\alpha^*\mathcal{G}\,(\gamma-a)^2+a^{\dagger}a\right]\vdots,\end{aligned} \tag{12}$$

where the symbol $\vdots\ \vdots$ means antinormal ordering, which requires that all the annihilation operators are to the left of all the creation operators, and we have used the integration formula (7), and $G = \left[(\beta+1)^2-4\,|\alpha|^2\right]^{-1}$. Therefore, substituting Eq. (12) into Eq. (2) and using Eq. (5) lead to

$$\begin{aligned}\rho_t &= \mathrm{e}^{\delta}\mathcal{G}^{1/2}\int \frac{\mathrm{d}^2 z}{\pi}\mathrm{e}^{\mathcal{T}|z|^2}\mathrm{e}^{-\kappa t a^{\dagger}a}\,|z\rangle\langle z|\,\mathrm{e}^{-\kappa t a^{\dagger}a}\\ &\quad\times\exp\Big[-\mathcal{G}\,(\beta+1)\,|\gamma-z|^2\\ &\quad+\alpha\mathcal{G}\,(\gamma^*-z^*)^2+\alpha^*\mathcal{G}\,(\gamma-z)^2+|z|^2\Big]\\ &= \mathrm{e}^{\delta}\mathcal{G}^{1/2}\int \frac{\mathrm{d}^2 z}{\pi}\left|z\mathrm{e}^{-\kappa t}\right\rangle\left\langle z\mathrm{e}^{-\kappa t}\right|\\ &\quad\times\exp\Big[-\mathcal{G}\,(\beta+1)\,|\gamma-z|^2\\ &\quad+\alpha\mathcal{G}\,(\gamma^*-z^*)^2+\alpha^*\mathcal{G}\,(\gamma-z)^2+|z|^2\Big],\end{aligned} \tag{13}$$

where we have used the identity

$$\mathrm{e}^{-ra^{\dagger}a}\,|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}\left(1-\mathrm{e}^{-2r}\right)|\alpha|^2}\left|\alpha\mathrm{e}^{-r}\right\rangle.$$

Further, using the normal-ordering product, i.e.,

$$\left|z\mathrm{e}^{-\kappa t}\right\rangle\left\langle z\mathrm{e}^{-\kappa t}\right| =: \mathrm{e}^{-|z|^2\mathrm{e}^{-2\kappa t}+z\mathrm{e}^{-\kappa t}a^{\dagger}+z^*\mathrm{e}^{-\kappa t}a-a^{\dagger}a}:$$

and the integral formula (7), we thus obtain

$$\rho_t = \mathrm{e}^{\delta_t}:\ \exp(\alpha_t a^{\dagger 2}+\beta_t a^{\dagger}a+\alpha_t^* a^2+\gamma_t a^{\dagger}+\gamma_t^* a):\ , \tag{14}$$

where we have used the normal-ordering product of a vacuum, i.e.,

$$|0\rangle\langle 0| =:\ \mathrm{e}^{-a^{\dagger}a}:\ , \tag{15}$$

and the parameters α_t, β_t, γ_t and δ_t are, respectively,

$$\begin{aligned}
\alpha_t &= \alpha\omega \mathrm{e}^{-2\kappa t},\\
\beta_t &= \omega\left[(\beta+1)\left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}\right)\right.\\
&\quad \left.+4|\alpha|^2\mathcal{T}\right]\mathrm{e}^{-2\kappa t}-1,\\
\gamma_t &= \omega\left[\left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}\right)\gamma+2\alpha\gamma^*\mathcal{T}\right]\mathrm{e}^{-\kappa t},\\
\mathrm{e}^{\delta_t} &= \mathrm{e}^{\delta}\omega^{1/2}\exp\left\{\omega\mathcal{T}\left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}\right)|\gamma|^2\right.\\
&\quad \left.+\omega\mathcal{T}^2\left(\alpha\gamma^{*2}+\alpha^*\gamma^2\right)\right\},\\
\omega &= \left[\left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}\right)^2-4|\alpha|^2\mathcal{T}^2\right]^{-1}.
\end{aligned} \tag{16}$$

Equation (14) shows that the evolved density operator ρ_t still has the same form as the initial density operator ρ_0.

Obviously, in order to make $\rho(t)$ a valid state in physics, we also need to check if the output state ρ_t satisfies the condition $\beta_t+2|\alpha_t|<0$. Since $\mathcal{T}=1-\mathrm{e}^{-2\kappa t}$ increases monotonically with time t, $\lim\limits_{t\to 0}\mathcal{T}=0$ and $\lim\limits_{t\to+\infty}\mathcal{T}=1$, that is $0\leqslant\mathcal{T}\leqslant 1$. Further, using $\beta+2|\alpha|<0$, we can find the following inequality:

$$(2|\alpha|+\beta+1)\mathcal{T}<\mathcal{T}\leqslant 1, \tag{17}$$

or

$$\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}-2|\alpha|\mathcal{T}\geqslant 0, \tag{18}$$

we thus have

$$\begin{aligned}
\omega^{-1} &= \left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}\right)^2-4|\alpha|^2\mathcal{T}^2\\
&= \left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}-2|\alpha|\mathcal{T}\right)\\
&\quad\times\left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}+2|\alpha|\mathcal{T}\right)>0.
\end{aligned} \tag{19}$$

It then follows that

$$\begin{aligned}
\beta_t+2|\alpha_t| &= \omega\left[(\beta+1)\left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}\right)+4|\alpha|^2\mathcal{T}\right]\mathrm{e}^{-2\kappa t}\\
&\quad+2|\alpha|\omega\mathrm{e}^{-2\kappa t}-1\\
&= \omega(\beta+2|\alpha|)\left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}+2|\alpha|\mathcal{T}\right)<0.
\end{aligned} \tag{20}$$

Especially, for the case of $\alpha=0$, we therefore have

$$\begin{aligned}
&\alpha_t=0,\quad \beta_t=\omega(\beta+1)\mathrm{e}^{-2\kappa t}-1,\\
&\gamma_t=\omega\gamma\mathrm{e}^{-\kappa t},\quad \mathrm{e}^{\delta_t}=\omega^{1/2}\mathrm{e}^{\delta+\omega\mathcal{T}|\gamma|^2},\\
&\omega=\left(\mathrm{e}^{-2\kappa t}-\beta\mathcal{T}\right)^{-1},
\end{aligned} \tag{21}$$

thus ρ_t becomes the evolution of the state ρ_{th} for amplitude decay. Further, taking $\beta=-1$ in Eq. (21), thus $\alpha_t=0$, $\beta_t=-1$, $\gamma_t=\gamma\mathrm{e}^{-\kappa t}$ and $\mathrm{e}^{\delta_t}=\exp(\mathrm{e}^{-2\kappa t}|\gamma|^2)$, we thus give $\rho_t=|\gamma\mathrm{e}^{-\kappa t}\rangle\langle\gamma\mathrm{e}^{-\kappa t}|$, corresponding to the evolution of the state ρ_{coh} in the amplitude decay channel. For a squeezed coherent state

$$\begin{aligned}
s(r)|z\rangle &= \mathcal{A}^{1/2}{:}\,\mathrm{e}^{-\mathcal{B}a^{\dagger 2}+(\mathcal{A}-1)a^\dagger a+\mathcal{B}a^2}\,{:}\,|z\rangle\\
&= \mathcal{A}^{1/2}\mathrm{e}^{-\mathcal{B}a^{\dagger 2}+(\mathcal{A}-1)a^\dagger z+\mathcal{B}z^2}|\rangle\\
&= \mathcal{A}^{1/2}\mathrm{e}^{-|z|^2/2}\mathrm{e}^{-\mathcal{B}a^{\dagger 2}+\mathcal{A}a^\dagger z+\mathcal{B}z^2}|0\rangle,
\end{aligned} \tag{22}$$

where we have used the normal-ordering product of the single-mode squeezing operator, that is $s(r)=A^{1/2}{:}\,\mathrm{e}^{-\mathcal{B}a^{\dagger 2}+(\mathcal{A}-1)a^\dagger a+\mathcal{B}a^2}{:}$, $A=\mathrm{sech}r$ and $B=(\tanh r)/2$. Further, using Eq. (15), we therefore have

$$\begin{aligned}
\rho_0 &= \mathcal{A}\mathrm{e}^{-|z|^2}{:}\,\exp\left[-\mathcal{B}\left(a^{\dagger 2}+a^2\right)-a^\dagger a\right.\\
&\quad\left.+\mathcal{A}\left(a^\dagger z+az^*\right)+\mathcal{B}\left(z^2+z^{*2}\right)\right]{:}\,.
\end{aligned} \tag{23}$$

Comparing Eq. (23) with Eq. (4) leads to

$$\begin{aligned}
&\alpha=-\mathcal{B},\quad \beta=-1,\quad \gamma=\mathcal{A}z,\\
&\mathrm{e}^{\delta}=\mathcal{A}\mathrm{e}^{-|z|^2+\mathcal{B}\left(z^2+z^{*2}\right)},
\end{aligned} \tag{24}$$

thus using Eqs. (16) and (24), we easily obtain the evolution of the state $s(r)|z\rangle$ for amplitude decay. On the other hand, for $t\to\infty$, thus $\omega\to\left(\beta^2-4|\alpha|^2\right)^{-1}$, $\alpha_t\to 0$, $\beta_t\to-1$, $\gamma_t\to 0$ and $\mathrm{e}^{\delta_t}\to 1$, so $\rho_t\to{:}\,\mathrm{e}^{-a^\dagger a}{:}\,=|0\rangle\langle 0|$, that is, the state ρ_t completely loses its non-classicality after undergoing the amplitude decay channel, and thereby becomes a vacuum.

4. Quantum statistical properties of ρ_t

4.1. Average photon number

The average photon number reflects the energy density of a given light field and is an important parameter in many fields, such as quantum optics, quantum information and quantum computation. For the state ρ_t, its average photon number, i.e., $\langle n\rangle=\mathrm{tr}\left(\rho_t a^\dagger a\right)=\mathrm{tr}\left(a^\dagger\rho_t a\right)-1$, is given by

$$\begin{aligned}
\langle n\rangle &= \mathrm{e}^{\delta_t}\mathrm{tr}\left[{:}\,a^\dagger\exp(\alpha_t a^{\dagger 2}+\beta_t a^\dagger a\right.\\
&\quad\left.+\alpha_t^* a^2+\gamma_t a^\dagger+\gamma_t^* a)a{:}\,\right]-1.
\end{aligned} \tag{25}$$

Inserting Eq. (5) into Eq. (25) and using the integral formula (7), we have

$$\begin{aligned}
\langle n\rangle &= \mathrm{e}^{\delta_t}\partial_s^1\partial_{s'}^1\int\frac{\mathrm{d}^2z}{\pi}\exp(\alpha_t z^{*2}+\beta_t|z|^2+\alpha_t^* z^2\\
&\quad+\left(\gamma_t+s'\right)z^*+\left(\gamma_t^*+s\right)z)\big|_{s=s'=0}-1\\
&= \frac{\mathrm{e}^{\delta_t}}{\sqrt{\beta_t^2-4|\alpha_t|^2}}\partial_s^1\partial_{s'}^1\exp\left(\frac{-\beta_t\left(\gamma_t^*+s\right)\left(\gamma_t+s'\right)}{\beta_t^2-4|\alpha_t|^2}\right.\\
&\quad\left.+\frac{\left(\gamma_t^*+s\right)^2\alpha_t+\left(\gamma_t+s'\right)^2\alpha_t^*}{\beta_t^2-4|\alpha_t|^2}\right)\Bigg|_{s=s'=0}-1.
\end{aligned} \tag{26}$$

Further, using a new two-index, three-variable special polynomial[22]

$$\begin{aligned}
&\mathcal{F}_{n,m}(x,y;\theta)\\
&=\partial_s^n\partial_\tau^m\exp\left(-s^2-\tau^2+\theta s\tau+sx+\tau y\right)\big|_{s=\tau=0}\\
&=\sum_{l=0}^{\min(n,m)}\binom{n}{l}\binom{m}{l}l!\theta^l\mathrm{H}_{n-l}\left(\frac{x}{2}\right)\mathrm{H}_{m-l}\left(\frac{y}{2}\right)
\end{aligned} \tag{27}$$

with the generating function $\mathrm{e}^{-s^2-\tau^2+\theta s\tau+sx+\tau y}$, we thus obtain

$$\langle n\rangle = \frac{|\alpha_t|\,\mathrm{e}^{\delta_t+\mathcal{A}_0}}{\left(\beta_t^2-4|\alpha_t|^2\right)^{3/2}}\mathcal{F}_{1,1}\left(\mathcal{A}_1,\mathcal{A}_1^*;-\frac{\beta_t}{|\alpha_t|}\right)-1, \quad (28)$$

where

$$\begin{aligned}\mathcal{A}_0 &= \frac{\alpha_t\gamma_t^{*2}-\beta_t|\gamma_t|^2+\alpha_t^*\gamma_t^2}{\beta_t^2-4|\alpha_t|^2},\\ \mathcal{A}_1 &= \frac{2\alpha_t\gamma_t^*-\beta_t\gamma_t}{\mathrm{i}\sqrt{\alpha_t/\left(\beta_t^2-4|\alpha_t|^2\right)}}.\end{aligned} \quad (29)$$

Obviously, the average photon number of ρ_t is exactly related to a $(1,1)$-order special polynomial.

In Fig. 1, we present the average photon number $\langle n\rangle$ of ρ_t as a function of time t for different values of α, β, γ and κ. Clearly, with the increase of t, $\langle n\rangle$ always decreases gradually for any values of α, β, κ and smaller γ and, especially, first decreases and then increases for larger γ. As κ increases, $\langle n\rangle$ always decreases for small time t and goes to a constant for sufficiently large t. In addition, $\langle n\rangle$ always increases with increasing α or γ for all time t, however it increases for smaller time t and increases for larger t with increasing β.

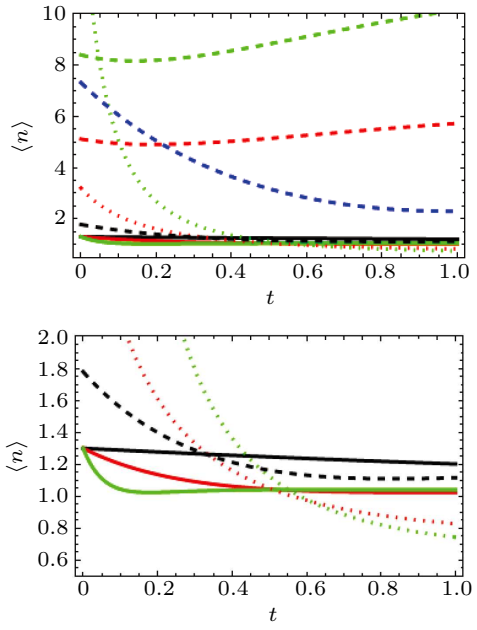


Fig. 1. The evolution of the average photon number of a general quadratic state for amplitude decay as a function of time t for different values of α, β, γ and κ, where the values of $(\alpha,\ \beta,\ \gamma,\ \kappa)$ are, respectively, $(0.2,-1,0.5,0.1)$ (thick black), $(0.2,-1,0.5,1)$ (thick red), $(0.2,-1,0.5,5)$ (thick green), $(0.2,-1,1,1)$ (dashed thick red), $(0.2,-1,1.2,1)$ (dashed thick green), $(0.2,-1.5,0.5,1)$ (dotted thick red), $(0.2,-2,0.5,1)$ (dotted thick green), $(0.3,-1,0.5,1)$ (dashed thick black), $(0.45,-1,0.5,1)$ (dashed thick blue).

4.2. Photon-counting distribution

The photon-counting distribution refers to the distribution of the number of photons measured by photon counting technology at different times or under different conditions.[23] This technology plays an important role in many fields, including scientific research, medical imaging, security detection, etc., due to its high time resolution and accuracy. For a given quantum state ρ, the formula of calculating its photon-counting distribution is defined as[24]

$$P(m,T) = \left(-\varsigma^{-2}\xi\right)^m\int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{-\xi|z|^2}\mathrm{L}_m\left(|z|^2\right)\langle\varsigma z|\,\rho\,|\varsigma z\rangle, \quad (30)$$

where $|\varsigma z\rangle$ is the coherent state, $\varsigma=\sqrt{1-\xi}$, and ξ is the quantum-detection efficiency within the time interval T, $\mathrm{L}_m(\cdot)$ is m-order Laguerre polynomial. Using Eqs. (14) and (30), we obtain the photon-counting distribution of ρ_t as

$$\begin{aligned}P(m,T) &= \mathrm{e}^{\delta_t}\left(-\varsigma^{-2}\xi\right)^m\int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{-\xi|z|^2}\\ &\quad\times\mathrm{L}_m\left(|z|^2\right)\exp(\alpha_t\varsigma^2z^{*2}+\beta_t\varsigma^2|z|^2\\ &\quad+\alpha_t^*\varsigma^2z^2+\gamma_t\varsigma z^*+\gamma_t^*\varsigma z)\\ &= \frac{\mathrm{e}^{\delta_t}\left(\varsigma^{-2}\xi\right)^m}{m!}\int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{-\xi|z|^2}\\ &\quad\times\mathrm{H}_{m,m}(z,z^*)\exp(\alpha_t\varsigma^2z^{*2}+\beta_t\varsigma^2|z|^2\\ &\quad+\alpha_t^*\varsigma^2z^2+\gamma_t\varsigma z^*+\gamma_t^*\varsigma z),\end{aligned} \quad (31)$$

where we have used the relation between the two-variable Hermite polynomial and the Laguerre polynomial, i.e., $\mathrm{L}_m\left(|z|^2\right)=\frac{(-1)^m}{m!}\mathrm{H}_{m,m}(z,z^*)$. Further, using the differential form of a two-variable Hermite polynomial, that is $\mathrm{H}_{n,m}(x,y)=\partial_s^n\partial_{s'}^m\exp(-ss'+sx+s'y)|_{s=s'=0}$ and the integral formula (7), as well as the new special polynomial (27), we therefore obtain

$$\begin{aligned}P(m,T) &= \frac{\varkappa^{1/2}\mathrm{e}^{\delta_t}\left(\varsigma^{-2}\xi\right)^m}{m!}\partial_s^m\partial_{s'}^m\mathrm{e}^{-ss'}\\ &\quad\times\exp\left[\varkappa\left(\xi-\beta_t\varsigma^2\right)\left(\gamma_t^*\varsigma+s\right)\left(\gamma_t\varsigma+s'\right)\right.\\ &\quad\left.+\varkappa\left(\gamma_t^*\varsigma+s\right)^2\alpha_t\varsigma^2+\varkappa\left(\gamma_t\varsigma+s'\right)^2\alpha_t^*\varsigma^2\right]\big|_{s=s'=0}\\ &= \frac{\varkappa^{1/2}\mathrm{e}^{\delta_t+\sigma}\left(\varsigma^{-2}\xi\right)^m}{m!}\\ &\quad\times\partial_s^m\partial_{s'}^m\mathrm{e}^{\mathcal{A}_2s^2+\mathcal{A}_2^*s'^2+\mathcal{A}_3ss'+\mathcal{A}_4s+\mathcal{A}_4^*s'}\big|_{s=s'=0}\\ &= \frac{\varkappa^{1/2}\mathrm{e}^{\delta_t+\sigma}\left(\varsigma^{-2}\xi\,|\mathcal{A}_2|\right)^m}{m!}\\ &\quad\times\mathcal{F}_{m,m}\left(\frac{\mathcal{A}_4}{\mathrm{i}\sqrt{\mathcal{A}_2}},\frac{\mathcal{A}_4^*}{-\mathrm{i}\sqrt{\mathcal{A}_2^*}};\frac{\mathcal{A}_3}{|\mathcal{A}_2|}\right),\end{aligned} \quad (32)$$

where

$$\begin{aligned}\mathcal{A}_2 &= \varkappa\varsigma^2\alpha_t,\\ \mathcal{A}_3 &= \varkappa\left(\xi-\beta_t\varsigma^2\right)-1,\\ \mathcal{A}_4 &= \varkappa\varsigma\left(\xi-\beta_t\varsigma^2\right)\gamma_t+2\varkappa\varsigma^3\alpha_t\gamma_t^*,\\ \sigma &= \varkappa\varsigma^2\left(\xi-\beta_t\varsigma^2\right)|\gamma_t|^2+\varkappa\varsigma^4\left(\alpha_t\gamma_t^{*2}+\alpha_t^*\gamma_t^2\right),\\ \varkappa &= \left[\left(\xi-\beta_t\varsigma^2\right)^2-4\varsigma^4|\alpha_t|^2\right]^{-1}.\end{aligned} \quad (33)$$

Therefore, the photon-counting distribution $P(m,T)$ is also expressed in the form of new three-variable special polynomial

$F_{n,m}(x,y;\theta)$. Especially, for $\xi=1$, the photon-counting distribution $P(m,T)$ reduces to the familiar photon number distribution, that is

$$P(m)=\frac{\mathrm{e}^{\delta_t}|\alpha_t|^m}{m!}\mathcal{F}_{m,m}\left(\frac{\gamma_t^*}{\mathrm{i}\sqrt{\alpha_t^*}},\frac{\gamma_t}{-\mathrm{i}\sqrt{\alpha_t}};\frac{\beta_t+1}{|\alpha_t|}\right). \quad (34)$$

Figure 2 shows the photon-counting distribution $P(m,T)$ of ρ_t as a function of m for different values of α, β, γ, κ, t and ξ. It is clear from Fig. 2 that, with increasing α, the distribution $P(m,T)$ obviously exhibits even–odd oscillation as an indicator of the non-classicality of ρ_t because α plays a role as a squeezing parameter, and as β or γ increases, the distribution $P(m,T)$ appears in more photon number states because β is a parameter describing thermal field and γ is a displaced parameter, and the maximum probability of finding photons decreases.

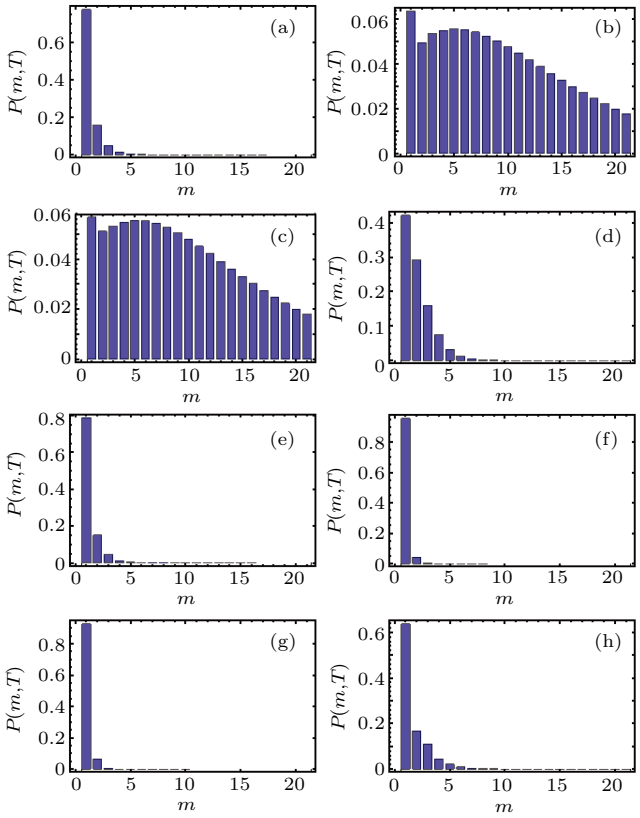


Fig. 2. The evolution of the photon-counting distribution of a general quadratic state for amplitude damping as a function of m for different values of α, β, γ, κ, t and ξ, where the values of (α, β, γ, κ, t, ξ) are, respectively, (a) $(0.2,-1,0.5,0.5,0.1,0.4)$, (b) $(0.45,-1,0.5,0.5,0.1,0.4)$, (c) $(0.2,-0.5,0.5,0.5,0.1,0.4)$, (d) $(0.2,-1,1,0.5,0.1,0.4)$, (e) $(0.2,-1,0.5,1,0.1,0.4)$, (f) $(0.2,-1,0.5,0.5,2,0.4)$, (g) $(0.2,-1,0.5,0.5,0.1,0.1)$, (h) $(0.2,-1,0.5,0.5,0.1,0.8)$.

However, with the increase of κ or t, the distribution $P(m,T)$ only appears in several smaller photon number states and is almost zero for others, and the probability of finding single-photon number state is maximum. Besides, with the increase of the detection efficiency ξ, the distribution $P(m,T)$ appears in more photon number states, but the impact of ξ on the probabilities of finding any m photons is not very significant.

5. Wigner distribution function for ρ_t

The Wigner distribution function for a quantum state, like its density matrix, contains its key information, such as probability distribution and phase, so studying the Wigner distribution function can more quickly and effectively obtain important information about the state in its evolution process. Using the coherent state representation of the Wigner operator, that is $\Delta(z)=\pi^{-2}\mathrm{e}^{|z|^2}\int\mathrm{d}^2z'\,|z'\rangle\langle -z'|\mathrm{e}^{2(zz'^*-z^*z')}$ [25–29] and Eq. (14), thus the Wigner distribution function $w(z,t)=\mathrm{tr}\,(\Delta(z)\rho_t)$ for the state ρ_t is obtained as

$$\begin{aligned}w(z,t)&=\mathrm{e}^{|z|^2+\delta_t}\int\frac{\mathrm{d}^2z'}{\pi^2}\exp\left[-(\beta_t+2)|z'|^2\right.\\&\quad\left.+(\gamma_t^*-2z^*)z'-(\gamma_t-2z)z'^*+\alpha_t^*z'^2+\alpha_tz'^{*2}\right]\\&=\frac{\mathcal{K}^{1/2}\mathrm{e}^{\delta_t}}{\pi}\mathrm{e}^{|z|^2}\exp\left[-(\beta_t+2)\mathcal{K}|\gamma_t-2z|^2\right.\\&\quad\left.+\alpha_t\mathcal{K}(\gamma_t^*-2z^*)^2+\alpha_t^*\mathcal{K}(\gamma_t-2z)^2\right]\\&=\frac{\mathcal{K}^{1/2}\mathrm{e}^{\mathcal{B}_0+\delta_t}}{\pi}\mathrm{e}^{\mathcal{B}_1|z|^2+\mathcal{B}_2z+\mathcal{B}_2^*z^*+\mathcal{B}_3z^2+\mathcal{B}_3^*z^{*2}},\end{aligned} \quad (35)$$

where we have used the integral formula (7), and

$$\begin{aligned}\mathcal{B}_0&=\mathcal{K}\alpha_t^*\gamma_t^2-\mathcal{K}(\beta_t+2)|\gamma_t|^2+\mathcal{K}\alpha_t\gamma_t^{*2},\\\mathcal{B}_1&=1-4\mathcal{K}(\beta_t+2),\\\mathcal{B}_2&=2\mathcal{K}[(\beta_t+2)\gamma_t^*-2\gamma_t\alpha_t^*],\\\mathcal{B}_3&=4\mathcal{K}\alpha_t^*,\quad\mathcal{K}=\left[(\beta_t+2)^2-4|\alpha_t|^2\right]^{-1}.\end{aligned} \quad (36)$$

Obviously, the Wigner distribution function $w(z,t)$ for the state ρ_t always remains Gaussian in the amplitude decay channel. Especially, substituting Eq. (21) into Eqs. (35) and (36), $w(z,t)$ becomes the Wigner distribution function evolution for the mixture of thermal states ρ_{th} in the amplitude decay channel; however, using Eqs. (16), (24) and (36), we easily obtain the evolved function $w(z,t)$ for the state $s(r)|z\rangle$ for amplitude decay.

Figure 3 shows the Wigner distribution function for the state ρ_t in the amplitude decay channel. With increasing α or β, the function $w(z,t)$ exhibits significantly greater squeezing and its peak value gradually becomes smaller; however, $w(z,t)$ shows obvious displacement in the case of constant peak value with the increase of γ. As the parameters κ or t increases, the initial state ρ_0 gradually loses its coherence and finally becomes vacuum owing to the evolution of the function $w(z,t)$ for amplitude decay in the phase space.

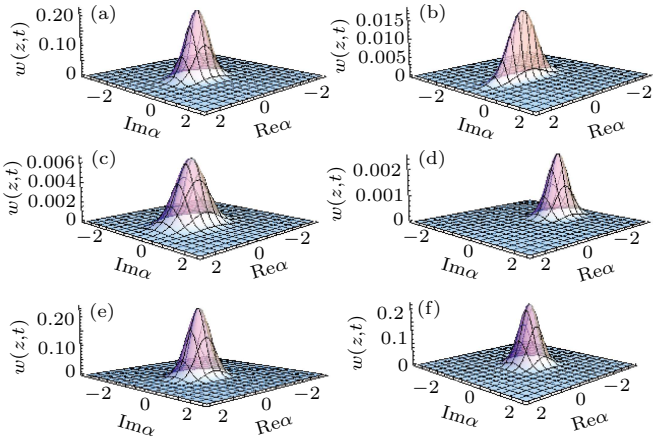


Fig. 3. The Wigner distribution function evolution of a general quadratic state in the amplitude decay channel for different values of α, β, γ, κ and t, where the values of $(\alpha,\ \beta,\ \gamma,\ \kappa,\ t)$ in (a), (b), (c), (d), (e), (f) are, respectively, $(0.2,-1,0.5,0.5,0.1)$, $(0.45,-1,0.5,0.5,0.1)$, $(0.2,-0.5,0.5,0.5,0.1)$, $(0.2,-1,2,0.5,0.1)$, $(0.2,-1,0.5,3,0.1)$, $(0.2,-1,0.5,0.5,2)$.

6. Conclusions

In summary, using the integration within ordered products method, we obtain the normalization of the general quadratic state ρ_0 and find that the conditions $\beta<0$ and $\beta+2|\alpha|<0$ can ensure that the state ρ_0 is valid in physics. According to the Kraus operator-sum form of the time-dependent density operator for amplitude decay, we obtain the analytical evolution law of the initial density operator ρ_0, which is just the same form as the initial operator ρ_0. In addition, the analytical results show that the evolved density operator ρ_t is determined by the decay factor κ of the amplitude decay channel and the coefficients α, β and γ of the initial quadratic state ρ_0, and the evolved operator ρ_t for amplitude decay is always valid in physics since we can also prove the conditions $\beta_t<0$ and $\beta_t+2|\alpha_t|<0$. As special cases, we obtain the evolutions of the mixture of thermal states, the coherent states and the squeezed coherent states. For a long time ($t\to\infty$), the initial quadratic state ρ_0 reduces to vacuum as a result of amplitude decay.

Additionally, the quantum statistical properties of the state ρ_0 in the amplitude decay channel are discussed, and the analytical results show that the average photon number and photon counting distribution are always related to the new three-variable special polynomials, and the evolved Wigner distribution function always remains Gaussian. Numerically speaking, the larger α or β may lead to the stronger nonclassicality for the state ρ_0 and, with increasing the parameters κ or t, the state ρ_0 undergoes decoherence in the amplitude decay channel, as shown in the above analytical results.

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