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Citation: Chin. Phys. B, 2025, 34 (**10**): 100202. DOI: 10.1088/1674-1056/ade666
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Generic stability of cooperative equilibria for multi-leader–follower-population mixed games

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(Received 1 April 2025; revised manuscript received 6 June 2025; accepted manuscript online 20 June 2025)

This paper proposes a mixed game with finitely many leaders and follower populations. In such a game, two types of equilibria are defined. First, a Nash equilibrium is introduced for the scenario in which leaders and follower populations are in perfect competition, each maximizing its own payoff. Second, a cooperative equilibrium is proposed for the case where leaders and follower populations, respectively, form coalitions and cooperate. Moreover, the existence of both Nash and cooperative equilibria is proved under the condition that the payoff functions are continuous and quasi-concave. Finally, we demonstrate the generic stability of cooperative equilibria in mixed games. More concretely, in the sense of Baire category, the cooperative equilibria in most mixed games are stable under perturbations of the payoff functions. In short, this paper presents two main contributions. On the one hand, we provide a novel mixed-game framework, which differs from both classical leader–follower games and leader–follower population games. On the other hand, the Nash and cooperative equilibria in our mixed games are distinct from those in existing leader–follower population games. The results are further illustrated with examples.

Keywords: mixed game, Nash equilibirum, cooperative equilibrium, existence, generic stability

PACS: 02.50.Le, 02.30.–f

DOI: 10.1088/1674-1056/ade666 **CSTR:** 32038.14.CPB.ade666

1. Introduction

The leader–follower game was initially proposed by Stackelberg[1] and has been extensively employed to model practical problems across various disciplines. For example, in the field of quantum game theory, Refs. [2,3] developed a quantum leader–follower game model to explore how quantum entanglement affects the decision-making process between leaders and followers. In power system research, Ref. [4] used a leader–follower game model to effectively address the technical problem of integrating new energy into hybrid alternating current/direct current (AC/DC) power grids. In complex networks, Ref. [5] applied the leader–follower game model to investigate network synchronization issues involving different types of network nodes. It is evident that leader–follower games have extensive applications in interdisciplinary areas of physics.

In classical leader–follower games, both types of decision-makers consist of a finite number of individuals. However, our research has found that, in many practical decision-making scenarios, a large number of followers form distinct populations due to their differing goals. Therefore, more attention should be paid to population-based decisions. For example, in Internet pricing[6], network users are divided into k follower populations with a large number of individuals according to the k links in the network. The payoff of each user depends not only on their own network usage behavior but also on the behaviors of other users in the network. In transportation, leaders are carrier companies allocating public transport routes, and follower populations represent transport flows of different types (e.g., individual or freight transport) choosing among city routes to minimize travel delays. Thus, a novel game model with finite leaders and follower populations is urgently needed. To the best of our knowledge, such a mixed game has not yet been proposed.

For classical leader–follower games, a Nash equilibrium characterizes the behavior of non-cooperative leaders and followers. Since the introduction of the leader–follower game model, scholars have conducted in-depth research on both the model and its equilibria. Başar and Olsder[7] defined multi-leader–follower games (MLFG) and studied the existence of their equilibria. Pang and Fukushima[8] analyzed MLFG via variational inequalities. Yu and Wang[9] presented an existence theorem of Nash equilibria for MLFG in infinite-dimensional spaces. Ding[10] introduced a more general MLFG and provided existence results in non-compact FC spaces. Jia *et al.*[11] proposed weakly Pareto–Nash equilibria for generalized MLFG and proved their existence and stability. In addition, uncertain leader–follower games have been widely investigated. Mallozzi *et al.*[12] studied the Nash equilibria of MLFG under aggregative uncertainty. Liu *et*

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al.[13,14] proposed single-leader–follower and multi-leader–follower games with fuzzy goals involving fuzzy parameters. They introduced the α-FNS equilibria and proved their existence. Wang *et al.*[15] defined robust weak Nash equilibria for multi-objective leader–follower games with uncertainty and proved their existence.

Population games were proposed by Sandholm[16], derived from the "mass action" interpretation of mixed strategies by Nash[17]. These games study strategic interactions among populations with a large number of anonymous agents. Compared with classical non-cooperative games with finitely many players that focus on individual payoffs, population games emphasize how many agents choose a pure strategy and the corresponding payoffs. Since their introduction, population games have been extensively studied[18–20]. From a population perspective, all leaders and followers can be viewed as a leader population and a follower population. Zhao[20] constructed leader–follower population games (LFPG) and proved the existence of Nash equilibria for such games.

Cooperation is a universal phenomenon in human society. Research on cooperative equilibria and cooperative mechanisms has become a cross-disciplinary hotspot spanning biology, physics, and mathematics. From a dynamical perspective, Xia *et al.*[21] investigated the evolution of cooperation under various reciprocity and reputation mechanisms using system modeling at the interface of physics and evolutionary game theory. Zhu *et al.*[22] studied the best-response dynamics of agents engaged in repeated multi-player games, where mutual cooperation represents the global optimum. Wang *et al.*[23] examined the effect of consistency on the evolution of cooperation in network games using a sampling learning approach. In Ref. [24], Wang *et al.* showed that conformity can promote cooperation in complex coordination games. From the perspective of static games, in normal-form games, a cooperative equilibrium means that agents cooperate through coalition formation, initially referred to as the α-core by Aumann.[25] Subsequently, Scarf[26] proved the existence of the α-core under the continuity and quasi-concavity of payoff functions. Ichiishi[27] extended Scarf's result to social coalitional equilibria, while Kajii[28] examined games with non-ordered preferences and established the existence of a generalized α-core. By treating cooperative players as leaders and non-cooperative players as followers in an n-person normal-form game, Mallozzi *et al.*[29] constructed leader–follower games. They studied the existence of partial cooperative equilibria and applied them to public goods problems. Subsequently, many scholars[30–33] further analyzed partial cooperative equilibria in non-cooperative games by constructing leader–follower frameworks and applying them to practical issues such as international pollution abatement and strategic network formation. Regarding cooperation among leaders and followers in MLFG, Yang and Ju[34] proposed the concept of cooperative equilibria and proved their existence and generic stability, showing that cooperative equilibria differ from non-cooperative ones. Focusing on cooperation among populations, Yang and Zhang[35] studied the cooperative equilibria of population games and proved their existence and essential stability, again showing the distinction between cooperative and non-cooperative equilibria. Zhang[36] introduced cooperative equilibria for multi-leader–multi-follower population games (MLFPG) and proved their existence. Chen *et al.*[37] introduced weakly and strongly cooperative equilibria for multi-objective population games and provided corresponding existence and stability results. They also analyzed the distinction between weakly cooperative and weakly non-cooperative equilibria. The concept of the α-core for multi-leader–follower games with set-valued payoffs was first introduced by Chen *et al.*[38], who also examined its existence and stability. Wu *et al.*[39] proposed cooperative equilibria for LFPG with one leader and finitely many follower populations, proving their existence and stability.

On the other hand, the stability of equilibria represents another key research focus. Considering perturbations of payoff functions in finite games, Wu and Jiang[40] introduced the notions of essential equilibria and essential games. Building on this result, many scholars have obtained numerous related findings.[11,18,19,34–36,39,41,42]

Inspired by Refs. [20,34–36], this paper first constructs a mixed leader–follower game with finitely many leaders and follower population games (LFMG). Then, a Nash equilibrium and a cooperative equilibrium are proposed for such LFMG, respectively. They differ from the Nash equilibria in Ref. [20] and the cooperative equilibria in Ref. [36], as illustrated by examples 1 and 2 in Section 3. The existence of these two types of equilibria constitutes the core of our work, while example 3 highlights the differences between the Nash and cooperative equilibria of LFMG. Finally, under perturbations of the payoff functions, the generic stability of the cooperative equilibrium is analyzed and established.

Additionally, our work differs from Ref. [36] in two main aspects. First, our model differs in structure: in our framework, the leaders play an n-player non-cooperative game, whereas in Ref. [36], the leaders form a population game. Second, we investigate the stability of cooperative equilibria, which Ref. [36] did not consider. The remainder of this paper is organized as follows. Section 2 introduces the preliminaries. Section 3 presents the LFMG model and defines the Nash and cooperative equilibria. Section 4 establishes the existence of these two equilibria. Section 5 focuses on the generic stability of the cooperative equilibrium. Section 6 concludes the paper.

2. Preliminaries

Definition 1[43] Let X and Y be two topological spaces, and let $F: X \to 2^Y$ be a correspondence.

(i) F is upper semi-continuous at $x \in X$ if, for any neighborhood $U(F(x))$ of $F(x)$, there exists a neighborhood $O(x)$ of x such that for all $x' \in O(x)$

$$F(x') \subset U(F(x)).$$

(ii) F is lower semi-continuous at $x \in X$ if, for any $y \in F(x)$ and any neighborhood $U(y)$ of y, there exists a neighborhood $O(x)$ of x such that for all $x' \in O(x)$

$$F(x') \cap U(y) \neq \emptyset.$$

(iii) F is continuous at $x \in X$ if it is both upper semi-continuous and lower semi-continuous.

Definition 2[44] Let X and Y be two topological spaces. A correspondence $F: X \to 2^Y$ is closed (or said to have a closed graph) if its graph

$$\text{graph}(F) = \{(x,y) \in X \times Y : y \in F(x)\},$$

is a closed subset of $X \times Y$.

Definition 3[44] (i) If $\text{cl}(Q) = \mathcal{M}$, then the subset Q of $\mathcal{M}$ is said to be dense.

(ii) If $Q \subset \mathcal{M}$ contains the intersection of a dense open set in $\mathcal{M}$, then Q is said to be a residual set in $\mathcal{M}$.

Next, we introduce several important theorems relevant to this study.

Lemma 1[44] (Berge's maximum theorem) Let X and Y be two topological spaces. Suppose that $F: X \to 2^Y$ is continuous with nonempty compact values, and let $f: X \times Y \to \mathbb{R}$ be continuous. Define $m: X \to \mathbb{R}$ by

$$m(x) = \sup_{y \in F(x)} f(x,y),$$

and define a correspondence $\mu: X \to 2^Y$ by

$$\mu(x) = \{y \in F(x) : f(x,y) = m(x)\}.$$

Then:

(i) The value function m is continuous;

(ii) The correspondence μ has nonempty compact values;

(iii) The correspondence μ is upper semi-continuous.

Lemma 2[44] (closed graph theorem) Let X and Y be two topological spaces. A correspondence $F: X \to 2^Y$ is closed if and only if it is upper semi-continuous and closed-valued.

Lemma 3[45] (Kakutani's fixed point theorem) Let X be a nonempty compact and convex subset of $\mathbb{R}^n$, and let $F: X \to 2^X$ be an upper semi-continuous correspondence with nonempty closed convex values. Then there exists $x^* \in X$ such that

$$x^* \in F(x^*).$$

Lemma 4[46] (Fort's theorem) Suppose that X is a complete metric space and Y is a topological space. If $F: X \to Y$ is a compact-valued and upper semi-continuous correspondence with $F(x) \neq \emptyset$ for all $x \in X$, then there exists a dense residual subset Q of X such that F is lower semi-continuous at every point in Q.

3. Nash equilibria and cooperative equilibria of LFMG

Consider the model of a LFMG, denoted by

$$\{\mathcal{N}, \mathcal{P}, \{X_i\}_{i\in\mathcal{N}}, \{Y^p\}_{p\in\mathcal{P}}, \{f_i\}_{i\in\mathcal{N}}, \{G^p\}_{p\in\mathcal{P}}\},$$

where $\mathcal{N} = \{1,2,\ldots,N\}$ denotes the set of leaders. For each $i \in \mathcal{N}$, the strategy set of leader i is $X_i \subseteq \mathbb{R}^{\ell_i}$, where ℓ_i is a positive integer. The joint strategy set of all leaders is $X = \prod_{i\in\mathcal{N}} X_i$, and for any given $i \in \mathcal{N}$, the strategy set of all other leaders is $X_{-i} = \prod_{t\in\mathcal{N}\setminus\{i\}} X_t$. $\mathcal{P} = \{1,2,\ldots,P\}$ denotes the set of follower populations. For each $p \in \mathcal{P}$, population p consists of a continuum of agents with total mass $m^p > 0$. Each agent in population p chooses a pure strategy from the finite set $S^p = \{1,2,\ldots,k^p\}$. The population state set of p is given by

$$Y^p = \left\{ y^p = (y_1^p, y_2^p, \ldots, y_{k^p}^p) \in \mathbb{R}_+^{k^p} \;\middle|\; \sum_{j\in S^p} y_j^p = m^p \right\},$$

where y_j^p represents the mass of agents in population p choosing strategy j. The overall follower-population state set is

$$Y = \prod_{p\in\mathcal{P}} Y^p = \left\{ y = (y^1, y^2, \ldots, y^P) \in \mathbb{R}_+^k : y^p \in Y^p \right\},$$

where $k = \sum_{p\in\mathcal{P}} k^p$. For each leader $i \in \mathcal{N}$, the payoff function is $f_i: X \times Y \to \mathbb{R}$. For each population $p \in \mathcal{P}$ and each strategy $j \in S^p$, the payoff function is $G_j^p: X \times Y \to \mathbb{R}$. The vector-valued payoff function of population p is $G^p = (G_1^p, G_2^p, \ldots, G_{k^p}^p)$, and the joint payoff function of all follower populations is

$$G = (G^1, G^2, \ldots, G^P).$$

3.1. Nash equilibria of LFMG

Assume that the leaders and follower populations are non-cooperative and exhibit complete rationality, aiming to maximize their own payoffs (they maintain an optimistic attitude when making decisions).

The leaders first choose their actions $x \in X$, after which the follower populations play a parametric population game. Define a correspondence $\mathcal{NC}: X \to 2^Y$, where $\mathcal{NC}(x)$ denotes the Nash equilibrium set of the follower populations' parametric games for any $x \in X$. Specifically, a Nash equilibrium for the follower populations, denoted by $y^* = (y^{*1}, y^{*2}, \ldots, y^{*P}) \in$

$\mathcal{NC}(x)$ with $y^{*p}=(y_1^{*p},y_2^{*p},\ldots,y_{k^p}^{*p})$, satisfies that for all $p\in\mathcal{P}$ and for all $j\in S^p$,

$$y_j^{*p}>0\Rightarrow G_j^p(x,y^*)\geq G_{j'}^p(x,y^*),\quad \forall j'\in S^p.$$

For each $i\in\mathcal{N}$, leader i seeks to maximize their payoff by choosing a strategy given $y\in\mathcal{NC}(x)$, i.e.,

$$\max_{x_i\in X_i}\left[\max_{y\in\mathcal{NC}(x)} f_i(x_i,x_{-i},y)\right].$$

Thus, we define the Nash equilibrium of the LFMG as follows.

Definition 4 A strategy $x^*\in X$ is a Nash equilibrium of the LFMG if, for each $i\in\mathcal{N}$, there exists $y_i^*\in\mathcal{NC}(x_i,x_{-i}^*)$ such that

$$f_i(x_i^*,x_{-i}^*,y_i^*)=\max_{u_i\in X_i}\max_{y_i\in\mathcal{NC}(x_i,x_{-i}^*)} f_i(x_i,x_{-i}^*,y_i),$$

for all $(x_i,y_i)\in X_i\times\mathcal{NC}(x_i,x_{-i}^*)$.

The Nash equilibrium of the LFMG is distinct from both the Nash equilibrium of MLFG[9] and LFPG.[20]

(i) **For the followers:** The Nash equilibrium of the follower populations in the LFMG describes the aggregate population behavior. It focuses on the mass of followers in each population choosing a given pure strategy and the corresponding payoffs. In contrast, the Nash equilibrium of the MLFG[9] concerns the individual strategy choices of followers and focuses on the payoffs received by each leader. Consequently, there are clear distinctions between the Nash equilibria of these two types of games.

(ii) **For the leaders:** The Nash equilibrium of the LFMG focuses on each leader's individual optimal choice and the corresponding maximum payoff. In contrast, the LFPG[20] emphasizes the optimal population state of the leader populations. Therefore, the Nash equilibrium of the LFMG differs conceptually and structurally from that of the LFPG[20].

The following example demonstrates that the Nash equilibrium in Definition 4 of the LFMG differs from that of the LFPG[20].

Example 1 Consider the following LFMG where $\mathcal{N}=\{1,2\}$, $X_1=X_2=[0,1]$, $\mathcal{P}=\{1,2\}$, $S^1=S^2=\{1,2\}$, and $m^1=m^2=1$,

$$Y^1=\{y^1=(y_1^1,y_2^1)\in\mathbb{R}_+^2\,|\,y_1^1+y_2^1=1\},$$
$$Y^2=\{y^2=(y_1^2,y_2^2)\in\mathbb{R}_+^2\,|\,y_1^2+y_2^2=1\}.$$

The leaders' payoff functions are given by

$$f_1(x_1,x_2,y^1,y^2)=2,\quad f_2(x_1,x_2,y^1,y^2)=x_2y_1^2,$$

and the follower populations' payoff functions are

$$G_1^1(x_1,x_2,y^1,y^2)=G_2^1(x_1,x_2,y^1,y^2)=-1,$$
$$G_1^2(x_1,x_2,y^1,y^2)=G_2^2(x_1,x_2,y^1,y^2)=2,$$

for all $(x_1,x_2,y^1,y^2)\in X\times Y$.

Clearly, the Nash equilibrium set of the follower populations is $\mathcal{NC}(x)=Y^1\times Y^2$. The Nash equilibrium set of the LFMG is $[0,1]\times\{1\}$, with $y^*\in Y^1\times\{(0,1)\}$.

The state $(y^1,y^2)=((1,0),(0,1))\in\mathcal{NC}(x)$ means that all followers in population 1 choose pure strategy 1 of S^1, and all followers in population 2 choose pure strategy 2 of S^2. The Nash equilibrium of the LFMG $(\frac{1}{2},1)$ implies that leader 1 chooses strategy $\frac{1}{2}\in X_1$ and leader 2 chooses strategy $1\in X_2$. The former reflects aggregate population behavior, while the latter represents individual strategic choice.

From the viewpoint of the LFPG,[20] leaders 1 and 2 in $\mathcal{N}=\{1,2\}$ are regarded as two anonymous agents in a single leader population with pure strategy set $S=\{a_1,a_2\}$, total mass $m^L=1$, and population state

$$\widetilde{X}=\{\widetilde{x}=(\widetilde{x}_{a_1},\widetilde{x}_{a_2})\in\mathbb{R}_+^2\,|\,\widetilde{x}_{a_1}+\widetilde{x}_{a_2}=1\}.$$

The corresponding pure-strategy payoff functions are

$$\widetilde{f}_{a_1}=2,\quad \widetilde{f}_{a_2}=x_{a_2}y_1^2.$$

The follower-population model remains the same as in the LFMG above.

The Nash equilibrium set of the follower populations is still $\mathcal{NC}(x)=Y^1\times Y^2$, but the Nash equilibrium set of the LFPG is $\{(1,0)\}\times Y^1\times Y^2$.

Therefore, the LFMG differs from the LFPG[20] in its game structure, and there is a clear distinction between their respective Nash equilibria.

3.2. Cooperative equilibria of LFMG

In reality, leaders or follower-populations cooperate through coalitions based on their interests. Now we suppose leaders cooperate with each other in a coalition, so for every coalition $B\subseteq\mathcal{N}$, $X^B=\prod_{i\in B}X_i$ is the strategy set for the coalition B. The strategy set of leaders outside the coalition B is denoted by $X^{-B}=\prod_{i\notin B}X_i$.

Similarly, follower-populations can cooperate via coalitions. The population coalition $S\subseteq\mathcal{P}$ corresponds to the feasible state set $\widehat{Z}_S$ as follows:

$$\hat{Z}_S=\left\{z_s\in\prod_{p\in S}R_+^{k^p}\;\middle|\;\sum_{p\in S}\sum_{j\in S^p}z_j^p=\sum_{p\in S}m^p\right\}.$$

Since there is a coalition among populations, the Cartesian product of population state space and coalition state space satisfies $Y\subset\widehat{Z}_{\mathcal{P}}\subset R_+^k$. Thus, the payoff function needs to be expanded to R_+^k, as stated in Ref. [27], to establish the existence of cooperative equilibria. For each $i\in\mathcal{N}$, $f_i:X\times R_+^k\to R$ is the payoff function of leader i. For any $p\in\mathcal{P}$, G_j^p :

$X \times R_+^k \to R$ is the payoff function of strategy j for follower-population p. Vector-valued function $G^p = (G_1^p, G_2^p, \ldots, G_{k^p}^p)$ represents the payoffs of follower-population p.

LFMG differs from MLFG[34] and MLFPG.[36]

(i) **With respect to followers:** LFMG mainly considers cooperation within follower-populations, in contrast to MLFG[34], which has finite leaders and followers and focuses on cooperation among followers.

(ii) **With respect to leaders:** Unlike MLFPG[36], which has finite leader-populations and follower-populations and focuses on cooperation among leader-populations, LFMG is aimed at studying the cooperation among individual leaders.

In LFMG, leaders first choose a strategy $x \in X$, and follower-populations subsequently react to the leaders' strategy. Essentially, follower-populations play a parametric population game.

The $\bar{y} \in R_+^k$ is a cooperative equilibrium of follower-populations for parametric population games if $\bar{y} \in \widehat{Z}_{\mathcal{P}}$, and for each $S \subseteq \mathcal{P}$, there is no $z_S \in \widehat{Z}_S$ such that

$$(z_S)^p G^p(x,\bar{y}) > \bar{y}^p G^p(x,\bar{y}), \quad \forall p \in S,$$

where $\bar{y}^p G^p(x,\bar{y}) = \sum_{j=1}^{k^p} \bar{y}_j^p G_j^p(x,\bar{y})$.

Clearly, the cooperative equilibrium set of follower-populations depends on the leaders' strategy x. Thus, we define the correspondence $\mathcal{C} : X \to R_+^k$, where $\mathcal{C}(x)$ is the cooperative equilibrium set of follower-populations responding to leaders' strategy $x \in X$.

Definition 5 A strategy $\bar{x} \in X$ is called a cooperative equilibrium of LFMG if, for each coalition $B \subseteq \mathcal{N}$, there is no $u^B \in X^B$ such that

$$f_i(u^B, X^{-B}, \bar{y}) > f_i(\bar{x}, \bar{y}), \quad \forall \bar{y} \in \mathcal{C}(\bar{x}), \quad \forall i \in B.$$

Next, the following example illustrates the cooperative equilibria of LFMG and the distinctions between the cooperative equilibria of LFMG and MLFPG.[36]

Example 2 Suppose that cooperative behaviors exist among leaders and follower-populations in example 1. Let $X^{12} = X_1 \times X_2$, $\widehat{Z}_{12} = \{(y_1, y_2) \in R_+^4 \mid y_1^1 + y_2^1 + y_1^2 + y_2^2 = 2\}$. All payoff functions are expanded to $X \times R_+^4$. For any $(x_1, x_2, y^1, y^2) \in X \times R_+^4$,

$$\begin{aligned}
&f_1(x_1, x_2, y^1, y^2) = 2,\\
&f_2(x_1, x_2, y^1, y^2) = x_2 y_1^2,\\
&G_1^1(x_1, x_2, y^1, y^2) = G_2^1(x_1, x_2, y^1, y^2) = -1,\\
&G_1^2(x_1, x_2, y^1, y^2) = G_2^2(x_1, x_2, y^1, y^2) = 2.
\end{aligned}$$

The cooperative equilibrium set of follower-populations is $C(x) = \{(y_1, y_2) \in R_+^4 \mid y_1^1 = y_2^1 = 0, y_1^2 + y_2^2 = 2\}$. The cooperative equilibrium set of LFMG is X^{12}.

From the viewpoint of MLFPG[36], suppose leaders 1 and 2 of $\mathcal{N} = \{1,2\}$ are anonymous agents of two leader populations, respectively. Pure strategy sets are denoted by $S_1 = S_2 = \{a_1, a_2\}$, with masses $m^{L_1} = m^{L_2} = 1$, and the leader populations' state sets are

$$\begin{aligned}
\widetilde{X}^1 &= \{\widetilde{x}^1 = (\widetilde{x}_{a_1}^1, \widetilde{x}_{a_2}^1) \in R_+^2 \mid \widetilde{x}_{a_1}^1 + \widetilde{x}_{a_2}^1 = 1\},\\
\widetilde{X}^2 &= \{\widetilde{x}^2 = (\widetilde{x}_{a_1}^2, \widetilde{x}_{a_2}^2) \in R_+^2 \mid \widetilde{x}_{a_1}^2 + \widetilde{x}_{a_2}^2 = 1\},
\end{aligned}$$

with payoff functions

$$\widetilde{f}_{a_1}^1 = \widetilde{f}_{a_2}^1 = 2, \quad \widetilde{f}_{a_1}^2 = x_{a_1}^2 y_1^2, \quad \widetilde{f}_{a_2}^2 = x_{a_2}^2 y_1^2.$$

If the two leader populations form a coalition, the coalition state set is

$$\begin{aligned}
\widetilde{X}^{12} = \{&\widetilde{x}^{12} = (\widetilde{x}_{a_1}^1, \widetilde{x}_{a_2}^1, \widetilde{x}_{a_1}^2, \widetilde{x}_{a_2}^2) \in\\
&R_+^4 \mid \widetilde{x}_{a_1}^1 + \widetilde{x}_{a_2}^1 + \widetilde{x}_{a_1}^2 + \widetilde{x}_{a_2}^2 = 2\}.
\end{aligned}$$

The model of follower-populations is the same as described above.

The cooperative equilibrium set of follower-populations is $C(x) = \{(y_1, y_2) \in R_+^4 \mid y_1^2 + y_2^2 = 2, y_1^1 = y_2^1 = 0\}$, and the cooperative equilibrium set of MLFPG is $\widetilde{X}^{12}$.

This example demonstrates that the LFMG and MLFPG[36] models are distinct. Their coalition strategy sets differ: the coalition strategy set in LFMG is the Cartesian product X^{12} of the two leaders' strategies, while in MLFPG it is expanded to a four-dimensional set $\widetilde{X}^{12}$. Finally, the meanings of cooperative equilibria differ: the cooperative equilibrium in LFMG characterizes the decisions of coalition individuals, whereas that in MLFPG describes the state distribution of coalition populations. Specifically, the cooperative equilibrium $(1,1) \in X^{12}$ of LFMG indicates that leader 1 chooses strategy $1 \in X_1$ and leader 2 chooses strategy $1 \in X_2$, whereas the cooperative equilibrium $(1,1,0,0) \in \widetilde{X}^{12}$ of MLFPG[36] represents that mass 1 of leader population 1 chooses a_1 and mass 1 of leader population 2 chooses a_2. Clearly, LFMG more fully captures the decision-making behavior of individual leaders compared to MLFPG.[36]

4. Existence of equilibria for LFMG

In this section, we present the existence theorems for Nash and cooperative equilibria of LFMG, respectively.

4.1. Existence theorem of Nash equilibria

According to Kakutani's fixed point theorem, the existence theorem of a Nash equilibrium for LFMG is given as follows.

Theorem 1 Suppose that X_i is a nonempty, compact, and convex subset of R^{ℓ_i} for any $i \in \mathcal{N}$, and the following conditions hold:

(i) $\forall i \in \mathcal{N}$, $f_i : X \times Y \to R$ is continuous on $X \times Y$;

(ii) $\forall i \in \mathcal{N}$, $\forall x_{-i} \in X_{-i}$, the mapping $(x_i, y_i) \to f_i(x_i, x_{-i}, y_i)$ is quasi-concave;

(iii) $\forall i \in \mathcal{N}$, the correspondence $\mathcal{NC} : X_i \times X_{-i} \to 2^Y$ is continuous, and $\forall x \in X$, $\mathcal{NC}(x)$ is a nonempty, bounded, compact subset of Y;

(iv) $\forall x_{-i} \in X_{-i}$, the correspondence $x_i \to \mathcal{NC}(x_i, x_{-i})$ is convex.

Then, for each $i \in \mathcal{N}$, there exists $(x_i^*, x_{-i}^*, y_i^*) \in X \times Y$, with $y_i^* \in \mathcal{NC}(x_i^*, x_{-i}^*)$, such that

$$f_i(x_i^*, x_{-i}^*, y_i^*) = \max_{x_i \in X_i, y_i \in \mathcal{NC}(x_i, x_{-i}^*)} f_i(x_i, x_{-i}^*, y_i).$$

That is, there exists a Nash equilibrium $(x_i^*, x_{-i}^*) \in X$ of LFMG.

Proof First, we define a correspondence $F_i : X_{-i} \times Y_{-i} \to 2^{X_i \times Y}$: for any $x_{-i} \in X_{-i}$ and $y_{-i} \in Y_{-i}$,

$$\mathcal{F}_i(x_{-i}, y_{-i}) = \left\{ (x_i, y_i) \in X_i \times \mathcal{NC}(x_i, x_{-i}) \;\middle|\; f_i(x_i, x_{-i}, y_i) = \max_{u_i \in X_i, v_i \in \mathcal{NC}(u_i, x_{-i})} f_i(u_i, x_{-i}, v_i) \right\},$$

for each $i \in \mathcal{N}$ ($Y_{-i} = \prod_{|\mathcal{N}\setminus\{i\}|}$, Y denotes the state set of the follower populations belonging to other leaders (excluding leader i). $|\mathcal{N}\setminus\{i\}|$ denotes the cardinality of set $|\mathcal{N}\setminus\{i\}|$).

Since X_i is compact and $\mathcal{NC}$ is a continuous correspondence with compact-valued images, $\mathcal{NC}(x_i, x_{-i})$ is bounded and closed for each $i \in \mathcal{N}$. By the continuity of f_i, $\mathcal{F}_i(x_{-i}, y_{-i})$ is nonempty and closed, hence bounded and closed.

Next, we show that $\mathcal{F}_i(x_{-i}, y_{-i})$ is convex. For any $(x_i', y_i'), (x_i'', y_i'') \in \mathcal{F}_i(x_{-i}, y_{-i})$, we have

$$f_i(x_i', x_{-i}, y_i') = \max_{u_i, v_i} f_i(u_i, x_{-i}, v_i),$$
$$f_i(x_i'', x_{-i}, y_i'') = \max_{u_i, v_i} f_i(u_i, x_{-i}, v_i).$$

By convexity of X_i and property (iv) of Theorem 1, for any $\lambda \in (0,1)$

$$\lambda x_i' + (1-\lambda) x_i'' \in X_i,$$
$$\lambda y_i' + (1-\lambda) y_i'' \in \mathcal{NC}(\lambda x_i' + (1-\lambda) x_i'', x_{-i}).$$

By quasi-concavity (ii)

$$f_i(\lambda x_i' + (1-\lambda) x_i'', x_{-i}, \lambda y_i' + (1-\lambda) y_i'')$$
$$\geq \min\{f_i(x_i', x_{-i}, y_i'), f_i(x_i'', x_{-i}, y_i'')\} \geq f_i(u_i, x_{-i}, v_i),$$

so $\mathcal{F}_i(x_{-i}, y_{-i})$ is convex.

Now, define $T(x_{-i}) = \{(u_i, v_i) \in X_i \times Y \mid u_i \in X_i, v_i \in \mathcal{NC}(u_i, x_{-i})\}$. T is continuous:

Firstly, let $\{x_{-i}^n\} \subset X_{-i}$, $x_{-i}^n \to x_{-i}$. For any $(u_i^n, v_i^n) \in T(x_{-i}^n)$ with $(u_i^n, v_i^n) \to (u_i, v_i)$, by upper semi-continuity and compactness of $\mathcal{NC}$, we have $(u_i, v_i) \in T(x_{-i})$, i.e., T is closed. By compactness of $X_i \times Y$ and Lemma 2, T is upper semi-continuous.

Furthermore, by lower semi-continuity of $\mathcal{NC}$, for any $(u_i, v_i) \in T(x_{-i})$ there exists $(u_i^n, v_i^n) \in T(x_{-i}^n)$ with $(u_i^n, v_i^n) \to (u_i, v_i)$. Therefore, T is lower semi-continuous, and thus continuous with compact-valued images.

By Lemma 1, $\mathcal{F}_i$ is upper semi-continuous with nonempty, convex, compact-valued images. Define

$$\mathcal{F} : X_1 \times Y \times \cdots \times X_N \times Y \to 2^{X_1 \times Y \times \cdots \times X_N \times Y},$$

$$\mathcal{F}(x_1, y_1, \ldots, x_N, y_N) = \mathcal{F}_1(x_{-1}, y_{-1}) \times \cdots \times \mathcal{F}_N(x_{-N}, y_{-N}).$$

Since $X_1 \times Y \times \cdots \times X_N \times Y$ is nonempty, convex, and compact, by Kakutani's fixed point theorem, there exists $(x_1^*, y_1^*, \ldots, x_N^*, y_N^*)$ such that

$$(x_1^*, y_1^*, \ldots, x_N^*, y_N^*) \in \mathcal{F}(x_1^*, y_1^*, \ldots, x_N^*, y_N^*).$$

Hence, for each $i \in \mathcal{N}$, $(x_i^*, y_i^*) \in \mathcal{F}_i(x_{-i}^*, y_{-i}^*)$, and

$$f_i(x_i^*, x_{-i}^*, y_i^*) = \max_{x_i \in X_i, y_i \in \mathcal{NC}(x_i, x_{-i}^*)} f_i(x_i, x_{-i}^*, y_i),$$

so (x_i^*, x_{-i}^*) is a Nash equilibrium of LFMG.

4.2. Existence theorem of cooperative equilibria

We investigate the existence of cooperative equilibria for LFMG as follows.

Theorem 2 Suppose that LFMG

$$\{\mathcal{N}, \mathcal{P}, \{X_i\}_{i \in \mathcal{N}}, \{Y^p\}_{p \in \mathcal{P}}, \{f_i\}_{i \in \mathcal{N}}, \{G^p\}_{p \in \mathcal{P}}\}$$

satisfies the following conditions:

(i) $\forall i \in \mathcal{N}$, X_i is a nonempty, compact, and convex subset of R^{ℓ_i};

(ii) $\forall i \in \mathcal{N}$, f_i is continuous on $X \times R_+^k$;

(iii) $\forall p \in \mathcal{P}$, G_j^p is continuous on $X \times R_+^k$;

(iv) $\forall i \in \mathcal{N}$, $\forall y \in Y$, $f_i(\cdot, y)$ is quasi-concave on X.

Then the LFMG has at least one cooperative equilibrium.

Proof We divide the proof into two steps. First, we construct the cooperative equilibrium correspondence of follower-populations' parametric games and show that it is nonempty, compact, and upper semi-continuous. Second, we prove the existence of a cooperative equilibrium for LFMG.

Step 1: Let $a \in R_+$ such that $a > \sum_{p \in \mathcal{P}} m^p$. Define

$$Y_a^p = \{y^p \in R_+^{k^p} \mid y_j^p \leq a, \forall j\}.$$

For any $x \in X$ and $p \in \mathcal{P}$, we construct a correspondence $F^p(x, \cdot) : Y^p \to Y^p$ for follower-population p by

$$F^p(x, y) = \{z \in Y_a^p \mid z^p G^p(x, y) > y^p G^p(x, y)\}.$$

Now we show that the generalized game $(\mathcal{P},(Y_a^p,F^p)_{p\in\mathcal{P}},(\hat{Z}_S)_{S\subseteq\mathcal{P}})$ satisfies assumptions $(A-1)-(A-4)$ in Proposition 2 of Ref. [28]:

(1) For any $p\in\mathcal{P}$, Y_a^p is nonempty, compact, and convex.

(2) By definition, F^p is irreflexive and convex. To show that F^p is open, suppose $\{(y_n,z_n)\}$ is a sequence in $Y_a^p\times Y_a^p$ with $(y_n,z_n)\to(y,z)$ and $z_n\notin F^p(x,y_n)$. If $z\in F^p(x,y)$, then

$$z^pG^p(x,y)>y^pG^p(x,y),\quad \forall p\in\mathcal{P}.$$

By continuity of G^p, for n large enough, we would have $z_n\in F^p(x,y_n)$, a contradiction. Hence F^p is open in $Y_a^p\times Y_a^p$.

(3) $\hat{Z}_S$ is independent of x for any $S\subseteq\mathcal{P}$ and $y\in Y_a^p$, so $(A-3)$ holds.

(4) By Theorem 2.2 of Ref. [35], assumption $(A-4)$ is satisfied.

According to Proposition 2 of Ref. [20], for any $x\in X$, there exists $y\in Y_a$ such that $y\in\widehat{Z}_{\mathcal{P}}$ and for any $S\subseteq\mathcal{P}$, there is no $z_S\in\widehat{Z}_S$ satisfying

$$(z_S)^pG^p(x,y)>y^pG^p(x,y),\quad \forall p\in S.$$

Therefore, $y\in\mathcal{C}(x)$, i.e., $\mathcal{C}(x)$ is nonempty.

Next, we prove that $\mathcal{C}$ is closed. Since R_+^k is compact, it suffices to show that graph$(\mathcal{C})$ is closed in $X\times R_+^k$. Let $\{(x_n,y_n)\}\subset X\times R_+^k$ with $(x_n,y_n)\to(x,y)$. Since $y_n\in\mathcal{C}(x_n)$, by continuity of G^p, we have $y\in\mathcal{C}(x)$. Hence $\mathcal{C}$ is upper semi-continuous with nonempty compact-valued images.

Step 2: For each $i\in\mathcal{N}$, define the preference correspondence

$$L_i(x)=\{u\in X\mid f_i(u,y)>f_i(x,y),\forall y\in\mathcal{C}(x)\}.$$

By Step 3 of the proof of Theorem 3.1 in Ref. [34], there exists $\bar{x}\in X$ such that for each $B\subseteq\mathcal{N}$, there is no $u^B\in X^B$ with

$$\{u^B\}\times X^{-B}\subset L_i(\bar{x}),\quad \forall i\in B.$$

Therefore, $\bar{x}$ is a cooperative equilibrium of LFMG.

Two special cases of Theorem 2: if $\mathcal{N}=\emptyset$, the theorem reduces to the existence of cooperative equilibria for population games;[35] if $\mathcal{N}$ is a singleton, it reduces to the existence result for single-leader-multi-follower-population games.[39]

Example 3 Consider two adjacent cities, city 1 and city 2, as leaders formulating environmental policies, such as carbon taxes or subsidies, to incentivize green industry investment. Local enterprise populations act as followers, deciding the proportion of investment in two emission reduction technologies, A and B, based on cities' policies.

Let $X_1=X_2=[0,1]$, where $x_i\in X_i$ represents city i's subsidy level. Follower enterprise strategy sets are $S^1=S^2=\{A,B\}$, with populations $m^1=m^2=1$. Their state sets are

$$Y^1=\{y^1=(y_A^1,y_B^1)\in R_+^2\mid y_A^1+y_B^1=1\},$$

$$Y^2=\{y^2=(y_A^2,y_B^2)\in R_+^2\mid y_A^2+y_B^2=1\},$$

where y_j^p is the mass of enterprises in population p choosing strategy j.

The payoffs are

$$f_1(x_1,x_2,y^1)=x_1+y_A^1,\quad f_2(x_1,x_2,y^2)=x_2+2y_A^2,$$
$$G_A^1=G_B^1=-1,\quad G_A^2=G_B^2=2.$$

Since payoffs for technologies A and B are identical for enterprises, any combination constitutes a Nash equilibrium. Hence $\mathcal{NC}(x)=Y^1\times Y^2$. The Nash equilibrium for cities is $\{(1,1)\}$ with all enterprises choosing A.

Suppose cooperation exists between both cities and enterprise populations. Let $X^{12}=X_1\times X_2$ and the coalition state set be

$$\widehat{Z}_{12}=\{(y^1,y^2)\in R_+^4\mid y_A^1+y_B^1+y_A^2+y_B^2=2\}.$$

Then there exists a cooperative equilibrium. For example, $\mathcal{C}(x)=\{(y^1,y^2)\in R_+^4\mid y_A^1=y_B^1=0,y_A^2+y_B^2=2\}$. The total payoff of cooperative equilibrium $(0,0,1,1)$ is 4, higher than the Nash equilibrium $(1,0)\times(1,0)$ with payoff 1.

Thus, cooperative equilibria can increase the total payoff of enterprise populations and improve cities' welfare. For instance, the city's total payoff in the cooperative equilibrium $(1,1)$ is 6, higher than 5 in the Nash equilibrium, illustrating the benefits of cooperation.

5. Generic stability of cooperative equilibria

From the results of example 3, the cooperative equilibrium set may not be a singleton. To refine the cooperative set of LFMG, we introduce the following concepts. Let $a\in R_+$ with $a>\sum\limits_{p\in\mathcal{P}}m^p$, and define

$$Y_a^p=\{y^p\in R_+^{n^p}\mid y_j^p\le a,\forall j\},\quad Y_a=\prod_{p\in\mathcal{P}}Y_a^p.$$

Let $\mathcal{M}$ denote the set of LFMG, and for any $\Upsilon=(f,G)\in\mathcal{M}$, assume Υ satisfies all conditions of Theorem 2. Denote the cooperative correspondence by $\mathcal{H}:\mathcal{M}\to X$, where $\mathcal{H}(\Upsilon)$ is the cooperative equilibrium set of Υ. By Theorem 2, $\mathcal{H}(\Upsilon)\neq\emptyset$.

Define the metric

$$\rho(\Upsilon,\Upsilon')=\sum_{i\in\mathcal{N}}\sup_{(x,y)\in X\times Y_a}|f_i(x,y)-f_i'(x,y)|$$
$$+\sum_{p\in\mathcal{P}}\sup_{(x,y)\in X\times Y_a}\sum_{j\in S^p}|G_j^p(x,y)-G_j'^p(x,y)|.$$

It follows that the metric space $(\mathcal{M},\rho)$ is complete.

Lemma 5 Given $(\Upsilon,x)\in\mathcal{M}\times X$, for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\mathcal{C}'(x')\subset U(\varepsilon,\mathcal{C}(x)):=\{y\in Y_a\mid d(y,\mathcal{C}(x))<\varepsilon\}$$

for any $(\Upsilon',x') \in \mathcal{M}\times X$ with $\rho(\Upsilon,\Upsilon')+d(x,x') < \delta$, where $\mathcal{C}'(x')$ is the cooperative equilibrium set of the follower-population for LFMG Υ' at leaders' strategy x'.

Proof Assume, to the contrary, that there exists $\varepsilon_0 > 0$ and a sequence $\{(\Upsilon^n,x^n)\} \subset \mathcal{M}\times X$ with $(\Upsilon^n,x^n) \to (\Upsilon,x)$, but $\mathcal{C}^n(x^n) \not\subset U(\varepsilon_0,\mathcal{C}(x))$. Then there exists a sequence $\{z^n\}$ with $z^n \in \mathcal{C}^n(x^n)$ and $z^n \notin U(\varepsilon_0,\mathcal{C}(x))$. By compactness of Y_a, assume $z^n \to z \in Y_a$. Then $z \notin U(\varepsilon_0,\mathcal{C}(x))$.

Suppose $z \notin \mathcal{C}(x)$. Then there exists a coalition $S \subseteq \mathcal{P}$ and $y_S \in \widehat{Z}_S$ such that

$$(y_S)^p G^p(x,z) > z^p G^p(x,z), \quad \forall p \in S.$$

Define

$$\varphi(x,z) = \min_{p\in S}\{[(y_S)^p - z^p]\cdot G^p(x,z)\} > 0.$$

By continuity, $\exists\sigma > 0$ such that $\varphi(x,z) > \sigma$. Since $\Upsilon^n \to \Upsilon$ and $(x^n,z^n) \to (x,z)$, $\exists n_0$ such that for all $n > n_0$,

$$\varphi^n(x^n,z^n) > \frac{\sigma}{2} > 0,$$

which implies $z^n \notin \mathcal{C}^n(x^n)$, a contradiction. Hence $z \in \mathcal{C}(x)$, contradicting $z \notin U(\varepsilon_0,\mathcal{C}(x))$.

Lemma 6 The correspondence $\mathcal{H}: \mathcal{M} \to X$ is upper semi-continuous and compact-valued.

Proof From the compactness of X and Lemma 2, it suffices to show that the graph of $\mathcal{H}$ is closed. Let $\{(\Upsilon^n,x^n)\}$ satisfy $x^n \in \mathcal{H}(\Upsilon^n)$ and $(\Upsilon^n,x^n) \to (\Upsilon,x)$. Suppose $x \notin \mathcal{H}(\Upsilon)$. Then there exists a coalition $B \subseteq \mathcal{N}$ and $\bar{u}^B \in X^B$ such that

$$f_i(\bar{u}^B,x^{-B},y) > f_i(x,y), \quad \forall y \in \mathcal{C}(x), \forall i \in B.$$

By continuity and compactness arguments (see Lemma 1), there exists $\varepsilon > 0$ and neighborhoods such that

$$\mathcal{V}^n(x^n,y^n) > \frac{\varepsilon}{2} > 0, \quad \forall y^n \in \mathcal{C}^n(x^n),$$

contradicting $x^n \in \mathcal{H}(\Upsilon^n)$. Hence $x \in \mathcal{H}(\Upsilon)$, showing the graph is closed.

Definition 6 A cooperative equilibrium $x \in \mathcal{H}(\Upsilon)$ is an essential equilibrium if for any open neighborhood $O(x)$ of x, $\exists\theta > 0$ such that $O(x) \cap \mathcal{H}(\Upsilon') \neq \emptyset$ for all $\Upsilon' \in \mathcal{M}$ with $\rho(\Upsilon,\Upsilon') < \theta$. LFMG Υ is essential relative to $\mathcal{M}$ if every cooperative equilibrium of Υ is essential.

Equivalently, x is essential iff the cooperative equilibrium correspondence $\mathcal{H}$ is lower semi-continuous at x.

Theorem 3 There exists a dense residual subset $A \subset \mathcal{M}$ such that every LFMG $\Upsilon \in A$ is essential relative to A.

Proof Since $(\mathcal{M},\rho)$ is complete and $\mathcal{H}$ is upper semi-continuous and compact-valued (Lemma 6), Lemma 4 implies that there exists a dense residual subset $A \subset \mathcal{M}$ such that $\mathcal{H}$ is lower semi-continuous at every $\Upsilon \in A$. Hence, each $\Upsilon \in A$ is essential.

Because A is dense in $\mathcal{M}$ and of the second category in the sense of Baire, the property of being essential is generic on $\mathcal{M}$.

6. Conclusion

This paper constructs a LFMG with finitely many leaders and follower-populations, based on the distinctive characteristics of non-cooperative games and population games. Compared with existing models such as MLFG,[9] LFPG,[20] and MLFPG[36], the proposed LFMG better captures practical modeling requirements.

For LFMG, we define both Nash equilibria and cooperative equilibria, corresponding to complete competition and coalitional cooperation, respectively. Example 1 illustrates the difference between the Nash equilibrium of LFMG and that of LFPG.[20] Example 2 highlights the distinctions between the cooperative equilibrium of LFMG and that of MLFPG,[36] considering model structure, coalition strategy sets, and equilibrium implications. Example 3 further shows that the Nash equilibrium set differs from the cooperative equilibrium set in LFMG.

Finally, we prove that, in the sense of Baire category, cooperative equilibria of the majority of LFMGs are generically stable under small perturbations of the payoff functions.

Future research will focus on the computation of cooperative and Nash equilibria for LFMG. We also plan to investigate multi-objective extensions of this framework and explore its applications in practical decision-making problems.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 11271098) and the Guizhou Provincial Basic Research Program (Grant No. MS[2025]676).

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