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Construction methods of nonlocal sets of orthogonal product states on multipartite quantum systems

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Nonlocal set of orthogonal product states (OPSs) can improve the confidentiality of information when it is used to design quantum cryptographic protocols. It is a difficult question how to construct a nonlocal set of OPSs on general multipartite and high dimensional quantum systems. Different from the previous works, we first present a novel method for constructing a nonlocal product set with $3d-2$ members on $\mathbb{C}^d\otimes\mathbb{C}^d\otimes\mathbb{C}^d$ quantum system for $d\geq 3$. Then, we extend this construction method to $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3}$ quantum system and $\otimes_{i=1}^{n}\mathbb{C}^{d_i}$ quantum system respectively, where $3\leq d_1\leq d_2\leq d_3\leq\cdots\leq d_n$ and $n\geq 3$. The nonlocal set of OPSs constructed by our method contains fewer elements than those constructed by the existing methods, except for one special case. More importantly, the set of states constructed by our method has a completely different structure from those constructed by the existing methods since our nonlocal set does not contain a "stopper" state. Our result is helpful to further understand the different structures of nonlocal sets on multipartite systems.

Keywords: nonlocal set, local indistinguishability, orthogonality-preserving measurement, quantum nonlocality without entanglement

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1. Introduction

In the field of quantum information theory, the local distinguishablity of orthogonal quantum states has been widely studied and discussed. If a set of orthogonal product states (OPSs) can be perfectly distinguished by local operations and classical communication (LOCC), then we say that the set is locally distinguishable. In 1999, Bennett *et al.*[1] first proposed a locally indistinguishable product basis with 9 OPSs on $\mathbb{C}^3\otimes\mathbb{C}^3$ quantum system, which was called quantum nonlocal without entanglement (QNWE) by them. Subsequently, Walgate *et al.*[2] proposed the sufficient and necessary conditions for a general set of quantum states to be locally distinguished on $\mathbb{C}^2\otimes\mathbb{C}^2$ quantum system, and the sufficient and necessary conditions for a general set of quantum states to be perfectly distinguished by LOCC on $\mathbb{C}^2\otimes\mathbb{C}^n$ quantum system. Inspired by these results, significant research interest has been generated, and consequently, substantial achievements have been made.[3–14]

In contrast to the construction of nonlocal sets of bipartite OPSs, the construction of nonlocal sets of multipartite OPSs turns out to be far more intricate and arduous. Nevertheless, in the face of such difficulties, scholars' persistent exploration and in-depth research have given rise to a multitude of significant discoveries and remarkable theoretical achievements.[15–21] Bennett *et al.*[22] first proposed a method to construct a nonlocal set of OPSs on $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ quantum system. Niset *et al.*[23] provided a method to construct a set of locally indistinguishable OPSs in an arbitrary-dimensional multipartite quantum system. Xu *et al.*[24] constructed a nonlocal set consisting of $2n$ OPSs on arbitrary n-partite quantum system. Zhu *et al.*[25] constructed a nonlocal set of OPSs with $3d-3$ elements on $\mathbb{C}^d\otimes\mathbb{C}^d\otimes\mathbb{C}^d$ quantum system, which is the smallest number of nonlocal sets constructed in a tripartite system to date. However, their construction method cannot be generalized to arbitrary multipartite quantum system.

As we know, it is more difficult to construct a nonlocal set of OPSs with different local dimensions than constructing a nonlocal set of OPSs with identical local dimensions. Wang *et al.*[26] constructed $2(d_1+d_3)-3$ OPSs that cannot be locally distinguished on $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3}$ quantum system for $4\leq d_1\leq d_2\leq d_3$. Zuo *et al.*[27] gave a method to construct $2(d_2+d_3-1)-d_1$ indistinguishable OPSs on $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3}$ quantum system for $3\leq d_1\leq d_2\leq d_3$. Jiang *et al.*[28] constructed $\sum_{j=1}^{n}(2d_j-3)+1$ indistinguishable OPSs on $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_n}$ system, where $n\geq 3$ and $d_j\geq 2$ for $j=1,2,\ldots,n$. Recently, Zhen *et al.*[29] constructed a

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set of OPSs with $\left(\sum_{i=2}^{n-1} d_i\right) + 2d_n - n + 1$ members that cannot be exactly distinguished on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$ system for $3 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ and $n \geq 3$, which is the smallest known number of nonlocal sets of OPSs on a given multipartite quantum system. All these results show that it is interesting to research the local distinguishability of OPSs on a general multipartite quantum system.

Different from the existing results, we first propose a novel method to construct a nonlocal set of OPSs on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ quantum system, and then extend this method to $\left(\mathbb{C}^d\right)^{\otimes n}$ quantum system, where $d \geq 4$ and $n \geq 3$. Next, we construct a nonlocal set of OPSs on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ quantum system, and then extend this method to $\otimes_{i=1}^{n} \mathbb{C}^{d_i}$ quantum system, where $3 \leq d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_n$ and $n \geq 3$. It is easy to see that the number of elements of the set that we construct on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ quantum system is less than that of Jiang *et al.*[28] and Zhen *et al.*[29] Compared with the results of Zuo *et al.*[27] and Zhang *et al.*,[30] the set that we constructed on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ quantum system has the same number of elements but a different structure. The rest of paper is organized as follow. In Section 2, two definitions are given. In Section 3, a method to construct a nonlocal set of OPSs is proposed. In Section 4, we generalize the construction method of nonlocal set of OPSs to a general tripartite quantum system and a more general quantum system, respectively. In Section 5, we compare and analyze our results with the existing ones. A brief conclusion is given in Section 6.

2. Preliminaries

In this section, we introduce two definitions that will be used in what follows.

Definition 1 If a set of OPSs can be perfectly distinguished by LOCC, we say that this set is locally distinguishable; otherwise, we say that it is locally indistinguishable.

Definition 2 A column vector $[N_{i1}, N_{i2}, ..., N_{in}]^{\mathrm{T}}$ is called orthogonal relation vector of the i-th party of a set of OPSs, where N_{ij} is the number of states that are orthogonal to the j-th state on the i-th subsystem, in the set.

3. Nonlocal set of OPSs on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ quantum system

In this section, we propose a new method for constructing locally indistinguishable OPSs on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ quantum system for $d \geq 3$. And then we generalize this method to $\left(\mathbb{C}^d\right)^{\otimes n}$ quantum system for $d \geq 3$ and $n \geq 4$. For convenience, all the constructed OPSs are not normalized in this paper.

Theorem 1 On $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ $(d \geq 3)$ quantum system, the following $3d-2$ OPSs cannot be exactly distinguished by LOCC.

$$
\begin{aligned}
&|\varphi_i\rangle = |i\rangle_A |0-i\rangle_B |\sum_{j=0}^{d-1} j\rangle_C, \quad 1 \leq i \leq d-1,\\
&|\varphi_{i+d-1}\rangle = |\sum_{j=0}^{d-1} j\rangle_A |i\rangle_B |0-i\rangle_C, \quad 1 \leq i \leq d-1,\\
&|\varphi_{i+2d-2}\rangle = |0-i\rangle_A |\sum_{j=0}^{d-1} j\rangle_B |i\rangle_C, \quad 1 \leq i \leq d-1,\\
&|\varphi_{3d-2}\rangle = |0\rangle_A |0\rangle_B |0\rangle_C,
\end{aligned} \tag{1}
$$

where $|0-i\rangle$ denotes $(1/\sqrt{2})(|0\rangle - |i\rangle)$ and $|\sum_{j=0}^{d-1} j\rangle$ denotes $(1/\sqrt{d})\sum_{j=0}^{d-1}|j\rangle$, etc.

Proof Based on the structure of the states in Eq. (1), we can infer that these states possess cyclic properties. Specifically, a quantum state can be transformed into another state by cyclically permuting its subsystems except the state $|\varphi_{3d-2}\rangle$ for which each subsystem is $|0\rangle$. Therefore, we only need to demonstrate that the first party, say Alice, can only perform a trivial orthogonality-preserving measurement. Assume that the POVM elements $M_A^\dagger M_A$ of Alice are written as follows:

$$
M_A^\dagger M_A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0(d-1)} \\ a_{10} & a_{11} & \cdots & a_{1(d-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(d-1)0} & a_{(d-1)1} & \cdots & a_{(d-1)(d-1)} \end{bmatrix}_{d\times d},
$$

under the basis $\{|0\rangle, |1\rangle, \ldots, |(d-1)\rangle\}$. It should be noted that the post-measurement states should remain pairwise orthogonal.

For the states $|\varphi_i\rangle$ and $|\varphi_{3d-2}\rangle$, $1 \leq i \leq d-1$, we can obtain

$$
\begin{aligned}
&\langle\varphi_i|(M_A^\dagger M_A)\otimes(I_B^\dagger I_B)\otimes(I_C^\dagger I_C)|\varphi_{3d-2}\rangle = 0,\\
&\langle\varphi_{3d-2}|(M_A^\dagger M_A)\otimes(I_B^\dagger I_B)\otimes(I_C^\dagger I_C)|\varphi_i\rangle = 0.
\end{aligned}
$$

Then we get $\langle i|M_A^\dagger M_A|0\rangle = 0$ and $\langle 0|M_A^\dagger M_A|i\rangle = 0$. Therefore, we have $a_{i0} = a_{0i} = 0$ for $1 \leq i \leq d-1$.

For the states $|\varphi_i\rangle$ and $|\varphi_j\rangle$, $1 \leq i \neq j \leq d-1$, we can obtain $\langle\varphi_i|(M_A^\dagger M_A)\otimes(I_B^\dagger I_B)\otimes(I_C^\dagger I_C)|\varphi_j\rangle = 0$ and $\langle\varphi_j|(M_A^\dagger M_A)\otimes(I_B^\dagger I_B)\otimes(I_C^\dagger I_C)|\varphi_i\rangle = 0$. Then we get $\langle i|M_A^\dagger M_A|j\rangle = 0$ and $\langle j|M_A^\dagger M_A|i\rangle = 0$. Therefore, we have $a_{ij} = a_{ji} = 0$ for $1 \leq i \neq j \leq d-1$.

For the states $|\varphi_{i+d-1}\rangle$ and $|\varphi_{i+2d-2}\rangle$, we can obtain $\langle\varphi_{i+d-1}|(M_A^\dagger M_A)\otimes(I_B^\dagger I_B)\otimes(I_C^\dagger I_C)|\varphi_{i+2d-2}\rangle = 0$ and $\langle\varphi_{i+2d-2}|(M_A^\dagger M_A)\otimes(I_B^\dagger I_B)\otimes(I_C^\dagger I_C)|\varphi_{i+d-1}\rangle = 0$. Then we get $\langle 0+1+\cdots+(d-1)|M_A^\dagger M_A|0-i\rangle = 0$ and $\langle 0-i|M_A^\dagger M_A|0+1+\cdots+(d-1)\rangle = 0$. Therefore, we have $a_{00} = a_{11} = \cdots = a_{(d-1)(d-1)}$ for $1 \leq i \leq d-1$.

From the above analyses, it is evident that each of the POVM elements, $M_A^\dagger M_A$, is proportional to the identity matrix. This implies that Alice can only perform a trivial measurement for the discrimination of these states. The same holds true for

Bob and Charlie. Consequently, the $3d-2$ OPSs in Eq. (1) cannot be perfectly distinguished by LOCC. This completes the proof.

To make Theorem 1 easier for the reader to understand, we give the case for $d=3$ in Appendix A.

Theorem 2. On $\left(\mathbb{C}^d\right)^{\otimes n}$ quantum system, the following $n(d-1)+1$ OPSs cannot be exactly distinguished by LOCC, where $d \geq 3$.

$$
\begin{aligned}
&|\varphi_i\rangle = |i\rangle_1|0-i\rangle_2|\sum_{j=0}^{d-1} j\rangle_3|\sum_{j=0}^{d-1} j\rangle_4|\sum_{j=0}^{d-1} j\rangle_5\cdots|\sum_{j=0}^{d-1} j\rangle_n, \quad 1\leq i\leq d-1,\\
&|\varphi_{i+d-1}\rangle = |\sum_{j=0}^{d-1} j\rangle_1|i\rangle_2|0-i\rangle_3|\sum_{j=0}^{d-1} j\rangle_4|\sum_{j=0}^{d-1} j\rangle_5\cdots|\sum_{j=0}^{d-1} j\rangle_n, \quad 1\leq i\leq d-1,\\
&|\varphi_{i+2(d-1)}\rangle = |\sum_{j=0}^{d-1} j\rangle_1|\sum_{j=0}^{d-1} j\rangle_2|i\rangle_3|0-i\rangle_4|\sum_{j=0}^{d-1} j\rangle_5\cdots|\sum_{j=0}^{d-1} j\rangle_n, \quad 1\leq i\leq d-1,\\
&\vdots\\
&|\varphi_{i+(n-2)(d-1)}\rangle = |\sum_{j=0}^{d-1} j\rangle_1|\sum_{j=0}^{d-1} j\rangle_2|\sum_{j=0}^{d-1} j\rangle_3\cdots|\sum_{j=0}^{d-1} j\rangle_{n-2}|i\rangle_{n-1}|0-i\rangle_n, \quad 1\leq i\leq d-1,\\
&|\varphi_{i+(n-1)(d-1)}\rangle = |0-i\rangle_1|\sum_{j=0}^{d-1} j\rangle_2|\sum_{j=0}^{d-1} j\rangle_3\cdots|\sum_{j=0}^{d-1} j\rangle_{n-2}|\sum_{j=0}^{d-1} j\rangle_{n-1}|i\rangle_n, \quad 1\leq i\leq d-1,\\
&|\varphi_{n(d-1)+1}\rangle = |0\rangle_1|0\rangle_2|0\rangle_3\cdots|0\rangle_{n-1}|0\rangle_n.
\end{aligned} \tag{2}
$$

Proof This set is composed of states generated by cyclic shifts. For the discrimination process to proceed, the post-measured states that have not been eliminated must remain orthogonality after each measurement. Without loss of generality, we assume that the first party makes the initial measurement. Each of the POVM elements $M_A^\dagger M_A$ can be written as a $d\times d$ matrix, i.e.,

$$
M_A^\dagger M_A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0(d-1)} \\ a_{10} & a_{11} & \cdots & a_{1(d-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(d-1)0} & a_{(d-1)1} & \cdots & a_{(d-1)(d-1)} \end{bmatrix}_{d\times d},
$$

under the basis $\{|0\rangle, |1\rangle, \ldots, |(d-1)\rangle\}$.

Since the post-measured states $(M_A\otimes I_B\otimes I_C\otimes\cdots\otimes I_N)|\varphi_i\rangle$ should retain orthogonal, any two states that are orthogonal only on one side are still orthogonal on this side after the measurement. By the symmetry of the set of these OPSs, we can derive that $M_A^\dagger M_A$ is proportional to an identity matrix. This means that the first party can only perform a trivial measurement. Thus the first party cannot get any useful information to be used to discriminate these states. So do other parties by Tables 1 and 2. Therefore, these $n(d-1)+1$ OPSs in Eq. (2) are locally indistinguishable. This completes the proof.

Table 1. Non-diagonal entries in $M_A^\dagger M_A, \ldots, M_N^\dagger M_N$.

POVM element	Pairs of states	Non-diagonal entries	Ranges
$M_A^\dagger M_A$	$\|\varphi_i\rangle$, $\|\varphi_j\rangle$	$a_{ij}=a_{ji}=0$	$1\leq i\neq j\leq d-1$
	$\|\varphi_i\rangle$, $\|\varphi_{n(d-1)+1}\rangle$	$a_{i0}=a_{0i}=0$	$1\leq i\neq j\leq d-1$
$M_B^\dagger M_B$	$\|\varphi_{i+(d-1)}\rangle$, $\|\varphi_{j+(d-1)}\rangle$	$b_{ij}=b_{ji}=0$	$1\leq i\neq j\leq d-1$
	$\|\varphi_{i+(d-1)}\rangle$, $\|\varphi_{n(d-1)+1}\rangle$	$b_{i0}=b_{0i}=0$	$1\leq i\neq j\leq d-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$M_N^\dagger M_N$	$\|\varphi_{i+(n-1)(d-1)}\rangle$, $\|\varphi_{j+(n-1)(d-1)}\rangle$	$n_{ij}=n_{ji}=0$	$1\leq i\neq j\leq d-1$
	$\|\varphi_{i+(n-1)(d-1)}\rangle$, $\|\varphi_{n(d-1)+1}\rangle$	$n_{i0}=n_{0i}=0$	$1\leq i\neq j\leq d-1$

Table 2. Diagonal entries in $M_A^\dagger M_A, \ldots, M_N^\dagger M_N$.

POVM element	Pairs of states	Non-diagonal entries	Ranges
$M_A^\dagger M_A$	$\|\varphi_{i+(n-1)(d-1)}\rangle$, $\|\varphi_{i+(n-2)(d-1)}\rangle$	$a_{00}=a_{11}=\cdots=a_{(d-1)(d-1)}$	$1\leq i\leq d-1$
$M_B^\dagger M_B$	$\|\varphi_i\rangle$, $\|\varphi_{i+(n-1)(d-1)}\rangle$	$b_{00}=b_{11}=\cdots=b_{(d-1)(d-1)}$	$1\leq i\leq d-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$M_N^\dagger M_N$	$\|\varphi_{i+(n-2)(d-1)}\rangle$, $\|\varphi_{i+(n-3)(d-1)}\rangle$	$n_{00}=n_{11}=\cdots=n_{(d-1)(d-1)}$	$1\leq i\leq d-1$

These two theorems offer a novel approach for constructing nonlocal sets of OPSs on multipartite quantum systems, distinct from the existing ones. Notably, when $d \geq 3$, the set of states constructed by our method contains fewer elements compared to those constructed by the existing methods. Currently, a natural query arises: whether this method can be generalized to multipartite quantum systems of arbitrary dimensions. In the subsequent chapters, we will provide an affirmative answer to this question.

4. Nonlocal set of OPSs on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ quantum system

In this section, we first extend the method of constructing a nonlocal set of OPSs from a tripartite system with the same local dimension to a more general case, *i.e.*, $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ quantum system, where $3 \leq d_1 \leq d_2 \leq d_3$. Subsequently, we further generalize this construction method from a tripartite system to an n-partite system, namely, the $\otimes_{i=1}^{n} \mathbb{C}^{d_i}$ quantum system, where $3 \leq d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_n$.

Theorem 3 On $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ quantum system, the following $d_2 + 2d_3 - 2$ OPSs cannot be reliably distinguished by LOCC, where $3 \leq d_1 \leq d_2 \leq d_3$.

$$
\begin{aligned}
&|\varphi_i\rangle = |i\rangle_A |0-i\rangle_B |\sum_{j=0}^{d_3-1} j\rangle_C,\ 1 \leq i \leq d_1 - 1,\\
&|\varphi_i\rangle = |m\rangle_A |(i-1)-i\rangle_B |\sum_{j=0}^{d_3-1} j\rangle_C,\ d_1 \leq i \leq d_2 - 1,\\
&|\varphi_{i+d_2-1}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_A |i\rangle_B |0-i\rangle_C,\ 1 \leq i \leq d_2 - 1,\\
&|\varphi_{i+d_2-1}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_A |m\rangle_B |(i-1)-i\rangle_C,\ d_2 \leq i \leq d_3 - 1,\\
&|\varphi_{i+d_2+d_3-2}\rangle = |0-i\rangle_A |\sum_{j=0}^{d_2-1} j\rangle_B |i\rangle_C,\ 1 \leq i \leq d_1 - 1,\\
&|\varphi_{i+d_2+d_3-2}\rangle = |0-(d_1-1)\rangle_A |\sum_{j=0}^{d_2-1} j\rangle_B |i\rangle_C,\ d_1 \leq i \leq d_3 - 1,\\
&|\varphi_{d_2+2d_3-2}\rangle = |0\rangle_A |0\rangle_B |0\rangle_C,
\end{aligned}
\tag{3}
$$

where $m = 0$ when i is odd while $m = 1$ when i is even.

Proof Since these states do not have the circularity as that of Theorem 1, we need to consider measuring these states from three parties separately to locally distinguish them. The proof method is similar to that of Theorem 1. When $d_1 \neq d_3$, the proof is shown as follows.

(i) Suppose Alice performs a measurement first, the POVM element $M_A^\dagger M_A$ can be written as a $d_1 \times d_1$ matrix

$$
M_A^\dagger M_A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0(d_1-1)} \\ a_{10} & a_{11} & \cdots & a_{1(d_1-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(d_1-1)0} & a_{(d_1-1)1} & \cdots & a_{(d_1-1)(d_1-1)} \end{bmatrix}_{d_1 \times d_1}
$$

under the basis $\{|0\rangle, |1\rangle, \ldots, |(d_1-1)\rangle\}$.

To enable the local discrimination to continue, the post-measurement states $\{(M_A \otimes I_B \otimes I_C)|\varphi_i\rangle\}$ should remain pairwise orthogonal, where $i = 1, \ldots, d_1 - 1$. For the states $|\varphi_i\rangle$ and $|\varphi_{d_2+2d_3-2}\rangle$, we have $\langle\varphi_i|(M_A^\dagger M_A) \otimes (I_B^\dagger I_B) \otimes (I_C^\dagger I_C)|\varphi_{d_2+2d_3-2}\rangle = 0$ and $\langle\varphi_{d_2+2d_3-2}|(M_A^\dagger M_A) \otimes (I_B^\dagger I_B) \otimes (I_C^\dagger I_C)|\varphi_i\rangle = 0$ for $1 \leq i \leq d_1 - 1$. Then we get $\langle i|M_A^\dagger M_A|0\rangle = 0$ and $\langle 0|M_A^\dagger M_A|i\rangle = 0$. Therefore, we have $a_{i0} = a_{0i} = 0$ for $1 \leq i \leq d_1 - 1$.

For the states $|\varphi_{i+d_2-1}\rangle$ and $|\varphi_{i+d_2+d_3-2}\rangle$, we can obtain $\langle\varphi_{i+d_2-1}|(M_A^\dagger M_A) \otimes (I_B^\dagger I_B) \otimes (I_C^\dagger I_C)|\varphi_{i+d_2+d_3-2}\rangle = 0$ and $\langle\varphi_{i+d_2+d_3-2}|(M_A^\dagger M_A) \otimes (I_B^\dagger I_B) \otimes (I_C^\dagger I_C)|\varphi_{i+d_2-1}\rangle = 0$. Thus, we get $\langle 0+1+\cdots+(d_1-1)|M_A^\dagger M_A|0-i\rangle = 0$ and $\langle 0-i|M_A^\dagger M_A|0+1+\cdots+(d_1-1)\rangle = 0$. Therefore, we have $a_{00} = a_{11} = \cdots = a_{(d_1-1)(d_1-1)}$ for $1 \leq i \leq d_1 - 1$.

Similarly, for the states $|\varphi_i\rangle$ and $|\varphi_j\rangle$, $1 \leq i \neq j \leq d_1 - 1$, we can obtain $\langle i|M_A^\dagger M_A|j\rangle = 0$ and $\langle j|M_A^\dagger M_A|i\rangle = 0$. Therefore, we have $a_{ij} = a_{ji} = 0$.

From the above analysis, it becomes clear that if Alice first measures, all of her POVM elements are proportional to the identity matrix. Consequently, Alice can only carry out a trivial measurement.

(ii) Suppose Bob performs a measurement first, the POVM elements $M_B^\dagger M_B$ can be written as a $d_2 \times d_2$ matrix

$$
M_B^\dagger M_B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0(d_2-1)} \\ b_{10} & b_{11} & \cdots & b_{1(d_2-1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(d_2-1)0} & b_{(d_2-1)1} & \cdots & b_{(d_2-1)(d_2-1)} \end{bmatrix}_{d_2 \times d_2}
$$

under the basis $\{|0\rangle, |1\rangle, \ldots, |(d_2-1)\rangle\}$.

The post-measurement states

$$\{(I_A \otimes M_B \otimes I_C)|\varphi_i\rangle, i = 1, \ldots, (d_2-1)\}$$

must remain orthogonality among themselves.

For the states $|\varphi_i\rangle$ and $|\varphi_{i+d_2+d_3-2}\rangle$, we can derive that $\langle\varphi_i|(IA^\dagger I_A) \otimes (M_B^\dagger M_B) \otimes (I_C^\dagger I_C)|\varphi_{i+d_2+d_3-2}\rangle = 0$ and $\langle\varphi_{i+d_2+d_3-2}|(I_A^\dagger I_A) \otimes (M_B^\dagger M_B) \otimes (I_C^\dagger I_C)|\varphi_i\rangle = 0$. This means $\langle 0+1+\cdots+(d_2-1)|M_B^\dagger M_B|0-i\rangle = 0$ and $\langle 0-i|M_B^\dagger M_B|0+1+\cdots+(d_2-1)\rangle = 0$. As a result, we conclude that $b_{00} = b_{11} = \cdots = b_{(d_1-1)(d_1-1)}$ for $1 \leq i \leq d_1 - 1$.

For the states $|\varphi_i\rangle$ and $|\varphi_{d_2+d_3-1}\rangle$, $d_1 \leq i \leq d_2 - 1$, we can show that $b_{(d_1-1)(d_1-1)} = b_{d_1 d_1} = \cdots = b_{(d_2-1)(d_2-1)}$. Hence, we can infer that $b_{00} = b_{11} = \cdots = b_{(d_2-1)(d_2-1)}$.

For the states $|\varphi_{i+d_2-1}\rangle$ and $|\varphi_{j+d_2-1}\rangle$, where $1 \le i \ne j \le d_2-1$, we can obtain $\langle i|M_B^\dagger M_B|j\rangle = 0$ and $\langle j|M_B^\dagger M_B|i\rangle = 0$. Thus, we have $b_{ij} = b_{ji} = 0$ for $1 \le i \ne j \le d_2-1$.

For the states $|\varphi_{i+d_2-1}\rangle$ and $|\varphi_{d_2+2d_3-2}\rangle$, where $1 \le i \le d_2-1$, we can obtain

$$\langle i|M_B^\dagger M_B|0\rangle = 0, \quad \langle 0|M_B^\dagger M_B|i\rangle = 0.$$

Consequently, we conclude that $b_{i0} = b_{0i} = 0, 1 \le i \le d_2-1$.

From the above analyses, it is evident that all of the POVM elements are proportional to the identity matrix if Bob first performs the measurement. Thus, Bob can only perform a trivial measurement.

From the above analysis, we can know that all of Bob's POVM elements are proportional to identity matrix if he makes the measurement first. So Bob can only make a trivial measurement.

(iii) Suppose that Charlie performs a measurement first, the POVM elements $M_C^\dagger M_C$ can be written as a $d_3 \times d_3$ matrix

$$M_C^\dagger M_C = \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0(d_3-1)} \\ c_{10} & c_{11} & \cdots & c_{1(d_3-1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(d_3-1)0} & c_{(d_3-1)1} & \cdots & c_{(d_3-1)(d_3-1)} \end{bmatrix}_{d_3\times d_3}$$

under the basis $\{|0\rangle, |1\rangle, \ldots, |(d_3-1)\rangle\}$.

The post-measurement states

$$\{(I_A \otimes I_B \otimes M_C)|\varphi_i\rangle, i = 1, \ldots, d_3-1\}$$

should remain orthogonal to each other.

For the states $|\varphi_i\rangle$ and $|\varphi_{i+d_2-1}\rangle$, $1 \le i \le d_2-1$, we can obtain $\langle\varphi_i|(I_A^\dagger I_A)\otimes(I_B^\dagger I_B)\otimes(M_C^\dagger M_C)|\varphi_{i+d_2-1}\rangle = 0$ and $\langle\varphi_{i+d_2-1}|(I_A^\dagger I_A)\otimes(I_B^\dagger I_B)\otimes(M_C^\dagger M_C)|\varphi_i\rangle = 0$. Then we get $\langle 0+1+\cdots+(d_2-1)|M_C^\dagger M_C|0-i\rangle = 0$ and $\langle 0-i|M_C^\dagger M_C|0+1+\cdots+(d_2-1)\rangle = 0$. Therefore, we have $c_{00} = c_{11} = \cdots = c_{(d_2-1)(d_2-1)}$. For the states $|\varphi_1\rangle$ and $|\varphi_{i+d_2-1}\rangle$, $d_2 \le i \le d_3-1$, we can obtain$c_{(d_2-1)(d_2-1)} = c_{d_2d_2} = \cdots = c_{(d_3-1)(d_3-1)}$. Therefore, $c_{00} = c_{11} = \cdots = c_{(d_3-1)(d_3-1)}$.

For the states $|\varphi_{i+d_2+d_3-2}\rangle$ and $|\varphi_{j+d_2+d_3-2}\rangle$, $1 \le i \ne j \le d_3-1$, we can obtain $\langle i|M_C^\dagger M_C|j\rangle = 0$ and $\langle j|M_C^\dagger M_C|i\rangle = 0$. Therefore, we have $c_{ij} = c_{ji} = 0$ for $1 \le i \ne j \le d_3-1$. For the states $|\varphi_{i+d_2+d_3-2}\rangle$ and $|\varphi_{d_2+2d_3-2}\rangle$, $1 \le i \le d_3-1$, we can obtain $\langle i|M_C^\dagger M_C|0\rangle = 0$ and $\langle 0|M_C^\dagger M_C|i\rangle = 0$. Therefore, we have $c_{i0} = c_{0i} = 0$ for $1 \le i \le d_3-1$.

From the above analysis, we can know that all of Charlie's POVM elements are proportional to identity matrix if he makes the measurement first. So Charlie can only make a trivial measurement.

In summary, the set of states on $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3}$ quantum system in Eq. (3) cannot be locally distinguished. This completes the proof.

To facilitate the reader's understanding of Theorem 3, we give one example in Appendix B.

Theorem 4 On $\otimes_{i=1}^n \mathbb{C}^{d_i}$ quantum system, the following $\sum_{j=2}^{n-1} d_j + 2d_n - n + 1$ OPSs cannot be locally distinguished, where $3 \le d_1 \le d_2 \le \cdots \le d_n$.

$$|\varphi_i\rangle = |i\rangle_1|0-i\rangle_2|\sum_{j=0}^{d_3-1} j\rangle_3|\sum_{j=0}^{d_4-1} j\rangle_4\cdots|\sum_{j=0}^{d_n-1} j\rangle_n,\ 1 \le i \le d_1-1,$$

$$|\varphi_i\rangle = |m\rangle_1|(i-1)-i\rangle_2|\sum_{j=0}^{d_3-1} j\rangle_3\cdots|\sum_{j=0}^{d_n-1} j\rangle_n,\ d_1 \le i \le d_2-1,$$

$$|\varphi_{i+d_2-1}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1|i\rangle_2|0-i\rangle_3|\sum_{j=0}^{d_4-1} j\rangle_4\cdots|\sum_{j=0}^{d_n-1} j\rangle_n,\ 1 \le i \le d_2-1,$$

$$|\varphi_{i+d_2-1}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1|m\rangle_2|(i-1)-i\rangle_3|\sum_{j=0}^{d_4-1} j\rangle_4\cdots|\sum_{j=0}^{d_n-1} j\rangle_n,\ d_2 \le i \le d_3-1,$$

$$|\varphi_{i+d_2+d_3-2}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1|\sum_{j=0}^{d_2-1} j\rangle_2|i\rangle_3|0-i\rangle_4|\sum_{j=0}^{d_5-1} j\rangle_5\cdots|\sum_{j=0}^{d_n-1} j\rangle_n,\ 1 \le i \le d_3-1,$$

$$|\varphi_{i+d_2+d_3-2}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1|\sum_{j=0}^{d_2-1} j\rangle_2|m\rangle_3|(i-1)-i\rangle_4|\sum_{j=0}^{d_5-1} j\rangle_5\cdots|\sum_{j=0}^{d_n-1} j\rangle_n,\ d_3 \le i \le d_4-1,$$

$$|\varphi_{i+d_2+d_3+d_4-3}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1|\sum_{j=0}^{d_2-1} j\rangle_2|\sum_{j=0}^{d_3-1} j\rangle_3|i\rangle_4|0-i\rangle_5|\sum_{j=0}^{d_6-1} j\rangle_6\cdots|\sum_{j=0}^{d_n-1} j\rangle_n,\ 1 \le i \le d_4-1,$$

$$|\varphi_{i+d_2+d_3+d_4-3}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1|\sum_{j=0}^{d_2-1} j\rangle_2|\sum_{j=0}^{d_3-1} j\rangle_3|m\rangle_4|(i-1)-i\rangle_5|\sum_{j=0}^{d_6-1} j\rangle_6\cdots|\sum_{j=0}^{d_n-1} j\rangle_n,\ d_4 \le i \le d_5-1,$$

$$|\varphi_{i+\sum_{j=2}^{5} d_j-4}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1 |\sum_{j=0}^{d_2-1} j\rangle_2 |\sum_{j=0}^{d_3-1} j\rangle_3 |\sum_{j=0}^{d_4-1} j\rangle_4 |i\rangle_5 |0-i\rangle_6 |\sum_{j=0}^{d_7-1} j\rangle_7 \cdots |\sum_{j=0}^{d_n-1} j\rangle_n,\ 1 \le i \le d_5-1,$$

$$|\varphi_{i+\sum_{j=2}^{5} d_j-4}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1 \cdots |\sum_{j=0}^{d_4-1} j\rangle_4 |m\rangle_5 |(i-1)-i\rangle_6 |\sum_{j=0}^{d_7-1} j\rangle_7 \cdots |\sum_{j=0}^{d_n-1} j\rangle_n,\ d_5 \le i \le d_6-1,$$

$$\vdots$$

$$|\varphi_{i+\sum_{j=2}^{n-1} d_j-n+2}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1 |\sum_{j=0}^{d_2-1} j\rangle_2 \cdots |\sum_{j=0}^{d_{(n-2)}-1} j\rangle_{n-2} |i\rangle_{n-1} |0-i\rangle_n,\ 1 \le i \le d_{(n-1)}-1,$$

$$|\varphi_{i+\sum_{j=2}^{n-1} d_j-n+2}\rangle = |\sum_{j=0}^{d_1-1} j\rangle_1 |\sum_{j=0}^{d_2-1} j\rangle_2 \cdots |\sum_{j=0}^{d_{(n-2)}-1} j\rangle_{n-2} |m\rangle_{n-1} |(i-1)-i\rangle_n,\ d_{(n-1)} \le i \le d_{(n)}-1,$$

$$|\varphi_{i+\sum_{j=2}^{n} d_j-n+1}\rangle = |0-i\rangle_1 |\sum_{j=0}^{d_2-1} j\rangle_2 |\sum_{j=0}^{d_3-1} j\rangle_3 \cdots |\sum_{j=0}^{d_{(n-1)}-1} j\rangle_{n-1} |i\rangle_n,\ 1 \le i \le d_1-1,$$

$$|\varphi_{i+\sum_{j=2}^{n} d_j-n+1}\rangle = |0-(d_1-1)\rangle_1 |\sum_{j=0}^{d_2-1} j\rangle_2 |\sum_{j=0}^{d_3-1} j\rangle_3 \cdots |\sum_{j=0}^{d_{(n-1)}-1} j\rangle_{n-1} |i\rangle_n,\ d_1 \le i \le d_n-1,$$

$$|\varphi_{\sum_{j=2}^{n-1} d_j+2d_n-n+1}\rangle = |0\rangle_1 |0\rangle_2 |0\rangle_3 \cdots |0\rangle_{n-1} |0\rangle_n,$$

where $m=0$ when i is odd while $m=1$ when i is even.

Since the proof of Theorem 4 is similar to that of Theorem 3, we omit it here.

5. Comparison and analysis

In this section, we will compare our results with the existing achievements.

From Table 3, we can easily see that the nonlocal set constructed by our method and that of Refs. [27,30] have smaller number of OPSs on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ quantum system.

Table 3. The comparison between our work and the existing results on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ quantum system.

References	Number of elements	Constraint conditions
Ref. [23]	d^3	$d \ge 2$
Ref. [26]	$4d-3$	$d \ge 4$
Ref. [27]	$3d-2$	$d \ge 3$
Ref. [28]	$6d-6/6d-8$	$d \ge 2$
Ref. [30]	$3d-2$	$d \ge 3$
Our result	$3d-2$	$d \ge 3$

By Table 4, we know that the sets constructed by Refs. [27,30] and that constructed by our method have smaller number of OPSs on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ quantum system. It should be noted that the sets construct by Ref. [27] and ours have identical numbers of members both on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ quantum system and $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ quantum system. However, the sets in Ref. [27] and ours have completely different structures. In fact, as more and more construction methods of nonlocal sets of OPSs are proposed, it is of great significance to research the structures of nonlocal sets constructed by different methods and to identify the essence of nonlocality that these sets of OPSs possess.

Table 4. The comparison between our work and the existing results on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ quantum system.

References	Number of elements	Constraint conditions
Ref. [23]	$d_1 d_2 d_3$	$d_j \ge 2$ for $j=1,2,3$
Ref. [26]	$2(d_1+d_3)-3$	$4 \le d_1 \le d_2 \le d_3$
Ref. [27]	$2(d_2+d_3-1)-d_1$	$3 \le d_1 \le d_2 \le d_3$
Ref. [28]	$2(d_1+d_2+d_3)-8$	$2 \le d_1 \le d_2 \le d_3$
Ref. [30]	d_2+2d_3-2	$3 \le d_1 \le d_2 \le d_3$
Our result	d_2+2d_3-2	$3 \le d_1 \le d_2 \le d_3$

According to Definition 2, we find that the orthogonal relation vectors of Alice's party of the sets of OPSs in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ of Refs. [27,30] are $[2,2,4,4,2,2,2]^{\mathrm{T}}$ and $[4,4,2,2,2,2,2]^{\mathrm{T}}$, respectively, while the orthogonal relation vectors of Alice's party of the set of OPSs in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ in this paper is $[3,3,2,2,3,3,2]^{\mathrm{T}}$. This means that the structure of our set is completely different from that of Refs. [27,30] on Alice's side. The corresponding graphs of the three orthogonal relation vectors are shown in Fig. 1.

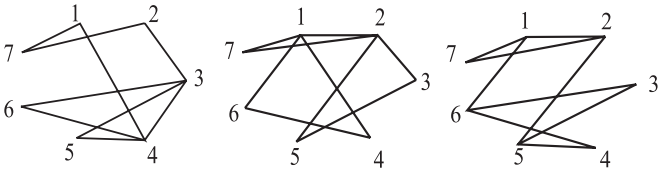


Fig. 1. The corresponding graphs of the three orthogonal relation vectors on Alice's side in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ quantum system in Refs. [27,30] and this paper.

Because of the symmetry of the sets of states, the structures of these sets of states are also different on Bob's side and

Charlie's side. Therefore, the set of OPSs constructed by our method has a completely different structure from those constructed by the existing methods.

6. Conclusion

QNWE has been one of the important research topics in the field of quantum information. It has attracted great attention from scholars since it was found. For a long time, it is a difficult problem to construct nonlocal sets of OPSs on multipartite quantum systems. In the past decade, this research has attracted widespread attention.[31–37] In this paper, we give a method to construct nonlocal set with $3d-2$ OPSs on $\mathbb{C}^d\otimes\mathbb{C}^d\otimes\mathbb{C}^d$ quantum system, and extend the method to $\left(\mathbb{C}^d\right)^{\otimes n}(d\geq 3)$ quantum system. Then we generalize the method for constructing nonlocal sets of OPSs to $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3}$ quantum system and $\otimes_{i=1}^{n}\mathbb{C}^{d_i}$ quantum system, respectively. Compared with the existing results, the nonlocal sets of OPSs constructed by our method have smaller numbers of elements. At the same time, the structures of the sets are relatively simple, which makes it easier for people to deepen their understanding and mastery of nonlocal sets on multipartite n-partite system.

At present, many achievements have been made in the research on the construction methods of nonlocal sets of OPSs. Meanwhile, in terms of the applications of nonlocal sets of OPSs, many beneficial attempts[38–43] have been carried out. However, the achievements regarding the applications of nonlocal sets are still very scarce. The applications of nonlocal sets of OPSs will be an important research content in the future.

Appendix A

Example 1

When $d=3$ in Theorem 1, the following 7 OPSs cannot be locally distinguished (see Fig. A1):

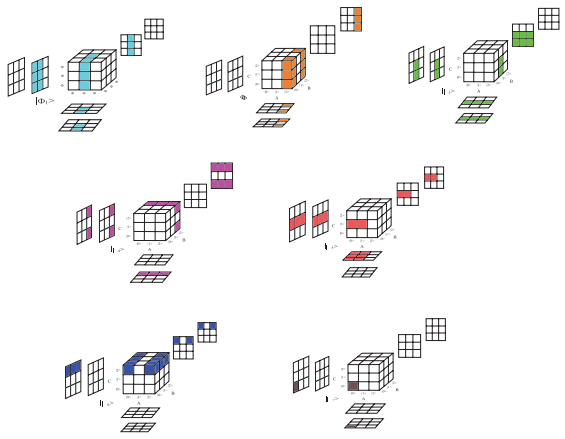


Fig. A1. The structure of 7 states on $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$.

$$
\begin{aligned}
|\Phi_1\rangle &= |1\rangle_A|0-1\rangle_B|0+1+2\rangle_C,\\
|\Phi_2\rangle &= |2\rangle_A|0-2\rangle_B|0+1+2\rangle_C,\\
|\Phi_3\rangle &= |0+1+2\rangle_A|1\rangle_B|0-1\rangle_C,\\
|\Phi_4\rangle &= |0+1+2\rangle_A|2\rangle_B|0-2\rangle_C,\\
|\Phi_5\rangle &= |0-1\rangle_A|0+1+2\rangle_B|1\rangle_C,\\
|\Phi_6\rangle &= |0-2\rangle_A|0+1+2\rangle_B|2\rangle_C,\\
|\Phi_7\rangle &= |0\rangle_A|0\rangle_B|0\rangle_C.
\end{aligned}
$$

Appendix B

Example 2

On $\mathbb{C}^3\otimes\mathbb{C}^4\otimes\mathbb{C}^4$ quantum system, the following 10 OPSs cannot be locally distinguished (see Fig. B1):

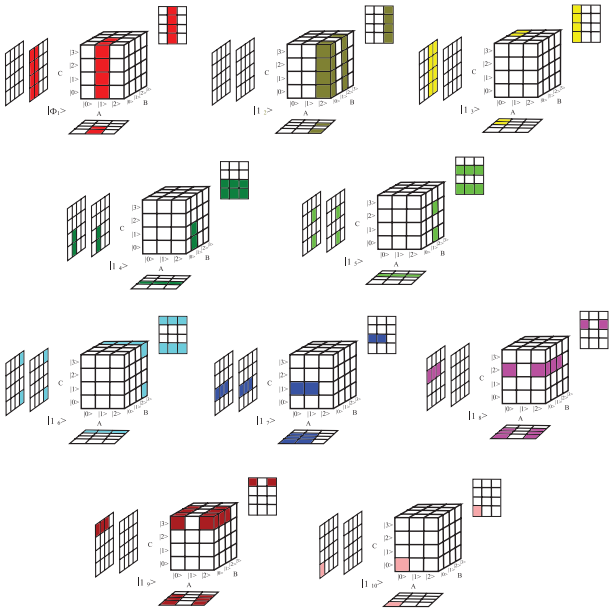


Fig. B1. The structure of 10 states on $\mathbb{C}^3\otimes\mathbb{C}^4\otimes\mathbb{C}^4$.

$$
\begin{aligned}
|\Phi_1\rangle &= |1\rangle_A|0-1\rangle_B|0+1+2+3\rangle_C,\\
|\Phi_2\rangle &= |2\rangle_A|0-2\rangle_B|0+1+2+3\rangle_C,\\
|\Phi_3\rangle &= |0\rangle_A|2-3\rangle_B|0+1+2+3\rangle_C,\\
|\Phi_4\rangle &= |0+1+2\rangle_A|1\rangle_B|0-1\rangle_C,\\
|\Phi_5\rangle &= |0+1+2\rangle_A|2\rangle_B|0-2\rangle_C,\\
|\Phi_6\rangle &= |0+1+2\rangle_A|3\rangle_B|0-3\rangle_C,\\
|\Phi_7\rangle &= |0-1\rangle_A|0+1+2+3\rangle_B|1\rangle_C,\\
|\Phi_8\rangle &= |0-2\rangle_A|0+1+2+3\rangle_B|2\rangle_C,\\
|\Phi_9\rangle &= |0-2\rangle_A|0+1+2+3\rangle_B|3\rangle_C,\\
|\Phi_{10}\rangle &= |0\rangle_A|0\rangle_B|0\rangle_C.
\end{aligned}
$$

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