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Observer-based prescribed-time time-varying output formation-containment control of heterogeneous multi-agent systems

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This paper investigates the observer-based prescribed-time time-varying output formation-containment (PT-TV-OFC) control problem for heterogeneous multi-agent systems in which the different agents have different state dimensions. The system comprises one tracking leader, multiple formation leaders, and followers, where two types of leaders are used to generate a reference trajectory for movement and achieve specific formation, respectively. Firstly, a prescribed-time dynamics observer is constructed for the formation leaders to estimate the tracking leader's dynamic model and state. On this basis, a prescribed-time control protocol is designed for the formation leaders to achieve time-varying output formation. Then, a prescribed-time convex hull observer is designed for the followers to estimate information regarding the convex hull formed by the formation leaders. Using the estimated convex hull information, a prescribed-time containment control protocol is designed to ensure the followers converge into the convex hull. Furthermore, using Lyapunov stability theory, the stability of systems is proved in detail, which implies that the heterogeneous multi-agent systems can achieve PT-TV-OFC control. Finally, numerical simulations validate the feasibility of the theoretical results.

Keywords: heterogeneous multi-agent systems, prescribed-time control, observers, time-varying output formation-containment control

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1. Introduction

In recent years, cooperative control of multi-agent systems (MASs) has garnered significant attention from various perspectives. Its research areas include consensus problems,[1–6] formation control problems,[7–11] containment control problems,[12–16] *etc*. Combining the formation control and the containment control, a more sophisticated formation-containment (FC) problem emerges.[17] This control problem requires that the leaders form a desired geometric formation, and the followers move into the convex hull spanned by the leaders. The authors in Ref. [18] investigated FC control for MASs with directed topology. The FC control of second-order nonlinear MASs with communication delays is studied.[19] In the disaster relief operation, leader robots equipped with detection devices can adjust their formation to adapt to the changing environment at any time, while follower robots move into the safe zone within the perimeter established by the leader robots. As a result, time-varying formation-containment (TV-FC) control can handle complex tasks more flexibly. The TV-FC control problem for MASs with second-order dynamics and time-varying delays is studied.[20] The TV-FC control of general linear MASs is analyzed.[21] Beyond achieving TV-FC, MASs often need to follow a desired trajectory to adapt to specific scenarios, such as dynamic environments or tasks requiring precise positioning. This is referred to as the TV-FC tracking control problem. The TV-FC tracking problem of MASs with unknown control inputs for tracking leaders is investigated, where the formation leaders track the trajectory of the tracking leader.[22] The observe-based adaptive TV-FC tracking problem with bounded unknown inputs is analyzed.[23] Considering the input saturation, the TV-FC tracking problem for linear MASs is investigated.[24] The above TV-FC tracking control problems are primarily focused on homogeneous MASs. However, in practical scenarios, agents may exhibit varied dynamics or state space dimensions due to specific requirements and physical constraints. Therefore, the time-varying output formation-containment (TV-OFC) control problem for the heterogeneous MASs is more valuable to research.

The output FC tracking problem of heterogeneous MASs is investigated in a learning-based framework using dynamic data.[25] Based on the output regulation framework, the distributed TV-OFC control problem of heterogeneous MASs is addressed.[26] In TV-FC tracking control problem, sometimes the state of leaders is unknown. In response to this, the above

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research work[23–26] and the authors in Refs. [27,28] have considered the design of the observers to estimate the unknown state, but it is usually considered that the system matrix information of the leader system is known. This is impractical in many applications.[3] Based on state feedback and output feedback, the distributed observer is designed to estimate the system matrix and state of the leader.[29] Although this approach improves the practicality of observer design, it still faces challenges in achieving rapid and precise estimation. The authors in Ref. [30] proposed a fixed-time observer to estimate the leader's system matrix and state in a fixed time. Compared to these approaches, the prescribed-time observer designed is more challenging and has distinct advantages in this paper.

The convergence time of the MASs is an important index to measure the control performance. The pioneering work of the finite-time stability analysis is presented.[31] Finite-time control is introduced to addresses the cooperative problem of heterogeneous MASs.[32–34] But it should be noted that the finite-time control depends on the initial states of agents. To address the limitation, fixed-time control is proposed.[6,12,35–37] However, these fixed-time control algorithms still rely on controller parameters to determine the upper bound of the convergence time. To overcome the challenge of convergence time, a new prescribed-time control algorithm is presented in the paper. It can set the prescribed convergence time at will, and the prescribed convergence time is irrelevant to the initial states of agents and controller parameters.

Inspired by the above discussion, this paper comprehensively investigates the observer-based prescribed-time time-varying output formation-containment control problem of heterogeneous MASs. The rest of this paper is organized as follows. Section 2 provides preliminaries and problem formulation. Section 3 presents the control protocols and stability analysis in detail. Abundant simulation results are presented in Section 4, and Section 5 concludes the paper.

2. Preliminaries

2.1. Graph theory

Let R^n and $R^{m\times n}$ represent the Euclidean space. This is the identity matrix denoted by I_n. Specifically, when the subscript size is evident, the subscript is removed. The symmetric positive-definite (negative-definite) matrix X is shown by $X>0$ ($X<0$). The respective maximum and minimum eigenvalues are represented by the notations $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$. The diagonal matrix is denoted as $\mathrm{diag}(\cdot)$. $\otimes$ represents the Kronecker product. Block diagonal matrix is represented by $\mathrm{blockdiag}(O_1,\ldots,O_n)$ with the matrices $O_i,\ldots,O_m$. The absolute value operation is represented by $|\cdot|$. For $O\in R^{m\times n}$, let $\mathrm{vec}(O)=\left[O_1^{\mathrm{T}},\ldots,O_\pi^{\mathrm{T}}\right]^{\mathrm{T}}$, where is the i-th column of O. Let $M_m^n(\mathrm{vec}(O))=[O_1,\ldots,O_n]$.

The information communication between agents is represented by the graph $\bar{G}=G\cup\{0\}$. The tracking leader is represented by 0; formation leaders and followers are represented by $1,2,\ldots,M+N$. $G=(\mathcal{V},\varepsilon,\mathcal{A})$ is used to describe the communication of information between formation leaders and followers. $\varepsilon=\{1,\ldots,N+M\}$ is the nodes set. The communication channels are represented by edges set $\mathcal{V}=\varepsilon\times\varepsilon$. The adjacency matrix $\mathcal{A}=[a_{ij}]_{(M+N)\times(M+N)}$ represents the connections between agents. If $a_{ij}>0$, it indicates that agent i can receive information from agent j; otherwise, $a_{ij}=0$. Particularly, indicates the absence of self-loops. The neighbor set of the agent i is $N_{\mathrm{i}}=\left\{j\in\mathcal{V}\mid a_{ij}\neq 0\right\}$. The Laplacian matrix $L=[l_{ij}]_{(M+N)\times(M+N)}$ is defined as $l_{ii}=\sum_{j=1}^{M+N}a_{ij}$, and $l_{ij}=-a_{ij}$ for $i\neq j$. Define $H=L+R$, where $R=\mathrm{diag}\left(r_{10},\ldots,r_{(M+N)0}\right)$.

2.2. Problem formulation

This paper explores the heterogeneous MASs with a single tracking leader and M formation leaders and N followers. The dynamic equations of the agents are described as follows:

$$\begin{aligned}&\dot{x}_i=A_ix_i+B_iu_i,\\&y_i=C_ix_i,\quad i=1,\ldots,M+N,\end{aligned}\tag{1}$$

where $x_i\in R^{n_i}$, $y_i\in R^q$, and $u_i\in R^{m_i}$ represent state, output, and control input. $A_i\in R^{n_i\times n_i}$, $B_i\in R^{n_i\times m_i}$, and $C_i\in R^{q\times n_i}$ are the given system matrices. Suppose there are M formation leaders indexed by $1,\ldots,M$ and N followers indexed by $M+1,\ldots,M+N$.

The dynamic equations of the tracking leader, indexed by 0, are described as follows:

$$\dot{x}_0=Sx_0,\quad y_0=Fx_0,\tag{2}$$

where $x_0\in R^{n_0}$, $y_0\in R^q$ are the state and output of the tracking leader. $S\in R^{n_0\times n_0}$ and $F\in R^{q\times n_0}$ are the given system matrices. The role of tracking leader is to generate a reference trajectory for formation leaders.

In this paper, all agents are divided into four types. The type-I agents include only tracking leaders. The type-II agents include m_1 formation leaders who can only get information from the tracking leader and not from other agents. Let $M=m_1+m_2$, the type-III agents include m_2 formation leaders. The type-III agents can get information from the type-II agents but not from the other types. The type-IV agents include all followers. The type-IV agents can get information from the type-II and type-III formation leaders. The information channel is directed between agents of different types and undirected between agents of the same type.

2.3. Some useful lemmas and definitions

We propose a time-varying scaling function as

$$\mu_l=\begin{cases}\left(\dfrac{T_l}{t_0+T_l-t}\right)^h, & t_0\le t<t_0+T_l,\\ 1, & t\ge t_0+T_l,\end{cases}\tag{3}$$

where h is a real number satisfying $h>1$, t_0 and T_l are the initial time and the prescribed time, respectively.[38]

Lemma 1[39] Consider the following system:

$$\dot{y}(t) = f(t, y(t)), \quad y(0) = y_0, \tag{4}$$

where $y(t) \in R^m$ is the state, $f : R_+ \times R^m \to R^m$ is a nonlinear vector bounded in time, $y(0)$ is the initial state. There exists a Lyapunov function $V(y)$ for system (4) such that

$$\dot{V}(y) + cV(y) + k\frac{\dot{\mu}_l}{\mu_l}V(y) \le 0, \quad t \in [t_0, t_0 + T_l), \tag{5}$$

where $c \ge 0$, $k > 0$. Then the origin of system (4) is globally prescribed-time stable with the prescribed time T_l. Especially, the following solution can be obtained

$$\begin{aligned} &V(y) \le \mu_l^{-k} \exp[-c(t - t_0)] V(y_0), \quad t \in [t_0, t_0 + T_l), \\ &V(y) \equiv 0, \quad t \in [t_0 + T_l, \infty). \end{aligned} \tag{6}$$

Assumption 1 For type-III (type-IV), there is at least one formation leader in type-II (type-III) that has a directed information channel to it. The communication topologies within the type-III and type-IV are undirected and connected.

Considering the mentioned-above conditions of MASs, the Laplacian matrix of the graph is denoted by

$$L = \begin{bmatrix} 0_{m_1 \times m_1} & 0_{m_1 \times m_2} & 0_{m_1 \times N} \\ L_1 & L_2 & 0_{m_2 \times N} \\ & L_3 & L_4 \end{bmatrix}, \tag{7}$$

where $L_1 \in R^{m_2 \times m_1}$, $L_2 \in R^{m_2 \times m_2}$, $L_3 \in R^{N \times M}$, $L_4 \in R^{N \times N}$.

For the formation leaders, the desired time-varying formation is designated by

$$\dot{l}_i = S_l l_i, \tag{8}$$

where S_l and S have the same dimensions, $l_i \in R^{n_0}$ is piecewise continuously differential offset vector with respect to the tracking leader. The desired output trajectory of the leader thus obtained can be expressed as

$$y_i = y_0 + F l_i. \tag{9}$$

Definition 1 A set $U \subseteq R^n$ is known as convex if for any $a \in U$, $b \in U$, and $\epsilon \in (0,1)$, $(1-\epsilon)a + \epsilon b \in U$ holds. Moreover, $co(A) = \left\{ \sum_{i=1}^{q} \varepsilon_i \tau_i \mid \tau_i \in A, \varepsilon_i \in R, \varepsilon_i \ge 0, \sum_{i=1}^{q} \varepsilon_i = 1 \right\}$ is the minimal convex set containing all points $A = \{\tau_1, \ldots, \tau_q\}$.

Definition 2 For heterogeneous MASs (1) and (2), it is said to achieve the PT-TV-OFC control if the following equation holds:

$$\begin{aligned} &\lim_{t \to t_0 + T_F} (y_i - y_0 - F l_i) = 0, \\ &\lim_{t \ge t_0 + T_F} (y_i - y_0 - F l_i) = 0, \quad i = 1, \ldots, M, \end{aligned} \tag{10}$$

$$\begin{aligned} &\lim_{t \to t_0 + T_C} \inf_{y \in co(y_l)} (y_i - y) = 0, \\ &\lim_{t \ge t_0 + T_C} \inf_{y \in co(y_l)} (y_i - y) = 0, \quad M+1, \ldots, M+N, \end{aligned} \tag{11}$$

where $y_i = [y_1^{\mathrm{T}}, \ldots, y_M^{\mathrm{T}}]^{\mathrm{T}}$, t_0 is the initial time, T_F and T_C are the prescribed times.

Remark 1. The PT-TV-OFC control problem will be addressed in two stage. When equation (10) is satisfied, the leaders can achieve the prescribed-time time-varying output formation (PT-TV-OF). When both equations (10) and (11) are all satisfied, the heterogeneous MASs can achieve PT-TV-OFC.

Lemma 2[40] Under Assumption 1, $-L_4^{-1}L_3$ has nonnegative entries, and each row sum of $-L_4^{-1}L_3$ equals one.

Lemma 3[41] Suppose that Assumption 1 holds. Then, there exists a positive-define matrix $\Xi = \mathrm{diag}(\xi_i) = \mathrm{diag}(y_i/x_i)$ such that $Q = \Xi H + H^{\mathrm{T}}\Xi$ is positive define, where $y = [y_1, y_2, \ldots, y_N]^{\mathrm{T}} = H^{-\mathrm{T}}\mathbf{1}_N$, $x = [x_1, x_2, \ldots, x_N]^{\mathrm{T}} = H^{-1}\mathbf{1}_N$, and H is positive-definite.

Assumption 2 For $i = 1, \ldots, M+N$, there exist solutions (X_i, U_i) and (X_i^I, U_t^l), such that the following equations hold:

$$\begin{aligned} &A_i X_i + B_i U_i = X_i S, C_i X_i - F = 0, \\ &A_i X_i^l + B_i U_i^l = X_i^l S_1, C_i X_i^l - F = 0. \end{aligned} \tag{12}$$

Lemma 4[3] Let

$$\begin{aligned} &\varpi_i^{j^*} = \mathrm{vec}(\Gamma_j), \quad \left(\Gamma_1 = \begin{bmatrix} X_i \\ U_i \end{bmatrix}; \Gamma_2 = \begin{bmatrix} X_i^l \\ U_i^l \end{bmatrix} \right) \\ &\varpi_i^j(t) = \mathrm{vec}(\Gamma_j(t)), \\ &\left(\Gamma_1(t) = \begin{bmatrix} X_i(t) \\ U_i(t) \end{bmatrix}; \Gamma_2(t) = \begin{bmatrix} X_i^l(t) \\ U_i^l(t) \end{bmatrix} \right), \\ &Q_i^j = Z_j^{\mathrm{T}} \otimes \begin{bmatrix} I_{n_i} & 0 \\ 0 & 0 \end{bmatrix} - I_{n_0} \otimes \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix}, \quad (Z_1 = S; Z_2 = S_l), \\ &Q_i^j(t) = Z_j(t)^{\mathrm{T}} \otimes \begin{bmatrix} I_{n_i} & 0 \\ 0 & 0 \end{bmatrix} - I_{n_0} \otimes \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix}, \\ &(Z_1(t) = S_i; Z_2(t) = S_{li}), \\ &b_i = \mathrm{vec}([0 - F]), \quad b_i(t) = \mathrm{vec}\left(\begin{bmatrix} 0 \\ -F_i \end{bmatrix} \right), \\ &i = 1, \ldots, M+N, \end{aligned} \tag{13}$$

where S_i and S_{li} are estimates of S and S_l respectively. If Assumption 2 holds and $\rho > 0$ is sufficiently large, the system

$$\begin{aligned} &\dot{\omega}_i^j(t) = -\rho Q_i^j(t)^{\mathrm{T}}(Q_i^j(t)\varpi_i^j(t) - b_i(t)), \\ &i = 1, \ldots, M+N, \quad j = 1, 2, \end{aligned} \tag{14}$$

has a unique bounded solution. Moreover, the following fact holds:

$$\lim_{t \to \infty} \left(M_{n_i + m_i}^{n_0}\left(\varpi_i^j(t)\right) - \Gamma_j \right) = 0. \tag{15}$$

3. Main result

3.1. PT-TV-OF control of heterogeneous MASs

In this section, we design a prescribed-time dynamics observer for the formation leaders to estimate all the information of the tracking leader and a prescribed-time control protocol to realize the PT-TV-OF control.

3.1.1. Prescribed-time dynamics observer

The prescribed-time dynamics observer is designed as follows:

$$\dot{S}_i = -k_1\left(1+\frac{\dot{\mu}_1}{\mu_1}\right)\phi_i,\quad \dot{F}_i = -k_1\left(1+\frac{\dot{\mu}_1}{\mu_1}\right)\phi_i^F,$$
$$\dot{\eta}_i = S_i\eta_i - k_2 P\varphi_i - k_3\frac{\dot{\mu}_2}{\mu_2}P\varphi_i,\quad i=1,\ldots,M, \tag{16}$$

where

$$\phi_i = \sum_{j=1}^{m} a_{ij}(S_i - S_j) + a_{i0}(S_i - S),$$
$$\phi_i^F = \sum_{j=1}^{m} a_{ij}(F_i - F_j) + a_{i0}(F_i - F),$$
$$\varphi_i = \sum_{j=1}^{M} a_{ij}(\eta_i - \eta_j) + a_{i0}(\eta_i - x_0),$$

S_i and F_i are estimates of S and F for the i-th formation leader, respectively, η_i is estimate of tracking leader's state x_0 for the i-th leader, $P \in R^{n_0\times n_0} > 0$, k_1, k_2, k_3 are parameters satisfying the following theorem. Let the observation errors $\bar{S}_i = S_i - S$, $\bar{F}_i = F_i - F$, and $\bar{\eta}_i = \eta_i - x_0$.

Theorem 1 Suppose that Assumption 1 holds. If $k_1 > 0$, $k_2 > 1/\lambda_1$, $k_3 > 0$, and P satisfies

$$S^{\mathrm{T}}P + PS - 2P^2 + cI_{n_0} = 0, \tag{17}$$

then observation errors of the observer (16) converge to 0 with the prescribed time T_1 and T_2, *i.e.*, when $t \to t_0 + T_1$, $\lim\limits_{t\to t_0+T_1}(S_i - S) = 0$, $\lim\limits_{t\to t_0+T_1}(F_i - F) = 0$, and when $t \to t_0 + T_2$, $\lim\limits_{t\to t_0+T_2}(\eta_i - x_0) = 0$, where $\lambda_1 = \lambda_{\min}(H)$, and $c > 0$.

Proof We prove the effectiveness of the prescribed-time dynamics observer (16) in two steps.

(i) For the observation error $\bar{S}_i = S_i - S$, it obtains $\bar{S} = \left[\bar{S}_1^{\mathrm{T}}, \bar{S}_2^{\mathrm{T}}, \ldots, \bar{S}_N^{\mathrm{T}}\right]^{\mathrm{T}}$. From the first equation of the observer (16) we can get

$$\dot{\bar{S}} = -k_1\left(1+\frac{\dot{\mu}_1}{\mu_1}\right)(H\otimes I_{n_0})\bar{S}, \tag{18}$$

where H is symmetric and positive-definite, as stated in Lemma 3. Define the operation $\langle S\rangle_j$, $j = 1,2,\ldots,n_0$, which denotes the j-th column of $\bar{S}$. Then, we obtain

$$\langle\dot{\bar{S}}\rangle_j = -k_1\left(1+\frac{\dot{\mu}_1}{\mu_1}\right)(H\otimes I_{n_i})\langle\bar{S}\rangle_j. \tag{19}$$

Considering the following Lyapunov candidate function:

$$V_1 = \frac{1}{2}\langle S\rangle_j^{\mathrm{T}}(\Xi\otimes I_{n_0})\langle S\rangle_j, \tag{20}$$

by Lemma 3, we get

$$\begin{aligned}\dot{V}_1 &= -k_1\left(1+\frac{\dot{\mu}_1}{\mu_1}\right)\langle S\rangle_j^{\mathrm{T}}(\Xi H\otimes I_{n_0})\langle S\rangle_j,\\ &\le -\frac{1}{2}k_1\lambda_{\min}(Q)\left(1+\frac{\dot{\mu}_1}{\mu_1}\right)\langle S\rangle_j^{\mathrm{T}}\langle S\rangle_j,\\ &\le -k_1\left(1+\frac{\dot{\mu}_1}{\mu_1}\right)V_1.\end{aligned} \tag{21}$$

It can be concluded that $\langle S\rangle_j$ converges to 0 with the prescribed-time T_1 based on Lemma 1. Further, $\bar{S} = 0$ is globally prescribed-time stable with T_1. So, the observation error $\bar{S}_i$ converges to 0 with T_1, *i.e.* when $t \to t_0 + T_1$, $\lim\limits_{t\to t_0+T_1}(S_i - S) = 0$. For the observation error $\bar{F}_i = F_i - F$, similarly, it can be concluded that $\bar{F}_i$ converges to 0 with T_1, *i.e.*, when $t \to t_0 + T_1$, $\lim\limits_{t\to t_0+T_1}(F_i - F) = 0$.

(ii) For the observation error $\bar{\eta}_i = \eta_i - x_0$, it obtains $\bar{\eta} = \left[\bar{\eta}_1^{\mathrm{T}}, \bar{\eta}_2^{\mathrm{T}}, \ldots, \bar{\eta}_N^{\mathrm{T}}\right]^{\mathrm{T}}$. From the second equation of the observer (16) we can get

$$\dot{\bar{\eta}} = I_M\otimes S - k_2(H\otimes P)\bar{\eta} - k_3\frac{\dot{\mu}_2}{\mu_2}(H\otimes P)\bar{\eta}. \tag{22}$$

Considering the following Lyapunov candidate function:

$$V_2 = \bar{\eta}^{\mathrm{T}}(H\otimes P)\bar{\eta}, \tag{23}$$

and taking the derivative of V_2, we obtain

$$\begin{aligned}\dot{V}_2 =\ & \bar{\eta}^{\mathrm{T}}\left[H\otimes\left(S^{\mathrm{T}}P + PS\right) - 2k_2\left(H^2\otimes P^2\right)\right]\bar{\eta}\\ & - 2k_3\bar{\eta}^{\mathrm{T}}(H\otimes P)\frac{\dot{\mu}_2}{\mu_2}\bar{\eta}.\end{aligned} \tag{24}$$

Let $\tilde{\eta} = (U\otimes I_n)\bar{\eta}$. Then, there exists an orthogonal $U \in R^{N\times N}$, where $UU^{\mathrm{T}} = UU^{-1} = I_N$, satisfying $UHU^{\mathrm{T}} = J = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_N)$, $k_2 > 1/\lambda_1$. According to Eq. (17), one has

$$\begin{aligned}&\tilde{\eta}^{\mathrm{T}}\left[J\otimes\left(S^{\mathrm{T}}P + PS\right) - 2k_2\left(J^2\otimes P^2\right)\right]\tilde{\eta}\\ &= \sum_{i=1}^{M}\tilde{\eta}^{\mathrm{T}}\lambda_i\left(S^{\mathrm{T}}P + PS - 2k_2\lambda_i P^2\right)\tilde{\eta}_i\\ &\le -c\lambda_1\sum_{i=1}^{M}\tilde{\eta}^{\mathrm{T}}\tilde{\eta}\\ &\le -\frac{c\lambda_1}{\lambda_{\max}(\Gamma)}V_2,\end{aligned} \tag{25}$$

where $\Gamma = H\otimes P$. Then we can get

$$\dot{V}_2 \le -\frac{c\lambda_1}{\lambda_{\max}(\Gamma)}V_2 - 2k_3\frac{\dot{\mu}_2}{\mu_2}V_2. \tag{26}$$

It can be concluded that $\bar{\eta}$ converges to 0 with the prescribed time T_2 based on Lemma 1. So, the observation error $\bar{\eta}_i$ converges to 0 with T_2, *i.e.*, when $t \to t_0 + T_2$, $\lim\limits_{t\to t_0+T_2}(\eta_i - x_0) = 0$.

The proof is completed.

3.1.2. PT-TV-OF control protocol

To design the control protocol, we need to solve the regulator equations (12). However, in the case where the leader's system matrix is not available to each agent, we cannot directly solve Eq. (12). So, we adopt the method in Lemma 4 to obtain the adaptive solutions $X_i(t)$, $U_i(t)$, and U_i^l, $X_i^l(t)$. Then, a prescribed-time control protocol based on the prescribed-time dynamics observer is designed as follows:

$$u_i = k_{i1}x_i + k_{i2}(t)\eta_i + k_{i3}(t)l_i - k_{i4}\frac{\dot{\mu}_{\rm F}}{\mu_{\rm F}}\bar{x}_i, \quad i = 1,\ldots,M, \tag{27}$$

where $\bar{x}_i = x_i - X_i(t)\eta_i - X_i^l(t)l_i$ and $k_{i2}(t) = U_i(t) - k_{i1}X_i(t)$, $k_{i3}(t) = U_i^l(t) - k_{i1}X_i^l(t)$, k_{i1} and k_{i4} are appropriate control gains. The time-varying scaling function $\mu_{\rm F}$ is as follows:

$$\mu_{\rm F} = \begin{cases} \left(\dfrac{T_{\rm F}}{t_0 + T_{\rm F} - t}\right)^h, & t_0 \le t < t_0 + T_{\rm F}, \\ 1, & t \ge t_0 + T_{\rm F}. \end{cases}$$

Theorem 2 Suppose that Assumptions 1 and 2 hold. For heterogeneous MASs (1) and (2), if k_{i1} satisfies that $A_i + B_i k_{i1}$ is Hurwitz, k_{i4} satisfies $B_i k_{i4} = I_{n_i \times n_i}$, then the PT-TV-OF (10) can be achieved under the prescribed-time control protocol (27).

Proof Define some error variables $\bar{k}_{i2}(t) = k_{i2}(t) - k_{i2}$, $\bar{k}_{i3}(t) = k_{i3}(t) - k_{i3}$, $\bar{X}_i(t) = X_i(t) - X_i$, and $\bar{X}_i^l(t) = X_i^l(t) - X_i^l$, then,

$$\begin{aligned}
\dot{\bar{x}}_i &= \dot{x}_i - X_i\dot{x}_0 - X_i^l\dot{l}_i \\
&= \bar{A}_i x_i + B_i k_{i2}(t)\left(\bar{\eta}_i + x_0\right) + B_i\left(\bar{k}_{i3}(t) + k_{i3}\right)l_i \\
&\quad - k_{i4}\frac{\dot{\mu}_F}{\mu_F}\left\{x_i - X_i(t)\left(\bar{\eta}_i + x_0\right) - \left(\bar{X}_i^l(t) + X_i^l\right)l_i\right\} \\
&\quad - X_i S x_0 - X_i^l S l_i \\
&= \bar{A}_i x_i + B_i k_{i2} x_0 + B_i k_{i3} l_i - X_i S x_0 - X_i^l S l_i + \Pi_1 \\
&\quad - B_i k_{i4}\frac{\dot{\mu}_F}{\mu_F}\left(\bar{x}_i - \Pi_2\right) \\
&= \bar{A}_i x_i + B_i\left(U_i - k_{i1}X_i\right)x_0 + B_i\left(U_i^l - k_{i1}X_i^l\right)l_i \\
&\quad - X_i S x_0 - X_i^l S l_i + \Pi_1 - \frac{\dot{\mu}_F}{\mu_F}\left(\bar{x}_i - \Pi_2\right) \\
&= \bar{A}_i\bar{x}_i + \Pi_1 - \frac{\dot{\mu}_F}{\mu_F}\left(\bar{x}_i - \Pi_2\right),
\end{aligned} \tag{28}$$

where $\bar{A}_i = A_i + B_i k_{i1}$, $\Pi_1 = B_i\bar{k}_{i2}(t)x_0 + B_i k_{i2}(t)\bar{\eta}_i + B_i\bar{k}_{i3}(t)l_i$, and $\Pi_2 = \bar{X}_i(t)x_0 + X_i(t)\bar{\eta}_i + \bar{X}_i^l(t)l_i$.

When $t \in (t_0, t_0 + T_2)$, $\bar{x}_i$ is bounded due to the fact that $\bar{\eta}_i$, $\bar{X}_i(t)$, $\bar{k}_{i2}(t)$, and $\bar{k}_{i3}(t)$ are bounded. When $t \in (t_0 + T_2, \infty)$, these four error variables are zero. Then,

$$\dot{\bar{x}}_i = \bar{A}_i\bar{x}_i - \frac{\dot{\mu}_F}{\mu_F}\bar{x}_i. \tag{29}$$

Considering Lyapunov candidate function $V_3 = \bar{x}_i^{\rm T}\bar{x}_i$, we can get

$$\begin{aligned}
\dot{V}_3 &= \bar{x}_i^{\rm T}\left(\bar{A}_i^{\rm T} + \bar{A}_i\right)\bar{x}_{\rm s} - 2\frac{\dot{\mu}_F}{\mu_F}\bar{x}_i^{\rm T}\bar{x}_i, \\
&\le \lambda_{\max}\left(\bar{A}_i^{\rm T} + \bar{A}_i\right)V_3 - 2\frac{\dot{\mu}_F}{\mu_F}V_3.
\end{aligned} \tag{30}$$

By Lemma 1, it can be obtained that $\bar{x}_i$ converges to 0 with the prescribed-time $T_{\rm F}$. Hence, when $t \to t_0 + T_{\rm F}$, $\lim\limits_{t\to t_0+T_{\rm F}}(y_i - y_0 - Fl_i) = \lim\limits_{t\to t_0+T_{\rm F}} C_i\bar{x}_i = 0$, *i.e.*, the heterogeneous MASs (1) and (2) can achieve the PT-TV-OF control.

The proof is completed.

3.2. PT-TV-OFC control of heterogeneous MASs

Next, we will design the observer and control protocol for the followers to achieve the PT-TV-OFC control. From Eq. (11) and Lemma 2, a convex hull formed by the formation leaders can be represented as $co(y_l) = \left(-L_4^{-1}L_3 \otimes I_q\right)y_l$. From above discussion, $\lim\limits_{t\to t_0+T_{\rm F}} y_l = (I_M \otimes F)(1_M \otimes x_0 + l)$ holds, where $l = \left[l_1^{\rm T},\ldots,l_M^{\rm T}\right]^{\rm T}$. Then, $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)(1_M \otimes x_0 + l)$ is key to achieve the PT-TV-OFC control. Hence, we will design the prescribed-time convex hull observers to estimate $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)(1_M \otimes x_0)$ and $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)l$, respectively.[30]

3.2.1. Prescribed-time convex hull observer for component $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)(1_M \otimes x_0)$

The prescribed-time convex hull observer for component $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)(1_M \otimes x_0)$ is designed as follows:

$$\begin{aligned}
\dot{S}_i &= -k_4\left(1 + \frac{\dot{\mu}_3}{\mu_3}\right)\xi_{i1}, \\
\dot{F}_i &= -k_4\left(1 + \frac{\dot{\mu}_3}{\mu_3}\right)\xi_{i1}^{\rm F}, \\
\dot{v}_i &= S_i v_i - k_5\zeta_i - k_6\left(1 + \frac{\dot{\mu}_4}{\mu_4}\right)\zeta_i, \\
i &= M+1,\ldots,M+N,
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\xi_{i1} &= \sum_{j=1}^{M} a_{ij}(S_i - S_j) + \sum_{j=M+1}^{M+N} a_{ij}(S_i - S_j), \\
\xi_{i1}^{\rm F} &= \sum_{j=1}^{M} a_{ij}(F_i - F_j) + \sum_{j=M+1}^{N+N} a_{ij}(F_i - F_j), \\
\zeta_i &= \sum_{j=1}^{M} a_{ij}(v_i - \eta_j) + \sum_{j=M+1}^{M+N} a_{ij}(v_i - v_j),
\end{aligned}$$

S_i and F_i are estimates of S and F for the i-th follower, v_i is estimate of component $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)(1_M \otimes x_0)$ for the i-th follower, k_4, k_5, and k_6 are appropriate parameters satisfying the following theorem.

Theorem 3 Suppose that Assumptions 1 and 2 hold. If $k_4 > 0$, $k_6 > 0$, and k_5 is sufficiently large, then observation errors of the observers (31) converge to 0 with the prescribed-time T_3 and T_4, *i.e.*, $t \to t_0 + T_3$, then $\lim\limits_{t\to t_0+T_3}(S_i - S) = 0$, $\lim\limits_{t\to t_0+T_3}(F_i - F) = 0$, and $t \to t_0 + T_4$, then

$$\lim_{t\to t_0+T_4} v = \left(-L_4^{-1}L_3 \otimes I_{n_0}\right)(1_M \otimes x_0), \tag{32}$$

where $\nu = \left[\nu_{M+1}^{\mathrm{T}}, \ldots, \nu_{M+N}^{\mathrm{T}}\right]^{\mathrm{T}}$.

Proof The proof of the part to estimate S and F is similar to the mentioned-above Subsection 3.1.1, then it is omitted here.

Let $\zeta = \left[\zeta_{M+1}^{\mathrm{T}}, \ldots, \zeta_{M+N}^{\mathrm{T}}\right]^{\mathrm{T}}$, then $\zeta = \left(L_3 \otimes I_{n_0}\right)\eta + \left(L_4 \otimes I_{n_0}\right)\nu$, $\nu = \left(L_4^{-1} \otimes I_{n_0}\right)\zeta - \left(L_4^{-1}L_3 \otimes I_{n_0}\right)\eta$. We take the derivative and get

$$\begin{aligned}\dot{\zeta} &= \left(L_3 \otimes I_{n_0}\right)\dot{\eta} + \left(L_4 \otimes I_{n_0}\right)\dot{\nu}\\ &= \left(L_3 \otimes I_{n_0}\right)\dot{\eta} + \left(L_4 \otimes I_{n_0}\right)\\ &\quad \times\Big\{\tilde{S}\nu - k_5\left(L_4 \otimes I_{n_0}\right)\zeta\\ &\quad - k_6\left(1 + \frac{\dot{\mu}_4}{\mu_4}\right)\left(L_4 \otimes I_{n_0}\right)\zeta\Big\}, \end{aligned} \tag{33}$$

where $\tilde{S} = \mathrm{blockdiag}\left(S_{M+1}, \ldots, S_{M+N}\right)$. When $t \geq t_0 + T_3$, obviously, $S_i - S = 0$ and $\bar{\eta}_0$ hold. It leads to $\check{S} = I_N \otimes S$, and $\dot{\eta} = 1_M \otimes \dot{x}_0 = 1_M \otimes Sx_0$. Then,

$$\begin{aligned}\dot{\zeta} &= \left(L_3 \otimes I_{n_0}\right)\left(I_m \otimes SX_0\right) + \left(I_N \otimes S\right)\zeta - \left(L_3 \otimes S\right)\left(I_M \otimes X_0\right)\\ &\quad - k_5\left(L_4^2 \otimes I_{n_0}^2\right)\zeta - k_6\left(1 + \frac{\dot{\mu}_5}{\mu_5}\right)\left(L_4^2 \otimes I_{n_0}^2\right)\zeta\\ &= \left(I_N \otimes S - k_5\left(L_4^2 \otimes I_{n_0}^2\right)\right)\zeta\\ &\quad - k_6\left(1 + \frac{\dot{\mu}_4}{\mu_4}\right)\left(L_4^2 \otimes I_{n_0}\right)\zeta. \end{aligned} \tag{34}$$

Considering Lyapunov candidate function $V_4 = \zeta^{\mathrm{T}}\zeta$, we obtain

$$\begin{aligned}\dot{V}_4 &= \zeta^{\mathrm{T}}\left[I_N \otimes \left(S^{\mathrm{T}} + S\right) - 2k_5\left(L_4^2 \otimes I_{n_0}^2\right)\right]\zeta\\ &\quad - 2k_6\left(1 + \frac{\dot{\mu}_4}{\mu_4}\right)\zeta^{\mathrm{T}}\left(L_4^2 \otimes I_{n_0}^2\right)\zeta. \end{aligned} \tag{35}$$

Since L_4 is symmetric, an orthogonal matrix $O \in R^{N\times N}$ must exist

$$OL_4^2O^{\mathrm{T}} = J_{L_4} = \mathrm{diag}\left(\lambda_{L_4^2(M+1)}, \ldots, \lambda_{L_4^2(M+N)}\right). \tag{36}$$

Let $\tilde{\zeta} = \left[\tilde{\zeta}_{M+1}^{\mathrm{T}}, \ldots, \tilde{\zeta}_{M+N}^{\mathrm{T}}\right]^{\mathrm{T}} = \left(O \otimes I_{n_0}\right)\zeta$. Then, we get

$$\begin{aligned}\dot{V}_4 &= \sum_{i=m+1}^{M+N}\left(\tilde{\zeta}_i^{\mathrm{T}}\left(S^{\mathrm{T}} + S - 2k_5\lambda_{L_4^2(i)}I_{n_0}\right)\tilde{\zeta}_i\right)\\ &\quad - 2k_6\left(1 + \frac{\dot{\mu}_4}{\mu_4}\right)\zeta^{\mathrm{T}}\left(L_4^2 \otimes I_{n_0}^2\right)\zeta. \end{aligned} \tag{37}$$

Since k_5 is sufficiently large, $S^{\mathrm{T}} + S - 2k_5\lambda_{L_4(i)}I_{n_0}$ can be Hurwitz. We can get

$$\dot{V}_4 \leq -2k_6\sigma\left(1 + \frac{\dot{\mu}_4}{\mu_4}\right)V_4, \tag{38}$$

where

$$\sigma \in \left(0, \max\left\{\sum_{j=1}^{M} a_{(M+1)j}, \ldots, \sum_{j=1}^{M} a_{(M+N)j}\right\}\right).$$

So, we can conclude that ζ converges to 0 with prescribed-time T_4 based on Lemma 1. Hence, when $t \to t_0 + T_4$, $\lim\limits_{t\to t_0+T_4} v = \left(-L_4^{-1}L_3 \otimes I_{n_0}\right)\left(1_M \otimes x_0\right)$.

The proof is completed.

3.2.2. Prescribed-time convex hull observer for component $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)l$

The prescribed-time convex hull observer for component $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)l$ is designed as follows:

$$\begin{aligned}\dot{S}_{li} &= -k_7\left(1 + \frac{\dot{\mu}_3}{\mu_3}\right)\xi_{i2},\\ \dot{w}_i &= S_{li}\omega_i - k_8\zeta_{i2} - k_9\left(1 + \frac{\dot{\mu}_4}{\mu_4}\right)\zeta_{i2},\\ i &= M+1, \ldots, M+N, \end{aligned} \tag{39}$$

where

$$\begin{aligned}\xi_{i2} &= \sum_{j=1}^{M} a_{ij}\left(S_{li} - S_{lj}\right) + \sum_{j=M+1}^{M+N} a_{ij}\left(S_{li} - S_{lj}\right),\\ \zeta_{i2} &= \sum_{j=1}^{M} a_{ij}\left(w_i - l_i\right) + \sum_{j=M+1}^{M+N} a_{ij}\left(w_i - w_j\right), \end{aligned}$$

S_{li} is estimate of S_l for the i-th follower, w_i is estimate of component for the i-th follower, k_7, k_8, and k_9 are parameters satisfying the following theorem.

Theorem 4 Suppose that Assumptions 1 and 2 hold. If $k_7 > 0$, $k_9 > 0$, and k_8 is sufficiently large, then observation errors of the observer (39) converge to 0 with the prescribed-times T_3 and T_4, *i.e.*, when $t \to t_0 + T_3$, $\lim\limits_{t\to t_0+T_3}\left(S_{li} - S_l\right) = 0$, and when $t \to t_0 + T_4$, there is

$$\lim_{t\to t_0+T_4} w = \left(-L_4^{-1}L_3 \otimes I_{n_0}\right)l, \tag{40}$$

where $w = \left[w_{M+1}^{\mathrm{T}}, \ldots, w_{M+N}^{\mathrm{T}}\right]^{\mathrm{T}}$.

Proof We can refer to Subsection 3.2.1 for the proof, which is omitted here.

3.2.3. PT-TV-OFC control protocol

Based on the above observers, we design control protocol for followers as follows:

$$\begin{aligned}u_i &= k_{i1}x_i + k_{i2}(t)\nu_i + k_{i3}(t)w_i\\ &\quad - k_{i4}\frac{\dot{\mu}_{\mathrm{C}}}{\mu_{\mathrm{C}}}\left(x_i - X_i(t)\nu_i - X_i^l(t)w_i\right),\\ &\quad i = M+1, \ldots, M+N, \end{aligned} \tag{41}$$

where $k_{i2}(t) = U_i(t) - k_{i1}X_i(t)$, $k_{i3}(t) = U_i^l(t) - k_{i1}X_i^l(t)$, k_{i1} and k_{i4} are control gains satisfying following theorem. $\left(X_i(t), U_i(t)\right)$ and $\left(X_i^l(t), U_i^l(t)\right)$ are calculated by Lemma 4. The time-varying scaling function μ_{C} is as follows:

$$\mu_{\mathrm{C}} = \begin{cases}\left(\frac{T_{\mathrm{C}}}{t_0 + T_{\mathrm{C}} - t}\right)^h, & t_0 \leq t < t_0 + T_{\mathrm{C}},\\ 1, & t \geq t_0 + T_{\mathrm{C}}.\end{cases}$$

Theorem 5 Suppose that Assumptions 1 and 2 hold. For heterogeneous MASs (1) and (2), if k_{i1} satisfies that $A_i + B_ik_{i1}$

is Hurwitz, k_{i4} satisfies $B_i k_{i4} = I_{n_i \times n_i}$, then the PT-TV-OFC can be achieved under the prescribed-time control protocol (41) and based on Theorem 2.

Proof First, we define some matrixes

$$\begin{aligned}
&x_{\mathrm{s}} = \left[x_{M+1}^{\mathrm{T}}, \ldots, x_{M+N}^{\mathrm{T}}\right]^{\mathrm{T}},\\
&X_{\mathrm{s}} = \mathrm{blockdiag}\left(X_{M+1}, \ldots, X_{M+N}\right),\\
&X_l = \mathrm{blockdiag}\left(X_{M+1}^l, \ldots, X_{M+N}^l\right),\\
&K_{\mathrm{s}1} = \mathrm{blockdiag}\left(k_{(M+1)1}, \ldots, k_{(M+N)1}\right),\\
&K_{\mathrm{s}2} = \mathrm{blockdiag}\left(k_{(M+1)2}, \ldots, k_{(M+N)2}\right),\\
&K_{\mathrm{s}3} = \mathrm{blockdiag}\left(k_{(M+1)3}, \ldots, k_{(M+N)3}\right),\\
&\Theta_{\mathrm{s}} = \mathrm{blockdiag}\left(\Theta_{M+1}, \ldots, \Theta_{M+N}\right),
\end{aligned}$$

where Θ can denote A, B, and C.

Let

$$\bar{x}_{\mathrm{s}} = x_{\mathrm{s}} - X_{\mathrm{s}}\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)\left(1_M \otimes x_0\right) - X_l\left(-L_4^{-1}L_3 \otimes I_{n_0}\right) l,$$

$\bar{X}_{\mathrm{s}}(t) = X_{\mathrm{s}}(t) - X_{\mathrm{s}}$, $\bar{X}_l(t) = X_l(t) - X_l$, $\bar{K}_{\mathrm{s}1}(t) = K_{\mathrm{s}1}(t) - K_{\mathrm{s}1}$, $\bar{K}_{\mathrm{s}2}(t) = K_{\mathrm{s}2}(t) - K_{\mathrm{s}2}$, and $\bar{K}_{\mathrm{s}3}(t) = K_{\mathrm{s}3}(t) - K_{\mathrm{s}3}$. Then, one has

$$\begin{aligned}
\dot{\bar{x}}_{\mathrm{s}} \;=\; & \left(A_{\mathrm{s}} + B_{\mathrm{s}}K_{\mathrm{s}1}\right)x_{\mathrm{s}} + B_{\mathrm{s}}K_{\mathrm{s}2}\nu + B_{\mathrm{s}}K_{\mathrm{s}3}w\\
& - X_{\mathrm{s}}\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)\left(I_M \otimes S\right)\left(1_M \otimes x_0\right)\\
& - X_l\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)\left(I_M \otimes S_l\right) l\\
& + \Pi_3 - \frac{\dot{\mu}_{\mathrm{C}}}{\mu_{\mathrm{C}}}\left\{x_{\mathrm{s}} - X_{\mathrm{s}}\nu - X_l w - \Pi_4\right\}, \qquad (42)
\end{aligned}$$

where $\Pi_3 = B_{\mathrm{s}}\bar{K}_{\mathrm{s}2}(t)\nu + B_{\mathrm{s}}\bar{K}_{\mathrm{s}3}(t)w$, $\Pi_4 = \bar{X}_{\mathrm{s}}(t)\nu + \bar{X}_l(t)w$. When $t \in (t_0, t_0+T_4)$, $\bar{x}_{\mathrm{s}}$ is bounded since the error variables $\bar{X}_{\mathrm{s}}(t)$, $\bar{X}_l(t)$, $\bar{K}_{\mathrm{s}2}(t)$, and $\bar{K}_{\mathrm{s}3}(t)$ are bounded. When $t \in (t_0+T_4, \infty)$, these four error variables are zero. Moreover, $v = \left(-L_4^{-1}L_3 \otimes I_{n_0}\right)\left(1_M \otimes x_0\right)$ and $w = \left(-L_4^{-1}L_3 \otimes I_{n_0}\right) l$ hold by Theorem 3 and Theorem 4. Using the regulation equations (12), equation (42) can be written as

$$\dot{\bar{x}}_{\mathrm{s}} = \left(A_{\mathrm{s}} + B_{\mathrm{s}}K_{\mathrm{s}1}\right)\bar{x}_{\mathrm{s}} - \frac{\dot{\mu}_{\mathrm{C}}}{\mu_{\mathrm{C}}}\bar{x}_{\mathrm{s}} = \bar{A}_{\mathrm{s}}\bar{x}_{\mathrm{s}} - \frac{\dot{\mu}_{\mathrm{C}}}{\mu_{\mathrm{C}}}\bar{x}_{\mathrm{s}}, \qquad (43)$$

where $\bar{A}_{\mathrm{s}} = A_{\mathrm{s}} + B_{\mathrm{s}}K_{\mathrm{s}1}$.

Let $V_5 = \bar{x}_{\mathrm{s}}^{\mathrm{T}}\bar{x}_{\mathrm{s}}$. Then, we get

$$\begin{aligned}
\dot{V}_5 \;&=\; \bar{x}_{\mathrm{s}}^{\mathrm{T}}\left(\bar{A}_i^{\mathrm{T}} + \bar{A}_i\right)\bar{x}_{\mathrm{s}} - 2\frac{\dot{\mu}_{\mathrm{C}}}{\mu_{\mathrm{C}}}\bar{x}_{\mathrm{s}}^{\mathrm{T}}\bar{x}_{\mathrm{s}}\\
&\leq\; \lambda_{\max}\left(\bar{A}_i^{\mathrm{T}} + \bar{A}_i\right)V_{\mathrm{s}} - 2\frac{\dot{\mu}_{\mathrm{C}}}{\mu_{\mathrm{C}}}V_{\mathrm{s}}. \qquad (44)
\end{aligned}$$

By Lemma 1, it can be obtained that $\bar{x}_{\mathrm{s}}$ converges to 0 with the prescribed-time T_{C}. Hence, when $t \to t_0 + T_{\mathrm{F}}$, we can get

$$\begin{aligned}
&\lim_{t \to t_0+T_{\mathrm{C}}}\left(y_{\mathrm{s}} - co(y_l)\right)\\
&= \lim_{t \to t_0+T_{\mathrm{C}}} y_{\mathrm{s}} - \left(I_N \otimes F\right)\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)\left(1_M \otimes x_0 + l\right)\\
&= \lim_{t \to t_0+T_{\mathrm{C}}} C_{\mathrm{s}}x_{\mathrm{s}} - C_{\mathrm{s}}X_{\mathrm{s}}\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)\\
&\quad \times \left(1_M \otimes x_0\right) - C_{\mathrm{s}}X_l\left(-L_4^{-1}L_3 \otimes I_{n_0}\right) l\\
&= 0.
\end{aligned}$$

Combining with Theorem 2, it can be concluded that the heterogeneous multi-agent systems can achieve PT-TV-OFC control.

The proof is completed.

4. Numerical simulations

In this example, consider a heterogeneous MASs consisting of a tracking leader labeled 0, four formation leaders labeled 1, 2, 3, 4, and two followers labeled 5, 6. The communication topology is shown in Fig. 1. The control target is to achieve PT-TV-OFC control under the designed control protocols (27) and (41).

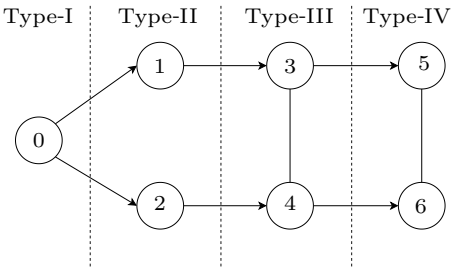


Fig. 1. Communication topology of heterogeneous MASs.

The expected time-varying formation shape is designed as

$$S_l = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -\omega^2\\ 0 & 0 & -\omega^2 & 0 \end{bmatrix},$$

$$l_i = \begin{bmatrix} r\sin(\omega t + (i-1)\pi/2) - r\cos(\omega t + (i-1)\pi/2)\\ r\sin(\omega t + (i-1)\pi/2)\\ r\omega\sin(\omega t + (i-1)\pi/2) + r\omega\cos(\omega t + (i-1)\pi/2)\\ r\omega\cos(wt + (i-1)\pi/2) \end{bmatrix},$$

where $\omega = 0.2$, $r = 10$. Let the initial time $t_0 = 0$, and initial states $x_i(0)$ $(i = 0, 1, \ldots, 6)$ of all agents be random values. Let $\eta_i(0) = [0,0,0,0]^{\mathrm{T}}$ $(i = 1, \ldots, 4)$, $v_i(0) = [0,0,0,0]^{\mathrm{T}}$, and $w_i(0) = [0,0,0,0]^{\mathrm{T}}$ $(i = 5, 6)$.

The system matrices of the tracking leader are defined as

$$S = \begin{bmatrix} 0.1 & 0 & 0 & 0\\ 0 & 0.1 & 0 & 0\\ 0 & 0 & 0.1 & 0\\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad F = \begin{bmatrix} 0.5 & 0.5 & 0 & 0\\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}.$$

The system matrix of the formation leader and follower are defined as

$$A_{1,2,5} = \begin{bmatrix} 0 & 1\\ -0.2 & -0.2 \end{bmatrix}, \quad B_{1,2,5} = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix},$$

$$C_{1,2,5} = \begin{bmatrix} 1 & 1\\ -2 & 2 \end{bmatrix},$$

$$A_{3,4,6} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0.5 & 1 & -2 \end{bmatrix}, \quad B_{3,4,6} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix},$$

$$C_{3,4,5} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$$

Figure 2 shows the output trajectories of all agents. Figure 3 shows the output snapshots of all agents at different moments. From Figs. 2 and 3, we can see that all formation leaders form a time-varying formation moving around the tracking leader and all followers lie in the convex hull of formation leaders.

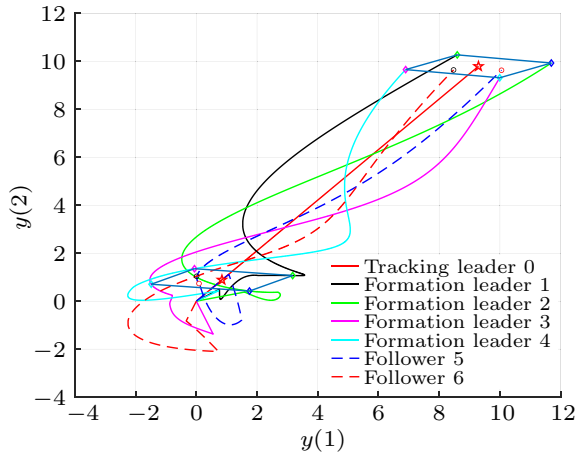


Fig. 2. Output trajectories of all agents.

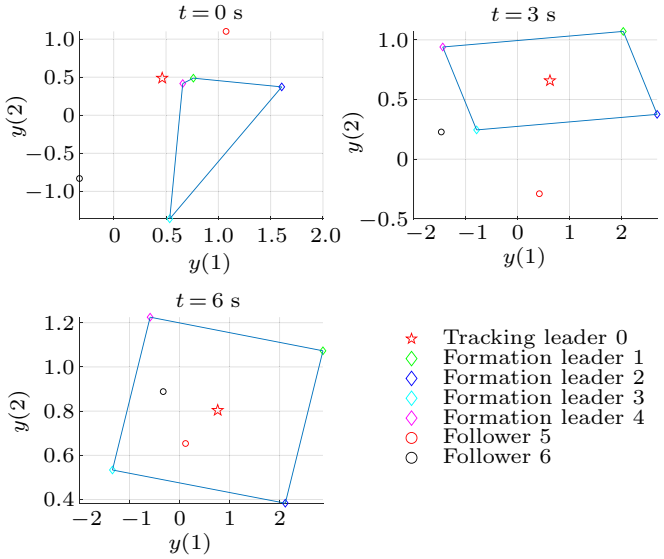


Fig. 3. Output snapshots of all agents at different moments.

According to Theorem 1, the relevant parameters are chosen as $k_1 = k_2 = k_3 = 0.5$, $T_1 = 1$ s, $T_2 = 2$ s. The time responses of observer estimation error of formation leaders $\eta_i - x_0$ are shown in Fig. 4. It is obvious that the observer estimation errors converge to 0 with the prescribed-time T_2.

According to Theorem 2, the necessary parameters are chosen as

$$k_{11} = \begin{bmatrix} -15.2 & -12.1 \\ -4.8 & -13.5 \end{bmatrix}, \quad k_{21} = \begin{bmatrix} -1.8 & -1.2 \\ -0.4 & -0.58 \end{bmatrix},$$

$$k_{31} = k_{41} = \begin{bmatrix} -1.8 & -1.2 & -1.5 \\ 1.5 & -0.4 & -0.5 \\ -0.4 & -0.58 & -1.1 \end{bmatrix},$$

$$k_{14} = k_{24} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad k_{44} = k_{64} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$k_{i2}(t) = U_i(t) - k_{i1}X_i(t)$, $k_{i3}(t) = U_i^l(t) - k_{i1}X_i^l(t)$, $i = 1, \ldots, 4$, where $U_i(t)$, $X_i(t)$, $U_i^l(t)$, and $X_i^l(t)$ are calculated by Lemma 4. Let $T_F = 3$ s. Figure 5 shows the formation errors of formation leaders converge to 0 with 3 s. It indicates that formation leaders can achieve PT-TV-OF with the prescribed-time T_F.

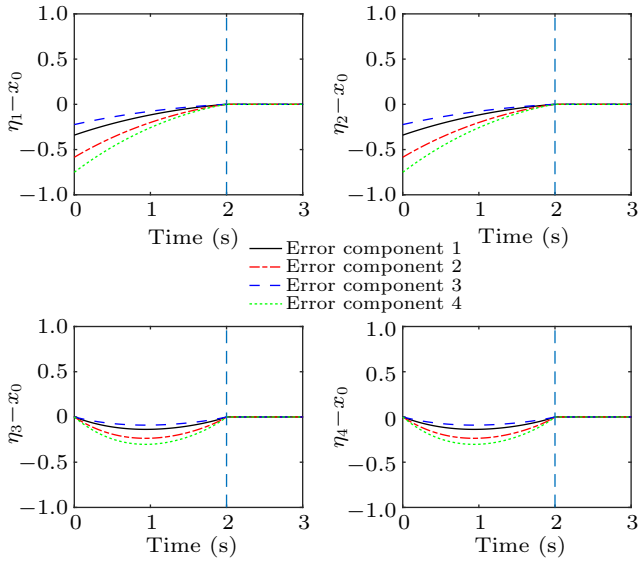


Fig. 4. Variation of the observer estimation errors of all formation leaders $\eta_i - x_0$, $i = 1, 2, 3, 4$.

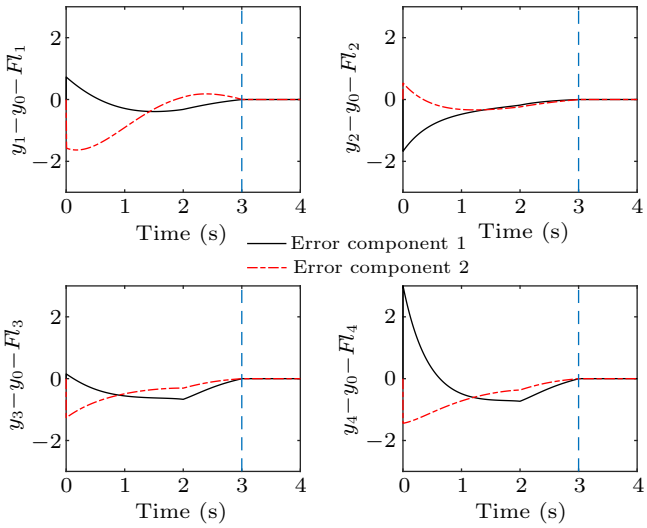


Fig. 5. Variation of formation errors of formation leaders $y_i - y_0 - Fl_i$, $i = 1, 2, 3, 4$.

According to Theorems 3 and 4, relevant parameters are set as $k_4 = k_5 = k_6 = 0.4$, $k_7 = k_8 = k_9 = 0.25$, $T_3 = 4$ s, $T_4 = 5$ s. The time responses of estimation error of the prescribed-time convex hull observer $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)(1_M \otimes x_0)$ and $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right) l$ are shown in Figs. 6 and 7. It is obvious that the observer estimation errors converge to 0 with the prescribed-time T_4.

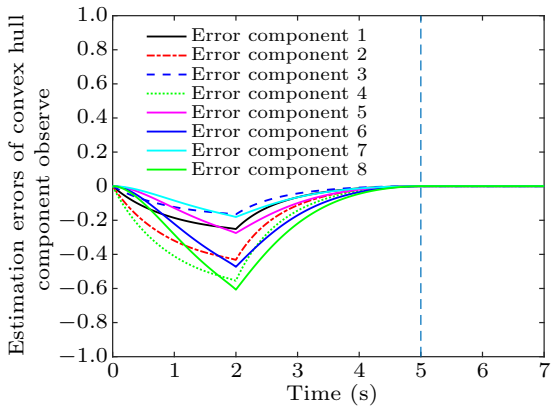


Fig. 6. Estimation errors variation of prescribed-time convex hull observer $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right)(1_M \otimes x_0)$.

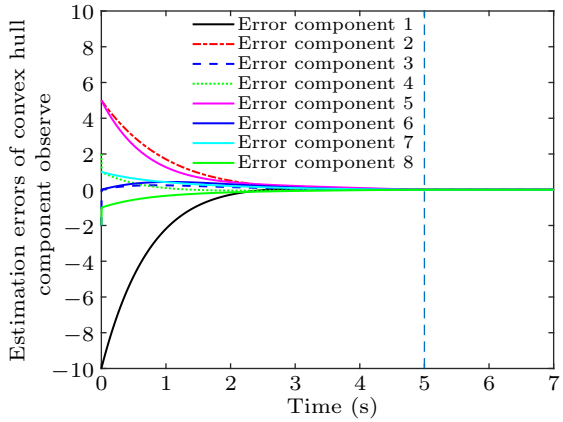


Fig. 7. Estimation errors variation of prescribed-time convex hull observer $\left(-L_4^{-1}L_3 \otimes I_{n_0}\right) l$.

According to Theorem 5, let $T_C = 6$ s, $k_{51} = k_{11}$, $k_{61} = k_{31}$, $k_{54} = k_{14}$, $k_{64} = k_{14}$, $k_{i2}(t) = U_i(t) - k_{i1}X_i(t)$, $k_{i3}(t) = U_i^l(t) - k_{i1}X_i^l(t)$, $i = 5, 6$, where $U_i(t)$, $X_i(t)$, $U_i^l(t)$, and $X_i^l(t)$ are calculated by Lemma 4. The containment error time response graph is shown in Fig. 8. We can see that the followers are able to enter the convex hull formed by the formation leaders with the prescribed time T_C. Combining with Fig. 5, it can be concluded that heterogeneous MASs can achieve PT-TV-OFC control under the designed control protocols.

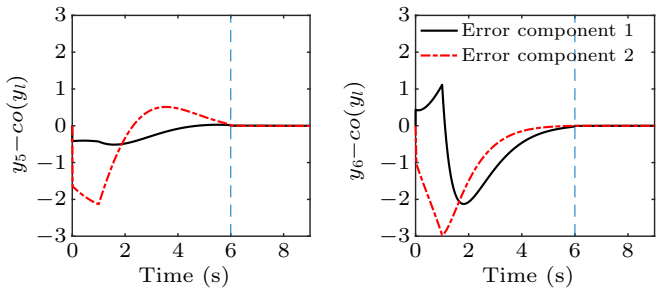


Fig. 8. Variation of containment errors $y_i - co(y_l)$, $i = 5, 6$.

5. Conclusion

This paper has addressed the observer-based PT-TV-OFC control problem of heterogeneous MASs. We have designed prescribed-time dynamics observer and convex hull observer to obtain the information of the tracking leader and the convex hull formed by the formation leaders, respectively. Based on these observers, we have designed prescribed-time control protocols for the formation leader and the follower, respectively. Using Lyapunov theory, we have proved that the heterogeneous MASs can realize the PT-TV-OFC control under these control protocols in detail. Finally, simulations have been given to verify the theoretical results.

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