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Entropy evolution law of a general linear state for diffusion noise

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Based on the Kraus operator-sum representation of the analytical solution of the diffusion equation, we obtain the evolution of a general linear state in the diffusion channel. Also, we study the quantum statistical properties of the initial general linear state and its von-Neumann entropy evolution in the diffusion channel, especially find that the entropy evolution is influenced by the diffusion noise and the thermal parameter but without the displacement.

Keywords: general linear state, diffusion channel, quantum statistical property, entropy evolution

PACS: 42.50.–p, 03.65.–w, 05.30.–d

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1. Introduction

The diffusion noise often occurs and can deteriorate the non-classicality of the systems when the systems are immersed in the realistic environments.[1–5] This is because that, under the influence of diffusion, there exists the exchange of the particles between the systems and their surrounding heat environments.

In quantum physics, the master equation describing the diffusion channel reads $\mathrm{d}\rho_t/\mathrm{d}t = -\kappa\left(\left[a^\dagger, a\rho_t\right] + \left[\rho_t a, a^\dagger\right]\right)$, κ is the diffusion factor.[6,7] Using the thermo entangled state $|\varsigma\rangle$ representation,[8–10] and the operators correspondence relation between the fictitious $(\tilde{a},\tilde{a}^\dagger)$ and real $(a,a^\dagger)$ modes in the state $|\varsigma = 0\rangle$, *i.e.*, $a \Leftrightarrow \tilde{a}^\dagger$, $a^\dagger \Leftrightarrow \tilde{a}$ and $(a^\dagger a)^n \Leftrightarrow (\tilde{a}^\dagger \tilde{a})^n$, we obtain the solution of the diffusion equation as[4,5]

$$\rho_t = \sum_{m,n=0}^{\infty} M_{m,n}\rho_0 M_{m,n}^\dagger, \tag{1}$$

which is the infinite Kraus operator-sum form of ρ_t, where

$$M_{m,n} = \sqrt{\frac{A^{m+n+1}(\kappa t)^{m+n}}{m!n!}}\, a^{\dagger m}\mathrm{e}^{a^\dagger a \ln A} a^n \tag{2}$$

is the Kraus operator, $A = 1/(1+\kappa t)$.

In this paper, inputting the general linear state with the thermal parameter β and the displaced parameter γ (where : : indicates normal ordering[11,12])

$$\rho_0 = \mathrm{e}^{\delta} :\, \exp\left(\gamma a^\dagger + \beta a^\dagger a + \gamma^* a\right) : \tag{3}$$

as an initial state into the diffusion channel, we investigate the evolution law of the state ρ_0 via the density operator, the statistical feature and the von-Neumann entropy. It should be emphasized that, the state ρ_0 has generality since it can be simplified as the coherent state, the thermal state and the mixture of thermal states, *etc*. Moreover, the von-Neumann entropy of the state ρ_0 is diffcult to be addressed as a result of the natural logarithm of ρ_0 in its original definition, however treating its entropy made sense since the entropy measures the disorder of the system.

The paper is arranged as follows. In Section 2, we obtain the normalization of the state ρ_0. In Section 3, we obtain the analytical evolution law of the state ρ_0 in the diffusion channel. In Sections 4 and 5, we devote to investigating the quantum statistical features and the von-Neumann entropy evolution of the state ρ_0 for the diffusion noise. Some main results are summarized in Section 6.

2. Normalization of ρ_0

Using the complete relation of coherent states

$$|z\rangle = \mathrm{e}^{-|z|^2/2}\mathrm{e}^{za^\dagger}|0\rangle$$

and the mathematical integral formula

$$\int \frac{\mathrm{d}^2\alpha}{\pi}\mathrm{e}^{\varsigma|\alpha|^2 + s\alpha + h\alpha^*} = -\frac{1}{\varsigma}\exp(-sh/\varsigma), \quad \mathrm{Re}\,\varsigma < 0, \tag{4}$$

as well as the integration within ordered products (IWOP), we obtain

$$\begin{aligned}\mathrm{tr}\,\rho_0 &= \mathrm{e}^{\delta}\mathrm{tr}\left(\int \frac{\mathrm{d}^2 z}{\pi}|z\rangle\langle z| :\, \mathrm{e}^{\gamma a^\dagger + \beta a^\dagger a + \gamma^* a} :\right)\\ &= \mathrm{e}^{\delta}\int \frac{\mathrm{d}^2 z}{\pi}\mathrm{e}^{\beta|z|^2 + \gamma z^* + \gamma^* z}\\ &= (-\beta)^{-1}\mathrm{e}^{\delta - \beta^{-1}|\gamma|^2} = 1.\end{aligned} \tag{5}$$

It then follows that the parameter δ is given by

$$\delta = \ln(-\beta) + |\gamma|^2/\beta, \tag{6}$$

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which means $\beta < 0$, coincided with the convergent condition in Eq. (4).

3. Evolution of ρ_0

Now we investigate the analytical evolution of the state ρ_0 in the diffusion channel. And for that, we need to use the following formula:

$$\rho = \int \frac{\mathrm{d}^2 z}{\pi} \langle -z| \rho |z\rangle \vdots \exp(|z|^2 + z^* a - z a^\dagger + a^\dagger a) \vdots \tag{7}$$

to calculate the anti-normal ordering product of ρ_0,[13,14] that is,

$$\begin{aligned}
\rho_0 &= \mathrm{e}^{\delta} \int \frac{\mathrm{d}^2 z}{\pi} \langle -z| : \mathrm{e}^{\beta a^\dagger a + \gamma a^\dagger + \gamma^* a} : |z\rangle \vdots \exp\left(|z|^2 + z^* a - z a^\dagger + a^\dagger a\right) \vdots \\
&= \mathrm{e}^{\delta} \int \frac{\mathrm{d}^2 z}{\pi} \vdots \exp\left(-(\beta+1)|z|^2 + (a-\gamma) z^* + \left(\gamma^* - a^\dagger\right) z\right) \mathrm{e}^{a^\dagger a} \vdots \\
&= \frac{\exp\left(\delta - \dfrac{1}{\beta+1}|\gamma|^2\right)}{\beta+1} \vdots \exp\left(\frac{\beta}{\beta+1} a^\dagger a + \frac{\gamma^*}{\beta+1} a + \frac{\gamma}{\beta+1} a^\dagger\right) \vdots ,
\end{aligned} \tag{8}$$

where $\vdots\ \vdots$ refers to anti-normal ordering, and we have used $\langle -z\,|z\rangle = \mathrm{e}^{-2|z|^2}$ and the integral formula in Eq. (4). Inserting Eq. (8) into Eq. (1), we obtain

$$\begin{aligned}
\rho_t = & \frac{\exp\left(\delta - \dfrac{1}{\beta+1}|\gamma|^2\right)}{\beta+1} \sum_{m,n=0}^{\infty} \frac{A^{m+n+1} (\kappa t)^{m+n}}{m! n!} \\
& \times a^{\dagger m} \mathrm{e}^{a^\dagger a \ln A} a^n \vdots \exp\left(\frac{\beta}{\beta+1} a^\dagger a + \frac{\gamma^*}{\beta+1} a + \frac{\gamma}{\beta+1} a^\dagger\right) \vdots a^{\dagger n} \mathrm{e}^{a^\dagger a \ln A} a^m .
\end{aligned} \tag{9}$$

Inserting the complete relation of $|z\rangle$ into Eq. (9) and using the identity $\mathrm{e}^{-\rho a^\dagger a} |z\rangle = \mathrm{e}^{-|z|^2/2} \|z\mathrm{e}^{-\rho}\rangle$, where $\|z\mathrm{e}^{-\rho}\rangle$ is unnormalized and ρ is a general parameter, we obtain

$$\begin{aligned}
\rho_t = & \frac{\exp\left(\delta - \dfrac{1}{\beta+1}|\gamma|^2\right)}{\beta+1} \sum_{m=0}^{\infty} \frac{A^{m+1} (\kappa t)^m}{m!} \int \frac{\mathrm{d}^2 z}{\pi} a^{\dagger m} \exp\left(a^\dagger a \ln A\right) |z\rangle \langle z| \exp\left(a^\dagger a \ln A\right) \\
& \times a^m \exp\left[\left(A\kappa t + \frac{\beta}{\beta+1}\right)|z|^2 + \frac{\gamma^*}{\beta+1} z + \frac{\gamma}{\beta+1} z^*\right] \\
= & \frac{\exp\left(\delta - \dfrac{1}{\beta+1}|\gamma|^2\right)}{\beta+1} \sum_{m=0}^{\infty} \frac{A^{m+1} (\kappa t)^m}{m!} \int \frac{\mathrm{d}^2 z}{\pi} a^{\dagger m} \|Az\rangle \langle Az\| a^m \exp\left[\left(A\kappa t - \frac{1}{\beta+1}\right)|z|^2 + \frac{\gamma^*}{\beta+1} z + \frac{\gamma}{\beta+1} z^*\right].
\end{aligned} \tag{10}$$

Further, using the integral formula (4) and the operator identity $|0\rangle\langle 0| = :\mathrm{e}^{-a^\dagger a}:$, we finally obtain

$$\begin{aligned}
\rho_t = & \frac{A \exp\left(\delta - \dfrac{1}{\beta+1}|\gamma|^2\right)}{\beta+1} \int \frac{\mathrm{d}^2 z}{\pi} \exp\left[-\left(\frac{1}{\beta+1} - A\kappa t\right)|z|^2\right] : \exp\left[\left(A a^\dagger + \frac{\gamma^*}{\beta+1}\right) z\right] \\
& \times \exp\left[\left(A a + \frac{\gamma}{\beta+1}\right)\right] z^* \exp\left[(A\kappa t - 1)\right] a^\dagger a : \ = \exp(\delta_t) : \exp\left(\beta_t a^\dagger a + \gamma_t a^\dagger + \gamma_t^* a\right) : ,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\gamma_t &= \frac{BA\gamma}{\beta+1}, \\
\beta_t &= BA^2 + A\kappa t - 1, \\
\mathrm{e}^{\delta_t} &= -\beta_t \exp\left(\beta_t^{-1} |\gamma_t|^2\right), \\
B &= \left(\frac{1}{\beta+1} - A\kappa t\right)^{-1}.
\end{aligned} \tag{12}$$

Further, using the identity

$$\exp(\rho a^\dagger a) =: \exp\left[(\mathrm{e}^\rho - 1) a^\dagger a\right] : , \tag{13}$$

to remove the symbol $: \ :$ from Eq. (11), we obtain

$$\rho_t = \mathrm{e}^{\delta_t} \mathrm{e}^{\gamma_t a^\dagger} \mathrm{e}^{a^\dagger a \ln(\beta_t + 1)} \mathrm{e}^{\gamma_t^* a}, \tag{14}$$

which shows that, after undergoing the diffusion channel, the resultant state ρ_t remains the form of the initial general linear

state state ρ_0. Further, we also prove the normalization of ρ_t, that is

$$\mathrm{tr}\rho_t = \mathrm{e}^{\delta_t}\int\frac{\mathrm{d}^2z}{\pi}\langle z|:\mathrm{e}^{\beta_t a^\dagger a+\gamma_t a^\dagger+\gamma_t^* a}:|z\rangle = 1. \tag{15}$$

For $\delta=-|z|^2$, $\gamma=z$ and $\beta=-1$, thus

$$B=0,\ \ \gamma_t=Az,\ \ \beta_t=-A,\ \ \mathrm{e}^{\delta_t}=A\mathrm{e}^{-A^{-1}|z|^2}, \tag{16}$$

ρ_t becomes the evolution of the coherent state $\rho_{|z\rangle}=|z\rangle\langle z|$ in the diffusion channel, the same as the result in Ref. [4]. For $\gamma=0$ and $\beta\neq-1$, thus $\gamma_t=0$, $\mathrm{e}^{\delta_t}=-\beta_t$, ρ_t refers to the evolution of the thermal state $\rho_{\mathrm{th}}=-\beta\,\mathrm{e}^{a^\dagger a\ln(\beta+1)}$, however $\gamma=0$ and $\beta=-1$, ρ_t reduces to the evolution of vacuum state $\rho_{\mathrm{vac}}=|0\rangle\langle 0|$.

4. Statistical properties of ρ_t

4.1. Average photon number

The average photon number (APN) represents the average value of all photon numbers and characterizes the energy density of quantum light fields. Based on the definition of the APN, and using the anti-normal ordering product of ρ_0 in Eq. (8), we obtain the APN of ρ_0 as

$$\begin{aligned}\mathrm{tr}\left(\rho_0 a^\dagger a\right) &= \frac{\exp\left(\delta-\frac{1}{\beta+1}|\gamma|^2\right)}{\beta+1}\mathrm{tr}\left(:\exp\left(\frac{\beta}{\beta+1}a^\dagger a+\frac{\gamma^*}{\beta+1}a+\frac{\gamma}{\beta+1}a^\dagger\right):a^\dagger a\right)\\ &= \frac{\exp\left(\delta-\frac{1}{\beta+1}|\gamma|^2\right)}{\beta+1}\mathrm{tr}\left(\int\frac{\mathrm{d}^2z}{\pi}\exp\left(\frac{\beta}{\beta+1}|z|^2\right)\exp\left(\frac{\gamma^*}{\beta+1}z+\frac{\gamma}{\beta+1}z^*\right)|z\rangle\langle z|a^\dagger a\right)\\ &= \frac{\exp\left(\delta-\frac{1}{\beta+1}|\gamma|^2\right)}{\beta+1}\int\frac{\mathrm{d}^2z}{\pi}\exp\left(\frac{\beta}{\beta+1}|z|^2+\frac{\gamma^*}{\beta+1}z+\frac{\gamma}{\beta+1}z^*\right)|z|^2\\ &= (\beta+1)|\gamma|^2-(\beta+1)^2/\beta,\end{aligned} \tag{17}$$

where we have used the mathematical integral formula

$$\int\frac{\mathrm{d}^2z}{\pi}z^n z^{*m}\exp(h|z|^2+sz+\eta z^*) = \mathrm{e}^{-s\eta/h}\sum_{l=0}^{\min(m,n)}\frac{m!n!s^{m-l}\eta^{n-l}}{l!(m-l)!(n-l)!(-h)^{m+n-l+1}}. \tag{18}$$

So, making the following substitutions $\gamma\to\gamma_t$ and $\beta\to\beta_t$, we easily obtain the APN evolution of ρ_0 in the diffusion channel. Further, letting $\gamma\to\gamma_t=Az$ (or 0), $\beta\to\beta_t=-A$ (or $BA^2+A\kappa t-1$), thus equation (17) becomes the APN evolution of the state $\rho_{|z\rangle}$ (or ρ_{th}) in the diffusion channel.

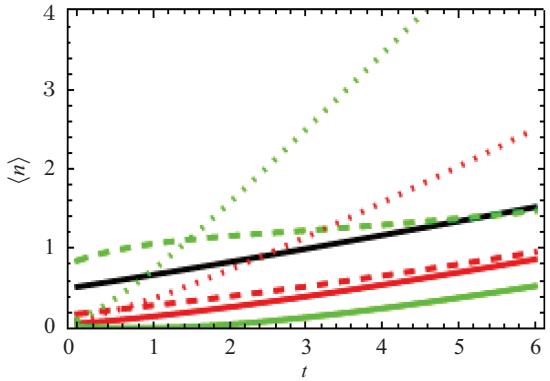


Fig. 1. The average photon number evolution of a general linear state in the diffusion channel over time t for different β,γ, and κ, where (β,γ,κ) correspond to, respectively, $(-0.5,0.2,0.2)$ (thick black), $(-0.8,0.2,0.2)$ (thick red), $(-1.2,0.2,0.2)$ (thick green), $(-0.8,0.8,0.2)$ (dashed thick red), $(-0.8,2,0.2)$ (dashed thick green), $(-0.8,0.5,0.5)$ (dotted thick red), and $(-0.8,0.5,1)$ (dotted thick green).

In Fig. 1, we plotted the APN of the state ρ_t with time t for different β,γ, and κ. Clearly, for all values of β,γ, and κ, the APN increases gradually with time t. For a fixed time t, the APN decreases with the increase of β, but increases as γ and κ increase. Besides, the parameters β and γ can change the initial APN at $t=0$, but κ has a more significant impact on the change rate of the APN over time t.

4.2. Photon distribution function

The photon distribution function (PDF) has wide applications in quantum information processing and quantum optical experiments, such as in quantum entanglement and quantum state preparation. Here, using Eq. (11) and the relation between the number state $|m\rangle$ and the unnormalized coherent state $\|z\rangle$, *i.e.*, $|m\rangle=(m!)^{-1/2}\mathrm{d}_z^m\|z\rangle|_{z=0}$, as well as the scalar product $\langle z'\|z\rangle=\mathrm{e}^{z'^*z}$, we obtain the PDF for the state ρ_t as follows:

$$\begin{aligned}p(m,t) &= \mathrm{tr}\left(|m\rangle\langle m|\rho_t\right)\\ &= \mathrm{e}^{\delta_t}\langle m|:\mathrm{e}^{\beta_t a^\dagger a+\gamma_t a^\dagger+\gamma_t^* a}:|m\rangle\\ &= \frac{\mathrm{e}^{\delta_t}}{m!}\partial_z^m\partial_{z'^*}^m\langle z'\|\,\mathrm{e}^{\beta_t z'^*z+\gamma_t z'^*+\gamma_t^* z}\|z\rangle\Big|_{z=z'^*=0}\\ &= \frac{\mathrm{e}^{\delta_t}}{m!}\partial_z^m\partial_{z'^*}^m\,\mathrm{e}^{(1+\beta_t)z'^*z+\gamma_t z'^*+\gamma_t^* z}\Big|_{z=z'^*=0}.\end{aligned} \tag{19}$$

Further, using the generating function for Hermite polynomial $\mathrm{H}_{n,m}(\cdot,\cdot)$, we have

$$p(m,t)=\frac{\mathrm{e}^{\delta_t}(-1)^m(1+\beta_t)^m}{m!}\times \mathrm{H}_{m,m}\left(\frac{\gamma_t}{\mathrm{i}\sqrt{1+\beta_t}},\frac{\gamma_t^*}{\mathrm{i}\sqrt{1+\beta_t}}\right), \tag{20}$$

which means that the PDF for the state $\rho(t)$ in the diffusion channel is just proportional to a two-variable Hermite polynomial of the order (m,m). Also, the PDF evolution of the state $\rho_{|z\rangle}$ (or $\rho_{\rm th}$) in the diffusion channel can be obtained by making the following substitutions $\gamma_t\to Az$ (or 0), $\beta_t\to -A$ $\left(\text{or } BA^2+A\kappa t-1\right)$, and $\mathrm{e}^{\delta_t}\to A\mathrm{e}^{-A^{-1}|z|^2}$ $(\text{or}-\beta_t)$ in Eq. (20).

In Fig. 2, we plotted the PDF for the state $\rho(t)$ with m for different β,γ, and κt. As β,γ, and κt increase, the PDF becomes nonzero for more photon number states and the probability peak of finding photons decreases, the PDF peak moves toward the larger m photon number. For sufficiently large diffusion time κt, the probability of finding any photon is nearly the same.

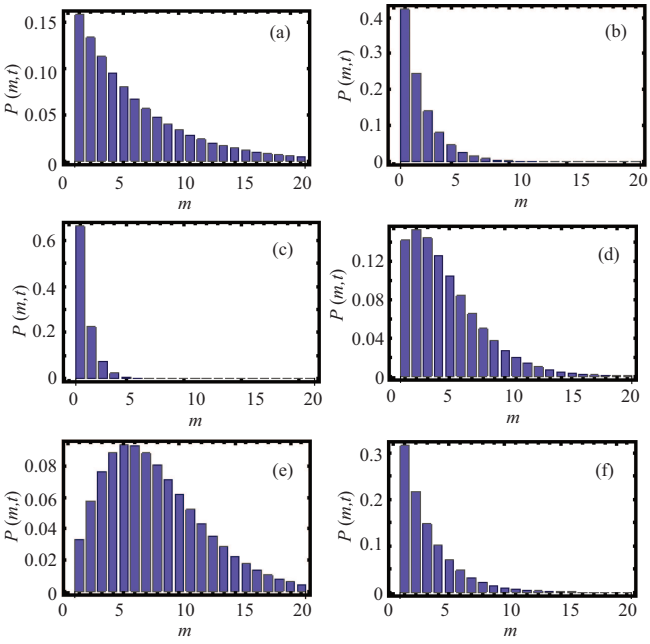


Fig. 2. The photon distribution function evolution of a general linear state in the diffusion channel with m for different β,γ, and κt. The values of $(\beta,\gamma,\kappa t)$ in panels (a), (b), (c), (d), (e), and (f) correspond to, respectively, $(-0.2,0.2,0.2)$, $(-0.5,0.2,0.2)$, $(-0.8,0.2,0.2)$, $(-0.5,0.8,0.2)$, and $(-0.5,1.2,0.2)$, $(-0.5,0.2,1)$.

4.3. Wigner distribution function

The Wigner distribution function (WDF) has a wide range of applications in quantum mechanics, quantum optics, and quantum information science, such as describing the statistical properties of the quantum systems, the quantum interference, and the quantum entanglement phenomena. Here we plan to study the WDF evolution of the initial state ρ_0 in the diffusion channel. Using the coherent state $|\alpha\rangle$ representation of the Wigner operator,[15–17] that is

$$\Delta(\alpha)=\mathrm{e}^{2|\alpha|^2}\int\frac{\mathrm{d}^2\alpha'}{\pi^2}\left|\alpha'\right\rangle\left\langle-\alpha'\right|\mathrm{e}^{2(\alpha\alpha'^*-\alpha^*\alpha')}, \tag{21}$$

we give the WDF for ρ_t as follows:

$$\begin{aligned}w(\alpha,t)&=\mathrm{tr}\left[\rho_t\Delta(\alpha)\right]\\&=\mathrm{e}^{\delta_t+2|\alpha|^2}\int\frac{\mathrm{d}^2\alpha'}{\pi^2}\left\langle-\alpha'\right|:\mathrm{e}^{\beta_t a^\dagger a}\\&\quad\times\mathrm{e}^{\gamma_t a^\dagger+\gamma_t^* a}:\left|\alpha'\right\rangle\mathrm{e}^{2(\alpha\alpha'^*-\alpha^*\alpha')}\\&=\mathrm{e}^{\delta_t+2|\alpha|^2}\int\frac{\mathrm{d}^2\alpha'}{\pi^2}\mathrm{e}^{-(2+\beta_t)|\alpha'|^2}\\&\quad\times\mathrm{e}^{-\gamma_t\alpha'^*+\gamma_t^*\alpha'}\mathrm{e}^{2(\alpha\alpha'^*-\alpha^*\alpha')}\\&=\frac{\mathrm{e}^{\delta_t+2|\alpha|^2}}{\pi(2+\beta_t)}\exp\left(-\frac{|\gamma_t-2\alpha|^2}{2+\beta_t}\right),\end{aligned} \tag{22}$$

which means that the WDF $w(\alpha,t)$ shows Gaussian in the diffusion channel. Especially, for $t=0$, $w(\alpha,t)$ becomes the WDF for the state ρ_0. Also, via making the following substitutions $\gamma_t\to Az$, $\beta_t\to -A$, and $\mathrm{e}^{\delta_t}\to A\mathrm{e}^{-A^{-1}|z|^2}$ in Eq. (22), we obtain the WDF for the coherent state $\rho_{|z\rangle}$ as

$$w_{|z\rangle}(\alpha,t)=\frac{A\mathrm{e}^{2|\alpha|^2-A^{-1}|z|^2}}{\pi(2-A)}\exp\left(-\frac{|Az-2\alpha|^2}{2-A}\right). \tag{23}$$

On the other hand, for $\gamma_t\to 0$ and $\mathrm{e}^{\delta_t}\to-\beta_t$ in Eq. (22), the WDF for the state ρ_{th} is obtained as

$$w_{\rm th}(\alpha,t)=\frac{1}{\pi(2+\beta_t)}\exp\left(\frac{2|\alpha|^2-\beta_t-2}{2\beta_t^{-1}+1}\right). \tag{24}$$

In Fig. 3, we plotted the WDF for the state $\rho(t)$ for different β,γ, and κt. For all values of β,γ, and κt, the WDF always maintains Gaussian characteristics. As β and κt increase, the peak value of the WDF decreases and the projection area of the WDF on the Reα–Imα plane slowly increases. Moreover, the parameter γ can lead to the fact that the Gaussian peak gradually moves away from the center of the phase space, which also confirms that γ is a displaced parameter.

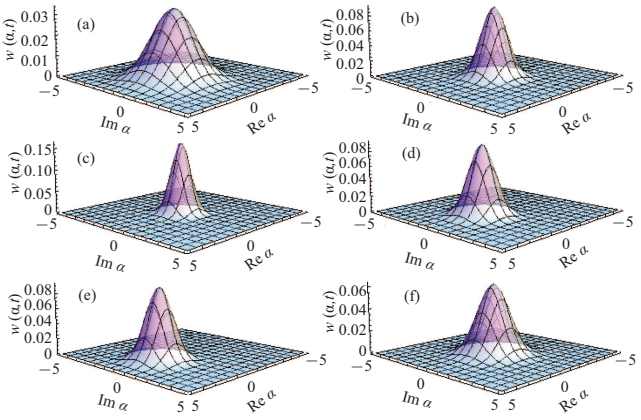


Fig. 3. The Wigner distribution function evolution of a general linear state in the diffusion channel is shown for different β,γ, and κt, where $(\beta,\gamma,\kappa t)$ in panels (a), (b), (c), (d), (e), and (f) correspond to, respectively, $(-0.2,0.2,0.2)$, $(-0.5,0.2,0.2)$, $(-0.8,0.2,0.2)$, $(-0.5,0.8,0.2)$, and $(-0.5,1.2,0.2)$, $(-0.5,0.2,1)$.

5. The von-Neumann entropy of ρ_t

The von-Neumann entropy is an indicator of how quantum information is stored in quantum states. Here, we investigate the von-Neumann entropy evolution of ρ_0 in the diffusion channel. Because of the natural logarithm of the state ρ_t, *i.e.*, $\ln\rho_t$, in its definition $S(\rho_t)=-k_{\rm B}{\rm tr}(\rho_t\ln\rho_t)$,[17,18] we first obtain the single exponential form of ρ_t. Using the identity $e^{-\kappa t a^\dagger a}a^\dagger e^{-\kappa t a^\dagger a}=a^\dagger e^{-\kappa t}$ and the formula $e^A e^B=\exp\left(A+\frac{\mu B+f}{1-e^{-\mu}}-\frac{f}{\mu}\right)$, where $[A,B]=\mu B+f$,[19] we have

$$\begin{aligned}\rho_t&=e^{\delta_t}e^{\gamma_t a^\dagger}e^{a^\dagger a\ln(1+\beta_t)}e^{\gamma_t^* a}=e^{\delta_t}e^{a^\dagger a\ln(1+\beta_t)}e^{-a^\dagger a\ln(1+\beta_t)}e^{\gamma_t a^\dagger}e^{a^\dagger a\ln(1+\beta_t)}e^{\gamma_t^* a}=e^{\delta_t}e^{a^\dagger a\ln(1+\beta_t)}\exp\left(-\frac{\beta_t d}{1+\beta_t}a^\dagger\right)e^{\gamma_t^* a}\\&=e^{\delta_t}e^{\left(a^\dagger a-da^\dagger\right)\ln(1+\beta(t))}e^{\gamma^*(t)a}=-\beta_t e^{(\beta_t^{-2}|\gamma_t|^2+\beta_t^{-1}\gamma_t a^\dagger+\beta_t^{-1}\gamma_t^* a+a^\dagger a)\ln(1+\beta_t)},\end{aligned}\tag{25}$$

thus we obtain $\ln\rho_t$ as

$$\ln\rho_t=\ln(-\beta_t)+(\beta_t^{-2}|\gamma_t|^2+\beta_t^{-1}\gamma_t a^\dagger+\beta_t^{-1}\gamma_t^* a+a^\dagger a)\ln(1+\beta_t).\tag{26}$$

Further, using Eqs. (14) and (26), the von-Neumann entropy of ρ_t is

$$\begin{aligned}S(\rho_t)/k_{\rm B}&=-{\rm tr}\left\{\rho_t\left[\ln(-\beta_t)+\beta_t^{-2}|\gamma_t|^2\ln(1+\beta_t)\right]\right\}+\beta_t e^{\beta_t^{-1}|\gamma_t|^2}\ln(1+\beta_t)\,{\rm tr}\left[e^{\gamma_t a^\dagger}e^{a^\dagger a\ln(1+\beta_t)}e^{\gamma_t^* a}\left(a^\dagger a+\beta_t^{-1}\gamma_t a^\dagger+\beta_t^{-1}\gamma_t^* a\right)\right]\\&=-\ln(-\beta_t)-\beta_t^{-2}|\gamma_t|^2\ln(1+\beta_t)+\beta_t e^{\beta_t^{-1}|\gamma_t|^2}\ln(1+\beta_t)\,{\rm tr}\left[e^{\gamma_t a^\dagger}e^{a^\dagger a\ln(1+\beta_t)}e^{\gamma_t^* a}a^\dagger a+\beta_t^{-1}\gamma_t a^\dagger+\beta_t^{-1}\gamma_t^* a\right].\end{aligned}\tag{27}$$

To obtain the entropy $S(\rho_t)$, we first calculate the part

$$\begin{aligned}e^{\gamma_t a^\dagger}e^{a^\dagger a\ln(1+\beta_t)}e^{\gamma_t^* a}\left(a^\dagger a+\beta_t^{-1}\gamma_t a^\dagger\right)&=e^{\gamma_t a^\dagger}e^{a^\dagger a\ln(1+\beta_t)}e^{\gamma_t^* a}\left(a^\dagger e^{-\gamma_t^* a}e^{\gamma_t^* a}a+\beta_t^{-1}\gamma_t a^\dagger e^{-\gamma_t^* a}e^{\gamma_t^* a}\right)\\&=e^{\gamma_t a^\dagger}e^{a^\dagger a\ln(1+\beta_t)}\left(a^\dagger+\gamma_t^*\right)e^{\gamma_t^* a}\left(a+\beta_t^{-1}\gamma_t\right)\\&=e^{\gamma_t a^\dagger}e^{a^\dagger a\ln(1+\beta_t)}\left(a^\dagger+\gamma_t^*\right)e^{-a^\dagger a\ln(1+\beta_t)}e^{a^\dagger a\ln(1+\beta_t)}e^{\gamma_t^* a}\left(a+\beta_t^{-1}\gamma_t\right)\\&=e^{\gamma_t a^\dagger}\left[(1+\beta_t)a^\dagger+\gamma_t^*\right]e^{a^\dagger a\ln(1+\beta_t)}e^{\gamma_t^* a}\left(a+\beta_t^{-1}\gamma_t\right),\end{aligned}\tag{28}$$

where we have used the operator identity $e^{\rho a}a^\dagger e^{-\rho a}=a^\dagger+\rho$, thus we have

$$\begin{aligned}{\rm tr}\left(e^{\gamma_t a^\dagger}e^{a^\dagger a\ln(1+\beta_t)}e^{\gamma_t^* a}a^\dagger a+\beta_t^{-1}\gamma_t a^\dagger+\beta_t^{-1}\gamma_t^* a\right)&=\int\frac{{\rm d}^2 z}{\pi}\langle z|:e^{\gamma_t a^\dagger+\gamma_t^* a+\beta_t a^\dagger a}\left\{\left[(1+\beta_t)a^\dagger+\gamma_t^*\right]\left(a+\beta_t^{-1}\gamma_t\right)+\beta_t^{-1}\gamma_t^* a\right\}:|z\rangle\\&=\int\frac{{\rm d}^2 z}{\pi}e^{\gamma_t z^*+\gamma_t^* z+\beta_t|z|^2}\left\{\left[(1+\beta_t)z^*+\gamma_t^*\right]\left(z+\beta_t^{-1}\gamma_t\right)+\beta_t^{-1}\gamma_t^* z\right\}\\&=\beta_t^{-2}\left(1+\beta_t+\beta_t^{-1}|\gamma_t|^2\right)e^{-\beta_t^{-1}|\gamma_t|^2}.\end{aligned}\tag{29}$$

Finally, using Eqs. (27) and (29), we obtain the entropy evolution $S(\rho_t)$ as

$$S(\rho_t)/k_{\rm B}=-\ln(-\beta_t)+\left(\beta_t^{-1}+1\right)\ln(1+\beta_t),\tag{30}$$

which is completely decided by the parameter β_t containing the diffusion factor κ and the thermal parameter β, and is unrelated to the displacement γ of the initial state ρ_0. This result is reasonable because the role of the displacement γ in the initial state ρ_0 is similar to that of the amplitude z in the coherent state $|z\rangle$, and the entropy of the state $|z\rangle$ is always zero without the amplitude z. Therefore, equation (30) also represents the entropy evolution of the state $\rho_{\rm th}$ for diffusion noise. Especially, when $\beta_t\to-A$, equation (30) becomes the von-Neumann entropy evolution of the coherent state $\rho_{|z\rangle}$ in the diffusion channel, that is,

$$S(\rho_t)/k_{\rm B}=-\ln\frac{1}{1+\kappa t}-\kappa t\ln\frac{\kappa t}{1+\kappa t},\tag{31}$$

which is the same as the result in Eq. (4).

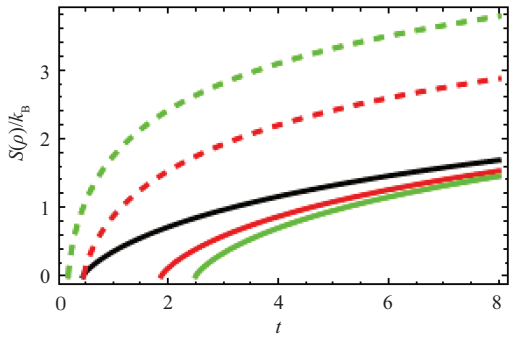


Fig. 4. The von-Neumann entropy $S(\rho_t)/k_{\rm B}$ of a general linear state in the diffusion channel with time t for different β and κ, where (β,κ) correspond to, respectively, $(-1.1,0.2)$ (thick black), $(-1.6,0.2)$ (thick red), $(-2,0.2)$ (thick green), $(-1.6,0.8)$ (dashed thick red), $(-1.6,2)$ (dashed thick green).

In Fig. 4, we plotted the von-Neumann entropy $S(\rho_t)/k_{\rm B}$ of $\rho(t)$ with time t for different β and κ. Obviously, the en-

tropy increases with time t for all values of β and κ, for a given time t, the entropy increases with the parameters β and κ. Besides, the parameters β and κ can lead to the change of the initial entropy at $t = 0$, and the entropy tends towards the same value for any β when time t is sufficiently large.

6. Conclusions

In summary, using the Kraus operator-sum representation of the analytical solution of the diffusion equation, and acting the general linear state ρ_0 as an initial input state, we obtained the analytical evolution law of the state ρ_0 for the diffusion channel. The result shows that the output state ρ_t remains the form of the initial linear state in the diffusion channel. Also, we investigated the quantum statistical properties of the initial state ρ_0 in the diffusion channel, and we found that the APN increases with time t, the PDF is proportional to a two-variable Hermite polynomial, and the WDF keeps Gaussian features. It is worth noting that, we obtained the analytical evolution of the von-Neumann entropy of the state ρ_0 in the diffusion channel, and found that the entropy evolution is completely governed by the diffusion and thermal noise, and is completely independent of the displacement of the state ρ_0. The von-Neumann entropy can usually be used as the core measurement tool of quantum information theory, so the results about the entropy are likely beneficial for evaluating channel capacity and information loss under noise interference in quantum communication, and may provide theoretical support for experimentally improving the fidelity of quantum storage and transmission in open systems.

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References

[1] Saleh B E A and Teich M C 1987 *Phys. Rev. Lett.* **58** 2656
[2] Weiss G H and Maradudin A A 1962 *J. Math. Phys.* **3** 771
[3] Aronson R and Corngold N 1999 *J. Opt. Soc. Am. A* **16** 1066
[4] Wu W F, Fang Y and Fu P 2024 *Chin. Phys. B* **33** 094202
[5] Yu H J and Fan H Y 2022 *Chin. Phys. B* **31** 020301
[6] Li G X, Tan H T, Wu S P and Huang G M 2006 *Phys. Rev. A* **74** 025801
[7] Fan H Y, Lou S Y, Pan X Y and Hu L Y 2014 *Sci. China Phys. Mech. Astron.* **57** 1649
[8] Fan H Y, Hu L Y and Fan Y 2006 *Ann. Phys.* **321** 480
[9] Fan H Y 2004 *Int. J. Mod. Phys. B* **18** 1387
[10] Meng X G, Wang J S, Liang B L and Han C X 2018 *Front. Phys.* **13** 130322
[11] Meng X G, Wang J S, Zhang X Y, Yang Z S, Liang B L and Zhang Z T 2020 *Ann. Phys. (Berlin)* **532** 2000219
[12] Meng X G, Li K C, Wang J S, Zhang X Y, Zhang Z T, Yang Z S and Liang B L 2020 *Ann. Phys. (Berlin)* **532** 1900585
[13] Fan H Y 1991 *Phys. Lett. A* **161** 1
[14] Fan H Y 2003 *J. Opt. B: Quantum Seciclass. Opt.* **5** R147
[15] Kano Y 1974 *J. Phys. Soc. Jpn.* **36** 39
[16] Vaccaro J 1995 *Phys. Rev. A* **52** 3474
[17] Guo Y, Zhang X, Du Q P, Yang Z S and Li J P 2023 *J. Liaocheng Univ. (Nat. Sci. Ed.)* **36** 39
[18] Chen F and Fan H Y 2013 *Ann. Phys.* **334** 272
[19] Meng X G, Liang B L, Liu J M and Zhou X C 2025 *Front. Phys.* **20** 022205
[20] Zhang J D and Wang S 2024 *J. Liaocheng Univ. (Nat. Sci. Ed.)* **37** 23