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The N-soliton solutions of the three-component coupled nonlinear Hirota equations based on Riemann–Hilbert method

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In order to more accurately and effectively consider the propagation process of solitons in electromagnetic pulse waves and make full use of wavelength division multiplexing, we study a class of high-order three-component Hirota equations by the Riemann–Hilbert method. Under zero boundary conditions and given initial conditions $q_j(x,0)$, the N-soliton solutions of the equations are obtained by constructing and solving Riemann–Hilbert problems based on matrix spectral problem. Specifically, we discuss the cases of $N=1,2$, analyze the dynamical properties of 1-soliton and 2-soliton solutions through numerical simulations, and summarize the effect of integrable perturbations and spectral parameters on soliton motion.

Keywords: coupled Hirota equation, soliton solutions, Riemann–Hilbert problem, matrix spectral problem

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1. Introduction

Nonlinear Schrödinger equation is a class of nonlinear equations with soliton solutions, which is closely related to many nonlinear problems in theoretical physics such as nonlinear optics and quantum mechanics, can be used to describe practical nonlinear phenomena. Nonlinear Schrödinger equation has always been a hot research issue for scholars and a wealth of relevant conclusions have been obtained.[1–5] Soliton solutions have also received much attention due to the special property that their shape and velocity remain unchanged before and after collision.

The study of soliton theory began in 1834 with the discovery of a strange phenomenon by British naval engineer Russel.[6] D. Korteweg and G. de Vries gave the famous KdV equation and obtained the analytic solutions of the equation.[7] Since then, research on soliton theory has flourished in academic circles.[8–12] In physics, solitons are used to describe solitary waves with elastic scattering properties. In mathematics, soliton theory provides a series of methods for solving nonlinear partial differential equations. Its research has been widely valued by scholars. At present, many subject areas contain problems related to soliton theory.[13–16] In these studies, it is very important to establish soliton equations and further construct its analytical solutions, especially soliton solutions. Among many solving methods for soliton equations, the most famous one is the inverse scattering method proposed by Gardner, Greene, Kruskal, Miura in 1967 to solve the initial value problem of KdV equation and its extension.[17] In addition, there are a variety of transformations and techniques for solving explicit exact solutions of soliton equations.[18–21] Such as Bäcklund transformation, Hirota bilinear transformation, Darboux transformation, and so on. The Riemann–Hilbert method is a more effective method, which can be used to construct the jump matrix by using the spectrum problem of integrable equations, and then analyze the symmetry and reconstruct the potential in turn. In recent years, the extension of this method has been carried out, and very important results have been obtained.[22]

With the development of nonlinear science, it is found that the classical nonlinear Schrödinger equation has limitations in describing some practical problems. In particular, the classical nonlinear Schrödinger equation may not provide an accurate description when describing systems with high nonlinear, multi-body interactions or significant quantum effects. In order to solve these problems, scholars have proposed various extended and modified models, such as adding more complex nonlinear terms, introducing coupled terms, and considering quantum field effects. These expanded and modified models can more accurately describe some complex physical phenomena. Hirota proposed the following integrable model in Ref. [23]:

$$\mathrm{i}u_t+\frac{1}{2}u_{xx}+|u|^2u-\mathrm{i}\alpha u_{xxx}-6\mathrm{i}\alpha|u|^2u_x=0, \tag{1}$$

where α is a real constant. As an integrable generalization of the nonlinear Schrödinger equation, this model can be used to describe the ultra-short pulse wave under the influence of high order dispersion and self-gradient effect, which is called Hirota equation.

In recent years, scholars have continuously promoted this classical model and obtained corresponding research results.[24–26] Among them, the authors in Ref. [24] studied the

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following class of generalized Hirota equations with bounded potential:

$$\mathrm{i}u_t+\alpha u_{xx}+\mathrm{i}\beta u_{xxx}+\mathrm{i}\gamma(|u|^2u)_x+\delta|u|^2u+\mathrm{i}\lambda u=0. \quad (2)$$

The soliton solution of the equation is obtained by the hypothetical method and the sub-equation method, and the dynamical properties of the solution are discussed. At the same time, coupled nonlinear Schrödinger equation, as a kind of classical generalized model, also has rich research achievements.[27–30] Among them,[30] the following two-component coupled modified Hirota equation is proposed and studied:

$$\begin{aligned}&\mathrm{i}u_t+\frac{1}{2}u_{xx}+(|u|^2+|v|^2)u+\mathrm{i}\varepsilon[u_{xxx}+(6|u|^2+3|v|^2)u_x+3uv^*v_x]=0,\\&\mathrm{i}v_t+\frac{1}{2}v_{xx}+(|u|^2+|v|^2)v+\mathrm{i}\varepsilon[v_{xxx}+(6|v|^2+3|u|^2)v_x+3vu^*u_x]=0,\end{aligned} \quad (3)$$

where ε is the small dimensionless real parameter. The equation can be used to describe the propagation of electromagnetic pulse waves in double optical waveguides. The soliton solution of the equation is obtained by inverse scattering method.

Compared with the low-order nonlinear model, the high-order nonlinear model has some obvious advantages and characteristics in describing the transmission process of electromagnetic pulse waves. For example, with more accurate description capabilities, higher-order nonlinear models can capture more complex nonlinear effects that may play a critical role in electromagnetic pulse transmission. Higher-order models have wider applicability. In some cases, especially when the medium parameters change greatly or the intensity of the electromagnetic pulse is high, the higher-order nonlinear model can provide a more accurate description. Therefore, we consider adding high-order terms to the equation. At the same time, in order to expand its wavelength division multiplexing, we extend the equation to discuss the simultaneous propagation of three-component electromagnetic pulse waves in a three-component coupled equation. Therefore, we study the following class of three-component coupled Hirota equations:

$$\begin{cases}\mathrm{i}q_{1,t}+\frac{1}{2}q_{1,xx}+(|q_1|^2+|q_2|^2+|q_3|^2)q_1+\mathrm{i}\varepsilon[q_{1,xxx}+3(2|q_1|^2+|q_2|^2+|q_3|^2)q_{1,x}+3q_1(q_2^*q_{2,x}+q_3^*q_{3,x})]=0,\\ \mathrm{i}q_{2,t}+\frac{1}{2}q_{2,xx}+(|q_1|^2+|q_2|^2+|q_3|^2)q_2+\mathrm{i}\varepsilon[q_{2,xxx}+3(2|q_1|^2+|q_2|^2+|q_3|^2)q_{2,x}+3q_2(q_1^*q_{1,x}+q_3^*q_{3,x})]=0,\\ \mathrm{i}q_{3,t}+\frac{1}{2}q_{3,xx}+(|q_1|^2+|q_2|^2+|q_3|^2)q_3+\mathrm{i}\varepsilon[q_{3,xxx}+3(2|q_1|^2+|q_2|^2+|q_3|^2)q_{3,x}+3q_3(q_1^*q_{1,x}+q_2^*q_{2,x})]=0,\end{cases} \quad (4)$$

where $q_j(x,t)$ $(j=1,2,3)$ is a complex envelope of three fields, each non-numerical subscript variable represents a partial differential, $*$ represents a complex conjugate, and ε represents an integrable perturbation of the coupled nonlinear Schrödinger equation, which is a small dimensionless real parameter. Equation (4) is an obvious generalization of Eq. (3) and can control the simultaneous propagation of three fields in the normal dispersion state of the fiber. In Ref. [31], the Lax pair and Darboux transform of the equation were constructed, and six mixed types of the first and second order local waves were discussed. In this paper, given the initial conditions $q_j(x,0)$ $(j=1,2,3)$, we obtain the soliton solutions $q_j(x,t)$ $(j=1,2,3)$ of Eq. (4) at any time t by Riemann–Hilbert method.

The structure of this paper is arranged as follows: The Lax pair of Eq. (4) is given in Section 2. In Section 3, the Riemann–Hilbert problem of Eq. (4) is constructed and solved. The N-soliton solutions of the equation is given in Section 4. In Section 5, the dynamic properties of 1, 2-soliton solutions are analyzed. Section 6 is conclusions.

2. Lax pair

To obtain the N-soliton solutions of Eq. (4), it is necessary to first give the Lax pair of the equation. After completing the positive scattering process, the expression of the N-soliton solutions can be obtained by constructing and solving the Riemann–Hilbert problem of the equation, combining the potential reconstruction results and the scattering data evolving over time.

According to Ref. [31], equation (4) is Lax integrable, and its corresponding linear spectrum problem is

$$\Phi_x=U\Phi=\left(\frac{\lambda}{12\varepsilon}\mathrm{i}\Lambda+U_1\right)\Phi, \quad (5)$$

$$\Phi_t=V\Phi=\left(\frac{\mathrm{i}}{192\varepsilon^2}(\lambda^3+2\lambda^2)\Lambda+V_1\right)\Phi, \quad (6)$$

where

$$\Lambda=\begin{pmatrix}-2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad U_1=\begin{pmatrix}-2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

$$V_1=\frac{\lambda^2}{16\varepsilon}U_1+\lambda V_2+V_3,$$

$$V_2=\frac{1}{4}\begin{pmatrix} \mathrm{i}e & -q_1/2\varepsilon-\mathrm{i}q_{1x} & -q_2/2\varepsilon-\mathrm{i}q_{2x} & -q_3/2\varepsilon-\mathrm{i}q_{3x}\\ q_1^*/2\varepsilon-\mathrm{i}q_{1x}^* & -\mathrm{i}|q_1|^2 & -\mathrm{i}q_1^*q_2 & -\mathrm{i}q_1^*q_3\\ q_2^*/2\varepsilon-\mathrm{i}q_{2x}^* & -\mathrm{i}q_2^*q_1 & -\mathrm{i}|q_2|^2 & -\mathrm{i}q_2^*q_3\\ q_3^*/2\varepsilon-\mathrm{i}q_{3x}^* & -\mathrm{i}q_3^*q_1 & -\mathrm{i}q_3^*q_2 & -\mathrm{i}|q_3|^2 \end{pmatrix},$$

$$V_3=\begin{pmatrix} \varepsilon(e_1+e_2+e_3)+(\mathrm{i}/2)e & \varepsilon e_4-(\mathrm{i}/2)q_{1x} & \varepsilon e_5-(\mathrm{i}/2)q_{2x} & \varepsilon e_6-(\mathrm{i}/2)q_{3x}\\ -\varepsilon e_4^*-(\mathrm{i}/2)q_{1x}^* & -\varepsilon e_1-(\mathrm{i}/2)|q_1|^2 & \varepsilon e_7-(\mathrm{i}/2)q_1^*q_2 & \varepsilon e_8-(\mathrm{i}/2)q_1^*q_3\\ -\varepsilon e_5^*-(\mathrm{i}/2)q_{2x}^* & -\varepsilon e_7^*-(\mathrm{i}/2)q_2^*q_1 & -\varepsilon e_2-(\mathrm{i}/2)|q_2|^2 & \varepsilon e_9-(\mathrm{i}/2)q_2^*q_3\\ -\varepsilon e_6^*-(\mathrm{i}/2)q_{3x}^* & -\varepsilon e_8^*-(\mathrm{i}/2)q_3^*q_1 & -\varepsilon e_9^*-(\mathrm{i}/2)q_3^*q_2 & -\varepsilon e_3-(\mathrm{i}/2)|q_3|^2 \end{pmatrix},$$

$e=|q_1|^2+|q_2|^2+|q_3|^2$, $e_1=q_1q_{1x}^*-q_{1x}q_1^*$, $e_2=q_2q_{2x}^*-q_{2x}q_2^*$, $e_3=q_3q_{3x}^*-q_{3x}q_3^*$,
$e_4=q_{1xx}+2eq_1$, $e_5=q_{2xx}+2eq_2$, $e_6=q_{3xx}+2eq_3$, $e_7=q_1^*q_{2x}-q_{1x}^*q_2$,
$e_8=q_1^*q_{3x}-q_{1x}^*q_3$, $e_9=q_2^*q_{3x}-q_{2x}^*q_3$.

3. The Riemann–Hilbert problem

In order to obtain the N-soliton solutions of Eq. (4), we construct its Riemann–Hilbert problem and solve it.

3.1. Positive scattering process

We assume that the potential function q_j $(j=1,2,3)$ decays to 0 at $x\to\pm\infty$ with sufficient speed. Without loss of generality, we assume that ε is a small positive number. For convenience, introduce the following notation:

$$J(x,t,\lambda)=\Phi(x,t,\lambda)E^{-1},$$
$$E=\exp\left[\frac{\mathrm{i}}{12\varepsilon}\lambda\Lambda x+\frac{\mathrm{i}}{192\varepsilon^2}(\lambda^3+2\lambda^2)\Lambda t\right]. \tag{7}$$

Therefore, the spectral equations (5) and (6) are equivalent to the following form:

$$J_x=\frac{\mathrm{i}\lambda}{12\varepsilon}[\Lambda,J]+U_1J, \tag{8}$$
$$J_t=\frac{\mathrm{i}}{192\varepsilon^2}(\lambda^3+2\lambda^2)[\Lambda,J]+V_1J, \tag{9}$$

where the bracket is poisson bracket, $[\Lambda,J]=\Lambda J-J\Lambda$.

In fact, we find that

$$U_1^\dagger=\begin{pmatrix} 0 & q_1 & q_2 & q_3\\ -q_1^* & 0 & 0 & 0\\ -q_2^* & 0 & 0 & 0\\ -q_3^* & 0 & 0 & 0 \end{pmatrix}=-U_1, \tag{10}$$

where "†" represents the Hermite conjugate of the matrix. In the scattering process, the Jost solution $J_\pm(x,\lambda)$ of the spectral problem satisfies the following asymptotic conditions:

$$J_+(x,t)\to I,\quad x\to+\infty, \tag{11}$$
$$J_-(x,t)\to I,\quad x\to-\infty, \tag{12}$$

where I is the identity matrix of 4×4. The subscripts in $J_\pm$ represent solutions that satisfy the corresponding boundary conditions as they go to plus or minus infinity along the x-axis.

Through the identity in matrix analysis

$$(\det J)_x=\det J\cdot\mathrm{tr}(J_xJ^{-1}),$$

where $\det(J)$ represents the determinant of matrix J and $\mathrm{tr}(\cdot)$ represents the trace of matrix. Combining $\mathrm{tr}(U_1)=0$, it can be obtained

$$\det J_\pm=1 \tag{13}$$

holds for all (x,λ). In addition, it is easy to verify that $J_\pm$ is uniquely determined by the following Volterra integral equation:

$$J_+(x,\lambda)=I-\int_x^{+\infty}\exp\left(\frac{\mathrm{i}\lambda}{12\varepsilon}\Lambda(x-\xi)\right)U_1J_+(\xi,\lambda)\times\exp\left(-\frac{\mathrm{i}\lambda}{12\varepsilon}\Lambda(x-\xi)\right)\mathrm{d}\xi, \tag{14}$$

$$J_-(x,\lambda)=I+\int_{-\infty}^{x}\exp\left(\frac{\mathrm{i}\lambda}{12\varepsilon}\Lambda(x-\xi)\right)U_1J_-(\xi,\lambda)\times\exp\left(-\frac{\mathrm{i}\lambda}{12\varepsilon}\Lambda(x-\xi)\right)\mathrm{d}\xi. \tag{15}$$

Therefore, as long as the above Volterra integral equation converges, there must be a matrix function $J_\pm(x,\lambda)$ analysis.

Dividing the matrix function $J_\pm$ by column, then we have

$$J_\pm=([J_\pm]_1,[J_\pm]_2,[J_\pm]_3,[J_\pm]_4). \tag{16}$$

By analyzing, the column vectors $[J_-]_1,[J_+]_2,[J_+]_3,[J_+]_4$ are analytic in the upper semi-complex plane $\lambda\in\mathbb{C}_+$ and continuous in $\lambda\in\mathbb{R}$. $[J_+]_1,[J_-]_2,[J_-]_3,[J_-]_4$ are analytic in the lower half complex plane $\lambda\in\mathbb{C}_-$ and are continuous in $\lambda\in\mathbb{R}$.

Through calculation, it can be seen that J_+E and J_-E are fundamental solution matrices of spectral problems, and they are related by scattering matrix $S(\lambda)$, namely,

$$J_-E=J_+ES(\lambda),\quad \lambda\in\mathbb{R}, \tag{17}$$

where $S(\lambda)=(s_{ij}(\lambda))_{4\times4}$. Using Eqs. (13) and (17), we can derive $\det S(\lambda)=1$. It is known by Eq. (17) that s_{11} is analytic in $\mathbb{C}_+$, $s_{22},s_{23},s_{24},s_{32},s_{33},s_{34},s_{42},s_{43},s_{44}$ are analytic in the lower half complex plane $\mathbb{C}_-$, and the remaining matrix elements $s_{12},s_{13},s_{14},s_{21},s_{31},s_{41}$ are continuous in $\lambda\in\mathbb{R}$.

Similarly, we consider the inverse matrix of $J_\pm$

$$J_\pm^{-1} = \begin{pmatrix} [J_\pm^{-1}]^1 \\ [J_\pm^{-1}]^2 \\ [J_\pm^{-1}]^3 \\ [J_\pm^{-1}]^4 \end{pmatrix}, \tag{18}$$

where $[J_\pm^{-1}]^l$ $(l=1,2,3,4)$ represent row l of $J_\pm^{-1}$ respectively. It is easy to know that $[J_-^{-1}]^1, [J_+^{-1}]^2, [J_+^{-1}]^3, [J_+^{-1}]^4$ are analytic in the lower half complex plane $\lambda \in \mathbb{C}_-$ and continuous in $\lambda \in \mathbb{R}$. $[J_+^{-1}]^1, [J_-^{-1}]^2, [J_-^{-1}]^3, [J_-^{-1}]^4$ are analytic in $\lambda \in \mathbb{C}_+$ and continuous in $\lambda \in \mathbb{R}$.

Besides, it is easy to know by Eq. (17)

$$E^{-1}J_-^{-1} = R(\lambda)E_{-1}J_+^{-1}, \quad \lambda \in \mathbb{R}, \tag{19}$$

where $R(\lambda) \equiv (r_{kj})_{4\times4} = S^{-1}(\lambda)$. Similar to the discussion of $S(\lambda)$, we know that r_{11} analyzes in the $\mathbb{C}_-$ of the lower half complex plane. Matrix elements r_{22}, r_{23}, r_{24}, r_{32}, r_{33}, r_{34}, r_{42}, r_{43}, and r_{44} are analytic in the upper semi-complex plane $\mathbb{C}_+$. $r_{12}, r_{13}, r_{14}, r_{21}, r_{31}$, and r_{41} are continuous in $\lambda \in \mathbb{R}$.

3.2. The Riemann–Hilbert problem

To construct the Riemann–Hilbert problem, we define a matrix function $P_1 = P_1(x,\lambda)$ as follows:

$$\begin{aligned} P_1 &= ([J_-]_1, [J_+]_2, [J_+]_3, [J_+]_4) \\ &= J_-H_1 + J_+H_2 + J_+H_3 + J_+H_4 \\ &= J_+(ES(\lambda)E^{-1}H_1 + H_2 + H_3 + H_4) \\ &= J_+ \begin{pmatrix} s_{11} & 0 & 0 & 0 \\ s_{21}\,\mathrm{e}^{3\mathrm{i}\lambda/12\varepsilon} & 1 & 0 & 0 \\ s_{31}\,\mathrm{e}^{3\mathrm{i}\lambda/12\varepsilon} & 0 & 1 & 0 \\ s_{41}\,\mathrm{e}^{3\mathrm{i}\lambda/12\varepsilon} & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{20}$$

where $H_1 = \mathrm{diag}(1,0,0,0)$, $H_2 = \mathrm{diag}(0,1,0,0)$, $H_3 = \mathrm{diag}(0,0,1,0)$, $H_4 = \mathrm{diag}(0,0,0,1)$, and $P_1(x,\lambda)$ in $\mathbb{C}_+$ is an analytic function of λ.

Next, we study the λ asymptotic behavior of P_1 and expand P_1 as follows:

$$P_1 = P_1^{(0)} + \lambda_{-1}P_1^{(1)} + \lambda_{-2}P_1^{(2)} + \cdots, \quad \lambda \in \infty. \tag{21}$$

Substitute it into Eq. (8) and compare λ to the same power to obtain

$$\frac{\mathrm{i}}{12\varepsilon}[\Lambda, P_1^0] = 0, \quad P_{1,x}^{(0)} = \frac{\mathrm{i}}{12\varepsilon}[\Lambda, P_1^{(1)}] + U_1P_1^{(0)},$$

mean $P_1^{(0)} = I$. That is,

$$P_1 \to I, \quad \lambda \in \mathbb{C}_+ \to \infty, \tag{22}$$

where I is the 4×4 identity matrix.

In order to obtain the Riemann–Hilbert problem of Eq. (4), it is also necessary to construct a matrix function P_2, which is analyzed in $\mathbb{C}_-$.

$$\begin{aligned} P_2 &= \begin{pmatrix} [J_-^{-1}]^1 \\ [J_+^{-1}]^2 \\ [J_+^{-1}]^3 \\ [J_+^{-1}]^4 \end{pmatrix} \\ &= H_1[J_-^{-1}] + H_2[J_+^{-1}] + H_3[J_+^{-1}] + H_4[J_+^{-1}] \\ &= (H_1ER(\lambda)E^{-1} + H_2 + H_3 + H_4)J_+^{-1} \\ &= \begin{pmatrix} r_{11} & r_{12}\,\mathrm{e}^{-3\mathrm{i}\lambda/12\varepsilon} & r_{13}\,\mathrm{e}^{-3\mathrm{i}\lambda/12\varepsilon} & r_{14}\,\mathrm{e}^{-3\mathrm{i}\lambda/12\varepsilon} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} J_+^{-1}. \end{aligned} \tag{23}$$

Similar to P_1, we can obtain the progressive state of P_2

$$P_2 \to I, \quad \lambda \in \mathbb{C}_- \to \infty. \tag{24}$$

We denote the limit of P_1 as $\lambda \in \mathbb{C}_+$ approaches $\mathbb{R}$ is P^+, and the limit of P_2 as $\lambda \in \mathbb{C}_-$ approaches $\mathbb{R}$ is P^-. So, using Eqs. (20) and (23) we get

$$P^-(x,\lambda)P^+(x,\lambda) = G(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{25}$$

where the matrix $G(x,\lambda)$ is

$$G(x,\lambda) = \begin{pmatrix} 1 & r_{12}\,\mathrm{e}^{-3\mathrm{i}\lambda/12\varepsilon} & r_{13}\,\mathrm{e}^{-3\mathrm{i}\lambda/12\varepsilon} & r_{14}\,\mathrm{e}^{-3\mathrm{i}\lambda/12\varepsilon} \\ s_{21}\,\mathrm{e}^{3\mathrm{i}\lambda/12\varepsilon} & 1 & 0 & 0 \\ s_{31}\,\mathrm{e}^{3\mathrm{i}\lambda/12\varepsilon} & 0 & 1 & 0 \\ s_{41}\,\mathrm{e}^{3\mathrm{i}\lambda/12\varepsilon} & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Equation (25) is the transition matrix Riemann–Hilbert problem and satisfies the corresponding specification conditions as follows:

$$P_\pm(x,\lambda) \to \mathrm{I}, \quad \lambda \to \infty. \tag{27}$$

3.3. Solution of the Riemann–Hilbert problem

In this part, we discuss how to solve the matrix Riemann–Hilbert problem (25) on the complex plane λ. In most discussions, t is a dummy variable and is therefore ignored.

First, we consider the solution of the regular Riemann–Hilbert problem. If $\det P_\pm(\lambda) \neq 0$ is on its corresponding analytic space, then the Riemann–Hilbert problem is called regular.

Theorem 1 In the case of the standard normalization condition (27), the solution of the regular Riemann–Hilbert problem (25) is unique.

Proof Let both $P_{\pm}(\lambda)$ and $\tilde{P}_{\pm}(\lambda)$ be solutions to the Riemann–Hilbert problem, then $P_{-}(\lambda)P_{+}(\lambda)=\tilde{P}_{-}(\lambda)\tilde{P}_{+}(\lambda)$. So we can get

$$P_{+}(\lambda)(\tilde{P}_{+}(\lambda))^{-1}=(P_{-}(\lambda))^{-1}\tilde{P}_{-}(\lambda),\quad \lambda\in\mathbb{R}. \tag{28}$$

Since the Riemann–Hilbert problem is regular, $\det P_{\pm}$ and $\det\tilde{P}_{\pm}$ have no zeros in the corresponding analytic plane. Therefore, $P_{+}(\lambda)(\tilde{P}_{+}(\lambda))^{-1}$ is analytic in $\mathbb{C}_{+}$, $(P_{-}(\lambda))^{-1}\tilde{P}_{-}(\lambda)$ is analytic in $\mathbb{C}_{-}$.

That is to say, we can define a complete $\mathbb{C}$ plane analytic function is as follows:

$$P(\lambda)=\begin{cases}P_{+}(\lambda)(\tilde{P}_{+}(\lambda))^{-1}, & \lambda\in\mathbb{C}_{+},\\ P_{+}(\lambda)(\tilde{P}_{+}(\lambda))^{-1}=(P_{-}(\lambda))^{-1}\tilde{P}_{-}(\lambda), & \lambda\in\mathbb{R},\\ (P_{-}(\lambda))^{-1}\tilde{P}_{-}(\lambda), & \lambda\in\mathbb{C}_{-},\end{cases} \tag{29}$$

meet the condition

$$P\to \mathrm{I},\lambda\to\infty. \tag{30}$$

According to Liouville's theorem, we can know that the function $P(\lambda)$ is a constant, and combined with the condition (30), we can get $P=\mathrm{I}$, *i.e.*,

$$P_{+}(\lambda)(\tilde{P}_{+}(\lambda))^{-1}=(P_{-}(\lambda))^{-1}\tilde{P}_{-}(\lambda)=\mathrm{I},\quad \lambda\in\mathbb{C}. \tag{31}$$

That is to say,

$$P_{+}=\tilde{P}_{+},\quad P_{-}=\tilde{P}_{-}. \tag{32}$$

Therefore, the solution to the regular Riemann–Hilbert problem is unique, and the proof ends.

On the basis that the solution to the Riemann–Hilbert problem is unique, applying Plemelj formula to the Riemann–Hilbert problem (25), the explicit solution can be obtained as follows:

$$(P_{+})^{-1}=\mathrm{I}+\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\hat{G}(P_{+})^{-1}}{\xi-\lambda}\mathrm{d}\xi,\quad \lambda\in\mathbb{C}_{+}, \tag{33}$$

where $\hat{G}=\mathrm{I}-G$.

Next, we examine the irregular Riemann–Hilbert problem, in which both $\det P_1$ and $\det P_2$ have zeros in the analytic region. Using Eqs. (20) and (23), we know

$$\det P_1(\lambda)=s_{11}(\lambda),\quad \lambda\in\mathbb{C}_{+}, \tag{34}$$

$$\det P_2(\lambda)=r_{11}(\lambda),\quad \lambda\in\mathbb{C}_{-}. \tag{35}$$

And because the potential matrix U_1 satisfies the symmetry condition

$$U_1^{\dagger}=-U_1. \tag{36}$$

We can easily verify that $J_{\pm}^{\dagger}(\lambda^{*})$ and $J_{\pm}^{-1}(\lambda)$ both satisfy the same x boundary conditions for the following equation about K:

$$K_x=\frac{\mathrm{i}\lambda}{12\varepsilon}[\Lambda,K]-KU_1. \tag{37}$$

Equation (37) is called the adjoint equation of the spectral problem Eq. (8). Therefore, there are

$$J_{\pm}^{\dagger}(\lambda^{*})=J_{\pm}^{-1}(\lambda). \tag{38}$$

The scattering relation Eq. (17) is used to know

$$S^{\dagger}(\lambda^{*})=S^{-1}(\lambda). \tag{39}$$

This means that

$$r_{12}(\lambda)=s_{21}^{*}(\lambda),\quad \lambda\in\mathbb{R}, \tag{40}$$

$$r_{13}(\lambda)=s_{31}^{*}(\lambda),\quad \lambda\in\mathbb{R}, \tag{41}$$

$$r_{14}(\lambda)=s_{41}^{*}(\lambda),\quad \lambda\in\mathbb{R}, \tag{42}$$

$$r_{11}^{*}(\lambda^{*})=s_{11}(\lambda),\quad \lambda\in\mathbb{C}_{+}. \tag{43}$$

Combined with the definitions of P_1 and P_2, we get

$$P_1^{\dagger}(\lambda^{*})=P_2(\lambda),\quad \lambda\in\mathbb{C}_{-}. \tag{44}$$

Through analysis, we know that if $\lambda_1\in\mathbb{C}_{+}$ is the zero point of s_{11}, then $\lambda_1^{*}\in\mathbb{C}_{-}$ must be the zero point of r_{11}. Therefore, we can consider the zero point structure of $\det P_1$ and $\det P_2$ as: $\det P_1$ has N single zero points $\{\lambda_j\}_1^N$ in $\mathbb{C}_{+}$. At this point, $\det P_1$ has N single zero points $\{\hat{\lambda}_j\}_1^N$ in $\mathbb{C}_{-}$, where $\hat{\lambda}_j=\lambda_j^{*}$.

Considering Eqs. (40)–(43), in order to get the solution of the Riemann–Hilbert problem (25), we need continuous scattering data $\{s_{21},s_{31},s_{41}\}$ and discrete scattering data $\{\lambda_j,v_j\}$, where $v_j=v_j(x,\lambda)$ is a nonzero column vector that satisfies the following conditions:

$$P_1(\lambda_j)v_j=0,\quad 1\le j\le N. \tag{45}$$

Taking the Hermite conjugate of the two ends of the equation, we can obtain

$$v_j^{\dagger}P_1^{\dagger}(\lambda_j)=0. \tag{46}$$

Considering Eq. (44), we can get

$$v_j^{\dagger}P_2(\lambda_j^{*})=0. \tag{47}$$

Simplicity, denote

$$\hat{v}_j:=v_j^{\dagger},\quad 1\le k\le N. \tag{48}$$

Taking the derivative of Eq. (45) with respect to x and combining Eq. (8) yields

$$P_1(\lambda_j)(v_{j,x}-\frac{\mathrm{i}\lambda_j}{12\varepsilon}\Lambda v_j)=0,\quad 1\le j\le N. \tag{49}$$

Without loss of generality, this expression means

$$v_j=\exp\left(\frac{\mathrm{i}\lambda_j}{12\varepsilon}\Lambda x\right)v_{j,0},\quad 1\le j\le N, \tag{50}$$

where $v_{j,0}$ is independent of x.

According to Ref. [32], we can transform the irregular Riemann–Hilbert problem (25) into a regular Riemann–Hilbert problem, and then find its formal solution expressed by Fredholm integral equation. To solve the soliton solution of Eq. (4), we need to consider the Riemann–Hilbert problem without reflection, that is, $s_{21} = s_{31} = s_{41} = 0$ in Riemann–Hilbert problem (26). In this case, the solution to the Riemann–Hilbert problem can be shown as

$$P_1(\lambda) = I - \sum_{k,j=1}^{N} \frac{v_k \hat{v}_j (M^{-1})_{kj}}{\lambda - \hat{\lambda}_j}, \tag{51}$$

$$P_2(\lambda) = I + \sum_{k,j=1}^{N} \frac{v_k \hat{v}_j (M^{-1})_{kj}}{\lambda - \lambda_k}, \tag{52}$$

where $M = (m_{kj})_{N\times N}$ is an invertible matrix with elements $m_{kj} = \hat{v}_k v_j/(\lambda_j - \hat{\lambda}_k)$. And $(M^{-1})_{kj}$ represents the (k,j) element of M^{-1}.

3.4. Reconstructed potential

From the display solution of the Riemann–Hilbert problem we can reconstruct potential q_1, q_2, q_3. In fact, according to Eq. (21) we can asymptotically expand P_1 to

$$P_1(\lambda) = I + \lambda_{-1} P_1^{(1)} + \lambda_{-2} P_1^{(2)} + \cdots, \quad \lambda \to \infty, \tag{53}$$

then substitute it into Eq. (8) to get

$$U_1 = -\frac{\mathrm{i}}{12\varepsilon}[\Lambda, P_1^{(1)}]. \tag{54}$$

Thus q_1, q_2, q_3 can be expressed respectively as

$$q_1 = -\frac{\mathrm{i}}{4\varepsilon}(P_1^{(1)})_{12}, \tag{55}$$

$$q_2 = -\frac{\mathrm{i}}{4\varepsilon}(P_1^{(1)})_{13}, \tag{56}$$

$$q_3 = -\frac{\mathrm{i}}{4\varepsilon}(P_1^{(1)})_{14}, \tag{57}$$

where $(P_1^{(1)})_{1l}$ $(l = 2,3,4)$ are the $(1,l)$ elements of the matrix function $P_1^{(1)}$.

Easy to get from Eq. (51),

$$P_1^{(1)} = -\sum_{k,j=1}^{N} v_k \hat{v}_j (M_{-1})_{kj}. \tag{58}$$

At this point, we have completed the potential reconstruction, that is, we have completed the inverse scattering transformation of Eq. (4).

3.5. The evolution of the scattering data

In the previous discussion, we actually considered the solutions of Eq. (4) at a fixed time, and we omitted the time term t. In order to find the solutions $q_1(x,t), q_2(x,t), q_3(x,t)$ to Eq. (4) at any time t, we also need scattering data at time t. It is necessary to investigate the evolution of scattering data over time.

Since the matrix function $J(x,t,\lambda)$ satisfies Eq. (9),

$$J_{-,t} = \frac{\mathrm{i}}{192\varepsilon^2}(\lambda^3 + 2\lambda^2)[\Lambda, J_-] + V_1 J_-. \tag{59}$$

Therefore,

$$(J_- E)_t = \frac{\mathrm{i}}{192\varepsilon^2}(\lambda^3 + 2\lambda^2)[\Lambda, J_- E] + V_1 J_- E.$$

Considering the relation Eq. (17), it is easy to know

$$(J_+ E S)_t = \frac{\mathrm{i}}{192\varepsilon^2}(\lambda^3 + 2\lambda^2)[\Lambda, J_+ E S] + V_1 J_+ E S. \tag{60}$$

Since the potential functions q_1, q_2, q_3 decay rapidly to zero at $x \to \infty$, it follows that the matrix V_1 tends to the zero matrix at $x \to \infty$.

Taking the limit $x \to \infty$ at both ends of Eq. (60), we have

$$S_t = \frac{\mathrm{i}}{192\varepsilon^2}(\lambda^3 + 2\lambda^2)[\Lambda, S]. \tag{61}$$

So,

$$s_{21,t} = \frac{\mathrm{i}}{64\varepsilon^2}(\lambda^3 + 2\lambda^2) s_{21}, \tag{62}$$

$$s_{31,t} = \frac{\mathrm{i}}{64\varepsilon^2}(\lambda^3 + 2\lambda^2) s_{31}, \tag{63}$$

$$s_{41,t} = \frac{\mathrm{i}}{64\varepsilon^2}(\lambda^3 + 2\lambda^2) s_{41}. \tag{64}$$

Similarly, there are

$$r_{12,t} = -\frac{\mathrm{i}}{64\varepsilon^2}(\lambda^3 + 2\lambda^2) r_{12}, \tag{65}$$

$$r_{13,t} = -\frac{\mathrm{i}}{64\varepsilon^2}(\lambda^3 + 2\lambda^2) r_{13}, \tag{66}$$

$$r_{14,t} = -\frac{\mathrm{i}}{64\varepsilon^2}(\lambda^3 + 2\lambda^2) r_{14}. \tag{67}$$

We can also get

$$\frac{\mathrm{d}\lambda_j}{\mathrm{d}t} = 0, \quad v_{j,t} = \frac{\mathrm{i}}{192\varepsilon^2}(\lambda^3 + 2\lambda^2)\Lambda v_j. \tag{68}$$

Therefore, we can obtain

$$v_j = \mathrm{e}^{\theta_j \Lambda} v_{j,0}, \quad \hat{v}_j = v_{j,0}^{\dagger} \mathrm{e}^{\theta_j^* \Lambda}, \tag{69}$$

where

$$\theta_j = \frac{\mathrm{i}\lambda_j}{12\varepsilon} x + \frac{\mathrm{i}}{192\varepsilon^2}(\lambda_j^3 + 2\lambda_j^2) t, \quad \lambda_j \in \mathbb{C}_+.$$

And $v_{j,0}$ is a nonzero constant sequence vector independent of x and t.

4. The N-soliton solutions

Theorem 2 Suppose that

$$\theta_j = \frac{\mathrm{i}\lambda_j}{12\varepsilon} x + \frac{\mathrm{i}}{192\varepsilon^2}(\lambda_j^3 + 2\lambda_j^2) t,$$

where

$$\lambda_j = \lambda_{jR} + \mathrm{i}\lambda_{jI}, \quad v_{j,0} = (\alpha_j, \beta_j, \gamma_j, \delta_j)^{\mathrm{T}},$$
$$(j = 1, 2, \ldots, N)$$

is any nonzero real constant series vector. Then we can obtain the N-soliton solutions of Eq. (4) as follows:

$$\begin{aligned} q_1 &= \frac{\mathrm{i}}{4\varepsilon}\sum_{k,j=1}^{N}(v_k\hat{v}_j)_{12}(M^{-1})_{kj} \\ &= \frac{\mathrm{i}}{4\varepsilon}\sum_{k,j=1}^{N}\alpha_k\beta_j^*\,\mathrm{e}^{-2\theta_k+\theta_j^*}(M^{-1})_{kj}, \qquad (70) \\ q_2 &= \frac{\mathrm{i}}{4\varepsilon}\sum_{k,j=1}^{N}(v_k\hat{v}_j)_{13}(M^{-1})_{kj} \\ &= \frac{\mathrm{i}}{4\varepsilon}\sum_{k,j=1}^{N}\alpha_k\gamma_j^*\,\mathrm{e}^{-2\theta_k+\theta_j^*}(M^{-1})_{kj}, \qquad (71) \\ q_3 &= \frac{\mathrm{i}}{4\varepsilon}\sum_{k,j=1}^{N}(v_k\hat{v}_j)_{14}(M^{-1})_{kj} \\ &= \frac{\mathrm{i}}{4\varepsilon}\sum_{k,j=1}^{N}\alpha_k\delta_j^*\,\mathrm{e}^{-2\theta_k+\theta_j^*}(M^{-1})_{kj}, \qquad (72) \end{aligned}$$

where $M=(m_{kj})_{N\times N}$, its element

$$\begin{aligned} m_{kj} &= \frac{\hat{v}_k v_j}{\lambda_j-\hat{\lambda}_k} \\ &= \frac{\alpha_k^*\alpha_j\,\mathrm{e}^{-2(\theta_k^*+\theta_j)}+(\beta_k^*\beta_j+\gamma_k^*\gamma_j+\delta_k^*\delta_j)\,\mathrm{e}^{\theta_k^*+\theta_j}}{\lambda_j-\hat{\lambda}_k}, \\ & 1\le k,j\le N. \qquad (73) \end{aligned}$$

Proof We define the non-zero row vector v_j and non-zero column vector $\hat{v}_j$ respectively

$$v_j=\mathrm{e}^{\theta_j\Lambda}v_{j,0},\quad \hat{v}_j=v_{j,0}^{\dagger}\,\mathrm{e}^{\theta_j^*\Lambda}. \qquad (74)$$

From this, the matrix function can be constructed

$$P_1^{(1)}=-\sum_{k,j=1}^{N}v_k\hat{v}_j(M^{-1})_{kj}, \qquad (75)$$

where $M=(m_{kj})_{N\times N}$ is an invertible matrix whose elements are $m_{kj}=\hat{v}_kv_j/(\lambda_j-\hat{\lambda}_k)$.

We define the matrix

$$U_1=-\frac{\mathrm{i}}{12\varepsilon}[\Lambda,P_1^{(1)}], \qquad (76)$$

then

$$\begin{aligned} &(U_1)_{11}=(U_1)_{2j}=(U_1)_{3j}=(U_1)_{4j}=0,\quad j=2,3,4, \\ &(U_1)_{12}=\frac{\mathrm{i}}{4\varepsilon}(P_1^{(1)})_{12},\quad (U_1)_{13}=\frac{\mathrm{i}}{4\varepsilon}(P_1^{(1)})_{13}, \\ &(U_1)_{14}=\frac{\mathrm{i}}{4\varepsilon}(P_1^{(1)})_{14}, \\ &(U_1)_{21}=-\frac{\mathrm{i}}{4\varepsilon}(P_1^{(1)})_{21},\quad (U_1)_{31}=-\frac{\mathrm{i}}{4\varepsilon}(P_1^{(1)})_{31}, \\ &(U_1)_{41}=-\frac{\mathrm{i}}{4\varepsilon}(P_1^{(1)})_{41}. \end{aligned}$$

Substituting Eq. (75) into Eq. (76), we can get

$$(U_1)_{12}=\frac{\mathrm{i}}{4\varepsilon}\sum_{k,j=1}^{N}\alpha_k\beta_j^*\,\mathrm{e}^{-2\theta_k+\theta_j^*}(M^{-1})_{kj}, \qquad (77)$$

$$(U_1)_{13}=\frac{\mathrm{i}}{4\varepsilon}\sum_{k,j=1}^{N}\alpha_k\gamma_j^*\,\mathrm{e}^{-2\theta_k+\theta_j^*}(M^{-1})_{kj}, \qquad (78)$$

$$(U_1)_{14}=\frac{\mathrm{i}}{4\varepsilon}\sum_{k,j=1}^{N}\alpha_k\delta_j^*\,\mathrm{e}^{-2\theta_k+\theta_j^*}(M^{-1})_{kj}, \qquad (79)$$

where $M=(m_{kj})_{N\times N}$, elements

$$\begin{aligned} &m_{kj}=\frac{\alpha_k^*\alpha_j\,\mathrm{e}^{-2(\theta_k^*+\theta_j)}+(\beta_k^*\beta_j+\gamma_k^*\gamma_j+\delta_k^*\delta_j)\,\mathrm{e}^{\theta_k^*+\theta_j}}{\lambda_j-\hat{\lambda}_k}, \\ &1\le k,j\le N. \end{aligned}$$

Similar to $(U_1)_{12},(U_1)_{13},(U_1)_{14}$, we can also obtain an explicit expression for $(U_1)_{21},(U_1)_{31},(U_1)_{41}$. And from some algebras, we know

$$\begin{aligned} &(M^{-1})_{kj}=(M^{-1})_{jk},\quad ((M^{-1})_{kj})^*=-(M^{-1})_{kj}, \\ &1\le k,j\le N. \qquad (80) \end{aligned}$$

That means

$$\begin{aligned} &(U_1)_{21}=-(U_1)_{12},\ (U_1)_{31}=-(U_1)_{13}, \\ &(U_1)_{41}=-(U_1)_{14}. \end{aligned}$$

In addition, we can prove that

$$J=I-\sum_{k,j=1}^{N}\frac{v_k\hat{v}_j(M^{-1})_{kj}}{\lambda_k-\hat{\lambda}_j},\quad \lambda\in\mathbb{C}_+, \qquad (81)$$

satisfies

$$\begin{aligned} J_x &= \frac{\mathrm{i}\lambda}{12\varepsilon}[\Lambda,J]+U_1J, \qquad (82) \\ J_t &= \frac{\mathrm{i}}{192\varepsilon^2}(\lambda^3+2\lambda^2)[\Lambda,J]+V_1J, \\ V_1 &= \frac{\lambda^2}{16\varepsilon}U_1+\lambda V_2+V_3, \qquad (83) \end{aligned}$$

where

$$V_2=\frac{1}{4}\begin{pmatrix} \mathrm{i}e & -\frac{q_1}{2\varepsilon}-\mathrm{i}q_{1x} & -\frac{q_2}{2\varepsilon}-\mathrm{i}q_{2x} & -\frac{q_3}{2\varepsilon}-\mathrm{i}q_{3x} \\ \frac{q_1^*}{2\varepsilon}-\mathrm{i}q_{1x}^* & -\mathrm{i}|q_1|^2 & -\mathrm{i}q_1^*q_2 & -\mathrm{i}q_1^*q_3 \\ \frac{q_2^*}{2\varepsilon}-\mathrm{i}q_{2x}^* & -\mathrm{i}q_2^*q_1 & -\mathrm{i}|q_2|^2 & -\mathrm{i}q_2^*q_3 \\ \frac{q_3^*}{2\varepsilon}-\mathrm{i}q_{3x}^* & -\mathrm{i}q_3^*q_1 & -\mathrm{i}q_3^*q_2 & -\mathrm{i}|q_3|^2 \end{pmatrix},$$

$$V_3=\begin{pmatrix} \varepsilon(e_1+e_2+e_3)+\frac{\mathrm{i}}{2}e & \varepsilon e_4-\frac{\mathrm{i}}{2}q_{1x} & \varepsilon e_5-\frac{\mathrm{i}}{2}q_{2x} & \varepsilon e_6-\frac{\mathrm{i}}{2}q_{3x} \\ -\varepsilon e_4^*-\frac{\mathrm{i}}{2}q_{1x}^* & -\varepsilon e_1-\frac{\mathrm{i}}{2}|q_1|^2 & \varepsilon e_7-\frac{\mathrm{i}}{2}q_1^*q_2 & \varepsilon e_8-\frac{\mathrm{i}}{2}q_1^*q_3 \\ -\varepsilon e_5^*-\frac{\mathrm{i}}{2}q_{2x}^* & -\varepsilon e_7^*-\frac{\mathrm{i}}{2}q_2^*q_1 & -\varepsilon e_2-\frac{\mathrm{i}}{2}|q_2|^2 & \varepsilon e_9-\frac{\mathrm{i}}{2}q_2^*q_3 \\ -\varepsilon e_6^*-\frac{\mathrm{i}}{2}q_{3x}^* & -\varepsilon e_8^*-\frac{\mathrm{i}}{2}q_3^*q_1 & -\varepsilon e_9^*-\frac{\mathrm{i}}{2}q_3^*q_2 & -\varepsilon e_3-\frac{\mathrm{i}}{2}|q_3|^2 \end{pmatrix},$$

$e=|q_1|^2+|q_2|^2+|q_3|^2$, $e_1=q_1q_{1x}^*-q_{1x}q_1^*$, $e_2=q_2q_{2x}^*-q_{2x}q_2^*$, $e_3=q_3q_{3x}^*-q_{3x}q_3^*$,

$e_4=q_{1xx}+2eq_1$, $e_5=q_{2xx}+2eq_2$, $e_6=q_{3xx}+2eq_3$, $e_7=q_1^*q_{2x}-q_{1x}^*q_2$,

$e_8=q_1^*q_{3x}-q_{1x}^*q_3$, $e_9=q_2^*q_{3x}-q_{2x}^*q_3$.

This is the Lax pair of Eq. (4). Thus $q_1=-(U_1)_{12}$, $q_2=-(U_1)_{13}$, and $q_3=-(U_1)_{14}$, that is, equations (70)–(72) are the N-soliton solutions of Eq. (4), which concludes the proof.

5. Dynamic property analysis

In this section, we discuss the evolution of 1, 2-soliton solutions and the influence of parameters on them.

Specifically, when $N=1$ we can obtain the expression for the 1-soliton solutions of Eq. (4) as follows:

$$q_1=-\frac{\lambda_{1\mathrm{I}}\alpha_1\beta_1^*}{2\varepsilon}\times\frac{\mathrm{e}^{-\theta_{1\mathrm{R}}-3\mathrm{i}\theta_{1\mathrm{I}}}}{|\alpha_1|^2\mathrm{e}^{-4\theta_{1\mathrm{R}}}+(|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2)\mathrm{e}^{2\theta_{1\mathrm{R}}}},\tag{84}$$

$$q_2=-\frac{\lambda_{1\mathrm{I}}\gamma_1\beta_1^*}{2\varepsilon}\times\frac{\mathrm{e}^{-\theta_{1\mathrm{R}}-3\mathrm{i}\theta_{1\mathrm{I}}}}{|\alpha_1|^2\mathrm{e}^{-4\theta_{1\mathrm{R}}}+(|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2)\mathrm{e}^{2\theta_{1\mathrm{R}}}},\tag{85}$$

$$q_3=-\frac{\lambda_{1\mathrm{I}}\delta_1\beta_1^*}{2\varepsilon}\times\frac{\mathrm{e}^{-\theta_{1\mathrm{R}}-3\mathrm{i}\theta_{1\mathrm{I}}}}{|\alpha_1|^2\mathrm{e}^{-4\theta_{1\mathrm{R}}}+(|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2)\mathrm{e}^{2\theta_{1\mathrm{R}}}}.\tag{86}$$

Actually, the 1-soliton solutions can be rewritten as follows:

$$q_1=-\frac{\lambda_{1\mathrm{I}}}{4\varepsilon}\frac{\alpha_1\beta_1^*}{|\alpha_1|\sqrt{|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2}}\times\mathrm{sech}\left[-\frac{\lambda_{1\mathrm{I}}}{4\varepsilon}\left(x+\frac{3\lambda_{1\mathrm{R}}^2+4\lambda_{1\mathrm{R}}-\lambda_{1\mathrm{I}}^2}{16\varepsilon}t+\Omega_0\right)\right]\times\exp\left\{-\frac{\mathrm{i}\lambda_{1\mathrm{R}}}{6\varepsilon}x-\frac{\mathrm{i}(\lambda_{1\mathrm{R}}^2-\lambda_{1\mathrm{I}})}{48\varepsilon^2}t-\mathrm{i}M_0\right\},\tag{87}$$

$$q_2=-\frac{\lambda_{1\mathrm{I}}}{4\varepsilon}\frac{\alpha_1\gamma_1^*}{|\alpha_1|\sqrt{|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2}}\times\mathrm{sech}\left[-\frac{\lambda_{1\mathrm{I}}}{4\varepsilon}\left(x+\frac{3\lambda_{1\mathrm{R}}^2+4\lambda_{1\mathrm{R}}-\lambda_{1\mathrm{I}}^2}{16\varepsilon}t+\Omega_0\right)\right]\times\exp\left\{-\frac{\mathrm{i}\lambda_{1\mathrm{R}}}{6\varepsilon}x-\frac{\mathrm{i}(\lambda_{1\mathrm{R}}^2-\lambda_{1\mathrm{I}})}{48\varepsilon^2}t-\mathrm{i}M_0\right\},\tag{88}$$

$$q_3=-\frac{\lambda_{1\mathrm{I}}}{4\varepsilon}\frac{\alpha_1\delta_1^*}{|\alpha_1|\sqrt{|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2}}\times\mathrm{sech}\left[-\frac{\lambda_{1\mathrm{I}}}{4\varepsilon}\left(x+\frac{3\lambda_{1\mathrm{R}}^2+4\lambda_{1\mathrm{R}}-\lambda_{1\mathrm{I}}^2}{16\varepsilon}t+\Omega_0\right)\right]\times\exp\left\{-\frac{\mathrm{i}\lambda_{1\mathrm{R}}}{6\varepsilon}x-\frac{\mathrm{i}(\lambda_{1\mathrm{R}}^2-\lambda_{1\mathrm{I}})}{48\varepsilon^2}t-\mathrm{i}M_0\right\},\tag{89}$$

with

$$\Omega_0=\frac{4\varepsilon}{\lambda_{1\mathrm{I}}}\ln\frac{|\alpha_1|}{\sqrt{|\beta_1|^2+|\gamma_1|^2+|\delta_1|^2}},\qquad M_0=\frac{\lambda_{1\mathrm{R}}}{96\varepsilon^2}(\lambda_{1\mathrm{R}}^2-3\lambda_{1\mathrm{I}}^2).\tag{90}$$

From the analysis of the above expression, it can be seen that the mode $|q_j|$ $(j=1,2,3)$ of the single soliton solutions of the three-component coupled Eq. (4) has the form of a hyperbolic secant function, and the maximum amplitude is $\lambda_{1\mathrm{I}}/4|\varepsilon|$ and the wave propagation speed is $-(3\lambda_{1\mathrm{R}}^2+4\lambda_{1\mathrm{R}}-\lambda_{1\mathrm{I}}^2)/16\varepsilon$. The phase linearity of the soliton solutions $q_j(x,t)$ depends on the space coordinate x and the time coordinate t, and the parameters Ω_0 and M_0 determine the initial position and phase of the soliton wave.

Since q_1, q_2, and q_3 have similar properties, here we take q_1 as an example for analysis. By taking the values $\alpha_1=1+\mathrm{i}$, $\beta_1=0.2+0.1\mathrm{i}$, $\gamma_1=0.02+0.02\mathrm{i}$, $\delta_1=0.1+0.2\mathrm{i}$, $\lambda=2+3\mathrm{i}$, we give figures of the 1-soliton solutions in Figs. 1 and 2.

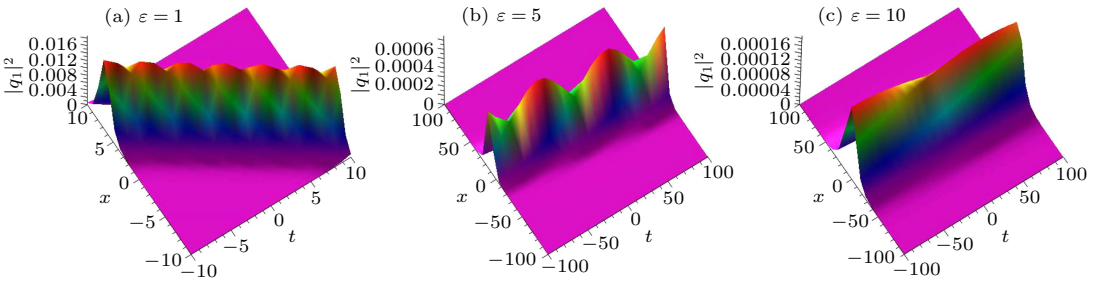


Fig. 1. Evolution of 1-soliton solution $|q_1|^2$.

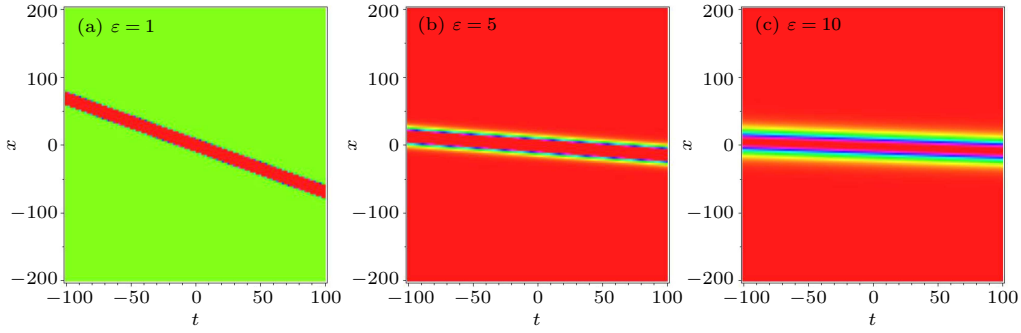


Fig. 2. Density figure of 1-soliton solutions $|q_1|^2$.

The figures show that the amplitude of the 1-soliton solutions decreases proportionally as the value of ε increases, while the width gradually increases. Therefore, it can be concluded that the value of ε affects the amplitude and width of 1-soliton solutions.

In addition, we also study the effect of initial values on 1-soliton solutions. We find that the amplitude, velocity, and other properties of solitons do not change significantly when the initial values are removed from real, complex, and pure imaginary numbers.

When $N = 2$, we can get the expression for the 2-soliton solution of Eq. (4) as follows:

$$\begin{aligned} q_1 =& \frac{\mathrm{i}}{4\varepsilon}\alpha_1\beta_1^*\,\mathrm{e}^{-2\theta_1+\theta_1^*}(M^{-1})_{11}+\frac{\mathrm{i}}{4\varepsilon}\alpha_1\beta_2^*\,\mathrm{e}^{-2\theta_1+\theta_2^*}(M^{-1})_{12}\\ &+\frac{\mathrm{i}}{4\varepsilon}\alpha_2\beta_1^*\,\mathrm{e}^{-2\theta_2+\theta_1^*}(M^{-1})_{21}\\ &+\frac{\mathrm{i}}{4\varepsilon}\alpha_2\beta_2^*\,\mathrm{e}^{-2\theta_2+\theta_2^*}(M^{-1})_{22}, \end{aligned} \tag{91}$$

$$\begin{aligned} q_2 =& \frac{\mathrm{i}}{4\varepsilon}\alpha_1\gamma_1^*\,\mathrm{e}^{-2\theta_1+\theta_1^*}(M^{-1})_{11}+\frac{\mathrm{i}}{4\varepsilon}\alpha_1\gamma_2^*\,\mathrm{e}^{-2\theta_1+\theta_2^*}(M^{-1})_{12}\\ &+\frac{\mathrm{i}}{4\varepsilon}\alpha_2\gamma_1^*\,\mathrm{e}^{-2\theta_2+\theta_1^*}(M^{-1})_{21}\\ &+\frac{\mathrm{i}}{4\varepsilon}\alpha_2\gamma_2^*\,\mathrm{e}^{-2\theta_2+\theta_2^*}(M^{-1})_{22},\\ q_3 =& \frac{\mathrm{i}}{4\varepsilon}\alpha_1\delta_1^*\,\mathrm{e}^{-2\theta_1+\theta_1^*}(M^{-1})_{11}\\ &+\frac{\mathrm{i}}{4\varepsilon}\alpha_1\delta_2^*\,\mathrm{e}^{-2\theta_1+\theta_2^*}(M^{-1})_{12}\\ &+\frac{\mathrm{i}}{4\varepsilon}\alpha_2\delta_1^*\,\mathrm{e}^{-2\theta_2+\theta_1^*}(M^{-1})_{21}\\ &+\frac{\mathrm{i}}{4\varepsilon}\alpha_2\delta_2^*\,\mathrm{e}^{-2\theta_2+\theta_2^*}(M^{-1})_{22}, \end{aligned} \tag{92}$$

where

$$M=\begin{pmatrix} \dfrac{\alpha_1\alpha_1^*\,\mathrm{e}^{-2(\theta_1^*+\theta_1)+(\beta_1\beta_1^*+\gamma_1\gamma_1^*+\delta_1\delta_1^*)\mathrm{e}^{\theta_1+\theta_1^*}}}{\lambda_1-\hat{\lambda}_1} & \dfrac{\alpha_2\alpha_1^*\,\mathrm{e}^{-2(\theta_1^*+\theta_2)+(\beta_2\beta_1^*+\gamma_2\gamma_1^*+\delta_2\delta_1^*)\mathrm{e}^{\theta_2+\theta_1^*}}}{\lambda_2-\hat{\lambda}_1}\\ \dfrac{\alpha_1\alpha_2^*\,\mathrm{e}^{-2(\theta_2^*+\theta_1)+(\beta_1\beta_2^*+\gamma_1\gamma_2^*+\delta_1\delta_2^*)\mathrm{e}^{\theta_1+\theta_2^*}}}{\lambda_1-\hat{\lambda}_2} & \dfrac{\alpha_2\alpha_2^*\,\mathrm{e}^{-2(\theta_2^*+\theta_2)+(\beta_2\beta_2^*+\gamma_2\gamma_2^*+\delta_2\delta_2^*)\mathrm{e}^{\theta_2+\theta_2^*}}}{\lambda_2-\hat{\lambda}_2} \end{pmatrix}. \tag{93}$$

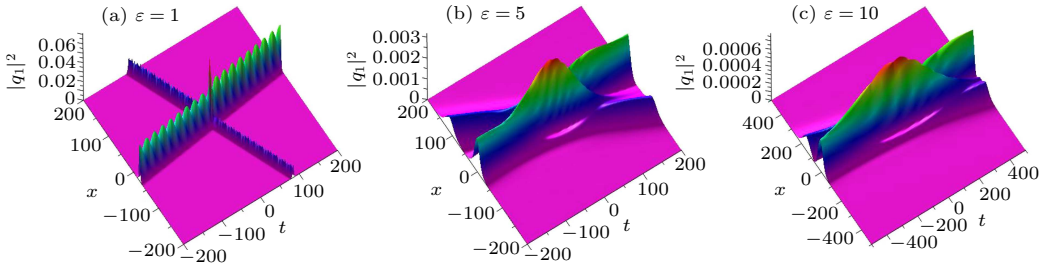


Fig. 3. Evolution of 2-soliton solutions $|q_1|^2$.

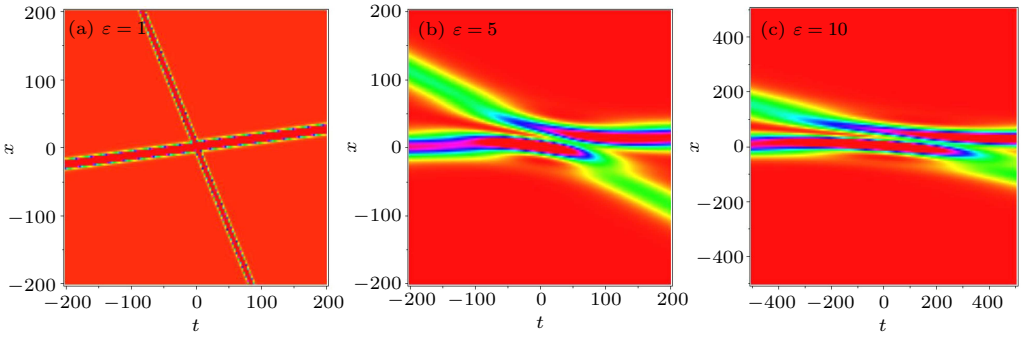


Fig. 4. Density figure of 2-soliton solutions $|q_1|^2$.

By taking the values $\alpha_1 = \mathrm{i}$, $\alpha_2 = 2\mathrm{i}$, $\beta_1 = \mathrm{i}$, $\beta_2 = 2\mathrm{i}$, $\gamma_1 = -\mathrm{i}$, $\gamma_2 = -2\mathrm{i}$, $\delta_1 = -\mathrm{i}$, $\delta_2 = 2\mathrm{i}$, and $\lambda_1 = 3+0.92\mathrm{i}$, $\lambda_2 = 0.05+1.5\mathrm{i}$, the evolution figures of the solutions are given with q_1 as an example in Figs. 3 and 4.

With the value of ε increases, the amplitude of the 2-soliton solutions decreases in proportion, and the width increases again. That is to say, the influence law of ε on 2-soliton solutions is consistent with that of 1-soliton solutions. It is worth noting that the time for the two components of the soliton to collide remains the same, but the time side of the interaction is longer. In addition, we can observe that the amplitude, velocity and direction of the two components do not change before and after the soliton collision, which means that the soliton has an elastic collision.

Next, we discuss the effect of the value of λ_j $(j = 1,2)$ on the 2-soliton solutions. The parameters are (a) $\lambda_1 = 3+0.92\mathrm{i}$, $\lambda_2 = 0.05+1.5\mathrm{i}$; (b) $\lambda_1 = 3+0.92\mathrm{i}$, $\lambda_2 = 3+1.5\mathrm{i}$; (c) $\lambda_1 = 3+0.92\mathrm{i}$, $\lambda_2 = 0.05+0.92\mathrm{i}$. The figures are shown in Figs. 5 and 6.

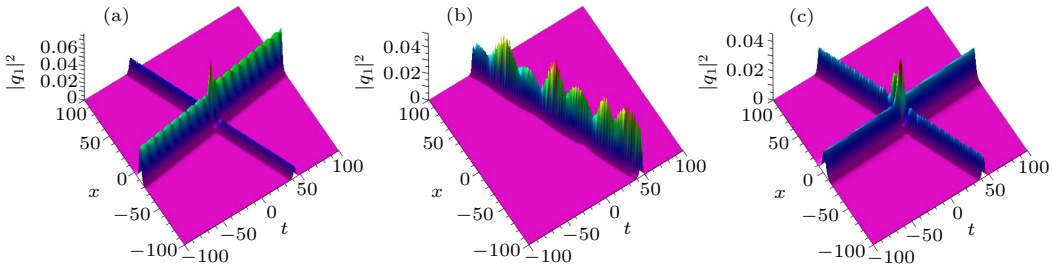


Fig. 5. Evolution of 2-soliton solutions $|q_1|^2$.

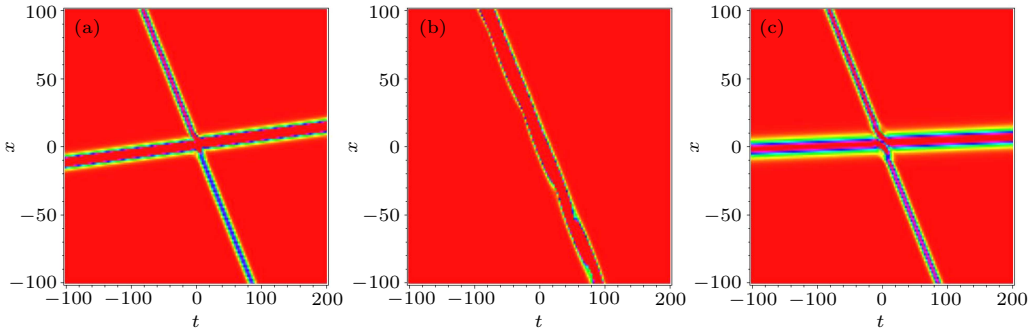


Fig. 6. Density figure of 2-soliton solutions $|q_1|^2$.

The figures reveal that when the real and imaginary parts are not equal, the two components of the soliton move forward with different amplitudes and have elastic collisions. When the real parts are equal but the imaginary parts are different, the two components merge into one component. When the real parts are different and the imaginary parts are equal, the two components of the soliton will still collide elastic, but the amplitude of the two components is the same. As can be seen from the expression, when the real and imaginary parts are equal, the solution is meaningless, so it is not discussed.

In addition, we find that the real, complex and pure imaginary values of the initial values have no obvious influence on the propagation process of 2-soliton solutions.

Remark 1 In fact, when $N \geq 2$, the N-soliton solutiond of the three-component coupled Eq. (4) can be expressed as

$$q_1 = -\frac{\mathrm{i}}{4\varepsilon}\frac{\det F_1}{\det M}, \quad q_2 = -\frac{\mathrm{i}}{4\varepsilon}\frac{\det F_2}{\det M}, \quad q_3 = -\frac{\mathrm{i}}{4\varepsilon}\frac{\det F_3}{\det M}, \tag{94}$$

where the $(N+1)\times(N+1)$ matrix F_1, F_2 and F_3 are defined as follows:

$$F_1 = \begin{pmatrix} 0 & \beta_1^* \mathrm{e}^{\theta_1^*} & \beta_2^* \mathrm{e}^{\theta_2^*} & \cdots & \beta_N^* \mathrm{e}^{\theta_N^*} \\ \alpha_1 \mathrm{e}^{-2\theta_1} & M_{11} & M_{12} & \cdots & M_{1N} \\ \alpha_2 \mathrm{e}^{-2\theta_2} & M_{21} & M_{22} & \cdots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_N \mathrm{e}^{-2\theta_N} & M_{N1} & M_{N2} & \cdots & M_{NN} \end{pmatrix}, \tag{95}$$

$$F_2 = \begin{pmatrix} 0 & \gamma_1^* \mathrm{e}^{\theta_1^*} & \gamma_2^* \mathrm{e}^{\theta_2^*} & \cdots & \gamma_N^* \mathrm{e}^{\theta_N^*} \\ \alpha_1 \mathrm{e}^{-2\theta_1} & M_{11} & M_{12} & \cdots & M_{1N} \\ \alpha_2 \mathrm{e}^{-2\theta_2} & M_{21} & M_{22} & \cdots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_N \mathrm{e}^{-2\theta_N} & M_{N1} & M_{N2} & \cdots & M_{NN} \end{pmatrix}, \tag{96}$$

$$F_3 = \begin{pmatrix} 0 & \delta_1^* \mathrm{e}^{\theta_1^*} & \delta_2^* \mathrm{e}^{\theta_2^*} & \cdots & \delta_N^* \mathrm{e}^{\theta_N^*} \\ \alpha_1 \mathrm{e}^{-2\theta_1} & M_{11} & M_{12} & \cdots & M_{1N} \\ \alpha_2 \mathrm{e}^{-2\theta_2} & M_{21} & M_{22} & \cdots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_N \mathrm{e}^{-2\theta_N} & M_{N1} & M_{N2} & \cdots & M_{NN} \end{pmatrix}. \tag{97}$$

6. Conclusions

In this paper, N-soliton solutions of a class of high-order three-component coupled nonlinear Hirota equations are ob-

tained by Riemann–Hilbert method. From the Lax pair of the equation, the equivalent form of the spectral equation is obtained by introducing auxiliary functions, the Riemann–Hilbert problem is constructed by analyzing the analytic property of the Jost solution, and the N-soliton solutions of the equation is obtained by solving the Riemann–Hilbert problem in regular and irregular cases.

In particular, we also studied the explicit solution of the $1,2$-soliton solutions of this equation, and gave the evolution figure of the soliton solutions through numerical simulation, analyzed the influence of integrable perturbations and other parameters on its dynamic properties, and summarized the corresponding laws as follows:

(i) The value of ε will affect the amplitude and width of the soliton solutions. With the increase of ε, the amplitude of the soliton solutions will decrease proportionally and the width will increase.

(ii) The value of spectral parameter λ_j $(j = 1,2)$ will affect the evolution state of 2-soliton solutions.

As a conclusion, we can control the physical quantity and evolution state of soliton solutions during propagation by changing the parameters.

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