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Citation: Chin. Phys. B, 2025, 34 (**7**): 070201. DOI: 10.1088/1674-1056/adcdf1
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Prolongation structure and Darboux transformation of nonlinear mixed gas equations

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(Received 26 February 2025; revised manuscript received 16 April 2025; accepted manuscript online 17 April 2025)

The widespread popularization and application of laser technology have provided a powerful tool for a deeper understanding of the material world and given birth to several emerging research fields. This study mainly focuses on the following three key aspects. First, the classical ensemble method is adopted to conduct a comprehensive and in-depth analysis of two-dimensional (2D) matter–wave pulses in Bose–Fermi mixed gases (including linear and nonlinear pulses). Second, under the strict constraints of unitary systems, a coupled KdV equation is successfully derived, and the prolongation structure theory is skillfully used to carry out detailed calculations and analyses on this equation. Thus, the prolongation algebra of this equation is accurately determined, and the corresponding Lax pair is rigorously derived. Finally, based on the carefully obtained Lax pair from the prolongation structure theory, the soliton solutions of this equation are further analyzed in depth, and intuitive images of each soliton solution are carefully drawn. This lays a solid foundation for subsequent detailed research on these soliton characteristics and provides great convenience.

Keywords: Bose–Fermi, mixture, prolongation structure, Lax pair, Darboux transformation

PACS: 02.30.Jr, 03.67.Bg, 03.75.Gg, 03.75.Nt

DOI: 10.1088/1674-1056/adcdf1 **CSTR:** 32038.14.CPB.adcdf1

1. Introduction

At present, solving the three-dimensional (3D) Schrödinger equation of a two-electron system numerically by quantum mechanics methods is facing tremendous computational challenges. Conversely, classical methods have provided effective solutions to this issue through the implementation of novel methodologies.[1,2] Therefore, this study adopts the classical ensemble method to explore the phenomenon of atomic ionization in elliptically polarized strong laser fields. In recent years, laser cooling and atomic trapping techniques have made significant breakthroughs in the field of mathematical physics. These breakthroughs have not only realized Bose–Einstein condensation (BEC), but also included phenomena such as mixtures of two types of bosons, Fermi degenerate atomic gases, and Bose–Fermi mixtures.[3,4] Subsequently, the academic community has carried out extensive and in-depth research on these phenomena at both experimental and theoretical levels. In particular, the Gross–Pitaevskii (GP) equation has been used for systematic research on BEC.[5–8] Furthermore, research on Bose–Fermi mixtures has been extended to the crossover regime between BEC and weakly correlated atomic BCS condensation, resulting in the establishment of numerous related parameters.

In 1975,Wahlquist and Estabrook established the prolongation structure theory of (1+1)-dimensional nonlinear evolution equations based on the system of exterior differential forms and Lie algebra representation theory, and systematically proposed an effective method for solving the linear spectral problems of nonlinear evolution equations. The core of this method is to express the (1+1)-dimensional integrable nonlinear differential equation as a set of exterior differential 2-forms and ensure that these exterior differential forms constitute a closed ideal. By introducing potentials or pseudopotentials and related exterior differential 1-forms, and requiring that these newly introduced exterior differential 1-forms together with the original exterior differential 2-forms form a new closed ideal, the Lax representation and Bäcklund transformation of the integrable equation are successfully derived. They applied this theory to the KdV equation and systematically obtained the linear spectral problem and Bäcklund transformation of the KdV equation. Since its inception, this theoretical approach has been extensively employed in the analysis of (1+1)-dimensional integrable nonlinear evolution equations.[9–11] Recently, researchers have used this theory to conduct in-depth analysis and research on the Heisenberg ferromagnetic chain equation,[12] higher-order nonlinear Schrödinger equation,[13] and reaction–diffusion equation.[14] With the in-depth study of soliton theory, in recent years, a series of systematic methods have been developed to find exact solutions of nonlinear evolution equations, including the inverse scattering method,[15] nonlinearization method,[16] Hirota bilinear method,[17] geometric algebra method,[18] Darboux transformation method,[19] Bäcklund transformation method,[20] and Painlevé analysis method,[21] etc. Among these methods, the Darboux transformation method has attracted much attention for its simplicity and effectiveness. The core idea of this method is to start from a known seed solution

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and obtain new and more meaningful solutions of the equation through a specific transformation process, thereby obtaining the exact solution of the nonlinear evolution equation. This process can also be described as follows: if a solution to a certain nonlinear partial differential equation is known, then further solutions to the differential equation can be found; if a solution to the same equation is known, then another solution to the integrable evolution equation can be obtained.[22] In addition, by using iteration techniques, higher-order Darboux transformations can be realized, and thus higher-order solutions of the equation can be obtained from simple solutions through higher-order transformations. So far, rich research results have been achieved in solving nonlinear evolution equations related to 2×2 matrix spectral problems using the Darboux transformation method.[23,24]

In recent years, the research on various types of coupled KdV equations has attracted widespread attention. This paper focuses on the properties of two-dimensional matter-wave pulses in Bose–Fermi mixed gases, covering linear and nonlinear Bose–Einstein condensates (BEC) and Bardeen–Cooper–Schrieffer (BCS) theory, as well as the derivation of solitary-wave equations under the limit conditions of unitary systems. Subsequently, this paper conducts an in-depth study on the coupled KdV equation using the prolongation structure theory, explores its prolongation algebraic structure, and successfully constructs the corresponding Lax pair. In this study, we employ the Darboux transformation method to systematically derive the single-soliton, double-soliton, triple-soliton, and general n-soliton solutions of the equation. Furthermore, graphical representations of these solutions are presented to facilitate a comprehensive analysis of their dynamical properties and interaction behaviors.

2. The evolution of dynamic equations

The spatial correlation properties of one-dimensional (1D) strongly interacting Bose–Fermi mixed gases were investigated at absolute zero and finite low temperatures. For Bose–Fermi mixed quantum gases in the Tonks–Girardeau atomic gas regime, the two-body spatial correlation function was calculated, and the complex many-body wave function spatial correlation function was transformed into an easily calculable matrix determinant form. By solving this equation exactly, the microscopic mechanism of interparticle interactions was revealed, and the critical conditions for quantum phase transitions in low-temperature environments were further verified, providing theoretical support for experimental control of the properties of mixed gases. In addition, the study found that as the temperature decreased, the correlation strength between particles significantly increased, causing significant changes in the macroscopic properties of the mixed gas. This finding is of great significance for understanding the unique behavior of low-temperature quantum systems. Therefore, for Bose–Fermi mixed gas systems containing a large number of particles, the following 2D nondimensionalized dynamic equation is satisfied at specific temperatures:[25]

$$\begin{cases} \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_{\mathrm{b}} = [-\frac{\hbar^2}{2m_{\mathrm{b}}}\nabla^2 + U_{\mathrm{b}} + \mu_{\mathrm{b}}(n_{\mathrm{b}},a_{\mathrm{b}}) + G_{\mathrm{bp}}|\Psi_{\mathrm{p}}|^2]\Psi_{\mathrm{b}}, \\ \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_{\mathrm{p}} = [-\frac{\hbar^2}{2m_{\mathrm{p}}}\nabla^2 + U_{\mathrm{p}} + \mu_{\mathrm{p}}(n_{\mathrm{p}},a_{\mathrm{p}}) + G_{\mathrm{bp}}|\Psi_{\mathrm{b}}|^2]\Psi_{\mathrm{p}}. \end{cases} \tag{1}$$

Here, $\mu_{\mathrm{b}}(n_{\mathrm{b}},a_{\mathrm{b}}) = c_{\mathrm{b}_0}n_{\mathrm{b}}^{r_{\mathrm{b}_0}}(1+c_{\mathrm{b}_1}n_{\mathrm{b}}^{r_{\mathrm{b}_1}})$ denotes the chemical formula for bosons, $\mu_{\mathrm{p}}(n_{\mathrm{p}},a_{\mathrm{p}}) = c_{\mathrm{p}_0}n_{\mathrm{p}}^{r_{\mathrm{p}_0}}(1+c_{\mathrm{p}_1}n_{\mathrm{p}}^{r_{\mathrm{p}_1}})$ represents the chemical formula for a fermion, The dependence of the propagation velocity, amplitude and the width of the solitary wave on the dimensionless interaction parameter $y = 1/(k_{\mathrm{F}}a_{\mathrm{sc}})$ is presented for the limited cases of the BEC regime and unitarity. $U_{\mathrm{b}} = U_{\mathrm{p}} = \frac{1}{2}m[\omega_\perp^2(x^2+y^2)+\omega_z^2z^2]$ denotes the external potential, $\Psi_{\mathrm{b}}(r,t) = \sqrt{n_{\mathrm{b}}(r,t)}\,\mathrm{e}^{\phi_{\mathrm{b}}(r,t)}$ constitutes an order parameter for bosons, $\Psi_{\mathrm{p}}(r,t) = \sqrt{n_{\mathrm{p}}(r,t)}\,\mathrm{e}^{\phi_{\mathrm{p}}(r,t)}$ constitutes an order parameter for fermions, n_j $(j=\mathrm{b},\mathrm{p})$ denotes the ionic density, The parameter G_{bp} characterizes the interaction between bosons and fermions.

Substituting $\Psi_{\mathrm{b}}(r,t) = \sqrt{n_{\mathrm{b}}(r,t)}\,\mathrm{e}^{\theta_{\mathrm{b}}(r,t)}$ into the first equation of Eq. (1) and $\Psi_{\mathrm{p}}(r,t) = \sqrt{n_{\mathrm{p}}(r,t)}\,\mathrm{e}^{\theta_{\mathrm{p}}(r,t)}$ into the second equation of Eq. (2), simplification yields

$$\begin{cases} \frac{\partial n_{\mathrm{b}}}{\partial t} + \frac{\hbar}{m_{\mathrm{b}}}\nabla\cdot(n_{\mathrm{b}},\bigtriangledown\phi_{\mathrm{b}}) = 0, \\ \mathrm{i}\frac{\partial\phi_{\mathrm{b}}}{\partial t} + \frac{\hbar}{2m_{\mathrm{b}}}[(\nabla\phi_{\mathrm{b}})^2 - \frac{1}{\sqrt{n_{\mathrm{b}}}}\nabla^2\sqrt{n_{\mathrm{b}}}] \\ +\frac{1}{\hbar}[U_{\mathrm{b}} + \mu_{\mathrm{b}}(n_{\mathrm{b}},a_{\mathrm{b}})] + G_{\mathrm{bp}}n_{\mathrm{p}} = 0, \\ \frac{\partial n_{\mathrm{p}}}{\partial t} + \frac{\hbar}{m_{\mathrm{p}}}\nabla\cdot(n_{\mathrm{p}},\bigtriangledown\phi_{\mathrm{p}}) = 0, \\ \mathrm{i}\frac{\partial\phi_{\mathrm{p}}}{\partial t} + \frac{\hbar}{2m_{\mathrm{p}}}[(\nabla\phi_{\mathrm{p}})^2 - \frac{1}{\sqrt{n_{\mathrm{p}}}}\nabla^2\sqrt{n_{\mathrm{p}}}] \\ +\frac{1}{\hbar}[U_{\mathrm{p}} + \mu_{\mathrm{p}}(n_{\mathrm{p}},a_{\mathrm{p}})] + G_{\mathrm{bp}}n_{\mathrm{b}} = 0. \end{cases} \tag{2}$$

Assuming a particle is confined within a potential well characterized by a disk-shaped potential, $U_{\mathrm{b}} = U_{\mathrm{p}} = \frac{1}{2}m[\omega_\perp^2(x^2+y^2)+\omega_z^2z^2]$, where $\omega_\perp$ represents the transverse frequency along the x or y axes, and ω_z denotes the potential well frequency along the z axis. To facilitate the nondimensionalization of variables, we define the following transformation of equation set (2):

$$\begin{cases} (x,y,z) = a_z(x',y',z'), \\ t = \omega_z^{-1}t', \\ n_j = n_{j0}n', \quad (j=\mathrm{b},\mathrm{p}), \\ a_z = [\frac{\hbar}{m\omega_z}]^{1/2}, \\ n_{j0} = N/a_z^3, \\ \mu_j = \hbar\omega_z\mu_j', \quad (j=\mathrm{b},\mathrm{p}). \end{cases} \tag{3}$$

Substituting these variables into Eq. (2), and given that the disk potential $\omega_\perp/\omega_z$ is exceedingly small, we can neglect the term $(\omega_\perp/\omega_Z)^2$. The subsequent discussion is as follows:

$$U_{\mathrm{b}} = U_{\mathrm{p}} = \frac{1}{2}m[\omega_\perp^2(x^2+y^2)+\omega_z^2z^2] \quad (\omega_\perp \ll \omega_z).$$

Consequently, equation (1) in equation set (2) and Eq. (3) can be transformed into

$$\begin{cases} \frac{\partial' n'_{\rm b}}{\partial' t'} + [\nabla' \cdot (n'_{\rm b} \nabla' \theta'_{\rm b})] = 0 \Longleftrightarrow \frac{\partial n_{\rm b}}{\partial t} + [\nabla \cdot (n_{\rm b} \nabla \theta_{\rm b})] = 0, \\ \frac{\partial' n'_{\rm p}}{\partial' t'} + [\nabla' \cdot (n'_{\rm p} \nabla' \theta'_{\rm p})] = 0 \Longleftrightarrow \frac{\partial n_{\rm p}}{\partial t} + [\nabla \cdot (n_{\rm p} \nabla \theta_{\rm p})] = 0. \end{cases} \tag{4}$$

Note: ($\nabla' = \frac{1}{a_z}\nabla$), In equation set (2), the second and fourth equations can be correspondingly transformed into the following equivalent forms:

$$\begin{cases} \mathrm{i}\frac{\partial' \theta'_{\rm b}}{\partial' t'} + \frac{1}{2}[(\nabla' \theta'_{\rm b})^2 - \frac{1}{\sqrt{n'_{\rm b}}}\nabla'^2\sqrt{n'_{\rm b}}] + \frac{1}{2}[\frac{\omega_\perp^2}{\omega_z^2}(x'^2 + y'^2) + z^2] + \frac{m}{\hbar^2}a_z^2\mu_{\rm b} + G_{\rm bp}n_{\rm p} = 0, \\ \mathrm{i}\frac{\partial' \theta'_{\rm p}}{\partial' t'} + \frac{1}{2}[(\nabla' \theta'_{\rm p})^2 - \frac{1}{\sqrt{n'_{\rm p}}}\nabla'^2\sqrt{n'_{\rm p}}] + \frac{1}{2}[\frac{\omega_\perp^2}{\omega_z^2}(x'^2 + y'^2) + z^2] + \frac{m}{\hbar^2}a_z^2\mu_{\rm p} + G_{\rm bp}n_{\rm b} = 0. \end{cases} \tag{5}$$

In this investigation,

$$\left(a_z = \left[\frac{\hbar}{m\omega_z}\right]^{1/2} \Longrightarrow a_z^2 = \left[\frac{\hbar}{m\omega_z}\right] \Longrightarrow \frac{m}{\hbar^2}a_z^2\mu_j = \frac{1}{\hbar\omega_z}\mu_j = \mu'_j\right). \tag{6}$$

Consequently, equation (5) can be reformulated as

$$\begin{cases} \mathrm{i}\frac{\partial \theta'_{\rm b}}{\partial' t'} + \frac{1}{2}[(\nabla' \theta'_{\rm b})^2 - \frac{1}{\sqrt{n'_{\rm b}}}\nabla'^2\sqrt{n'_{\rm b}}] + \frac{1}{2}[\frac{\omega_\perp^2}{\omega_z^2}(x'^2 + y'^2) + z^2] + \mu'_{\rm b} + G_{\rm bp}n_{\rm p} = 0, \\ \mathrm{i}\frac{\partial \theta'_{\rm p}}{\partial' t'} + \frac{1}{2}[(\nabla' \theta'_{\rm p})^2 - \frac{1}{\sqrt{n'_{\rm p}}}\nabla'^2\sqrt{n'_{\rm p}}] + \frac{1}{2}[\frac{\omega_\perp^2}{\omega_z^2}(x'^2 + y'^2) + z^2] + \mu'_{\rm p} + G_{\rm bp}n_{\rm b} = 0. \end{cases} \tag{7}$$

Equation (7) is thus transformed into the following equivalent form:

$$\begin{cases} \mathrm{i}\frac{\partial \theta_{\rm b}}{\partial t} + \frac{1}{2}[(\nabla \theta_{\rm b})^2 - \frac{1}{\sqrt{n_{\rm b}}}\nabla^2\sqrt{n_{\rm b}}] + \frac{1}{2}[\frac{\omega_\perp^2}{\omega_z^2}(x^2 + y^2) + z^2] + \mu_{\rm b} + G_{\rm bp}n_{\rm p} = 0, \\ \mathrm{i}\frac{\partial \theta_{\rm p}}{\partial t} + \frac{1}{2}[(\nabla \theta_{\rm p})^2 - \frac{1}{\sqrt{n_{\rm p}}}\nabla^2\sqrt{n_{\rm p}}] + \frac{1}{2}[\frac{\omega_\perp^2}{\omega_z^2}(x^2 + y^2) + z^2] + \mu_{\rm p} + G_{\rm bp}n_{\rm b} = 0. \end{cases} \tag{8}$$

Consequently, the system of Eq. (2) can be recast in the following form:

$$\begin{cases} \frac{\partial n_{\rm b}}{\partial t} + [\nabla \cdot (n_{\rm b} \bigtriangledown \theta_{\rm b})] = 0, \\ \mathrm{i}\frac{\partial \theta_{\rm b}}{\partial t} + \frac{1}{2}[(\nabla \theta_{\rm b})^2 - \frac{1}{\sqrt{n_{\rm b}}}\nabla^2\sqrt{n_{\rm b}}] + \frac{1}{2}[\frac{\omega_\perp^2}{\omega_z^2}(x^2 + y^2) + z^2] + \mu_{\rm b} + G_{\rm bp}n_{\rm p} = 0, \\ \frac{\partial n_{\rm p}}{\partial t} + [\nabla \cdot (n_{\rm p} \bigtriangledown \theta_{\rm p})] = 0, \\ \mathrm{i}\frac{\partial \theta_{\rm p}}{\partial t} + \frac{1}{2}[(\nabla \theta_{\rm p})^2 - \frac{1}{\sqrt{n_{\rm p}}}\nabla^2\sqrt{n_{\rm p}}] + \frac{1}{2}[\frac{\omega_\perp^2}{\omega_z^2}(x^2 + y^2) + z^2] + \mu_{\rm p} + G_{\rm bp}n_{\rm b} = 0. \end{cases} \tag{9}$$

Hypothesis:

$$\begin{cases} \sqrt{n_j} = A_j(x,y,z)G_j(z), \quad (j = {\rm b}, {\rm p}) \\ \theta_j = -\xi t + \varphi_j(x,y,z) \end{cases} \tag{10}$$

and assuming:

$$-\frac{1}{2}\frac{{\rm d}^2 G_j}{{\rm d}z^2} + \frac{1}{2}z^2 G_j = \nu G_j. \tag{11}$$

For $\nu = -\frac{1}{2}$, $G_j = {\rm e}^{-z^2/2}$ (eigenvalue). Substituting Eqs. (10) and (11) into the system of Eq. (9) yields

$$\begin{cases} \frac{\partial A_{\rm b}}{\partial t} + \frac{\partial A_{\rm b}}{\partial x}\frac{\partial \varphi_{\rm b}}{\partial x} + \frac{\partial A_{\rm b}}{\partial y}\frac{\partial \varphi_{\rm b}}{\partial y} + \frac{1}{2}A_{\rm b}(\frac{\partial^2 \varphi_{\rm b}}{\partial x^2} + \frac{\partial^2 \varphi_{\rm b}}{\partial y^2}) = 0, \\ -\frac{1}{2}\mathrm{i}(\frac{\partial^2 A_{\rm b}}{\partial x^2} + \frac{\partial^2 A_{\rm b}}{\partial y^2}) + (-\xi + \frac{1}{2})A_{\rm b} + A_{\rm b}\{\frac{\partial \varphi_{\rm b}}{\partial t} + \frac{1}{2}[(\frac{\partial \varphi_{\rm b}}{\partial x})^2 + (\frac{\partial \varphi_{\rm b}}{\partial y})^2]\} + \mu_{\rm b}A_{\rm b} + G_{\rm bp}A_{\rm b}A_{\rm p}^2G_{\rm p}^2 = 0, \\ \frac{\partial A_{\rm p}}{\partial t} + \frac{\partial A_{\rm p}}{\partial x}\frac{\partial \varphi_{\rm p}}{\partial x} + \frac{\partial A_{\rm p}}{\partial y}\frac{\partial \varphi_{\rm p}}{\partial y} + \frac{1}{2}A_{\rm p}(\frac{\partial^2 \varphi_{\rm p}}{\partial x^2} + \frac{\partial^2 \varphi_{\rm p}}{\partial y^2}) = 0, \\ -\frac{1}{2}\mathrm{i}(\frac{\partial^2 A_{\rm p}}{\partial x^2} + \frac{\partial^2 A_{\rm p}}{\partial y^2}) + (-\xi + \frac{1}{2})A_{\rm p} + A_{\rm p}\{\frac{\partial \varphi_{\rm p}}{\partial t} + \frac{1}{2}[(\frac{\partial \varphi_{\rm p}}{\partial x})^2 + (\frac{\partial \varphi_{\rm p}}{\partial y})^2]\} + \mu_{\rm p}A_{\rm p} + G_{\rm bp}A_{\rm p}A_{\rm b}^2G_{\rm b}^2 = 0. \end{cases} \tag{12}$$

In this derivation, given $\omega_\perp \ll \omega_z$, it follows that $\frac{\omega_\perp^2}{\omega_z^2}(x^2 + y^2) \to 0$. Based on the established conditions:

$$\begin{cases} \mu_j(n_j, a_j) = c_{j0}n_j^{r_{j0}}(1 + c_{j1}n_j^{r_{j1}}), \\ n_j = A_j^2 G_j^2, \quad (j = {\rm b}, {\rm p}). \end{cases} \tag{13}$$

Substituting Eq. (13) into the second and fourth equations of system (12) yields

$$\begin{cases} -\frac{1}{2}\mathrm{i}\left(\frac{\partial^2 A_\mathrm{b}}{\partial x^2}+\frac{\partial^2 A_\mathrm{b}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{b}+A_\mathrm{b}\left\{\frac{\partial \varphi_\mathrm{b}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \varphi_\mathrm{b}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{b}}{\partial y}\right)^2\right]\right\} \\ +A_\mathrm{b}c_{\mathrm{b}0}n_\mathrm{b}^{r_{\mathrm{b}0}}+A_\mathrm{b}c_{\mathrm{b}0}n_\mathrm{b}^{r_{\mathrm{b}0}}c_{\mathrm{b}1}n_\mathrm{b}^{r_{\mathrm{b}1}}+G_{\mathrm{bp}}A_\mathrm{b}A_\mathrm{p}^2G_\mathrm{p}^2=0, \\ -\frac{1}{2}\left(\frac{\partial^2 A_\mathrm{p}}{\partial x^2}+\frac{\partial^2 A_\mathrm{p}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{p}+A_\mathrm{p}\left\{\frac{\partial \varphi_\mathrm{p}}{\partial t}+\frac{1}{2}\mathrm{i}\left[\left(\frac{\partial \varphi_\mathrm{p}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{p}}{\partial y}\right)^2\right]\right\} \\ +A_\mathrm{p}c_{\mathrm{p}0}n_\mathrm{p}^{r_{\mathrm{p}0}}+A_\mathrm{p}c_{\mathrm{p}0}n_\mathrm{p}^{r_{\mathrm{p}0}}c_{\mathrm{p}1}n_\mathrm{p}^{r_{\mathrm{p}1}}+G_{\mathrm{bp}}A_\mathrm{p}A_\mathrm{b}^2G_\mathrm{b}^2=0. \end{cases} \tag{14}$$

The preceding equation is thus transformed into the following equivalent form:

$$\begin{cases} -\frac{1}{2}\mathrm{i}\left(\frac{\partial^2 A_\mathrm{b}}{\partial x^2}+\frac{\partial^2 A_\mathrm{b}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{b}+A_\mathrm{b}\left\{\frac{\partial \varphi_\mathrm{b}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \varphi_\mathrm{b}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{b}}{\partial y}\right)^2\right]\right\} \\ +c_{\mathrm{b}0}A_\mathrm{b}^{2r_{\mathrm{b}0}+1}G_\mathrm{b}^{2r_{\mathrm{b}0}}+c_{\mathrm{b}0}c_{\mathrm{b}1}A_\mathrm{b}^{2r_{\mathrm{b}0}+2r_{\mathrm{b}1}+1}G_\mathrm{b}^{2r_{\mathrm{b}0}+2r_{\mathrm{b}1}}+G_{\mathrm{bp}}A_\mathrm{b}A_\mathrm{p}^2G_\mathrm{p}^2=0, \\ -\frac{1}{2}\mathrm{i}\left(\frac{\partial^2 A_\mathrm{p}}{\partial x^2}+\frac{\partial^2 A_\mathrm{p}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{p}+A_\mathrm{p}\left\{\frac{\partial \varphi_\mathrm{p}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \varphi_\mathrm{p}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{p}}{\partial y}\right)^2\right]\right\} \\ +c_{\mathrm{p}0}A_\mathrm{p}^{2r_{\mathrm{p}0}+1}G_\mathrm{p}^{2r_{\mathrm{p}0}}+c_{\mathrm{p}0}c_{\mathrm{p}1}A_\mathrm{p}^{2r_{\mathrm{p}0}+2r_{\mathrm{p}1}+1}G_\mathrm{p}^{2r_{\mathrm{p}0}+2r_{\mathrm{p}1}}+G_{\mathrm{bp}}A_\mathrm{p}A_\mathrm{b}^2G_\mathrm{b}^2=0. \end{cases} \tag{15}$$

Moreover, for $\nu=-\frac{1}{2}$, $G_j=\mathrm{e}^{-z^2/2}$ (eigenvalue), which reduces the governing equation to

$$\begin{cases} -\frac{1}{2}\mathrm{i}\left(\frac{\partial^2 A_\mathrm{b}}{\partial x^2}+\frac{\partial^2 A_\mathrm{b}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{b}+A_\mathrm{b}\left\{\frac{\partial \varphi_\mathrm{b}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \varphi_\mathrm{b}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{b}}{\partial y}\right)^2\right]\right\} \\ +c_{\mathrm{b}0}A_\mathrm{b}^{2r_{\mathrm{b}0}+1}\mathrm{e}^{-r_{\mathrm{b}0}z^2}+c_{\mathrm{b}0}c_{\mathrm{b}1}A_\mathrm{b}^{2r_{\mathrm{b}0}+2r_{\mathrm{b}1}+1}\mathrm{e}^{-(r_{\mathrm{b}0}+r_{\mathrm{b}1})z^2}+G_{\mathrm{bp}}A_\mathrm{b}A_\mathrm{p}^2\mathrm{e}^{-2z^2}=0, \\ -\frac{1}{2}\mathrm{i}\left(\frac{\partial^2 A_\mathrm{p}}{\partial x^2}+\frac{\partial^2 A_\mathrm{p}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{p}+A_\mathrm{p}\left\{\frac{\partial \varphi_\mathrm{p}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \varphi_\mathrm{p}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{p}}{\partial y}\right)^2\right]\right\} \\ +c_{\mathrm{p}0}A_\mathrm{p}^{2r_{\mathrm{p}0}+1}\mathrm{e}^{-r_{\mathrm{p}0}z^2}+c_{\mathrm{p}0}c_{\mathrm{p}1}A_\mathrm{p}^{2r_{\mathrm{p}0}+2r_{\mathrm{p}1}+1}\mathrm{e}^{-(r_{\mathrm{p}0}+r_{\mathrm{p}1})z^2}+G_{\mathrm{bp}}A_\mathrm{p}A_\mathrm{b}^2\mathrm{e}^{-2z^2}=0. \end{cases} \tag{16}$$

Upon multiplying both sides of the aforementioned equation by e^{-z^2}, and integrating with respect to z over the domain ($\int_{-\infty}^{\infty}\mathrm{e}^{-z^2}\,\mathrm{d}z=\sqrt{\pi}$), we obtain

$$\begin{cases} -\frac{1}{2}\mathrm{i}\left(\frac{\partial^2 A_\mathrm{b}}{\partial x^2}+\frac{\partial^2 A_\mathrm{b}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{b}+A_\mathrm{b}\left\{\frac{\partial \varphi_\mathrm{b}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \varphi_\mathrm{b}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{b}}{\partial y}\right)^2\right]\right\} \\ +\frac{c_{\mathrm{b}0}}{\sqrt{r_{\mathrm{b}0}+1}}A_\mathrm{b}^{2r_{\mathrm{b}0}+1}+\frac{c_{\mathrm{b}0}c_{\mathrm{b}1}}{\sqrt{r_{\mathrm{b}0}+r_{\mathrm{b}1}+1}}A_\mathrm{b}^{2r_{\mathrm{b}0}+2r_{\mathrm{b}1}+1}+\sqrt{\pi}G_{\mathrm{bp}}A_\mathrm{b}A_\mathrm{p}^2=0, \\ -\frac{1}{2}\mathrm{i}\left(\frac{\partial^2 A_\mathrm{p}}{\partial x^2}+\frac{\partial^2 A_\mathrm{p}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{p}+A_\mathrm{p}\left\{\frac{\partial \varphi_\mathrm{p}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \varphi_\mathrm{p}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{p}}{\partial y}\right)^2\right]\right\} \\ +\frac{c_{\mathrm{p}0}}{\sqrt{r_{\mathrm{p}0}+1}}A_\mathrm{p}^{2r_{\mathrm{p}0}+1}+\frac{c_{\mathrm{p}0}c_{\mathrm{p}1}}{\sqrt{r_{\mathrm{p}0}+r_{\mathrm{p}1}+1}}A_\mathrm{p}^{2r_{\mathrm{p}0}+2r_{\mathrm{p}1}+1}+\sqrt{\pi}G_{\mathrm{bp}}A_\mathrm{p}A_\mathrm{b}^2=0. \end{cases} \tag{17}$$

Consequently, the system of Eq. (17) can ultimately be reduced to

$$\begin{cases} \frac{\partial A_\mathrm{b}}{\partial t}+\frac{\partial A_\mathrm{b}}{\partial x}\frac{\partial \varphi_\mathrm{b}}{\partial x}+\frac{\partial A_\mathrm{b}}{\partial y}\frac{\partial \varphi_\mathrm{b}}{\partial y}+\frac{1}{2}A_\mathrm{b}\left(\frac{\partial^2 \varphi_\mathrm{b}}{\partial x^2}+\frac{\partial^2 \varphi_\mathrm{b}}{\partial y^2}\right)=0, \\ -\frac{1}{2}\mathrm{i}\left(\frac{\partial^2 A_\mathrm{b}}{\partial x^2}+\frac{\partial^2 A_\mathrm{b}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{b}+A_\mathrm{b}\left\{\frac{\partial \varphi_\mathrm{b}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \varphi_\mathrm{b}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{b}}{\partial y}\right)^2\right]\right\} \\ +\frac{c_{\mathrm{b}0}}{\sqrt{r_{\mathrm{b}0}+1}}A_\mathrm{b}^{2r_{\mathrm{b}0}+1}+\frac{c_{\mathrm{b}0}c_{\mathrm{b}1}}{\sqrt{r_{\mathrm{b}0}+r_{\mathrm{b}1}+1}}A_\mathrm{b}^{2r_{\mathrm{b}0}+2r_{\mathrm{b}1}+1}+\sqrt{\pi}G_{\mathrm{bp}}A_\mathrm{b}A_\mathrm{p}^2=0, \\ \frac{\partial A_\mathrm{p}}{\partial t}+\frac{\partial A_\mathrm{p}}{\partial x}\frac{\partial \varphi_\mathrm{p}}{\partial x}+\frac{\partial A_\mathrm{p}}{\partial y}\frac{\partial \varphi_\mathrm{p}}{\partial y}+\frac{1}{2}A_\mathrm{p}\left(\frac{\partial^2 \varphi_\mathrm{p}}{\partial x^2}+\frac{\partial^2 \varphi_\mathrm{p}}{\partial y^2}\right)=0, \\ -\frac{1}{2}\mathrm{i}\left(\frac{\partial^2 A_\mathrm{p}}{\partial x^2}+\frac{\partial^2 A_\mathrm{p}}{\partial y^2}\right)+(-\xi+\frac{1}{2})A_\mathrm{p}+A_\mathrm{p}\left\{\frac{\partial \varphi_\mathrm{p}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \varphi_\mathrm{p}}{\partial x}\right)^2+\left(\frac{\partial \varphi_\mathrm{p}}{\partial y}\right)^2\right]\right\} \\ +\frac{c_{\mathrm{p}0}}{\sqrt{r_{\mathrm{p}0}+1}}A_\mathrm{p}^{2r_{\mathrm{p}0}+1}+\frac{c_{\mathrm{p}0}c_{\mathrm{p}1}}{\sqrt{r_{\mathrm{p}0}+r_{\mathrm{p}1}+1}}A_\mathrm{p}^{2r_{\mathrm{p}0}+2r_{\mathrm{p}1}+1}+\sqrt{\pi}G_{\mathrm{bp}}A_\mathrm{p}A_\mathrm{b}^2=0. \end{cases} \tag{18}$$

Assuming, concurrently, that $A_j=u_{j0}+\varepsilon^2a_{j0}+\varepsilon^4a_{j1}$, $\varphi_j=\varepsilon\varphi_{j0}+\varepsilon^3\varphi_{j1}$, $(j=\mathrm{b},\mathrm{p})$, $\xi=\varepsilon(z-c_jt)$ $(j=\mathrm{b},\mathrm{p})$, $\tau=\varepsilon^3t$, and substituting this expression into equation set (18), we obtain, after computation and simplification,

$$\begin{cases} \mathrm{i}\frac{\partial a_{\mathrm{b}0}}{\partial \tau}-\frac{1}{8c_\mathrm{b}}\frac{\partial^3 a_{\mathrm{b}0}}{\partial \xi^3}+\frac{3c_\mathrm{b}}{2u_{\mathrm{b}0}}a_{\mathrm{b}0}\frac{\partial a_{\mathrm{b}0}}{\partial \xi}+\pi G_{\mathrm{bp}}(2u_{\mathrm{p}0}a_{\mathrm{p}0})\frac{\partial a_{\mathrm{b}0}}{\partial \xi}+\pi G_{\mathrm{bp}}(2u_{\mathrm{b}0}a_{\mathrm{p}0}+2u_{\mathrm{p}0}a_{\mathrm{b}0})\frac{\partial a_{\mathrm{p}0}}{\partial \xi}=0, \\ \mathrm{i}\frac{\partial a_{\mathrm{p}0}}{\partial \tau}-\frac{1}{8c_\mathrm{p}}\frac{\partial^3 a_{\mathrm{p}0}}{\partial \xi^3}+\frac{3c_\mathrm{p}}{2u_{\mathrm{p}0}}a_{\mathrm{p}0}\frac{\partial a_{\mathrm{p}0}}{\partial \xi}+\pi G_{\mathrm{bp}}(2u_{\mathrm{b}0}a_{\mathrm{b}0})\frac{\partial a_{\mathrm{p}0}}{\partial \xi}+\pi G_{\mathrm{bp}}(2u_{\mathrm{p}0}a_{\mathrm{b}0}+2u_{\mathrm{b}0}a_{\mathrm{p}0})\frac{\partial a_{\mathrm{b}0}}{\partial \xi}=0. \end{cases} \tag{19}$$

Consequently, we derive the coupled KdV equations

$$\begin{cases} \mathrm{i}\frac{\partial a_{\mathrm{b}0}}{\partial \tau}-\frac{1}{8c_\mathrm{b}}\frac{\partial^3 a_{\mathrm{b}0}}{\partial \xi^3}+\frac{3c_\mathrm{b}}{2u_{\mathrm{b}0}}a_{\mathrm{b}0}\frac{\partial a_{\mathrm{b}0}}{\partial \xi}+2\pi G_{\mathrm{bp}}u_{\mathrm{b}0}a_{\mathrm{p}0}\frac{\partial a_{\mathrm{p}0}}{\partial \xi}+2\pi G_{\mathrm{bp}}u_{\mathrm{p}0}\frac{\partial (a_{\mathrm{b}0}a_{\mathrm{p}0})}{\partial \xi}=0, \\ \mathrm{i}\frac{\partial a_{\mathrm{p}0}}{\partial \tau}-\frac{1}{8c_\mathrm{p}}\frac{\partial^3 a_{\mathrm{p}0}}{\partial \xi^3}+\mathrm{i}\frac{3c_\mathrm{p}}{2u_{\mathrm{p}0}}a_{\mathrm{p}0}\frac{\partial a_{\mathrm{p}0}}{\partial \xi}+2\pi G_{\mathrm{bp}}u_{\mathrm{p}0}a_{\mathrm{b}0}\frac{\partial a_{\mathrm{b}0}}{\partial \xi}+2\pi G_{\mathrm{bp}}u_{\mathrm{b}0}\frac{\partial (a_{\mathrm{b}0}a_{\mathrm{p}0})}{\partial \xi}=0. \end{cases} \tag{20}$$

3. Prolongation structure of the coupled KdV equations

The KdV equation, a cornerstone in the study of the dynamics of Bose–Fermi mixtures, provides a precise description of the behavior of certain non-ideal fluids. Through the exact analytical solutions of the KdV equation, researchers can predict phenomena such as solitons and shock waves in fluid dynamics. These phenomena have significant applications in diverse fields including astrophysics, oceanography, and acoustics. In the realm of mathematics, the development of soliton theory has revealed the nature of the KdV equation as a nonlinear evolution equation with soliton solutions, leading to the development of various methods within mathematical physics for solving such nonlinear partial differential equations. This paper aims to solve the Lax pair of this nonlinear evolution equation via prolongation structures.

Proposed hypothesis: $\frac{1}{8c_{\rm b}} = \frac{1}{8c_{\rm p}} = 1$, $2\pi G_{\rm bp} u_{\rm b0} = 2\pi G_{\rm bp} u_{\rm p0} = 4$, $\partial a_{\rm b_0} = u$, $\partial a_{\rm p_0} = v$, $\frac{\partial a_{\rm b_0}}{\partial \tau} = u_t$, $\frac{\partial a_{\rm b_0}}{\partial \xi} = u_x$, $\frac{\partial a_{\rm p_0}}{\partial \tau} = v_t$, $\frac{\partial a_{\rm p_0}}{\partial \xi} = v_x$ Therefore, the coupled KdV system obtained in the second section admits the following transformation:

$$\begin{cases} {\rm i}u_t + 4uu_x - u_{xxx} + 4vv_x + 4(uv)_x = 0, \\ {\rm i}v_t + 4vv_x - v_{xxx} + 4uu_x + 4(uv)_x = 0. \end{cases} \tag{21}$$

In the case where $v = 0$, the coupled KdV equations presented herein simplify to the canonical KdV equation.

We now define the following transformations: $u_x = p$, $u_{xx} = p_x = q$, $v_x = \varphi$, and $v_{xx} = \varphi_x = \psi$. Consequently, equation (21) can be reformulated as an equivalent first-order system of partial differential equations

$$\begin{cases} u_x = p, \\ u_{xx} = p_x, \\ v_x = \varphi, \\ v_{xx} = \varphi_x, \\ {\rm i}u_t = q_x - 4up - 4(u+v)\varphi - 4vp, \\ {\rm i}v_t = \psi_x - 4v\varphi - 4(u+v)p - 4u\varphi. \end{cases} \tag{22}$$

On the manifold $U = \{x, t, u, v, p, q, \varphi, \psi\}$, we define a set of exterior 2-forms

$$\begin{cases} \alpha_1 = {\rm d}t \wedge {\rm d}u + {\rm d}x \wedge {\rm d}t\ p, \\ \alpha_2 = {\rm d}t \wedge {\rm d}p + {\rm d}x \wedge {\rm d}t\ q, \\ \alpha_3 = {\rm d}t \wedge {\rm d}v + {\rm d}x \wedge {\rm d}t\varphi, \\ \alpha_4 = {\rm d}t \wedge {\rm d}\varphi + {\rm d}x \wedge {\rm d}t\psi, \\ \alpha_5 = {\rm i}\,{\rm d}x \wedge {\rm d}u + {\rm d}t \wedge {\rm d}q - {\rm d}x \wedge {\rm d}t 4(u\varphi + v\varphi + vp), \\ \alpha_6 = {\rm i}\,{\rm d}x \wedge {\rm d}v + {\rm d}t \wedge {\rm d}\psi - {\rm d}x \wedge {\rm d}t 4(up + vp + u\varphi). \end{cases} \tag{23}$$

In this context, d signifies the exterior derivative, and $\wedge$ symbolizes the wedge product. Applying the exterior derivative to Eq. (23) results in

$$\begin{cases} {\rm d}\alpha_1 = {\rm d}x \wedge \alpha_2, \\ {\rm d}\alpha_2 = {\rm d}x \wedge \alpha_5, \\ {\rm d}\alpha_3 = {\rm d}x \wedge \alpha_4, \\ {\rm d}\alpha_4 = {\rm d}x \wedge \alpha_6, \\ {\rm d}\alpha_5 = {\rm d}x \wedge (\alpha_1 4p + \alpha_2 u + \alpha_6 4u), \\ {\rm d}\alpha_6 = {\rm d}x \wedge (\alpha_3 4\varphi + \alpha_5 4v). \end{cases} \tag{24}$$

Therefore, $I = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ forms a closed ideal on the manifold U. Upon restriction of the 2-forms α_i to zero on the solution manifold $U = \{x, t, u(x,t), v(x,t), p(x,t), q(x,t), \varphi(x,t), \psi(x,t)\}$, equation (21) is derived. We now introduce n 1-forms ω^k:

$$\begin{aligned} w^k \ =\ & {\rm d}y^k + F^k(x,t,u,v,p,q,\varphi,\psi,y^i)\,{\rm d}x \\ & + G^k(x,t,u,v,p,q,\varphi,\psi,y^i)\,{\rm d}t, \quad (k = 1,\ldots,n). \end{aligned} \tag{25}$$

In this context, y^i denotes the augmented variables, and we stipulate that they, in conjunction with α_i, define a novel closed ideal. Specifically, we impose the following condition on ω^k:

$${\rm d}\omega^k = \sum_{j=1}^{6} f_j^i \alpha^j + \sum^{n} \eta_j^i \wedge \omega^j. \tag{26}$$

Given that f_j^i represents a 0-form and η_j^i represents a 1-form, the following system of partial differential equations is satisfied by $F^k(u,v,p,q,\varphi,\psi,y^i)$ and $G^k(u,v,p,q,\varphi,\psi,y^i)$.

$$F_{\rm p} = F_q = F_\varphi = F_\psi = 0, \tag{27}$$

$$G_q = F_u, \quad G_\psi = F_v. \tag{28}$$

$$\begin{aligned} & pG_u + qG_{\rm p} + \varphi G_v + \psi G_\varphi \\ & + (4u\varphi + 4v\varphi + 4vp)F_u \\ & + (4up + 4vp + 4u\varphi)F_v + [G,F] = 0. \end{aligned} \tag{29}$$

Integrating Eq. (29) over the variables q and ψ produces the subsequent outcome

$$G = qF_u + \psi F_v + G_1(u,v,p,\varphi,y^i). \tag{30}$$

Substituting expression (30) into Eq. (29) yields the following result:

$$\begin{aligned} G_1 \ =\ & -{\rm i}\Big(\frac{1}{2}p^2 F_{uu} + 2p\varphi F_{vu} + \frac{1}{2}\varphi^2 F_{vv}\Big) \\ & - p[F_u, F] - \varphi[F_v, F] + G_2(u,v,y^i). \end{aligned} \tag{31}$$

Inserting expression (31) into Eq. (30) and performing the subsequent derivation produces the following system of equations:

$$\begin{cases} F_{uuu} = 0, \quad [F_{uu}, F] = 0, \\ F_{vvv} = 0, \quad [F_{vv}, F] = 0, \\ F_{vuu} = 0, \quad [F_{vu}, F] = 0, \\ F_{uv} = 0, \quad [F_u, F_v] = 0, \\ G_{2u} = -4vF_u - 4uF_v - 4vF_v + {\rm i}[[F_u, F], F], \\ G_{2v} = -4uF_u - 4vF_u - 4uF_v + {\rm i}[[F_v, F], F], \quad [G_2, F] = 0. \end{cases} \tag{32}$$

Based on Eqs. (27), (30), (31), and (32), we posit that

$$F = {\rm i}(X_0 + uX_1 + vX_2). \tag{33}$$

Here, we posit that the variables X_i $(i = 0, 1, 2,)$ are functions exclusively of the prolonged variables y^j. Given that $[X_1, X_0] = X_4$, $[X_1, X_2] = X_5$, $[X_2, X_0] = X_6$, substitution of F

into Eq. (10) allows us to derive the subsequent relations via application of the Jacobi identity:

$$\begin{cases} [X_4,X_1]-4X_2=0, \quad [X_6,X_2]=-4X_1, \\ [X_4,X_2]=4X_2, \quad [X_5,X_2]=0, \\ [X_6,X_0]=-4X_2, \quad [X_7,X_2]=0, \\ [X_6,X_1]+[X_0,X_5]=4X_1, \\ [X_2,X_5]+[X_0,X_5]=0, \quad X_1=-X_2, \\ G_2=\mathrm{i}u[X_6,X_0]+u^2[X_4,X_1]-4uvX_1 \\ \qquad +v^2X_2+\mathrm{i}v[X_6,X_0]+\mathrm{i}X_3. \end{cases} \tag{34}$$

Substituting G_2 into the equation $[G_2,F]=0$, we obtain the following relation:

$$\begin{cases} X_4+[[X_4,X_1],X_0]+[[X_6,X_0],X_1]=0, \\ [[X_4,X_1],X_2]+[[X_4,X_0],X_1]+[[X_6,X_1],X_1]=0, \\ X_6+[[X_6,X_2],X_0]-2X_5+[[X_6,X_0],X_2]=0, \\ -2X_4+[[X_6,X_0],X_0]-2X_5+[X_3,X_2]=0, \\ -2X_6+[[X_6,X_0],X_0]+[X_3,X_1]=0, \\ [[X_6,X_2],X_2]=0, \\ [[X_4,X_1],X_1]=0, \quad [X_3,X_0]=0. \end{cases} \tag{35}$$

Assuming that $[X_4,X_0]=X_8$, $[X_4,X_1]=X_9$, $[X_4,X_2]=X_{10}$, $[X_6,X_0]=X_{11}$, $[X_6,X_1]=X_{12}$, and $[X_6,X_2]=X_{13}$, the subsequent relations are derived from Eq. (35):

$$\begin{cases} [X_9,X_1]=0, \quad X_4+[X_9,X_0]=-2X_5-[X_{11},X_0], \\ -2X_6+[X_{11},X_0]+[X_3,X_1]=0, \\ -4X_6+[X_8,X_0]-4X_4+[X_{12},X_0] \\ +[X_{11},X_2]+[X_{11},X_1]=0, \\ X_5+[X_9,X_2]+[X_8,X_1]+[X_{12},X_1]=0, \\ [X_8,X_2]-4X_5-X_5-[X_5,X_6]=0, \\ X_6+[X_{13},X_0]-2X_5+[X_{11},X_2]=0, \\ -2X_4+[X_{11},X_0]+[X_3,X_2]=0, \\ X_5+[X_{12},X_2]=0, \quad [X_3,X_0]=0. \end{cases} \tag{36}$$

By combining Eqs. (35) and (36), we obtain

$$\begin{cases} [X_8,X_1]=[X_8,X_2], \quad [X_8,X_0]=X_4, \\ [X_9,X_0]=[X_9,X_3]=[X_0,X_2], \\ [X_9,X_1]=[X_9,X_2]=0, \\ [X_{10},X_0]=[X_0,X_4], \quad [X_{10},X_1]=[X_{10},X_2]=0, \\ [X_{11},X_0]=[X_0,X_2], \quad [X_{11},X_1]=[X_{11},X_2]=0, \\ [X_{12},X_0]=[X_0,X_2], \quad [X_{12},X_1]=[X_{12},X_2]=0. \end{cases} \tag{37}$$

From Eq. (34), we find that $[X_4,X_1]=-X_1$ and $[X_1,X_0]=X_4$, which implies that X_1 is a nilpotent element, whereas X_4 is a central element within the algebraic structure. Moreover, equation (21) displays a particular scaling symmetry, where λ denotes the spectral parameter.

$$x\to\lambda^{-1}x, \quad t\to\lambda^{-3}t, \quad u\to\lambda^2u, \quad v\to\lambda^2v. \tag{38}$$

If we impose the invariance of ω^i under the aforementioned scaling transformation, we obtain

$$F\to\lambda F, \quad G\to\lambda^3G. \tag{39}$$

And the condition on X_i is

$$\begin{array}{l} X_0\to\mathrm{i}\lambda X_0, \quad X_1\to\lambda^{-1}X_1, \quad X_2\to\lambda^{-1}X_2, \\ X_3\to\mathrm{i}\lambda^3X_3, \quad X_4\to X_4, \quad X_5\to X_5, \quad X_6\to\mathrm{i}X_6. \end{array} \tag{40}$$

Employing the basis of generators $\{X_i,i=1,2,\ldots,6\}$, by

$$X_6=-X_1, \quad [X_4,X_1]=-X_1, \quad [X_1,X_0]=X_4. \tag{41}$$

Within this investigation, we effect an embedding of the algebra y^i into the Lie algebra $SL(2,c)$, subject to the subsequent constraints:

$$X_1=e_-, \quad X_4=h. \tag{42}$$

Then, we have

$$e_-=\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad h=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_+=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

To explicitly define X_0,X_1,X_2, this study derives a closed Lie algebra generated by $(X_0,\ldots,X_6)$ based on Eqs. (35) and (36)

$$\begin{cases} [X_2,X_1]=0, \quad [X_2,X_0]=X_6, \quad [X_3,X_1]=-4X_4, \\ [X_3,X_2]=-4X_4, \quad [X_3,X_4]=4X_3, \quad [X_4,X_0]=X_0, \\ [X_5,X_0]=0, \quad [X_5,X_1]=0, \quad [X_5,X_2]=0, \\ [X_6,X_0]=-X_1, \quad [X_6,X_1]=X_1, \quad [X_6,X_2]=-X_2. \end{cases} \tag{43}$$

Consequently, we establish the corresponding generators, where λ denotes the spectral parameter

$$\begin{cases} X_0=e_++\mathrm{i}\lambda^2e_-, \quad X_2=e_-, \\ X_3=2\mathrm{i}\lambda^2e_+-4\mathrm{i}\lambda^4e_-. \end{cases} \tag{44}$$

Substituting the aforementioned generators into Eqs. (31)–(34) yields the explicit forms of F and G,

$$F=\begin{pmatrix} 0 & -1 \\ \mathrm{i}\lambda^2+u+v & 0 \end{pmatrix}, \quad G=\begin{pmatrix} -p-\varphi & 2\mathrm{i}\lambda^2-2u-2v \\ 2u^2+2v^2-2\mathrm{i}\lambda^2u-2\mathrm{i}\lambda^2v-4\mathrm{i}\lambda^4+4uv+q+\psi & p+\varphi \end{pmatrix}. \tag{45}$$

Assuming $\omega^k|_U=0$, we can derive the Lax representation of Eq. (21)

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}_x=-F\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, \quad \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}_t=-G\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}. \tag{46}$$

Employing the symmetry relation $y_{xt}=y_{tx}$, the original equation (21) is immediately recovered.

In the investigation of the coupled KdV equations, we have successfully employed the prolongation structure method, which is crucial for revealing the intrinsic algebraic structure of nonlinear partial differential equations. Through this approach, we derived the Lax pair for the equation, a significant mathematical result. This not only confirms the integrability of the equation but also provides a powerful tool for solving it. The existence of the Lax pair is fundamental to constructing the Darboux transformation, an effective method for obtaining exact solutions of nonlinear evolution equations. Utilizing the Darboux transformation, we can systematically generate new, more complex solutions, such as soliton solutions, from known seed solutions. These solutions are particularly important in describing physical phenomena like the nonlinear propagation and interaction of waves. Furthermore, the Darboux transformation allows us to explore the algebraic structure and geometric properties of the solutions, as well as their underlying physical significance.

Subsequent research will concentrate on utilizing the obtained Lax pair to construct the Darboux transformation for the coupled KdV equations. This step is not only a critical component of the theoretical derivation but also a key bridge for translating abstract mathematical structures into concrete physical applications.

4. Employing the Darboux transformation to ascertain exact solutions

Within the Lax pair formalism, we have successfully derived the single-soliton solutions of a nonlinear evolution equation system. This was achieved by introducing seed solutions, employing variable transformations, and utilizing the Darboux transformation method, all facilitated by symbolic computation. This approach not only simplifies the solution process for complex equations but also elucidates the intrinsic structure of soliton solutions, thereby providing a robust foundation for subsequent investigations into multi-soliton solutions and their interactions. Subsequently, by constructing a Darboux matrix and iteratively applying the Darboux transformation method, we computed and obtained the explicit expressions for two-soliton and three-soliton solutions, which validated the reliability of the method. Through iterative calculations, we ultimately derived the n-soliton solutions of the equation.

Initially, based on the prolongation structure, we obtained the Lax pair for the linear spectral problem, which served as the basis for the precise solution of the studied equation

$$\begin{cases} \Phi_x = F\Phi, \\ \Phi_t = G\Phi. \end{cases} \tag{47}$$

Here, λ denotes the spectral parameter, and we define $w = u + v$. Consequently, the original Lax pair (45) can be reformulated as follows:

$$\begin{aligned} F &= \mathrm{i}\lambda^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ w & 0 \end{pmatrix} = \mathrm{i}\lambda^2 J + P, \\ G &= 4\mathrm{i}\lambda^4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\mathrm{i}\lambda^2 \begin{pmatrix} 0 & -1 \\ w & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} -w_x & -2w \\ 2w^2 + w_x & w_x \end{pmatrix} \\ &= 4\mathrm{i}\lambda^4 J + 2\mathrm{i}\lambda^2 P + Q. \end{aligned} \tag{48}$$

The vector $\Phi = (f, g)^{\mathrm{T}}$ constitutes an eigenfunction associated with the spectral parameter λ and the potential function $u(x,t)$. This eigenfunction, composed of two complex-valued functions $f = f(x,t)$ and $g = g(x,t)$, satisfies the compatibility condition of the zero-curvature equation, which can be verified as

$$u_t - v_x + [u, v] = 0.$$

We introduce the single Darboux transformation $\Phi_{[1]}$, a canonical transformation of a countable nature. This transformation is expressed as

$$\Phi_{[1]} = D_{[1]}\Phi.$$

Here, the Darboux transformation $D_{[1]}$ is defined as $D_{[1]} = \lambda E - S_{[0]}$, where E represents the $2\times$ identity matrix, and $S_{[0]} = S_{[0]}(x,t)$ is a $2\times$ matrix dependent on the variables x and t.

$$\begin{cases} \Phi_{[1]} = D_{[1]}\Phi, \\ \Phi_x = F\Phi, \\ \Phi_{[1]x} = F_{[1]}\Phi_{[1]}. \end{cases} \tag{49}$$

Considering $F_{[1]} = \mathrm{i}\lambda^2 J + P_{[1]}$, $D_{[1]} = \lambda E + S_{[0]}$, and $F = \mathrm{i}\lambda^2 J + P$, we can then deduce the formula for a single Darboux transformation, and subsequently derive the expression for the single-soliton solution

$$\begin{aligned} \Phi_{[1]x} &= F_{[1]}\Phi_{[1]} = F_{[1]}D_{[1]}\Phi \\ &= (\mathrm{i}\lambda^2 J + P_{[1]})(\lambda E + S_{[0]})\Phi \\ &= (\mathrm{i}\lambda^3 J - \mathrm{i}\lambda^2 J S_{[0]} + \lambda P_{[1]} - P_{[1]}S_{[0]})\Phi, \end{aligned} \tag{50}$$

$$\begin{aligned} \Phi_{[1]x} &= D_{[1]x}\Phi + D_{[1]}\Phi_x \\ &= S_{[0]x}\Phi + (\lambda E - S_{[0]})(I\lambda^2 J + P)\Phi \\ &= (S_{[0]x} + \mathrm{i}\lambda^3 J - \mathrm{i}\lambda^2 S_{[0]}J + \lambda P - S_{[0]}P)\Phi. \end{aligned} \tag{51}$$

From the difference between Eqs. (50) and (51), we obtain

$$P_{[1]}(\lambda - S_{[0]}) = S_{[0]x} - S_{[0]}P - \mathrm{i}\lambda^2[S_{[0]}, J] + \lambda P\,. \tag{52}$$

Suppose $S_{[0]} = H \wedge H^{-1}$, where $\wedge = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ is a diagonal matrix, and $H = H(x,t)$ is a nonsingular matrix. Given the Lax pair (45) with eigenfunction $(f_1, g_1)^{\mathrm{T}}$ at $\lambda = \lambda_1$, it is straightforward to demonstrate that $(-g_1, f_1)^{\mathrm{T}}$ is also an eigenfunction of the Lax pair (45) at λ_1. Consequently, we select the matrix function

$$H = \begin{pmatrix} f_1 & -\overline{g_1} \\ g_1 & \overline{f_1} \end{pmatrix},$$

$$S_{[0]} = \begin{pmatrix} S_{11[0]} & S_{12[0]} \\ S_{21[0]} & S_{22[0]} \end{pmatrix} = \frac{1}{|f_1|^2 + |g_1|^2}$$

$$\times \begin{pmatrix} \lambda_1|f_1|^2+\overline{\lambda_1}|g_1|^2 & (\lambda_1-\overline{\lambda_1})f_1\overline{g_1} \\ (\lambda_1-\overline{\lambda_1})\overline{f_1}g_1 & \lambda_1|g_1|^2+\overline{\lambda_1}|f_1|^2 \end{pmatrix}. \quad (53)$$

For computational convenience, we define $a=\lambda_1|f_1|^2+\overline{\lambda_1}|g_1|^2$, $b=(\lambda_1-\overline{\lambda_1})f_1\overline{g_1}$, $c=(\lambda_1-\overline{\lambda_1})\overline{f_1}g_1$, $b=\lambda_1|g_1|^2+\overline{\lambda_1}|f_1|^2$, where a and d are constants. Consequently, $S_{[0]}$ and $S_{[0]x}$ can be simplified to the following form:

$$S_{[0]}=\frac{1}{|f_1|^2+|g_1|^2}\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
$$S_{[0]x}=\frac{1}{|f_1|^2+|g_1|^2}\begin{pmatrix} 0 & b_x \\ c_x & 0 \end{pmatrix}. \quad (54)$$

By computation of Eq. (52), we find

$$\begin{aligned} P_{[1]} &= (S_{[0]x}-S_{[0]}P-\mathrm{i}\lambda^2[S_{[0]},J]+\lambda P)(\lambda-S_{[0]})^{-1} \\ &= \begin{pmatrix} -b(w+\mathrm{i}\lambda^2) & b_x+a-\lambda \\ c_x-w(\lambda-d)+\mathrm{i}\lambda^2(a-d) & c-\mathrm{i}\lambda^2 b \end{pmatrix} \\ &\quad \times \begin{pmatrix} \lambda-d & b-\lambda \\ c-\lambda & \lambda-a \end{pmatrix} \\ &= \begin{pmatrix} P_{11[1]} & P_{12[1]} \\ P_{21[1]} & P_{22[1]} \end{pmatrix}. \end{aligned} \quad (55)$$

To simplify the expression of matrix $P_{[1]}$, we will now elucidate its constituent elements as follows:

$$\begin{aligned} P_{11[1]} &= (d-\lambda)b(w+\mathrm{i}\lambda^2)+(c-\lambda)(b_x+a-\lambda), \\ P_{12[1]} &= (\lambda-b)b(w+\mathrm{i}\lambda^2)+(\lambda-a)(b_x+a-\lambda), \\ P_{21[1]} &= (\lambda-d)(c_x-w(\lambda-d) \\ &\quad +\mathrm{i}\lambda^2(a-d))+(c-\lambda)(c-\mathrm{i}\lambda^2 b), \\ P_{22[1]} &= (b-\lambda)(c_x-w(\lambda-d) \\ &\quad +\mathrm{i}\lambda^2(a-d))(\lambda-a)(c-\mathrm{i}\lambda^2 b). \end{aligned}$$

By comparing $P_{[1]}$ and P, the single-soliton solution of the KdV equation can be derived

$$\begin{aligned} w_{[1]} &= w(\lambda-d)^2+(\lambda-d)[c_x+\mathrm{i}\lambda(a-d)] \\ &\quad +(c-\lambda)(c-\mathrm{i}\lambda^2 b). \end{aligned}$$

Assuming $z=(\lambda-d)[c_x+\mathrm{i}\lambda(a-d)]+(c-\lambda)(c-\mathrm{i}\lambda^2 b)$, we obtain

$$w_{[1]}=w(\lambda-d)^2+z. \quad (56)$$

Based on the initial implementation of the Darboux transformation, we can deduce that

$$\Phi_{[2]}=D_{[2]}\Phi_{[1]}=(\lambda E-S_{[1]})\Phi_{[1]}=D_{[2]}D_{[1]}\Phi. \quad (57)$$

Here $L_2=(f_2,g_2)^{\mathrm{T}}$

$$\begin{aligned} L_{[1]} &= \begin{pmatrix} f_{2[1]} \\ g_{2[1]} \end{pmatrix}=(\lambda_2 E-S_{[0]})L_2 \\ &= \begin{pmatrix} \lambda_2-S_{11[0]} & -S_{12[0]} \\ -S_{21[0]} & \lambda_2-S_{22[0]} \end{pmatrix}\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \\ &= \begin{pmatrix} (\lambda_2-S_{11[0]})f_2-S_{12[0]})g_2 \\ \lambda_2-S_{22[0]})g_2-S_{12[0]})f_2 \end{pmatrix}. \end{aligned} \quad (58)$$

Thus, within the framework of the $H_{[1]}$ space:

$$f_{2[1]}=\lambda_2-S_{11[0]})f_2-S_{12[0]})g_2,$$
$$g_{2[1]}=\lambda_2-S_{22[0]})g_2-S_{12[0]})f_2.$$

From $S_{[1]}=H_{[1]}\wedge H_{[1]}^{-1}$, we can deduce

$$\begin{aligned} S_{[1]} &= \begin{pmatrix} S_{11[1]} & S_{12[1]} \\ S_{21[1]} & S_{22[1]} \end{pmatrix}=\frac{1}{|f_{2[1]}|^2+|g_{2[1]}|^2} \\ &\quad \times \begin{pmatrix} \lambda_2|f_{2[1]}|^2+\overline{\lambda_2}|g_{2[1]}|^2 & (\lambda_2-\overline{\lambda_2})f_{2[1]}\overline{g_{2[1]}} \\ (\lambda_2-\overline{\lambda})\overline{f_{2[1]}}g_{2[1]} & \lambda_2|g_{2[1]}|^2+\overline{\lambda_2}|f_{2[1]}|^2 \end{pmatrix}. \end{aligned} \quad (59)$$

Employing a comparative analysis of $P_{[2]}$ and $P_{[1]}$, we can derive the two-soliton solution of the KdV equation

$$\begin{aligned} w_{[2]} &= w_{[1]}(\lambda-d_{[1]})^2+(\lambda-d_{[1]})[c_{[1]x}+\mathrm{i}\lambda(a_{[1]}-d_{[1]})] \\ &\quad +(c_{[1]}-\lambda)(c_{[1]}-\mathrm{i}\lambda^2 b_{[1]}). \end{aligned}$$

Given $z_{[1]}=(\lambda-d_{[1]})[c_{[1]x}+\mathrm{i}\lambda(a_{[1]}-d_{[1]})]+(c_{[1]}-\lambda)(c_{[1]}-\mathrm{i}\lambda^2 b_{[1]})$, it follows that

$$w_{[2]}=w(\lambda-d_{[1]})^2+z_{[1]}. \quad (60)$$

Employing an iterative approach, in conjunction with a second-order Darboux transformation, we obtain

$$\Phi_{[3]}=D_{[3]}\Phi_{[2]}=(\lambda E-S_{[2]})\Phi_{[2]}=D_{[3]}D_{[2]}D_{[1]}\Phi. \quad (61)$$

Here, L_3 is defined as $(f_3,g_3)^{\mathrm{T}}$.

$$\begin{aligned} L_{[2]} &= \begin{pmatrix} f_{[2]} \\ g_{[2]} \end{pmatrix}=(\lambda_3 E-S_{[1]})\times L_3 \\ &= \begin{pmatrix} \lambda_3-S_{11[1]} & -S_{12[1]} \\ -S_{21[1]} & \lambda_3-S_{22[1]} \end{pmatrix}\times\begin{pmatrix} f_3 \\ g_3 \end{pmatrix} \\ &= \begin{pmatrix} (\lambda_3-S_{11[1]})f_3-S_{12[1]})g_3 \\ \lambda_3-S_{22[1]})g_3-S_{12[1]})f_3 \end{pmatrix}. \end{aligned} \quad (62)$$

Thus, within the framework of $H_{[2]}$:

$$\begin{aligned} f_{3[2]} &= f_{[2]}f_{2[1]}=(\lambda_3-S_{11[1]})f_3-S_{12[1]})g_3 \\ &\quad \times(\lambda_2-S_{11[0]})f_2-S_{12[0]})g_2), \\ g_{3[2]} &= g_{[2]}g_{2[1]}=(\lambda_3-S_{22[1]})g_3-S_{12[1]})f_3) \\ &\quad \times\lambda_2-S_{22[0]})g_2-S_{12[0]})f_2). \end{aligned}$$

From the relation $S_{[2]}=H_{[2]}\wedge H_{[2]}^{-1}$, it follows that

$$\begin{aligned} S_{[2]} &= \begin{pmatrix} S_{11[2]} & S_{12[2]} \\ S_{21[2]} & S_{22[2]} \end{pmatrix} \\ &= \begin{pmatrix} f_{3[2]} & -\overline{g_{3[2]}} \\ g_{3[2]} & \overline{f_{22[2]}} \end{pmatrix}\times\begin{pmatrix} \lambda_3 & 0 \\ 0 & \overline{\lambda_3} \end{pmatrix}\begin{pmatrix} f_{3[2]} & -\overline{g_{3[2]}} \\ g_{3[2]} & \overline{f_{22[2]}} \end{pmatrix}^{-1} \\ &= \frac{1}{|f_{3[2]}|^2+|g_{3[2]}|^2} \\ &\quad \times\begin{pmatrix} \lambda_3|f_{3[2]}|^2+\overline{\lambda_3}|g_{3[2]}|^2 & (\lambda_3-\overline{\lambda_3})f_{3[2]}\overline{g_{3[2]}} \\ (\lambda_3-\overline{\lambda_3})\overline{f_{3[2]}}g_{3[2]} & \lambda_3|g_{3[2]}|^2+\overline{\lambda_3}|f_{3[2]}|^2 \end{pmatrix}. \end{aligned} \quad (63)$$

By comparing $P_{[3]}$ and $P_{[2]}$, the three-soliton solution of the KdV equation can be derived

$$\begin{aligned} w_{[3]} &= w_{[2]}(\lambda - d_{[2]})^2 + (\lambda - d_{[2]})[c_{[2]x} + \mathrm{i}\lambda(a_{[2]} - d_{[2]})] \\ &\quad + (c_{[2]} - \lambda)(c_{[2]} - \mathrm{i}\lambda^2 b_{[2]}). \end{aligned}$$

Given $z_{[2]} = (\lambda - d_{[2]})[c_{[2]x} + \mathrm{i}\lambda(a_{[2]} - d_{[2]})] + (c_{[2]} - \lambda)(c_{[2]} - \mathrm{i}\lambda^2 b_{[2]})$, it follows that

$$w_{[3]} = w(\lambda - d_{[2]})^2 + z_{[2]}. \tag{64}$$

Therefore, new Lax pairs can be formulated and calculated through mathematical induction

$$\begin{aligned} \Phi_{[N]} &= D_{[N]}\Phi_{[N-1]} = (\lambda E - S_{[N-1]})\Phi_{[N-1]} \\ &= \begin{pmatrix} \lambda - S_{11[N-1]} & -S_{12[N-1]} \\ -S_{21[N-1]} & \lambda - S_{22[N-1]} \end{pmatrix} \Phi_{[N-1]}. \end{aligned} \tag{65}$$

That is $\Phi_{[N]} = D_{[N]}D_{[N-1]}\cdots D_{[1]}\Phi$.

From $S_{[2]} = H_{[2]} \wedge H_{[2]}^{-1}$, we can deduce

$$\begin{aligned} S_{[N-1]} &= \begin{pmatrix} S_{11[N-1]} & S_{12[N-1]} \\ S_{21[N-1]} & S_{22[N-1]} \end{pmatrix} = \begin{pmatrix} f_{N[N-1]} & -\overline{g_{N[N-1]}} \\ g_{N[N-1]} & \overline{f_{N[N-1]}} \end{pmatrix} \times \begin{pmatrix} \lambda_N & 0 \\ 0 & \overline{\lambda_N} \end{pmatrix} \times \begin{pmatrix} f_{N[N-1]} & -\overline{g_{N[N-1]}} \\ g_{N[N-1]} & \overline{f_{N[N-1]}} \end{pmatrix}^{-1} \\ &= \frac{1}{|f_{N[N-1]}|^2 + |g_{N[N-1]}|^2} \begin{pmatrix} \lambda_N|f_{N[N-1]}|^2 + \overline{\lambda_N}|g_{N[N-1]}|^2 & (\lambda_N - \overline{\lambda_N})f_{N[N-1]}\overline{g_{N[N-1]}} \\ (\lambda_N - \overline{\lambda_N})\overline{f_{N[N-1]}}g_{N[N-1]} & \lambda_N|g_{N[N-1]}|^2 + \overline{\lambda_N}|f_{N[N-1]}|^2 \end{pmatrix}, \end{aligned} \tag{66}$$

where $j = 0, 1, \ldots, N-2$

$$\begin{aligned} f_{N[N-1]} &= \prod(\lambda_N - S_{11[j]})f_N - S_{12[j]})g_N), \\ g_{N[N-1]} &= \prod(\lambda_N - S_{22[j]})g_N - S_{21[j]})f_N). \end{aligned}$$

Employing an $(n-1)$-fold iterative procedure and n applications of the Darboux transformation, we derive the n-soliton solution of the KdV equation

$$\begin{aligned} w_{[N]} &= w_{[N-1]}(\lambda - d_{[N-1]})^2 + (\lambda - d_{[N-1]}) \\ &\quad \times [c_{[N-1]x} + \mathrm{i}\lambda(a_{[N-1]} - d_{[N-1]})] \\ &\quad + (c_{[N-1]} - \lambda)(c_{[N-1]} - \mathrm{i}\lambda^2 b_{[N-1]}). \end{aligned}$$

Given $z_{[N-1]} = (\lambda - d_{[N-1]})[c_{[N-1]x} + \mathrm{i}\lambda(a_{[N-1]} - d_{[N-1]})] + (c_{[N-1]} - \lambda)(c_{[N-1]} - \mathrm{i}\lambda^2 b_{[N-1]})$, it follows that

$$w_{[N]} = w(\lambda - d_{[N-1]})^2 + z_{[N-1]}. \tag{67}$$

In this research work, we employ the Darboux transformation method to construct exact multi-soliton solutions for the coupled KdV equations. Considering the remarkable efficiency of the Darboux transformation in solving complex nonlinear partial differential equations, we will subsequently present detailed graphical representations to deeply analyze and demonstrate the dynamic behavior characteristics of nonlinear waves within the system.

Now, our research focus shifts to solving the coupled KdV equation set, specifically including Eqs. (20), (56), (60), (64), and (67). To simplify the analysis process, we propose an assumption that there exists a solution in a specific form $\omega_{\mathrm{p}_0} = \chi\omega_{\mathrm{b}_0}$, where χ is an undetermined constant, and for all $i = 1, 2, 3, 4$, $a_i = b_i$, besides, it is also assumed that $u_{\mathrm{b}_0} = u_{\mathrm{p}_0}$. Based on these premises, we can derive and organize to obtain a stationary solitary wave solution of the coupled KdV equation, and this result has been verified and supported in Ref. [26].

$$\begin{cases} \omega_{\mathrm{b}_0} = R_m \cdot \mathrm{sech}^2\left(\frac{\xi - u_{\mathrm{b}_0}\tau}{W}\right), & \text{if} \quad a_1 < 0, \\ \omega_{\mathrm{b}_0} = R_m \cdot \mathrm{sech}^2\left(\frac{\xi - u_{\mathrm{b}_0}\tau}{W}\right), & \text{if} \quad a_1 > 0, \end{cases} \tag{68}$$

where $R_m = \frac{3u_{\mathrm{b}_0}}{a_2 + a_3\chi^2 + 2a_4\chi}$ and $W = 2\sqrt{|a_1|/u_{\mathrm{b}_0}}$ refer to the amplitude and the width of soliton, respectively.

5. Conclusion

By means of the exact expressions of single-soliton, double-soliton, triple-soliton, and n-soliton solutions obtained through the Darboux transformation method, this paper presents the graphical representation of soliton solutions, which exhibit different propagation characteristics due to the different parameter selections. By systematically adjusting and analyzing specific parameter values, we have carried out a comprehensive study on dynamic behavior. This research work provides profound and valuable insights into the propagation dynamics of solitary waves in various integrable systems. Especially in the fields of wave force research and ocean engineering applications, the results are helpful for better understanding and explaining complex physical phenomena observed in reality, thus further promoting the development of related theories and applications.

Based on the (1+1)-dimensional coupled KdV Eq. (68), Figures 1 and 2 in the article show in detail the propagation and evolution process of solitons over time. These figures not only reveal the fine structure of soliton waves, but also provide an intuitive basis for exploring the stability and changing trends of soliton behavior under different parameter conditions.

Through the analysis of Eq. (68), we can find that the amplitude and width of solitary waves are affected by multiple parameters, including $c_{\mathrm{b}0}$, $c_{\mathrm{b}1}$, γ, $\gamma_{\mathrm{b}0}$, $\gamma_{\mathrm{b}1}$, a_{sc}, c_{b}, and $u_{\mathrm{b}0}$, *etc*. In the BEC region of the BEC–BCS crossover system, figures 1 and 2 specifically show the dependence relationship between the soliton width W and the scattering length a_{sc} as well as the parameter $u_{\mathrm{b}0}$. It is worth noting that, with the gradual increase in the values of $u_{\mathrm{b}0}$ and a_{sc}, the width of the

soliton shows a decreasing trend. This phenomenon indicates that under the influence of these specific parameters, the soliton exhibits particular dynamic behavior during its propagation, which has profound implications for understanding and regulating the application of solitons in different physical environments.

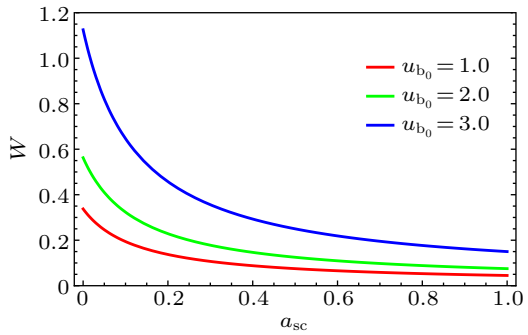


Fig. 1. The dependence of the width, W, of the soliton on the scattering length, a_{sc}, for different parameter u_{b_0}, i.e., $u_{b_0}=1.0$ (solid line), $u_{b_0}=2.0$ (dash line), $u_{b_0}=3.0$ (dot line), where $c_{b0}=3.77$, $c_{b1}=3.10$, $\alpha=21.71$, $\gamma=2.83$, $\gamma_{b0}=1$, $\gamma_{b1}=1/2$, $\chi=0.2$, $\nu=1.15$, $c_b=4.50$.

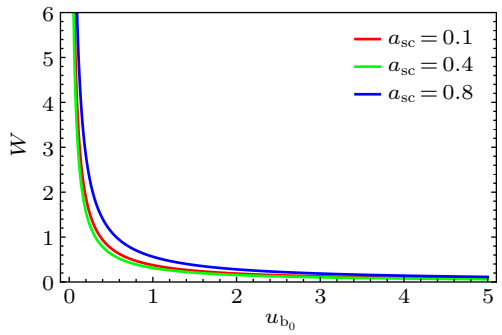


Fig. 2. The dependence of the width of soliton, W, on system parameter, u_{b_0}, for different parameter a_{sc}, i.e., $a_{sc}=0.1$ (solid line), $a_{sc}=0.4$ (dash line), $a_{sc}=0.8$ (dot line), where $c_{b0}=3.77$, $c_{b1}=3.10$, $\alpha=21.71$, $\gamma=2.83$, $\gamma_{b0}=1$, $\gamma_{b1}=1/2$, $\chi=0.2$, $\nu=1.15$, $c_b=4.49$.

Based on Eq. (68), the propagation and evolution of three-soliton and n-soliton solutions for the (1+1)-dimensional coupled second-order KdV equation (21) with respect to time t are presented in Figs. 3–5, for parameter values ξ, γ, a_{sc}, ω_{b_0}, χ, and ν. Utilizing the aforementioned methodology, we have extended our derivation to encompass three-soliton solutions and higher-order soliton solutions. This approach primarily employs the well-established similarity balance method, coupled with iterative techniques, to ascertain multi-soliton solutions for nonlinear evolution equations. Through the solution process, we have elucidated the phase diagram's evolution from the BCS regime to the unitarity regime, identifying the coupling coefficient ω_{b_0} and a_{sc} scattering length as pivotal factors influencing the unitarity regime's phase diagram. By analyzing Eqs. (56), (60), (64), and (66), we have systematically investigated the propagation characteristics of solitons and their underlying physical mechanisms. Our research reveals the direct influence of parameters ξ, γ, a_{sc}, ω_{b_0}, χ, ν, and λ on the soliton's propagation velocity and width. In the deep BEC regime Fig. 3, the soliton amplitude R_m decreases with the increase of the scattering length a_{sc}, while it increases with the increase of the parameter u_{b0}. Figure 4 further shows the characteristics of the soliton waveform changing with u_{b0}. That is, as the value of u_{b0} increases, the soliton amplitude increases and its width decreases, which indicates that the soliton amplitude increases with the increase of the Bose–Einstein condensate number density, and at the same time, its width decreases with the increase of the number density. On the other hand, figure 5 shows the dependence of the soliton waveform on a_{sc}. As the value of a_{sc} increases, both the soliton amplitude and width decrease, which means that both the soliton amplitude and width decrease with the increase of the scattering length between bosons and fermions. In conclusion, the soliton amplitude and width are affected by the parameters a_{sc} and u_{b0}, where the amplitude is inversely proportional to a_{sc} and directly proportional to u_{b0}, while the width has an opposite trend to the changes of these two parameters. During the propagation process, the energy distribution of solitons may undergo dynamic changes, where the decrease in amplitude may indicate a gradual process of energy diffusion or transfer. These meticulous observations not only enhance our understanding of the complex interrelationship between soliton behavior and parameter values, but also provide crucial guidance for the precise control of soliton applications in physical systems.

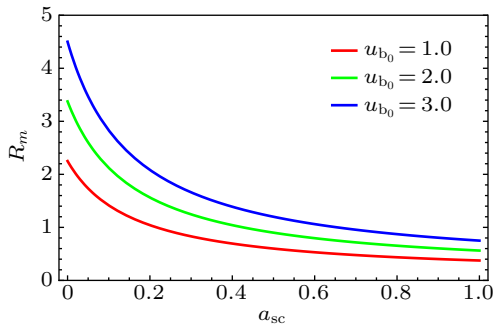


Fig. 3. The dependence of the amplitude of soliton, R_m, on the scattering length, a_{sc}, for different parameter u_{b_0}, i.e., $u_{b_0}=1.0$ (solid line), $u_{b_0}=2.0$ (dash line), $u_{b_0}=3.0$ (dot line), where $c_{b0}=3.77$, $c_{b1}=3.10$, $\alpha=21.71$, $\gamma=2.83$, $\gamma_{b0}=1$, $\gamma_{b1}=1/2$, $\chi=0.2$, $\nu=1.15$, $c_b=7.42$.

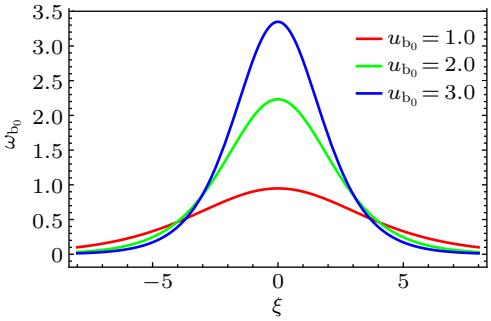


Fig. 4. The waveform of soliton, ω_{b_0}, versus ξ for different parameter u_{b_0}, i.e., $u_{b_0}=1.0$ (solid line), $u_{b_0}=2.0$ (dash line), $u_{b_0}=3.0$ (dot line), where $c_{b0}=3.77$, $c_{b1}=3.10$, $\alpha=21.71$, $\gamma=2.83$, $\gamma_{b0}=1$, $\gamma_{b1}=1/2$, $\chi=0.2$, $\nu=1.15$, $c_b=2.01$.

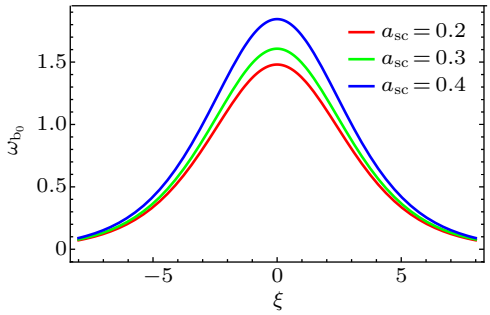


Fig. 5. The waveform of soliton, ω_{b_0}, versus ξ for different scattering length a_{sc}, i.e., $a_{sc} = 0.2$ (solid line), $a_{sc} = 0.3$ (dash line), $a_{sc} = 0.4$ (dot line), where $c_{b0} = 3.77$, $c_{b1} = 3.10$, $\alpha = 21.71$, $\gamma = 2.83$, $\gamma_{b0} = 1$, $\gamma_{b1} = 1/2$, $\chi = 0.2$, $\nu = 1.15$, $c_b = 2.47$.

Based on the above findings, the future research direction can focus on the following key areas: First, it is necessary to deeply explore how these parameters specifically affect the stability and propagation characteristics of solitons under different physical conditions. Second, attention should be paid to the behavior of solitons in complex media, especially considering the inhomogeneity and non-linearity of the medium. Finally, the research on parameter modulation is crucial to optimize the application of solitons in fields such as optical communication and matter-wave control. By carrying out these in-depth studies, we expect to build a more comprehensive and integrated understanding framework of soliton physics, thereby laying a solid theoretical foundation for the progress and development of related technologies.

In short, this study not only reveals the core factors affecting the propagation characteristics of solitons, but also opens up new ways and possibilities for future scientific research and technical applications in the field of soliton physics. These advances herald that in the near future, we will see the practical applications and far-reaching impacts of soliton physics in many frontier scientific and technological fields.

Acknowledgement

Project supported by the National Natural Science Foundation of China (Grant No. 12261072).

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