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An N-breather solution and hybrid solutions of rogue wave and breather for complex mKdV equation

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A compact Grammian form for N-breather solution to the complex mKdV equation is derived using the bilinear Kadomtsev–Petviashvili hierarchy reduction method. The propagation trajectory, period, maximum points, and peak value of the 1-breather solution are calculated. Additionally, through the asymptotic analysis of 2-breather solution, we show that two breathers undergo an elastic collision. By applying the generalized long-wave limit method, the fundamental and second-order rogue wave solutions for the complex mKdV equation are obtained from the 1-breather and 2-breather solutions, respectively. We also construct the hybrid solution of a breather and a fundamental rogue wave for the complex mKdV equation from the 2-breather solution. Furthermore, the hybrid solution of two breathers and a fundamental rogue wave as well as the hybrid solution of a breather and a second-order rogue wave for the complex mKdV equation are derived from the 3-breather solution via the generalized long-wave limit method. By controlling the phase parameters of breathers, the diverse phenomena of interaction between the breathers and the rogue waves are demonstrated.

Keywords: complex mKdV equation, hybrid solutions of breather and rogue wave, KP hierarchy reduction method, generalized long-wave limit method

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1. Introduction

Nonlinear integrable equations constitute an important part of the nonlinear wave theory. These equations arise in various physical areas encompassing fluid dynamics, nonlinear optics, plasma physics, mathematical biology, and quantum field.[1,2] The KdV equation is a fundamental mathematical model for describing weakly nonlinear wave propagation in dispersive media. It was originally established by Korteweg and de Vries in 1895 while developing a model for shallow water waves.[3] A complex modified version of the KdV equation is the complex mKdV equation

$$q_t + 6\varepsilon|q|^2 q_x + q_{xxx} = 0, \tag{1}$$

where $\varepsilon = \pm 1$. When $\varepsilon = 1$, it is referred to as the focusing complex mKdV equation, and when $\varepsilon = -1$, it is called the defocusing complex mKdV equation. The complex mKdV equation (1) is completely integrable, which belongs to the member of the nonlinear Schrödinger hierarchy.[3] It is important to clarify that confusion must be avoided between Eq. (1) and the non-integrable complex mKdV equation,

$$q_t + \alpha(|q|^2 q)_x + q_{xxx} = 0, \tag{2}$$

which describes the propagation of few-cycle optical pulses in cubic nonlinear media.[4,5] The complex mKdV equation (3) is a universal asymptotic equation that governs the evolution of quasi-monochromatic unidirectionally propagating wave packets in strongly dispersive weakly nonlinear media. In 2003, Rodríguez *et al*. considered a model for a cylindric nematic waveguide and derived the basic dynamic equations for the electromagnetic modes and for orientation field. By using a multiple scale technique to the coupled basic equations, they derived equations which govern the evolution of light pulses in liquid crystals at different scales. The nonlinear Schrödinger equation was obtained by scaling to the third order, while the complex mKdV equation (1) arose from scaling up to the fourth order.[6,7] Compared with the nonlinear Schrödinger equation, the complex mKdV equation has more stable solitons.[7] Additionally, Karpman's research reveals that the complex mKdV equation (1) can also be obtained from the extended third-order cubic NLS equation through a Galilean transformation.[8]

Nonlinear localized waves are ubiquitous in various branches of physics including hydrodynamics, plasma physics, acoustics, condensed matter physics, nonlinear optics, and photonics.[1,2] Breathers and rogue waves are important components of the nonlinear localized waves. Rogue waves, also referred to as "freak waves", "killer waves" or "monster waves" are characterized by their spatial and temporal localization, exhibiting amplitudes several times greater than the surrounding background.[9,10] Due to their unique properties, rogue waves have received extensive research.[11–15] Rogue waves appear in a variety of physical fields including oceanography,[16,17] parametrically driven

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capillary waves,[18] plasmas,[19] liquid helium,[20] and nonlinear optics.[21,22] Since Solli *et al*. first observed optical rogue waves in nonlinear optical systems in 2007,[21] there has been a growing interest in rogue waves within the fields of optics and lasers.[23–26] Breathers, also referred to as periodic solitons, are named 'breathers' due to their localized and periodic oscillation patterns. The concept of breathers was first proposed by Ablowitz while investigating the initial value problem for the sine-Gordon equation using the inverse scattering method.[27] The most well-known types of breathers are Kuznetsov–Ma breathers and Akhmediev breathers. Kuznetsov–Ma breathers, which display temporal periodicity and spatial localization, are first discovered in the NLS equation by Kuznetsov in 1977.[28] In contrast, Akhmediev breathers, also derived in the NLS equation, are discovered in 1986 by Akhmediev and exhibit spatial periodicity and temporal localization.[29] Subsequently, more general breather solutions for the NLS equation were derived in Ref. [30]. Experiments in optical fibers have demonstrated significant connections among breathers,[31] modulation instability (MI)[32] and the Fermi–Pasta–Ulam recurrence.[33] Additionally, breathers are widely utilized in practical applications such as optical microresonators,[34] ultrafast fiber lasers,[35] and quantized superfluid vortices.[36] Breathers are potential prototypes for rogue waves, and they can transform into rogue waves by taking the long-wave limit.[37–39] It is essential to emphasize that the interactions between breathers and rogue waves can produce complex hybrid wave structures, leading to interesting dynamic behaviors.

Over the past few decades, various types of solutions of the complex mKdV equation (1) have been studied. The inverse scattering method is performed to derive bound-state soliton solutions of the complex mKdV equation.[40] A variety of exact traveling wave solutions of the complex mKdV equation are investigated by using the tanh method and the sine-cosine method.[41] Superregular breathers in the complex mKdV equation are discussed in Ref. [42]. Moreover, the Nth-order rogue wave solution for the complex mKdV equation has been derived using the generalized Darboux transformations method in Refs. [43,44]. Besides, the 1-breather solution, 2-breather solution, and hybrid breather-rogue wave solutions for the complex mKdV equation were derived using the Darboux transformation method in Refs. [45,46]. In addition, the Nth-order rogue wave solution for complex mKdV equation is derived by the bilinear Kadomtsev–Petviashvili hierarchy reduction method (KP hierarchy reduction method) in Ref. [47].

The KP hierarchy reduction method is one of the highly efficient techniques for deriving nonlinear localized wave solutions to the nonlinear wave equations. In the 1970s, the Kyoto school developed this method to derive soliton solutions in integrable systems.[48] In 2012, Ohta and Yang applied the KP hierarchy reduction method to derive the rogue wave solutions for the NLS equation.[49] Chen *et al*. applied this method to derive the breather solutions for the Yajima–Oikawa system in 2017.[50] From then on, breather solutions and rogue wave solutions of a variety of integrable wave equations were constructed using the KP hierarchy reduction method.[51–58] The advantage of this method is that N-breather solution and N-th-order rogue wave solution can be compactly expressed in terms of the Gram determinants. Moreover, the rogue wave patterns associated with the root structures of various polynomials can be derived from the rogue wave solutions expressed in the Gram determinant form.[59–61] In this paper, starting from the proper bilinear equations from the KP hierarchy and their Gram determinant solution, we derive the N-breather solution for the complex mKdV equation through dimensional reduction and complex conjugate reduction. By taking the long-wave limit to the breather solutions, we derive the rogue wave solutions and the hybrid solutions of breather and rogue wave for the complex mKdV equation. In contrast to the hybrid solutions of breather and rogue wave constructed in Ref. [46], each hybrid solution constructed through the generalized long-wave limit method incorporates all free parameters from the individual rogue waves and breathers involved in this solution, thereby revealing a rich variety of interaction phenomena between breathers and rogue waves.

In this paper, we focus on the complex mKdV equation with $\varepsilon = 1$, represented by the form

$$q_t + 6|q|^2 q_x + q_{xxx} = 0. \tag{3}$$

The rest of this paper is structured as follows. In Section 2, we present the tau functions in Grammian form for the complex mKdV equation (3). In Section 3, based on the tau functions, we derive the N-breather solution for the complex mKdV equation (3) by selecting proper determinant elements. In Section 4, we discuss the dynamics of the 1-breather solution and the interaction between two breathers in the complex mKdV equation (3). In Section 5, by applying the method of the generalized long-wave limit to the breather solutions, we derive the rogue wave solutions as well as the hybrid solutions of breather and rogue wave for the complex mKdV equation (3). Finally, we summarize and discuss the results of this paper in Section 6.

2. Tau functions of the complex mKdV equation

Through the transformation

$$q = \frac{g}{f} \mathrm{e}^{\mathrm{i}(k(x-6t)+k^3 t)}, \tag{4}$$

equation (3) is transformed into the following bilinear form:

$$\left(D_x^2 + 2\right) f \cdot f = 2gg^*, \tag{5}$$

$$\left(D_t - 3(k^2-2)D_x + 3k\mathrm{i}D_x^2 + D_x^3\right) g \cdot f = 0. \tag{6}$$

Here, f is a real function, g is a complex function, k is a real constant, $*$ denotes the complex conjugate, and D is Hirota's bilinear differential operator defined by[62]

$$D_x^m D_t^n f\cdot g = \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n \left[f(x,t)g(x',t')\right]\Big|_{x=x',t=t'}.$$

To obtain the Gram determinant solutions for the bilinear complex mKdV equations (5)–(6), we consider some higher-dimensional bilinear equations in the KP hierarchy given in Lemma 1.

Lemma 1 The following higher-dimensional bilinear equations in the KP hierarchy

$$\left(D_{x_1}D_{x_{-1}}-2\right)\tau_n\cdot\tau_n=-2\tau_{n+1}\cdot\tau_{n-1}, \tag{7a}$$
$$\left(D_{x_{-1}}(D_{x_1}^2-D_{x_2})-4D_{x_1}\right)\tau_{n+1}\cdot\tau_n=0, \tag{7b}$$
$$\left(D_{x_1}^3-4D_{x_3}+3D_{x_1}D_{x_2}\right)\tau_{n+1}\cdot\tau_n=0, \tag{7c}$$
$$\left(D_{x_1}^2-D_{x_2}\right)\tau_{n+1}\cdot\tau_n=0 \tag{7d}$$

have the following tau function expressed in terms of Gram determinant:

$$\tau_n=\det_{1\le i,j\le N}(m_{i,j}^{(n)}). \tag{8}$$

Here, $m_{i,j}^{(n)}$, $\phi_i^{(n)}$, and $\psi_j^{(n)}$ are functions with respect to x_{-1}, x_1, x_2, and x_3 satisfying the following differential and difference relations:

$$\partial_{x_1}m_{i,j}^{(n)}=\phi_i^{(n)}\psi_j^{(n)}, \tag{9a}$$
$$\partial_{x_2}m_{i,j}^{(n)}=\phi_i^{(n+1)}\psi_j^{(n)}+\phi_i^{(n)}\psi_j^{(n-1)}, \tag{9b}$$
$$\partial_{x_3}m_{i,j}^{(n)}=\phi_i^{(n+2)}\psi_j^{(n)}+\phi_i^{(n+1)}\psi_j^{(n-1)}+\phi_i^{(n)}\psi_j^{(n-2)}, \tag{9c}$$
$$\partial_{x_{-1}}m_{i,j}^{(n)}=-\phi_i^{(n-1)}\psi_j^{(n+1)}, \tag{9d}$$
$$m_{i,j}^{(n+1)}=m_{i,j}^{(n)}+\phi_i^{(n)}\varphi_j^{(n+1)}, \tag{9e}$$
$$m_{i,j}^{(n-1)}=m_{i,j}^{(n)}-\phi_i^{(n-1)}\varphi_j^{(n)}, \tag{9f}$$
$$\partial_{x_k}\phi_i^{(n)}=\phi_i^{(n+k)}, \tag{9g}$$
$$\partial_{x_k}\psi_i^{(n)}=-\psi_i^{(n-k)},\quad (k=-1,1,2,3). \tag{9h}$$

The Gram determinant solution (8) of bilinear equations (7a)–(7d) have been proved in Refs. [49,58] and we omit their proof process here. In order to reduce the higher-dimensional bilinear equations (7a)–(7d) to a lower-dimensional one, τ_n in Eq. (8) needs to satisfy the following dimensional reduction condition:

$$(\partial_{x_{-1}}+\partial_{x_1})\tau_n=C\tau_n. \tag{10}$$

Under the condition (10), equations (7a) and (7b) become

$$(D_{x_1}^2+2)\tau_n\cdot\tau_n=2\tau_{n+1}\cdot\tau_n, \tag{11a}$$
$$(D_{x_1}^3-D_{x_1}D_{x_2}+4D_{x_1})\tau_{n+1}\cdot\tau_n=0. \tag{11b}$$

Substituting Eq. (7c) into Eq. (11b), we get equation

$$(D_{x_1}^3-D_{x_3}+3D_{x_1})\tau_{n+1}\cdot\tau_n=0. \tag{12}$$

Then by taking $3k\mathrm{i}\times$Eq. (7d) + Eq. (12), we obtain

$$3k\mathrm{i}(D_{x_1}^2-D_{x_2})\tau_{n+1}\cdot\tau_n +(D_{x_1}^3-D_{x_3}+3D_{x_1})\tau_{n+1}\cdot\tau_n=0. \tag{13}$$

By implementing the coordinate transformations

$$x_1=x+3(k^2-1)t,\quad x_2=-3k\mathrm{i}t,\quad x_3=-t, \tag{14}$$

and defining $f=\tau_0$, $g=\tau_1$, and $h=\tau_{-1}$, equations (11a) and (13) can be converted into the following equations:

$$(D_x^2+2)f\cdot f=2gh, \tag{15a}$$
$$\left(D_t-3(k^2-2)D_x+3k\mathrm{i}D_x^2+D_x^3\right)g\cdot f=0. \tag{15b}$$

Furthermore, if the complex conjugate condition

$$\tau_n^*(x,t)=\tau_{-n}(x,t) \tag{16}$$

is satisfied, then $h=g^*$, and the bilinear equations (15a)–(15b) would reduce to the bilinear equations (5)–(6). When we construct N-breather solution for the complex mKdV equation (3) in the following section, dimensional reduction condition (10) and complex conjugate condition (16) can be satisfied by properly constraining the parameters.

3. The N-breather solutions for complex mKdV equation

In this section, we derive the N-breather solution for the complex mKdV equation (3).

Theorem 1 The complex mKdV equation (3) admits the following N-breather solution:

$$q_N(x,t)=\frac{g_N}{f_N}\mathrm{e}^{\mathrm{i}(k(x-6t)+k^3t)}, \tag{17}$$

where the real function $f_N(x,t)$ and the complex function $g_N(x,t)$ are given by

$$f_N=\tau_0,\quad g_N=\tau_1, \tag{18}$$

and τ_n is defined as the following determinant:

$$\tau_n=\det_{1\le i,j\le N}\left(\overline{m}_{i,j}^{(n)}\right), \tag{19}$$

$$\overline{m}_{i,j}^{(n)}=\frac{1}{p_i^{[1]}+p_j^{[1]*}}\left(-\frac{p_i^{[1]}}{p_j^{[1]*}}\right)^n\mathrm{e}^{\zeta_i+\zeta_j^*} +\frac{1}{p_i^{[1]}+p_j^{[2]*}}\left(-\frac{p_i^{[1]}}{p_j^{[2]*}}\right)^n\mathrm{e}^{\zeta_i} +\frac{1}{p_i^{[2]}+p_j^{[1]*}}\left(-\frac{p_i^{[2]}}{p_j^{[1]*}}\right)^n\mathrm{e}^{\zeta_j^*}$$

$$+\frac{1}{p_i^{[2]}+p_j^{[2]*}}\left(-\frac{p_i^{[2]}}{p_j^{[2]*}}\right)^n, \tag{20}$$

$$\zeta_i = (p_i^{[1]}-p_i^{[2]})x+\left[3(k^2-1)(p_i^{[1]}-p_i^{[2]}) -3k\mathrm{i}({p_i^{[1]}}^2-{p_i^{[2]}}^2)-({p_i^{[1]}}^3-{p_i^{[2]}}^3)\right]t+\xi_{i0}. \tag{21}$$

In the above expression, $p_i^{[\alpha]}$, ξ_{i0} are arbitrary complex constants and the parameters $p_i^{[1]}$, $p_i^{[2]}$ satisfy the following constraint:

$$p_i^{[2]}=\frac{1}{p_i^{[1]}}. \tag{22}$$

Proof We construct N-breather solution of the complex mKdV equation (3) by choosing the matrix elements $m_{i,j}^{(n)}$ of τ_n in Eq. (8) as follows:

$$m_{i,j}^{(n)}=\sum_{\alpha,\beta=1}^{2}\frac{1}{p_i^{[\alpha]}+q_j^{[\beta]}}\left(-\frac{p_i^{[\alpha]}}{q_j^{[\beta]}}\right)^n \mathrm{e}^{\xi_i^{[\alpha]}+\eta_j^{[\beta]}}, \tag{23}$$

$$\phi_i^{(n)}=\sum_{\alpha=1}^{2}{p_i^{[\alpha]}}^n\mathrm{e}^{\xi_i^{[\alpha]}},\quad \psi_j^{(n)}=\sum_{\beta=1}^{2}\left(-q_j^{[\beta]}\right)^{-n}\mathrm{e}^{\eta_j^{[\beta]}}, \tag{24}$$

where

$$\xi_i^{[\alpha]}=\frac{1}{p_i^{[\alpha]}}x_{-1}+p_i^{[\alpha]}x_1+{p_i^{[\alpha]}}^2x_2+{p_i^{[\alpha]}}^3x_3+\xi_{i0}^{[\alpha]},$$
$$\eta_j^{[\beta]}=\frac{1}{q_j^{[\beta]}}x_{-1}+q_j^{[\beta]}x_1-{q_j^{[\beta]}}^2x_2+{q_j^{[\beta]}}^3x_3+\eta_{j0}^{[\beta]}.$$

Taking $\Lambda=\Pi_{i,j=1}^{N}e^{\xi_i^{[2]}+\eta_j^{[2]}}$, equation (23) can be expressed by

$$\tau_n=\Lambda\det_{1\le i,j\le N}\left(\overline{m}_{i,j}^{(n)}\right), \tag{25}$$

where

$$\overline{m}_{i,j}^{(n)} = \frac{1}{p_i^{[1]}+q_j^{[1]}}\left(-\frac{p_i^{[1]}}{q_j^{[1]}}\right)^n\mathrm{e}^{\zeta_i+\overline{\zeta}_j}+\frac{1}{p_i^{[1]}+q_j^{[2]}}\left(-\frac{p_i^{[1]}}{q_j^{[2]}}\right)^n\mathrm{e}^{\zeta_i} +\frac{1}{p_i^{[2]}+q_j^{[1]}}\left(-\frac{p_i^{[2]}}{q_j^{[1]}}\right)^n\mathrm{e}^{\overline{\zeta}_j}+\frac{1}{p_i^{[2]}+q_j^{[2]}}\left(-\frac{p_i^{[2]}}{q_j^{[2]}}\right)^n, \tag{26}$$

in which $\zeta_i=\xi_i^{[1]}-\xi_i^{[2]}$, $\overline{\zeta}_j=\eta_j^{[1]}-\eta_j^{[2]}$, $\xi_{i0}=\xi_{i0}^{[1]}-\xi_{i0}^{[2]}$ and $\eta_{j0}=\eta_{j0}^{[1]}-\eta_{j0}^{[2]}$.

In order to satisfy the dimensional reduction condition (10) and the complex conjugate condition (16), we will impose constraints on $p_i^{[\alpha]}$, $q_j^{[\alpha]}$, ζ_i, and $\overline{\zeta_j}$ $(\alpha=1,2;i,j=1,2,\ldots,N)$. By defining $\mathcal{Q}(p)=p_i^{[1]}-p_i^{[2]}+\frac{1}{p_i^{[1]}}-\frac{1}{p_i^{[2]}}$ and $\widehat{\mathcal{Q}}(q)=q_j^{[1]}-q_j^{[2]}+\frac{1}{q_j^{[1]}}-\frac{1}{q_j^{[2]}}$, we have

$$\begin{aligned}&\left(\partial_{x_{-1}}+\partial_{x_1}\right)\overline{m}_{i,j}^{(n)}\\ &=\left(\mathcal{Q}(p)+\widehat{\mathcal{Q}}(q)\right)\left(-\frac{p_i^{[1]}}{q_j^{[1]}}\right)^n\left(\frac{1}{p_i^{[1]}+q_j^{[1]}}\right)\mathrm{e}^{\zeta_i+\overline{\zeta}_j}\\ &\quad+\mathcal{Q}(p)\left(-\frac{p_i^{[1]}}{q_j^{[2]}}\right)^n\left(\frac{1}{p_i^{[1]}+q_j^{[2]}}\right)\mathrm{e}^{\zeta_i}\\ &\quad+\widehat{\mathcal{Q}}(q)\left(-\frac{p_i^{[2]}}{q_j^{[1]}}\right)^n\left(\frac{1}{p_i^{[2]}+q_j^{[1]}}\right)\mathrm{e}^{\overline{\zeta}_j}.\end{aligned} \tag{27}$$

If $\mathcal{Q}(p)=0$, $\widehat{\mathcal{Q}}(q)=0$, then

$$\begin{aligned}\partial_{x_{-1}}\tau_n &= \sum_{i,j=1}^{N}\Delta_{i,j}\partial_{x_{-1}}\overline{m}_{i,j}^{(n)}\\ &= -\sum_{i,j=1}^{N}\Delta_{i,j}\partial_{x_1}\overline{m}_{i,j}^{(n)}=-\partial_{x_1}\tau_n,\end{aligned} \tag{28}$$

thus the dimensional reduction condition (10) is satisfied. Therefore, we impose the following constraints on $p_i^{[\alpha]}$ and $q_j^{[\alpha]}$ $(\alpha=1,2;i,j=1,2,\ldots,N)$:

$$p_i^{[2]}=\frac{1}{p_i^{[1]}},\quad q_j^{[2]}=\frac{1}{q_j^{[1]}}, \tag{29}$$

such that $\mathcal{Q}(p)=0$, $\widehat{\mathcal{Q}}(q)=0$, and the dimensional reduction condition (10) is satisfied. Besides, setting the following complex conjugate conditions on the parameters:

$$q_j^{[\alpha]}={p_j^{[\alpha]}}^*,\quad \eta_{j0}=\xi_{j0}^*,\quad (\alpha=1,2;j=1,2,\ldots,N), \tag{30}$$

we have $\zeta_i^*=\overline{\zeta}_i$ and $\overline{m}_{i,j}^{(n)*}=\overline{m}_{j,i}^{(-n)}$. Therefore, the complex conjugate condition (16) is also satisfied.

Thus, under the parameter constraints (29)–(30) and the coordinate transformations (14), the τ_n function (25) reduces to the N-breather solution of the complex mKdV equation (3) for $f=\tau_0, g=\tau_1$. The factor Λ in Eq. (25) can be eliminated from τ_1/τ_0. Consequently, the N-breather solution of the complex mKdV equation is derived, and Theorem 1 is proved.

Remark 1 To avoid singularities in the N-breather solution (17) for complex mKdV equation, $p_i^{[\alpha]}$ $(i=1,2,\ldots,N;\alpha=1,2)$ cannot be pure imaginary numbers, because the denominator of the solution has a factor $p_i^{[\alpha]}+{p_i^{[\alpha]}}^*$, which will become zero if $p_i^{[\alpha]}$ are purely imaginary.

4. Dynamics of breather solutions in complex mKdV equation

In this section, we discuss the dynamics of 1-breather solution and 2-breather solution in complex mKdV equation (3).

4.1. Dynamics of 1-breather solution

By taking $N=1$ in Eq. (19), 1-breather solution can be expressed as the following form:

$$q_1(x,t)=A\frac{\sqrt{A_{11}}\cosh(\zeta_{1\mathrm{R}}+\mathrm{i}\phi_{c_1}+\ln(\sqrt{A_{11}}\rho_1))+\cos(\zeta_{1\mathrm{I}}+\varphi_1-\mathrm{i}\ln\rho_{c_1})}{\sqrt{A_{11}}\cosh(\zeta_{1\mathrm{R}}+\ln(\sqrt{A_{11}}\rho_1))+\cos(\zeta_{1\mathrm{I}}+\varphi_1)}, \tag{31}$$

where

$$\begin{aligned}
&\zeta_1=\zeta_{1\mathrm{R}}+\mathrm{i}\zeta_{1\mathrm{I}},\quad (p_1^{[1]})^2=\rho_{c_1}\mathrm{e}^{\mathrm{i}\phi_{c_1}},\\
&\frac{p_1^{[1]}+p_1^{[1]*}}{p_1^{[1]}(p_1^{[1]}p_1^{[1]*}+1)}=\rho_1\mathrm{e}^{\mathrm{i}\varphi_1},\\
&A_{11}=\left(\frac{p_1^{[1]}p_1^{[1]*}+1}{p_1^{[1]}+p_1^{[1]*}}\right)^2,\\
&A=-\frac{p_1^{[1]*}}{p_1^{[1]}}\mathrm{e}^{\mathrm{i}[k(x-6t)+k^3t+\phi_{c_1}]}.
\end{aligned} \tag{32}$$

The peak $|q_1|_{\max}$ and the maximum points $(x_{\max},t_{\max})$ are calculated as follows:

$$|q_1(x_{\max},t_{\max})|_{\max}=\left|\frac{\cos(\phi_{c_1})\cdot\sqrt{A11}-\cosh(\ln\rho_{c_1})}{\sqrt{A11}-1}\right|, \tag{33}$$

$$\begin{aligned}
&x_{\max}=\frac{(\pi-\varphi_1)\cdot\alpha_{12}+\ln(\sqrt{A_{11}}\rho_1)\beta_{21}}{-\alpha_{11}\beta_{21}+\alpha_{12}\beta_{11}}+kT_{[x]},\\
&k=0,\pm1,\pm2,\ldots,
\end{aligned} \tag{34}$$

$$\begin{aligned}
&t_{\max}=\frac{(\pi-\varphi_1)\cdot\alpha_{11}+\ln(\sqrt{A_{11}}\rho_1)\beta_{11}}{\alpha_{11}\beta_{21}-\alpha_{12}\beta_{11}}+kT_{[t]},\\
&k=0,\pm1,\pm2,\ldots,
\end{aligned} \tag{35}$$

where $T_{[x]}$ and $T_{[t]}$ are the periods of 1-breather in the x direction and in the t direction, respectively, expressed as

$$T_{[t]}=\frac{2\pi\alpha_{11}}{\beta_{21}\alpha_{11}-\beta_{11}\alpha_{12}}, \tag{36}$$

$$T_{[x]}=-\frac{2\pi\alpha_{12}}{\beta_{21}\alpha_{11}-\beta_{11}\alpha_{12}}. \tag{37}$$

The distance between two closest peaks of $|q_1|$ is given by

$$T=2\pi\sqrt{\frac{\alpha_{11}^2+\alpha_{12}^2}{(\beta_{21}\alpha_{11}-\beta_{11}\alpha_{12})^2}}, \tag{38}$$

where α_{11}, α_{12}, β_{11}, β_{21} are defined as

$$\zeta_{1\mathrm{R}}=\alpha_{11}x+\alpha_{12}t+\xi_{10\mathrm{R}}, \tag{39}$$

$$\zeta_{1\mathrm{I}}=\beta_{11}x+\beta_{21}t+\xi_{10\mathrm{I}} \tag{40}$$

with

$$\begin{aligned}
\alpha_{11} &= p_{1\mathrm{R}}-\frac{p_{1\mathrm{R}}}{p_{1\mathrm{I}}^2+p_{1\mathrm{R}}^2},\quad \beta_{11}=p_{1\mathrm{I}}+\frac{p_{1\mathrm{I}}}{p_{1\mathrm{I}}^2+p_{1\mathrm{R}}^2},\\
\alpha_{12} &= \frac{p_{1\mathrm{R}}(3p_{1\mathrm{I}}^2-p_{1\mathrm{R}}^2)}{(p_{1\mathrm{R}}^2+p_{1\mathrm{I}}^2)^3}+\frac{6kp_{1\mathrm{R}}p_{1\mathrm{I}}}{(p_{1\mathrm{R}}^2+p_{1\mathrm{I}}^2)^2}\\
&\quad -\frac{3(k^2-1)p_{1\mathrm{R}}}{p_{1\mathrm{R}}^2+p_{1\mathrm{I}}^2}+p_{1\mathrm{R}}(3p_{1\mathrm{I}}^2-p_{1\mathrm{R}}^2)\\
&\quad +6kp_{1\mathrm{R}}p_{1\mathrm{I}}+3(k^2-1)p_{1\mathrm{R}},\\
\beta_{21} &= \frac{p_{1\mathrm{I}}(p_{1\mathrm{I}}^2-3p_{1\mathrm{R}}^2)}{(p_{1\mathrm{R}}^2+p_{1\mathrm{I}}^2)^3}\\
&\quad +\frac{3k(p_{1\mathrm{R}}^2-p_{1\mathrm{I}}^2)}{(p_{1\mathrm{R}}^2+p_{1\mathrm{I}}^2)^2}+\frac{3(k^2-1)p_{1\mathrm{I}}}{p_{1\mathrm{R}}^2+p_{1\mathrm{I}}^2}+p_{1\mathrm{I}}(p_{1\mathrm{I}}^2\\
&\quad -3p_{1\mathrm{R}}^2)-3k(p_{1\mathrm{R}}^2-p_{1\mathrm{I}}^2)+3(k^2-1)p_{1\mathrm{I}},
\end{aligned}$$

in which $p_1^{[1]}=p_{1\mathrm{R}}+\mathrm{i}p_{1\mathrm{I}}$, $\xi_{10}=\xi_{10\mathrm{R}}+\mathrm{i}\xi_{10\mathrm{I}}$. Therefore, equations (38) is the period of the 1-breather solution in the direction of trajectory. From Eqs. (33)–(35), we know that the propagation trajectory of the breather is $\zeta_{1\mathrm{R}}+\ln(\sqrt{A_{11}})\rho_1=0$. Figure 1 displays the modulus of the 1-breather solution $|q_1|$ by setting the parameter values $k=1$, $p_1^{[1]}=\frac{1}{3}-\frac{3}{4}\mathrm{i}$, $\xi_{10}=0$. If $\frac{\partial\zeta_{1\mathrm{R}}}{\partial x}=0$, 1-breather solution (31) is cyclical in space and localized in time, which is called Akhmediev breather. If $\frac{\partial\zeta_{1\mathrm{R}}}{\partial t}=0$, the 1-breather solution (31) is cyclical in time and localized in space, which is called Kuznetsov–Ma breather.

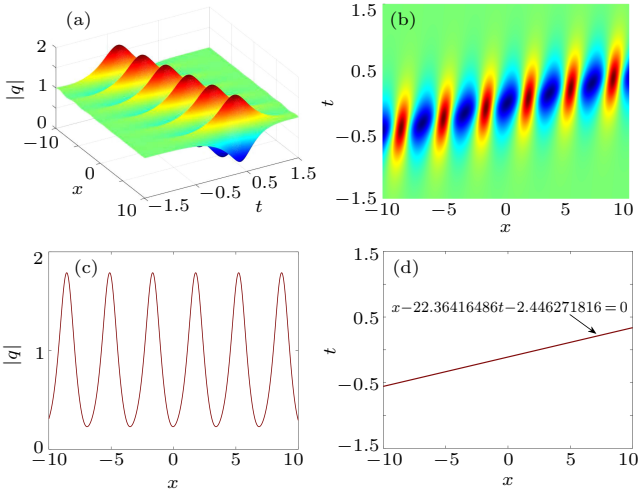


Fig. 1. (a) The 1-breather solution with parameters $\xi_{10}=0$, $k=1$, and $p_1^{[1]}=\frac{1}{3}-\frac{3}{4}\mathrm{i}$. (b) The corresponding density plot of panel (a). (c) The intersection between panel (a) and the plane $x-22.36416486t-2.446271816=0$. (d) The trajectory line of panel (a) in the x, t plane: $x-22.36416486t-2.446271816=0$.

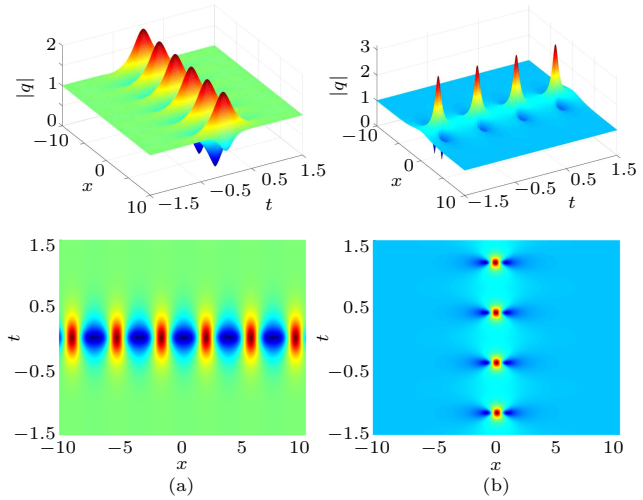


Fig. 2. (a) Akhmediev breather with $p_1^{[1]}=\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$, $k=1$, $\xi_{10}=0$. (b) Kuznetsov–Ma breather with $p_1^{[1]}=1.5$, $k=1.5$, $\xi_{10}=0$. In all panels, $-10\le x\le 10$ and $-1.5\le t\le 1.5$.

Akhmediev breather and Kuznetsov–Ma breather are displayed in Fig. 2 by taking specific parameters.

4.2. The interaction between two breathers

In this section, we study the interaction between two breathers. When $N=2$, by setting $\zeta_1=\zeta_{1\mathrm{R}}+\mathrm{i}\zeta_{1\mathrm{I}}$, $\zeta_2=\zeta_{2\mathrm{R}}+\mathrm{i}\zeta_{2\mathrm{I}}$, and $\xi_{20}=\xi_{20\mathrm{R}}+\mathrm{i}\xi_{20\mathrm{I}}$, the equation (18) has the following form:

$$f_2=\begin{vmatrix}\overline{m}_{11}^{(0)} & \overline{m}_{12}^{(0)}\\ \overline{m}_{21}^{(0)} & \overline{m}_{22}^{(0)}\end{vmatrix},\quad g_2=\begin{vmatrix}\overline{m}_{11}^{(1)} & \overline{m}_{12}^{(1)}\\ \overline{m}_{21}^{(1)} & \overline{m}_{22}^{(1)}\end{vmatrix}, \tag{41}$$

where $\overline{m}_{i,j}^{(n)}$ $(n=0;i,j=1,2)$ are defined in Theorem 1. We introduce

$$\begin{aligned}
&(p_2^{[1]})^2=\rho_{c_2}\mathrm{e}^{\mathrm{i}\phi_{c_2}},\quad \frac{p_2^{[1]}+p_2^{[1]^*}}{p_2^{[1]}(p_2^{[1]}p_2^{[1]^*}+1)}=\rho_2\mathrm{e}^{\mathrm{i}\varphi_2},\\
&\frac{(1-p_1^{[1]}p_2^{[1]})(p_1^{[1]}+p_2^{[1]^*})}{(1+p_1^{[1]}p_2^{[1]^*})(p_1^{[1]}-p_2^{[1]})}=\rho_{12}\mathrm{e}^{\mathrm{i}\phi_1},\\
&\frac{(p_1^{[1]}p_2^{[1]}-1)(p_1^{[1]^*}+p_2^{[1]})}{(p_1^{[1]^*}p_2^{[1]}+1)(p_1^{[1]}-p_2^{[1]})}=\rho_{12}\mathrm{e}^{\mathrm{i}\phi_2},\\
&A_{22}=\left(\frac{p_2^{[1]}p_2^{[1]^*}+1}{p_2^{[1]}+p_2^{[1]^*}}\right)^2,
\end{aligned}$$

$$\widehat{A}=-\frac{p_2^{[1]^*}}{p_2^{[1]}}\mathrm{e}^{\mathrm{i}[k(x-6t)+k^3t+\phi_{c_2}]}, \tag{42}$$

$$\zeta_{2\mathrm{R}}=\alpha_{21}x+\alpha_{22}t+\xi_{20\mathrm{R}}, \tag{43}$$

$$\omega_{12}=\frac{\alpha_{12}}{\alpha_{11}}-\frac{\alpha_{22}}{\alpha_{21}}, \tag{44}$$

where

$$\begin{aligned}
p_2^{[1]}&=p_{2\mathrm{R}}+\mathrm{i}p_{2\mathrm{I}},\quad \alpha_{21}=p_{2\mathrm{R}}-\frac{p_{2\mathrm{R}}}{p_{2\mathrm{I}}^2+p_{2\mathrm{R}}^2},\\
\alpha_{22}&=\frac{p_{2\mathrm{R}}(3p_{2\mathrm{I}}^2-p_{2\mathrm{R}}^2)}{(p_{2\mathrm{R}}^2+p_{2\mathrm{I}}^2)^3}+\frac{6kp_{2\mathrm{R}}p_{2\mathrm{I}}}{(p_{2\mathrm{R}}^2+p_{2\mathrm{I}}^2)^2}\\
&\quad-\frac{3(k^2-1)p_{2\mathrm{R}}}{p_{2\mathrm{R}}^2+p_{2\mathrm{I}}^2}+p_{2\mathrm{R}}(3p_{2\mathrm{I}}^2-p_{2\mathrm{R}}^2)\\
&\quad+6kp_{2\mathrm{R}}p_{2\mathrm{I}}+3(k^2-1)p_{2\mathrm{R}}.
\end{aligned}$$

From Eqs. (39), (43), and (44), we obtain

$$\frac{\zeta_{1\mathrm{R}}}{\alpha_{11}}-\frac{\zeta_{2\mathrm{R}}}{\alpha_{21}}=\omega_{12}t. \tag{45}$$

We consider the asymptotic behavior of 2-breather solution under the constraints $\alpha_{11}>0$, $\alpha_{21}>0$, and $\omega_{12}<0$. By applying Eq. (45), we obtain the following asymptotic forms for the 2-breather solution (41)

(i) Before collision $(t\to-\infty)$:

(i)-1 Breather 1 $(\zeta_{1\mathrm{R}}\approx 0,\zeta_{2\mathrm{R}}\to-\infty)$:

$$q\approx A\cdot\frac{p_2^{[1]^*}}{p_2^{[1]}}\frac{\sqrt{A_{11}}\cosh(\zeta_{1\mathrm{R}}+\mathrm{i}\phi_{c_1}+\ln(\sqrt{A_{11}}\rho_1\rho_{12}))+\cos(\zeta_{1\mathrm{I}}+\varphi_1+\phi_1-\mathrm{i}\ln\rho_{c_1})}{\sqrt{A_{11}}\cosh(\zeta_{1\mathrm{R}}+\ln(\sqrt{A_{11}}\rho_1\rho_{12}))+\cos(\zeta_{1\mathrm{I}}+\varphi_1+\phi_1)}. \tag{46}$$

(i)-2 Breather 2 $(\zeta_{2\mathrm{R}}\approx 0,\zeta_{1\mathrm{R}}\to+\infty)$:

$$q\approx\widehat{A}\cdot\frac{p_1^{[1]^*}}{p_1^{[1]}}\frac{\sqrt{A_{22}}\cosh(\zeta_{2\mathrm{R}}+\mathrm{i}\phi_{c_2}+\ln(\frac{\sqrt{A_{22}}\rho_2}{\rho_{12}}))+\cos(\zeta_{2\mathrm{I}}+\varphi_2-\phi_2-\mathrm{i}\ln\rho_{c_2})}{\sqrt{A_{22}}\cosh(\zeta_{2\mathrm{R}}+\ln(\frac{\sqrt{A_{22}}\rho_2}{\rho_{12}}))+\cos(\zeta_{2\mathrm{I}}+\varphi_2-\phi_2)}. \tag{47}$$

(ii) After collision $(t\to+\infty)$:

(ii)-1 Breather 1 $(\zeta_{1\mathrm{R}}\approx 0,\zeta_{2\mathrm{R}}\to+\infty)$:

$$q\approx A\cdot\frac{p_2^{[1]}}{p_2^{[1]^*}}\frac{\sqrt{A_{11}}\cosh(\zeta_{1\mathrm{R}}+\mathrm{i}\phi_{c_1}+\ln(\frac{\sqrt{A_{11}}\rho_1}{\rho_{12}}))+\cos(\zeta_{1\mathrm{I}}+\varphi_1-\phi_1-\mathrm{i}\ln\rho_{c_1})}{\sqrt{A_{11}}\cosh(\zeta_{1\mathrm{R}}+\ln(\frac{\sqrt{A_{11}}\rho_1}{\rho_{12}}))+\cos(\zeta_{1\mathrm{I}}+\varphi_1-\phi_1)}. \tag{48}$$

(ii)-2 Breather 2 $(\zeta_{2\mathrm{R}}\approx 0,\zeta_{1\mathrm{R}}\to-\infty)$:

$$q\approx\widehat{A}\cdot\frac{p_1^{[1]}}{p_1^{[1]^*}}\frac{\sqrt{A_{22}}\cosh(\zeta_{2\mathrm{R}}+\mathrm{i}\phi_{c_2}+\ln(\sqrt{A_{22}}\rho_2\rho_{12}))+\cos(\zeta_{2\mathrm{I}}+\varphi_2+\phi_2-\mathrm{i}\ln\rho_{c_2})}{\sqrt{A_{22}}\cosh(\zeta_{2\mathrm{R}}+\ln(\sqrt{A_{22}}\rho_2\rho_{12}))+\cos(\zeta_{2\mathrm{I}}+\varphi_2+\phi_2)}. \tag{49}$$

From the above asymptotic analysis, we observe that the two breathers undergo an elastic collision with only changes in the phases of the two breathers. Specifically, the first breather has the phase shifts $(-2\ln\rho_{12},-2\phi_1)$ and the second one has the phase shifts $(2\ln\rho_{12},2\phi_2)$. Furthermore, when $\frac{\alpha_{12}}{\alpha_{11}}\neq\frac{\alpha_{22}}{\alpha_{21}}$, the breathers exhibit an oblique interaction. Conversely, when $\frac{\alpha_{12}}{\alpha_{11}}=\frac{\alpha_{22}}{\alpha_{21}}$, the breathers display a parallel interaction. Figures 3 and 4 exhibit the interactions among two breathers and three breathers by selecting specific parameters, respectively.

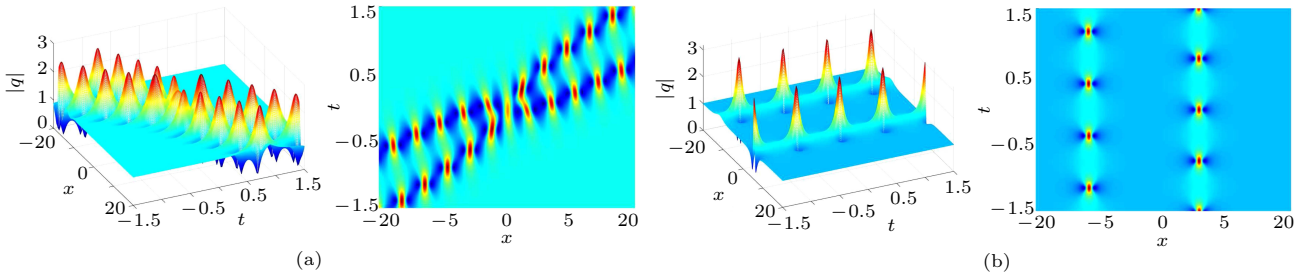


Fig. 3. (a) An oblique interaction between two breathers with parameters $k=1.5$, $p_1^{[1]}=1-\mathrm{i}$, $p_2^{[1]}=1-0.85\mathrm{i}$, $\xi_{10}=0$. (b) A parallel interaction between two breathers with parameters of $k=1.5$, $p_1^{[1]}=1.5$, $p_2^{[1]}=1.51$, $\xi_{10}=5$.

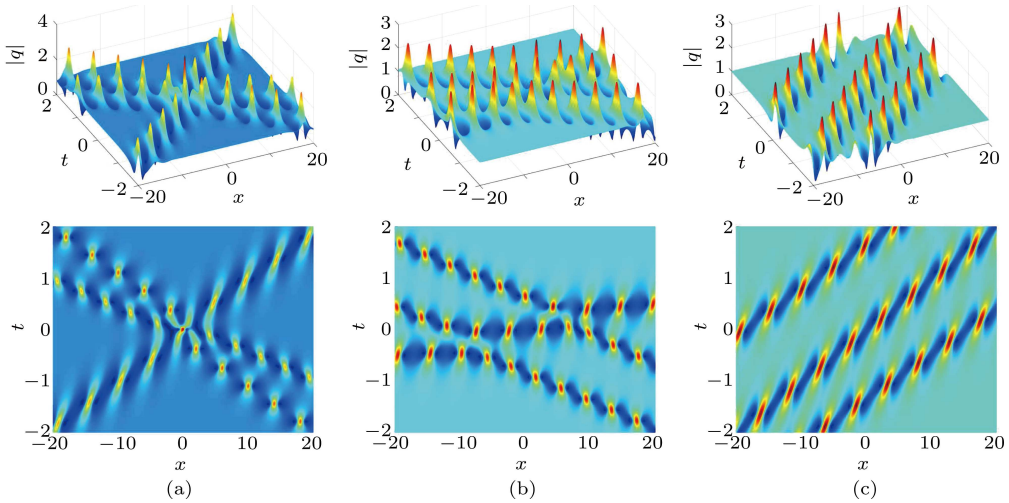


Fig. 4. 3-breather solution with parameters $k=1$, $\xi_{20}=0$, $\xi_{30}=0$ and (a) $p_1^{[1]}=1-\mathrm{i}$, $p_2^{[1]}=1+\mathrm{i}$, $p_3^{[1]}=1.2+0.75\mathrm{i}$, $\xi_{10}=0$; (b) $p_1^{[1]}=1+\mathrm{i}$, $p_2^{[1]}=1+1.01\mathrm{i}$, $p_3^{[1]}=0.8-0.65\mathrm{i}$, $\xi_{10}=2$; (c) $p_1^{[1]}=1-\mathrm{i}$, $p_2^{[1]}=1-1.1\mathrm{i}$, $p_2^{[1]}=1-1.01\mathrm{i}$, $\xi_{10}=2$.

5. The hybrid solution of breather and rogue wave for the complex mKdV equation

In this section, we construct the hybrid solutions of breather and rogue wave for the complex mKdV equation (3) by taking the long-wave limit to the breather solutions. To achieve this goal, we first obtain the fundamental rogue wave and second-order rogue wave solutions from the 1-breather and 2-breather solutions, respectively.

5.1. Rogue wave solutions for the complex mKdV equation

The long-wave limit method for obtaining rogue wave solutions to a nonlinear wave equation from its breather solutions relies on the flexibility of choosing the phase constants.[38] In Theorem 1, we have obtained the N-breather solution (17) by using the KP hierarchy reduction method. The N-th-order rogue wave solution can be derived from N-breather solution (17) by taking the long-wave limits $p_i^{[1]}$, $p_i^{[2]}\to 1$ $(1\leqslant i\leqslant N)$. By taking $p_i^{[1]}=\mathrm{e}^{\lambda_i}$, $p_i^{[1]^*}=\mathrm{e}^{\lambda_i{}^*}$, $p_i^{[2]}=\mathrm{e}^{-\lambda_i}$, $p_i^{[2]^*}=\mathrm{e}^{-\lambda_i{}^*}$, the determinant elements $\overline{m}_{i,j}^{(n)}$ $(1\leqslant i,j\leqslant N)$ of τ_n function in N-breather solution (19) are represented by

$$\begin{aligned}\overline{m}_{i,j}^{(n)} &= (-1)^n\left(\frac{1}{\mathrm{e}^{\lambda_i}+\mathrm{e}^{\lambda_j{}^*}}\mathrm{e}^{\zeta_i+\zeta_j^*+n(\lambda_i-\lambda_j^*)}\right.\\ &\quad+\frac{1}{\mathrm{e}^{\lambda_i}+\mathrm{e}^{-\lambda_j{}^*}}\mathrm{e}^{\zeta_i+n(\lambda_i+\lambda_j^*)}\\ &\quad+\frac{1}{\mathrm{e}^{-\lambda_i}+\mathrm{e}^{\lambda_j{}^*}}\mathrm{e}^{\zeta_j^*+n(-\lambda_i-\lambda_j^*)}\\ &\quad\left.+\frac{1}{\mathrm{e}^{-\lambda_i}+\mathrm{e}^{-\lambda_j{}^*}}\mathrm{e}^{n(-\lambda_i+\lambda_j^*)}\right),\end{aligned}\tag{50}$$

where

$$\begin{aligned}\zeta_i &= 2\sinh(\lambda_i)x+[6(k^2-1)\sinh(\lambda_i)\\ &\quad-6k\mathrm{i}(\sinh(2\lambda_i)-\sinh(3\lambda_i))]t+\xi_{i0}.\end{aligned}\tag{51}$$

When $N=1$ in Eq. (17), 1-breather solution can be expressed as

$$q_1 = \mathrm{e}^{\mathrm{i}[k(x-6t)+k^3t]-\lambda_1+\lambda_1{}^*}\frac{f_1}{g_1},\tag{52}$$

$$f_1 = 1+\mathrm{e}^{\zeta_1+\zeta_1{}^*+\theta_1}+\mathrm{e}^{\zeta_1+\theta_2}+\mathrm{e}^{\zeta_1{}^*+\theta_2^*},\tag{53}$$

$$\begin{aligned}g_1 &= 1+\mathrm{e}^{\zeta_1+\zeta_1{}^*+2(\lambda_1-\lambda_1{}^*)+\theta_1}\\ &\quad+\mathrm{e}^{\zeta_1+2\lambda_1+\theta_2}+\mathrm{e}^{\zeta_1{}^*-2\lambda_1{}^*+\theta_2{}^*},\end{aligned}\tag{54}$$

in which

$$\frac{\mathrm{e}^{-\lambda_1}+\mathrm{e}^{-\lambda_1{}^*}}{\mathrm{e}^{\lambda_1}+\mathrm{e}^{\lambda_1{}^*}}=\mathrm{e}^{\theta_1},\quad \frac{\mathrm{e}^{-\lambda_1}+\mathrm{e}^{-\lambda_1{}^*}}{\mathrm{e}^{\lambda_1}+\mathrm{e}^{-\lambda_1{}^*}}=\mathrm{e}^{\theta_2},\tag{55}$$

$$\begin{aligned}\zeta_1 &= 2\sinh(\lambda_1)x+[6(k^2-1)\sinh(\lambda_1)\\ &\quad-6k\mathrm{i}(\sinh(2\lambda_1)-\sinh(3\lambda_1))]t+\xi_{10}.\end{aligned}\tag{56}$$

Taking $\xi_{10}=\ln(-1)+c_1\lambda_1$, we can rewrite Eqs. (53)–(54) as

$$f_1 = 1+\mathrm{e}^{\xi_1+\xi_1^*+2(\lambda_1-\lambda_1^*)}-\mathrm{e}^{\xi_1+2\lambda_1}-\mathrm{e}^{\xi_1^*+2\lambda_1^*},$$

$$g_1 = 1+\mathrm{e}^{\xi_1+\xi_1^*+2(\lambda_1-\lambda_1^*)+\theta_1} + \mathrm{e}^{\xi_1+2\lambda_1+\theta_2} - \mathrm{e}^{\xi_1^*-2\lambda_1^*+\theta_2^*}, \tag{57}$$

where

$$\xi_1 = 2\sinh(\lambda_1)x+[6(k^2-1)\sinh(\lambda_1) - 6k\mathrm{i}(\sinh(2\lambda_1)-\sinh(3\lambda_1))]t+c_1\lambda_1, \tag{58}$$

in which c_1 is a complex constant. By taking the long-wave limit $\lambda_1\to 0, \lambda_1^*\to 0$ on Eq. (57), we have

$$f_1 = f_1^{(1)}\lambda_1\lambda_1^* + o(\lambda_1\lambda_1^*), \tag{59}$$

$$g_1 = g_1^{(1)}\lambda_1\lambda_1^* + o(\lambda_1\lambda_1^*), \tag{60}$$

where

$$f_1^{(1)} = \frac{1}{2}|\psi-1|^2+\frac{1}{2}, \tag{61}$$

$$g_1^{(1)} = \frac{1}{2}(\psi-3)(\psi^*+1)+\frac{1}{2}, \tag{62}$$

$$\psi = 2x+6\left(k^2+2k\mathrm{i}-2\right)t+c_1. \tag{63}$$

By taking the coefficient of the leading order terms in Eqs. (59) and (60), we derive the following fundamental rogue wave solution for complex mKdV equation (3):

$$q_1 = \mathrm{e}^{\mathrm{i}(k(x-6t)+k^3t)}\frac{g_1^{(1)}}{f_1^{(1)}}. \tag{64}$$

We can determine the extremum points of the fundamental rogue wave solution (64) as follows:

$$(x_{\max},t_{\max}) = \left(\frac{c_{1\mathrm{I}}k^2-2c_{1\mathrm{R}}k-2c_{1\mathrm{I}}+2k}{4k}, \frac{c_{1\mathrm{I}}}{12k}\right), \tag{65}$$

$$(x_{\min},t_{\min}) = \left(\frac{c_{1\mathrm{I}}k^2-2c_{1\mathrm{R}}k-2c_{1\mathrm{I}}+2k}{4k}+\frac{\sqrt{3}k}{2}, \frac{c_{1\mathrm{I}}}{12k}\right), \left(\frac{c_{1\mathrm{I}}k^2-2c_{1\mathrm{R}}k-2c_{1\mathrm{I}}+2k}{4k}-\frac{\sqrt{3}k}{2}, \frac{c_{1\mathrm{I}}}{12k}\right), \tag{66}$$

where $c_{1\mathrm{R}}$ and $c_{1\mathrm{I}}$ are the real and imaginary parts of parameter c_1, respectively. The fundamental rogue wave solution (64) is displayed in Fig. 5 by selecting parameters $k=1$ and $c_1=0$. Due to the complexity of the calculations involved in the process of the long-wave limits, the subsequent Taylor expansions are supported by mathematical software.

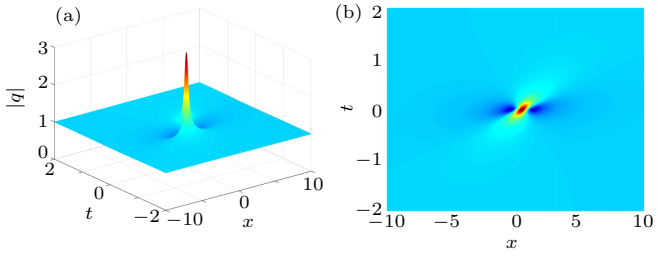


Fig. 5. (a) The three-dimensional (3D) plot of the fundamental rogue wave solution with parameters $k=1$ and $c_1=0$. (b) The corresponding density plot of panel (a).

We now proceed to derive the second-order rogue wave solution from the 2-breather solution (41). In Eq. (41), we take

$$p_1^{[1]} = \mathrm{e}^{\lambda_1}, \quad p_1^{[1]^*} = \mathrm{e}^{\lambda_1^*}, \quad p_1^{[2]} = \mathrm{e}^{-\lambda_1}, \quad p_1^{[2]^*} = \mathrm{e}^{-\lambda_1^*},$$
$$p_2^{[1]} = \mathrm{e}^{\lambda_2}, \quad p_2^{[1]^*} = \mathrm{e}^{\lambda_2^*}, \quad p_2^{[2]} = \mathrm{e}^{-\lambda_2}, \quad p_2^{[2]^*} = \mathrm{e}^{-\lambda_2^*}, \tag{67}$$
$$\xi_{10} = \ln(-1)+c_1\lambda_1,$$
$$\xi_{20} = \ln(-1)+c_1\lambda_2+c_3\lambda_2^3, \tag{68}$$

where c_1 and c_3 are complex constants. By performing Taylor expansions for f_2, g_2 with respect to λ_1, λ_1^* as $\lambda_1\to 0, \lambda_1^*\to 0$, we obtain

$$f_2 = f_2^{(1)}\lambda_1\lambda_1^* + o(\lambda_1\lambda_1^*), \tag{69}$$

$$g_2 = g_2^{(1)}\lambda_1\lambda_1^* + o(\lambda_1\lambda_1^*). \tag{70}$$

The coefficient of the leading order terms in Eqs. (69) and (70) are $f_2^{(1)}$ and $g_2^{(1)}$, respectively. Furthermore, by expanding $f_2^{(1)}$, $g_2^{(1)}$ in terms of the Taylor series of λ_2, λ_2^* as $\lambda_2\to 0$, $\lambda_2^*\to 0$, we have

$$f_2^{(1)} = f_2^{(2)}{\lambda_2}^3{\lambda_2^*}^3 + o(\lambda_2^3\lambda_2^{*3}), \tag{71}$$

$$g_2^{(1)} = g_2^{(2)}{\lambda_2}^3{\lambda_2^*}^3 + o(\lambda_2^3\lambda_2^{*3}), \tag{72}$$

where

$$f_2^{(2)} = \left(\frac{1}{576}\psi^3-\frac{1}{192}\psi^2+\frac{1}{288}\psi-\frac{1}{48}\psi_1\right)\psi^{*3} + \left(-\frac{1}{192}\psi^3+\frac{1}{32}\psi^2-\frac{1}{24}\psi+\frac{1}{16}\psi_1+\frac{1}{64}\right)\psi^{*2} + \left(\frac{1}{288}\psi^3-\frac{1}{24}\psi^2+\frac{29}{288}\psi-\frac{1}{24}\psi_1-\frac{1}{16}\right)\psi^* + \left(-\frac{1}{48}\psi^3+\frac{1}{16}\psi^2-\frac{1}{24}\psi+\frac{1}{4}\psi_1\right){\psi_1}^* + \frac{\psi^2}{64}-\frac{\psi}{16}+\frac{1}{16}, \tag{73}$$

$$g_2^{(2)} = \left(\frac{1}{576}\psi^3-\frac{1}{64}\psi^2+\frac{13}{288}\psi-\frac{1}{48}\psi^*-\frac{1}{24}\right)\psi^{*3} + \left(\frac{1}{192}\psi^3-\frac{1}{32}\psi^2+\frac{1}{24}\psi-\frac{1}{16}\psi^*+\frac{1}{64}\right)\psi^{*2} + \left(\frac{1}{288}\psi^3-\frac{19}{288}\psi-\frac{1}{24}\psi^*+\frac{5}{48}\right)\psi^* + \left(-\frac{1}{48}\psi^3+\frac{3}{16}\psi^2-\frac{13}{24}\psi+\frac{1}{4}\psi^*+\frac{1}{2}\right)\psi_1^* + \frac{\psi^2}{64}-\frac{\psi}{16}+\frac{1}{16}, \tag{74}$$

$$\psi_1 = \frac{1}{3}x+\left(k^2+8k\mathrm{i}-10\right)t+c_3, \tag{75}$$

ψ is given in Eq. (63). Therefore, the second-order rogue wave solution for the complex mKdV equation (3) is given by

$$q_2 = \mathrm{e}^{\mathrm{i}(k(x-6t)+k^3t)}\frac{g_2^{(2)}}{f_2^{(2)}}. \tag{76}$$

We display the second-order rogue wave solution (76) in Fig. 6 by selecting specific parameters.

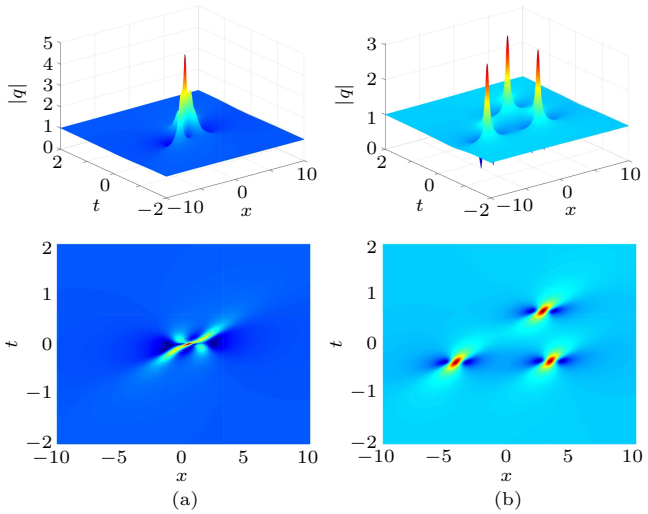


Fig. 6. The second-order rogue wave solution with parameters (a) $k=1$, $c_1=0$, $c_3=0$; (b) $k=1$, $c_1=0$, $c_3=50\mathrm{i}$.

5.2. The hybrid solutions of breather and rogue wave for complex mKdV equation

Based on the method for obtaining the fundamental rogue wave solution to the complex mKdV equation (3) from 1-breather solution, we can obtain the hybrid solution of a breather and a fundamental rogue wave from 2-breather solution. In the 2-breather solutions f_2 and g_2 of Eq. (41), we set $p_1^{[1]}=\mathrm{e}^{\lambda_1}$, $p_1^{[1]^*}=\mathrm{e}^{\lambda_1^*}$, $p_1^{[2]}=\mathrm{e}^{-\lambda_1}$, $p_1^{[2]^*}=\mathrm{e}^{-\lambda_1^*}$, and $\xi_{10}=\ln(-1)+c_1\lambda_1$, while treating $p_2^{[1]}$, $p_2^{[2]}$, and ξ_{20} as arbitrary complex parameters. By expanding f_2 and g_2 in terms of Taylor series of λ_1, λ_1^* as $\lambda_1\to 0$, $\lambda_1^*\to 0$, we have

$$f_2=\widehat{f}_2^{(1)}\lambda_1\lambda_1^*+o(\lambda_1\lambda_1^*), \tag{77}$$

$$g_2=\widehat{g}_2^{(1)}\lambda_1\lambda_1^*+o(\lambda_1\lambda_1^*), \tag{78}$$

where

$$\begin{aligned}\widehat{f}_2^{(1)}&=\overline{m}_{2,2}^{(0)}f_1^{(1)}-\Lambda_2\Lambda_2^*,\\ \widehat{g}_2^{(1)}&=-\overline{m}_{2,2}^{(1)}g_1^{(1)}+\Gamma_2\widehat{\Gamma}_2,\end{aligned} \tag{79}$$

and $\overline{m}_{i,j}^{(n)}$ $(n=0,1)$ is defined in Eq. (20), $f_1^{(1)}$ and $g_1^{(1)}$ are given in Eqs. (61) and (62), Λ_i, Γ_i, $\widehat{\Gamma}_i$ $(i,j=1,2,3)$ are defined as follows:

$$\Lambda_i=\frac{\psi\mathrm{e}^{\zeta_i}}{p_i^{[1]}+1}+\frac{\psi}{p_i^{[2]}+1}-\frac{2\mathrm{e}^{\zeta_i}}{(p_i^{[1]}+1)^2}-\frac{2}{(p_i^{[2]}+1)^2}, \tag{80}$$

$$\begin{aligned}\Gamma_i=&\frac{\mathrm{e}^{\zeta_i}(2-\psi)}{p_i^{[2]}\left(p_i^{[1]}+1\right)}+\frac{2-\psi}{p_i^{[1]}\left(p_i^{[2]}+1\right)}\\&+\frac{2\mathrm{e}^{\zeta_i}}{p_i^{[2]}\left(p_i^{[1]}+1\right)^2}+\frac{2}{p_i^{[1]}\left(p_i^{[2]}+1\right)^2},\end{aligned} \tag{81}$$

$$\begin{aligned}\widehat{\Gamma}_i=&\frac{\mathrm{e}^{\zeta_i^*}(\psi^*+2)}{p_i^{[1]^*}\left(p_i^{[1]^*}+1\right)}+\frac{\psi^*+2}{p_i^{[2]^*}\left(p_i^{[2]^*}+1\right)}\\&-\frac{2\mathrm{e}^{\zeta_i^*}}{p_i^{[1]^*}\left(p_i^{[1]^*}+1\right)^2}-\frac{2}{p_i^{[2]^*}\left(p_i^{[2]^*}+1\right)^2},\end{aligned} \tag{82}$$

where $p_i^{[\alpha]}$ $(\alpha=1,2)$, ζ_i are given in Eqs. (21) and (22), and ψ is given in Eq. (63). Therefore, we obtain the hybrid solution of a breather and a fundamental rogue wave

$$q=\mathrm{e}^{\mathrm{i}(k(x-6t)+k^3t)}\frac{\widehat{g}_2^{(1)}}{\widehat{f}_2^{(1)}}. \tag{83}$$

To more significantly elucidate the interaction between a breather and a fundamental rogue wave in the hybrid solutions, we consider the asymptotic behavior of the obtained hybrid solution (83). Equation (65) shows that the position of the fundamental rogue wave is controlled by the real and imaginary parts of the parameter c_1, while the phase of the breather is determined by the phase parameter ξ_{20}. For the asymptotic analysis of breather, $\zeta_{2\mathrm{R}}=\alpha_{21}x+\alpha_{22}t+\xi_{20\mathrm{R}}\approx 0$, we have $\psi=2x+6\left(k^2+2k\mathrm{i}-2\right)t+c_1=\psi_\mathrm{R}+\mathrm{i}\psi_\mathrm{I}$, $\psi_\mathrm{R}\to\pm\infty$ as $c_{1\mathrm{I}}\to\pm\infty$. For the asymptotic analysis of fundamental rogue wave, $x\approx\frac{c_{1\mathrm{I}}k^2-2c_{1\mathrm{R}}k-2c_{1\mathrm{I}}+2k}{4k}$, $t\approx\frac{c_{1\mathrm{I}}}{12k}$, thus $\psi_\mathrm{R}\approx 0$, $\psi_\mathrm{I}\approx 0$. As a result, we have $\zeta_{2\mathrm{R}}\to\pm\infty$ as $\zeta_{20\mathrm{R}}\to\pm\infty$. Based on these analyses, we derive the following asymptotic forms for the hybrid solution of a breather and a fundamental rogue wave (83)

(i) Asymptotic forms of the fundamental rogue wave

(i)-1 Before collision $(\psi_\mathrm{R}\approx 0,\psi_\mathrm{I}\approx 0,\zeta_{2\mathrm{R}}\to-\infty)$

$$q\approx\mathrm{e}^{\mathrm{i}(k(x-6t)+k^3t)}\frac{p_2^{[1]^*}}{p_2^{[1]}}\frac{(\psi+\Delta-3)(\psi^*+\Delta^*+1)+1}{|\psi+\Delta-1|^2+1}. \tag{84}$$

(i)-2 After collision $(\psi_\mathrm{R}\approx 0,\psi_\mathrm{I}\approx 0,\zeta_{2\mathrm{R}}\to+\infty)$

$$q\approx\mathrm{e}^{\mathrm{i}(k(x-6t)+k^3t)}\frac{p_2^{[1]}}{p_2^{[1]^*}}\frac{(\psi-\Delta-3)(\psi^*-\Delta^*+1)+1}{|\psi-\Delta-1|^2+1}. \tag{85}$$

(ii) Asymptotic forms of the breather

(ii)-1 Before collision $(\zeta_{2\mathrm{R}}\approx 0,\psi_\mathrm{I}\to-\infty)$

$$q\approx\widehat{A}\cdot\frac{\sqrt{A_{22}}\cosh(\zeta_{2\mathrm{R}}+\mathrm{i}\phi_{c_2}+\ln(\sqrt{A_{22}}\rho_2))+\cos(\zeta_{2\mathrm{I}}+\varphi_2+\pi-\mathrm{i}\ln\rho_{c_2})}{\sqrt{A_{22}}\cosh(\zeta_{2\mathrm{R}}+\ln(\sqrt{A_{22}}\rho_2))+\cos(\zeta_{2\mathrm{I}}+\varphi_2+\pi)}. \tag{86}$$

(ii)-2 After collision $(\zeta_{2\mathrm{R}}\approx 0,\psi_\mathrm{I}\to+\infty)$

$$q\approx\widehat{A}\cdot\frac{\sqrt{A_{22}}\cosh(\zeta_{2\mathrm{R}}+\mathrm{i}\phi_{c_2}+\ln(\sqrt{A_{22}}\rho_2))+\cos(\zeta_{2\mathrm{I}}+\varphi_2+\pi-\mathrm{i}\ln\rho_{c_2})}{\sqrt{A_{22}}\cosh(\zeta_{2\mathrm{R}}+\ln(\sqrt{A_{22}}\rho_2))+\cos(\zeta_{2\mathrm{I}}+\varphi_2+\pi)}. \tag{87}$$

In the above asymptotic analysis, $\widehat{A}$, A_{22}, ρ_{c_2}, ϕ_{c_2}, ρ_2, φ_2 are given in Eq. (42), and the expression for Δ is given by

$$\Delta = \frac{2(p_2^{[1]} + p_2^{[1]^*})(p_2^{[1]^*} - 1)}{(p_2^{[1]^*} + 1)(|p_2^{[1]^*} + 1|^2 - 2(p_2^{[1]} + p_2^{[1]^*}))}. \tag{88}$$

From the above asymptotic analysis, it can be observed that when the phase parameter ξ_{20} of the breather is altered while keeping the parameter c_1 of the fundamental rogue wave fixed, the fundamental rogue wave undergoes a phase shift of 2Δ before and after its interaction with the breather. Conversely, when the parameter c_1 of the fundamental rogue wave is altered while keeping the phase parameter ξ_{20} fixed, the breather remains unchanged before and after its interaction with the rogue wave.

Figures 7 and 8 illustrate the interaction between a fundamental rogue wave and a breather. In Fig. 7, the value of the parameter c_1 for the rogue wave is kept constant, while the phase parameter of the breather is varied. When $\xi_{20} < 0$, the Akhmediev breather is positioned on the positive time axis. When $\xi_{20} > 0$, the Akhmediev breather is positioned on the negative time axis. As the value of ξ_{20} increases, the breather moves along the negative direction of the t-axis. When $\xi_{20} = 0$, the breather merges with the fundamental rogue wave. The maximum points $(x_{\max}, t_{\max})$ of the fundamental rogue wave in Figs. 7(a) and 7(c) are $(\frac{1}{2} - \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{18})$ and $(\frac{1}{2} + \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{18})$, respectively. Therefore, the fundamental rogue wave undergoes a phase shift after its interaction with the breather. In Fig. 8, the phase parameter ξ_{20} of the breather remains unchanged, while the value of the parameter $c_{1\mathrm{I}}$ for the fundamental rogue wave in Eq. (83) is varied. As the value of $c_{1\mathrm{I}}$ decreases, the fundamental rogue wave moves along the negative direction of the t-axis and collides with the Akhmediev breather when $c_1 = 0$. Meanwhile, the shape and phase of the Akhmediev breather remain unchanged after its interaction with the fundamental rogue wave.

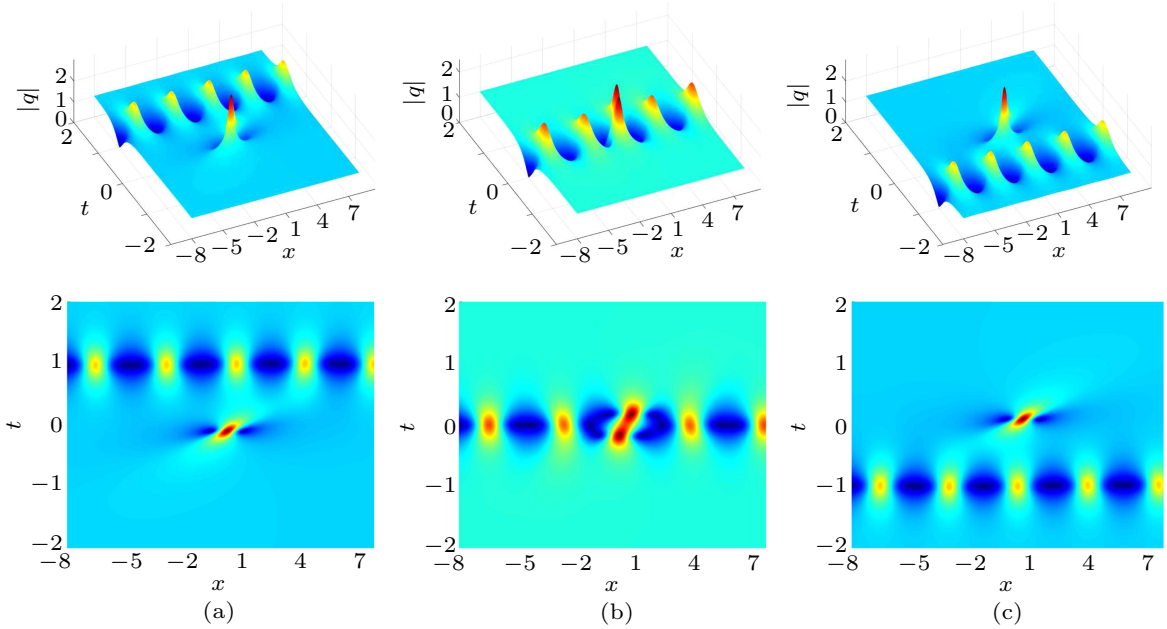


Fig. 7. The interaction between a breather and a fundamental rogue wave with parameters $k = 1$, $p_2^{[1]} = \mathrm{e}^{\mathrm{i}\pi/3}$, $c_1 = 0$ and (a) $\xi_{20} = -5$, (b) $\xi_{20} = 0$, (c) $\xi_{20} = 5$.

Similarly, we can obtain the hybrid solution of two breathers and a fundamental rogue wave from 3-breather solution. We take $p_1^{[1]} = \mathrm{e}^{\lambda_1}$, $p_1^{[1]^*} = \mathrm{e}^{\lambda_1^*}$, $p_1^{[2]} = \mathrm{e}^{-\lambda_1}$, $p_1^{[2]^*} = \mathrm{e}^{-\lambda_1^*}$, $\xi_{10} = \ln(-1) + c_1\lambda_1$ and $p_2^{[1]}$, $p_2^{[2]}$, ξ_{20}, $p_3^{[1]}$, $p_3^{[2]}$, ξ_{30}, as arbitrary complex parameters in 3-breather solution f_3, g_3 given by Eq. (18) with $N = 3$. The Taylor series expansions of f_3 and g_3 in terms of λ_1, λ_1^* as $\lambda_1 \to 0$, $\lambda_1^* \to 0$ give

$$\begin{aligned} f_3 &= f_3^{(1)}\lambda_1\lambda_1^* + o(\lambda_1\lambda_1^*), \\ g_3 &= g_3^{(1)}\lambda_1\lambda_1^* + o(\lambda_1\lambda_1^*), \end{aligned} \tag{89}$$

where

$$\begin{aligned} f_3^{(1)} &= f_1^{(1)}\left(\overline{m}_{2,2}^{(0)}\overline{m}_{3,3}^{(0)} - \overline{m}_{2,3}^{(0)}\overline{m}_{3,2}^{(0)}\right) + \overline{m}_{2,3}^{(0)}\Lambda_2^*\Lambda_3 \\ &\quad + \overline{m}_{3,2}^{(0)}\Lambda_2\Lambda_3^* - \overline{m}_{3,3}^{(0)}\Lambda_2\Lambda_2^* - \overline{m}_{2,2}^{(0)}\Lambda_3\Lambda_3^*, \end{aligned} \tag{90}$$

$$\begin{aligned} g_3^{(1)} &= g_1^{(1)}\left(\overline{m}_{2,2}^{(1)}\overline{m}_{3,3}^{(1)} - \overline{m}_{2,3}^{(1)}\overline{m}_{3,2}^{(1)}\right) + \overline{m}_{2,3}^{(1)}\widehat{\Gamma}_2\Gamma_3 \\ &\quad + \overline{m}_{3,2}^{(1)}\Gamma_2\widehat{\Gamma}_3 - \overline{m}_{3,3}^{(1)}\Gamma_2\widehat{\Gamma}_2 - \overline{m}_{2,2}^{(1)}\Gamma_3\widehat{\Gamma}_3, \end{aligned} \tag{91}$$

and $\overline{m}_{i,j}^{(n)}$ $(n = 0, 1)$ are given in Eq. (20), Λ_i, Γ_i, $\widehat{\Gamma}_i$ are given in Eqs. (80)–(82). Therefore, we obtain the hybrid solution of two breathers and a fundamental rogue wave for the complex mKdV equation (3)

$$q = \mathrm{e}^{\mathrm{i}(k(x-6t)+k^3t)}\frac{g_3^{(1)}}{f_3^{(1)}}. \tag{92}$$

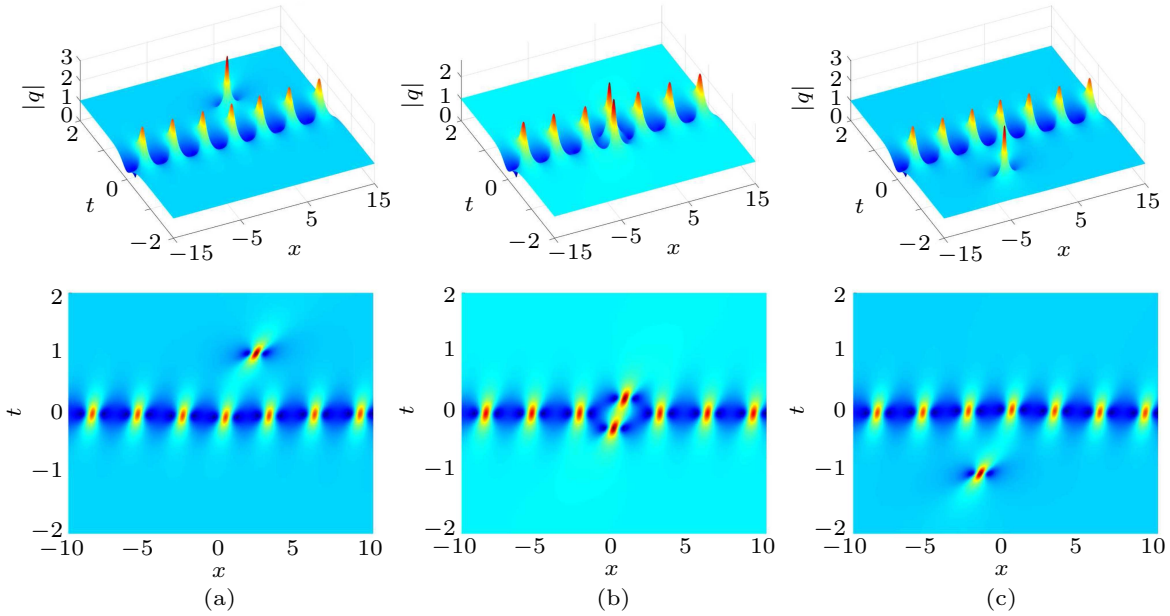


Fig. 8. The interaction between a breather and a fundamental rogue wave with parameters $k = 1$, $p_2^{[1]} = e^{i\pi/4}$, $\xi_{20} = 0$ and (a) $c_1 = 10i$, (b) $c_1 = 0$, (c) $c_1 = -10i$.

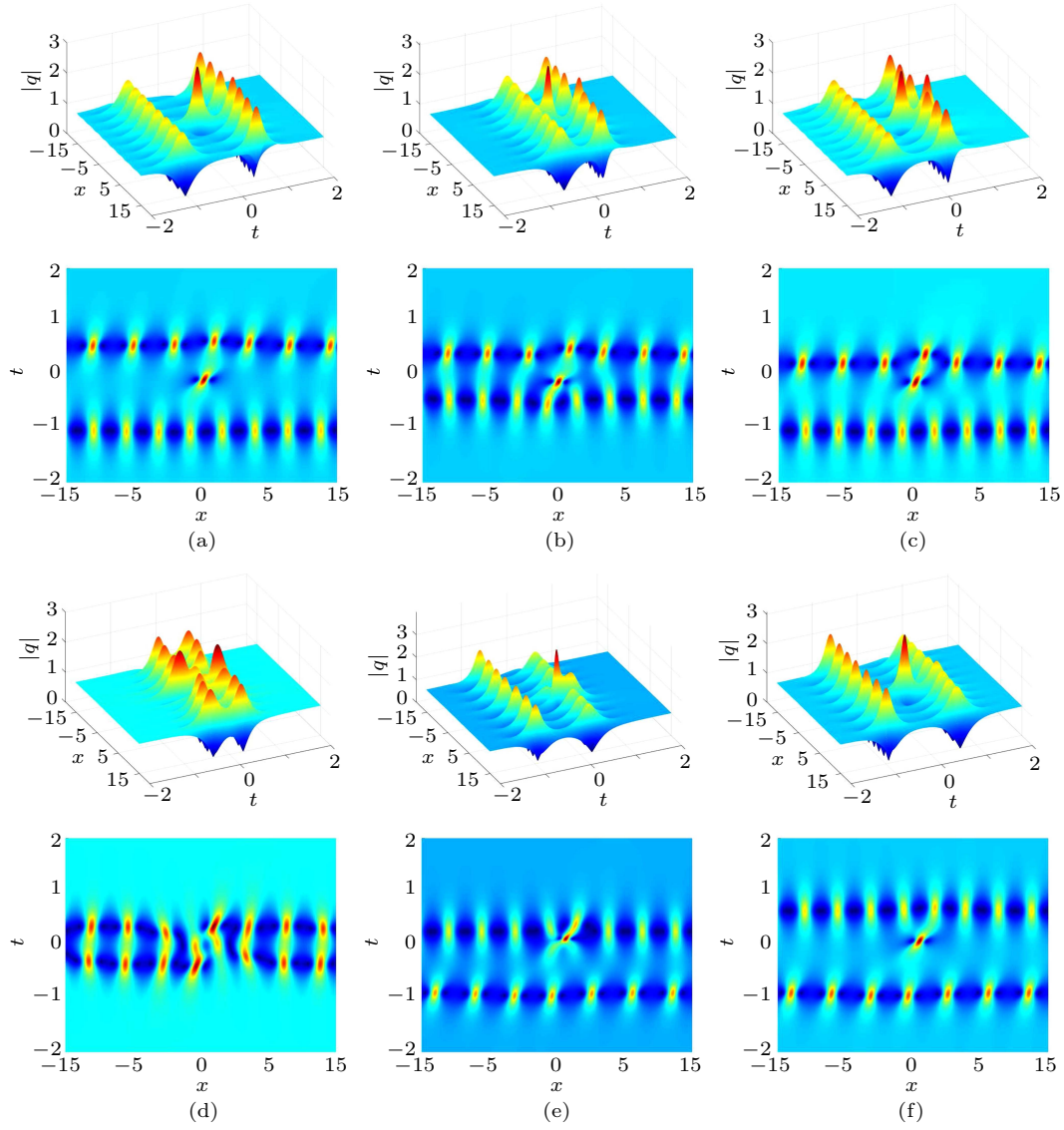


Fig. 9. The interaction between two breathers and a fundamental rogue wave with parameters $k = 1$, $p_2^{[1]} = e^{i\pi/3}$, $p_3^{[1]} = e^{i\pi/4}$, $c_1 = 0$ and (a) $\xi_{20} = 4$, $\xi_{30} = -2$; (b) $\xi_{20} = 1$, $\xi_{30} = -1$; (c) $\xi_{20} = 4$, $\xi_{30} = 0$; (d) $\xi_{20} = 0$, $\xi_{30} = 0$; (e) $\xi_{20} = 0$, $\xi_{30} = 4$; (f) $\xi_{20} = -2$, $\xi_{30} = 4$.

Figure 9 shows the interaction between a rogue wave and two breathers by choosing different values for the phase parameters ξ_{20}, ξ_{30} of the breathers. In Fig. 9, as parameters $|\xi_{20}|$ and $|\xi_{30}|$ decrease, two Akhmediev breathers approach the rogue wave, and when $\xi_{20}=\xi_{30}=0$, the Akhmediev breathers merge with the rogue wave. Furthermore, when the parameter ξ_{20} decreases from 4 to -2 and the parameter ξ_{30} increases from -2 to 4, the two breathers pass through the rogue wave and exchange positions. But their shapes remain unchanged after the collisions.

To obtain the hybrid solution of a breather and a second-order rogue wave for the complex mKdV equation (3), we follow the procedure of deriving second order rogue wave solution from 2-breather solution. We take $p_1^{[1]}=\mathrm{e}^{\lambda_1}$, $p_1^{[1]^*}=\mathrm{e}^{\lambda_1^*}$, $p_1^{[2]}=\mathrm{e}^{-\lambda_1}$, $p_1^{[2]^*}=\mathrm{e}^{-\lambda_1^*}$, $p_2^{[1]}=\mathrm{e}^{\lambda_2}$, $p_2^{[1]^*}=\mathrm{e}^{\lambda_2^*}$, $p_2^{[2]}=\mathrm{e}^{-\lambda_2}$, $p_2^{[2]^*}=\mathrm{e}^{-\lambda_2^*}$, $\xi_{10}=\ln(-1)+c_1\lambda_1$, $\xi_{20}=\ln(-1)+c_1\lambda_2+c_3\lambda_2^3$ and $p_3^{[1]}$, $p_3^{[2]}$, ξ_{30} as arbitrary complex parameters in the 3-breather solution f_3, g_3 from Eq. (18) with $N=3$. The expansion of 3-breather solution f_3 and g_3 in terms of the Taylor series of λ_1, λ_1^* as $\lambda_1\to 0,\lambda_1^*\to 0$ gives

$$f_3=\widehat{f}_3^{(1)}\lambda_1\lambda_1^*+o(\lambda_1\lambda_1^*), \tag{93}$$

$$g_3=\widehat{g}_3^{(1)}\lambda_1\lambda_1^*+o(\lambda_1\lambda_1^*). \tag{94}$$

The coefficients of the leading order terms in Eqs. (93) and (94) are $\widehat{f}_3^{(1)}$ and $\widehat{g}_3^{(1)}$, respectively. Furthermore, by expanding $\widehat{f}_3^{(1)}$, $\widehat{g}_3^{(1)}$ in terms of the Taylor series of λ_2, λ_2^* as $\lambda_2\to 0$, $\lambda_2^*\to 0$, we have

$$\widehat{f}_3^{(1)}=f_3^{(2)}{\lambda_2}^3{\lambda_2^*}^3+o(\lambda_2^3\lambda_2^{*3}), \tag{95}$$

$$\widehat{g}_3^{(1)}=g_3^{(2)}{\lambda_2}^3{\lambda_2^*}^3+o(\lambda_2^3\lambda_2^{*3}), \tag{96}$$

where

$$\begin{aligned} f_3^{(2)} &= -m_{3,3}^{(0)}f_2^{(2)}+\Lambda_3(\psi_2\Lambda_3^*-\psi_3^*\widehat{\Omega}^*)\\ &\quad+\widehat{\Omega}(\widehat{\Omega}^*f_1^{(1)}-\psi_3\Lambda_3^*), \end{aligned} \tag{97}$$

$$\begin{aligned} g_3^{(2)} &= -m_{3,3}^{(1)}g_2^{(2)}+\Gamma_3(\psi_4\Omega^*-\psi_6\widehat{\Gamma}_3)\\ &\quad+\Omega(\Omega^*g_2^{(2)}-\psi_5\widehat{\Gamma}_3), \end{aligned} \tag{98}$$

and

$$\begin{aligned}
\psi_2 &= \left(-\frac{1}{72}\psi^3+\frac{1}{48}\psi^2-\frac{1}{12}\psi_1-\frac{1}{144}\right)\psi^{*3}+\left(\frac{1}{48}\psi^3-\frac{1}{16}\psi^2+\frac{1}{32}\psi+\frac{1}{8}\psi_1+\frac{1}{48}\right)\psi^{*2}\\
&\quad+\left(-\frac{1}{12}\psi^3+\frac{1}{8}\psi^2-\frac{1}{2}\psi_1-\frac{1}{24}\right)\psi_1^*+\left(\frac{1}{32}\psi^2-\frac{1}{16}\psi+\frac{1}{48}\right)\psi^*-\frac{\psi^3}{144}+\frac{\psi^2}{48}+\frac{\psi}{48}-\frac{\psi_1}{24}-\frac{5}{72},\\
\psi_3 &= \left(-\frac{1}{12}\psi+\frac{1}{12}\right)\psi^{*3}+\left(-\frac{1}{4}+\frac{1}{8}\psi\right)\psi^{*2}+\left(-\frac{1}{24}-\frac{1}{2}\psi_1^*\right)\psi+\frac{1}{8}\psi^*+\frac{1}{2}\psi_1^*+\frac{1}{12},\\
\psi_4 &= \left(\frac{1}{12}\psi^3-\frac{3}{8}\psi^2+\frac{1}{2}\psi+\frac{1}{2}\psi_1-\frac{3}{8}\right)\psi^*+\frac{\psi^3}{12}-\frac{\psi^2}{4}+\frac{\psi}{8}+\frac{\psi_1}{2}+\frac{1}{12},\\
\psi_5 &= \left(\frac{1}{12}\psi^{*3}+\frac{1}{8}\psi^{*2}+\frac{1}{2}\psi_1^*-\frac{1}{24}\right)\psi+\frac{\psi^{*3}}{4}-\frac{\psi^{*2}}{4}+\frac{\psi^*}{8}-\frac{3\psi_1^*}{2}+\frac{1}{12},\\
\psi_6 &= \left(\frac{1}{72}\psi^3-\frac{1}{16}\psi^2+\frac{1}{12}\psi+\frac{1}{12}\psi_1-\frac{1}{16}\right)\psi^{*3}+\left(\frac{1}{48}\psi^3-\frac{1}{16}\psi^2+\frac{1}{32}\psi+\frac{1}{8}\psi_1+\frac{1}{48}\right)\psi^{*2}\\
&\quad+\left(\frac{1}{32}\psi^2-\frac{1}{16}\psi+\frac{1}{48}\right)\psi^*+\left(\frac{1}{12}\psi^3-\frac{3}{8}\psi^2+\frac{1}{2}\psi+\frac{1}{2}\psi_1-\frac{3}{8}\right)\psi_1^*-\frac{\psi^3}{144}+\frac{\psi^2}{48}+\frac{\psi}{48}-\frac{\psi_1}{24}-\frac{5}{72},\\
\widehat{\Omega} &= \frac{1}{6(p_3^{[2]}+1)^4(p_3^{[1]}+1)^4}(((\psi^3+6\psi_1){p_3^{[1]}}^3+(3\psi^3-3\psi^2-3\psi+18\psi_1-2){p_3^{[1]}}^2\\
&\quad+(3\psi^3-6\psi^2+18\psi_1+8)p_3^{[1]}+\psi^3-3\psi^2+3\psi+6\psi_1-2)(p_3^{[2]}+1)^4\mathrm{e}^{\xi_3}+((\psi^3+6\psi_1){p_3^{[2]}}^3\\
&\quad+(3\psi^3-3\psi^2-3\psi+18\psi_1-2){p_3^{[2]}}^2+(3\psi^3-6\psi^2+18\psi_1+8)p_3^{[2]}+\psi^3-3\psi^2+3\psi+6\psi_1-2)(p_3^{[1]}+1)^4),\\
\Omega &= \frac{1}{6(p_3^{[2]}+1)^4(p_3^{[1]}+1)^4}(((\psi^3-3\psi^2+3\psi+6\psi_1-2){p_3^{[2]}}^3+(3\psi^3-12\psi^2+12\psi+18\psi_1-2){p_3^{[2]}}^2\\
&\quad+(3\psi^3-15\psi^2+21\psi+18\psi_1+2)p_3^{[2]}+\psi^3-6\psi^2+12\psi+6\psi_1-10)(p_3^{[1]}+1)^4p_3^{[2]}\\
&\quad+((\psi^3-3\psi^2+3\psi+6\psi_1-2){p_3^{[1]}}^3+(3\psi^3-12\psi^2+12\psi+18\psi_1-2){p_3^{[1]}}^2+(3\psi^3-15\psi^2+21\psi\\
&\quad+18\psi_1+2)p_3^{[1]}+\psi^3-6\psi^2+12\psi+6\psi_1-10)(p_3^{[2]}+1)^4\mathrm{e}^{\xi_3}p_3^{[1]}),
\end{aligned}$$

in which $f_1^{(1)}$ and $g_1^{(1)}$ are given in Eqs. (61) and (62), $f_2^{(2)}$ and $g_2^{(2)}$ are given in Eqs. (73)–(74). Besides, ψ_2, ψ_3, ψ_4, ψ_5, ψ_6, $f_1^{(1)}$, $g_1^{(1)}$, $f_2^{(2)}$, and $g_2^{(2)}$ have the following relations:

$$\psi_2 f_1^{(1)} - \psi_3 \psi_3^* = f_2^{(2)}, \tag{99}$$

$$\psi_6 g_1^{(1)} - \psi_4 \psi_5 = g_2^{(2)}. \tag{100}$$

Therefore, we obtain the hybrid solution of a breather and a second-order rogue wave for the complex mKdV equation (3)

$$q = \mathrm{e}^{\mathrm{i}(k(x-6t)+k^3 t)} \frac{g_3^{(2)}}{f_3^{(2)}}. \tag{101}$$

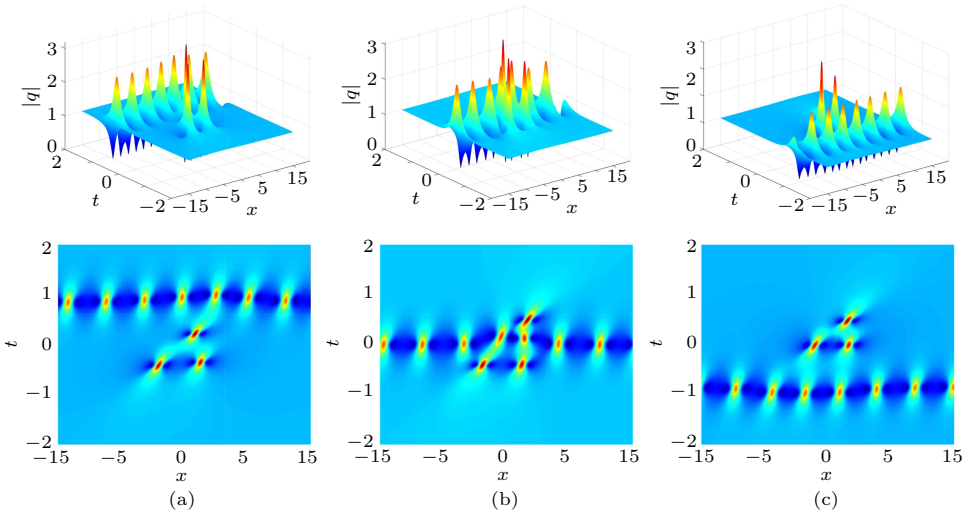


Fig. 10. The interaction between a second-order rogue wave and a breather with parameters $k = 1$, $p_3^{[1]} = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$, $c_1 = 0$, $c_3 = 10\mathrm{i}$ and (a) $\xi_{30} = -5$, (b) $\xi_{30} = 0$, (c) $\xi_{30} = 5$.

Figure 10 shows the interaction between a breather and a second-order rogue wave by altering the phase parameter ξ_{30}. As the value of ξ_{30} increases from -5 to 5, the Akhmediev breather moves in the negative direction along the t-axis, and passes through the second-order rogue wave, but their shapes do not change after the collision.

6. Conclutions

In this paper, we have derived a compact Gram determinant form for N-breather solution to the complex mKdV equation (3) using the KP hierarchy reduction method. The asymptotic behavior of the 2-breather solution has been analyzed, revealing that the two breathers undergo an elastic collision. Furthermore, the fundamental and second-order rogue wave solutions have been derived from the 1-breather solution and 2-breather solution through the generalized long-wave limit method, respectively. Applying this method to 2-breather solution and 3-breather solution, we have derived the hybrid solutions of breather and rogue wave. By managing the phase parameter of the breathers, the interaction phenomena involving a breather and a rogue wave, two breathers and a rogue wave, along with a breather and a second-order rogue wave, have been demonstrated. The generalized long-wave limit method can still be applied to construct the hybrid solutions of multiple breathers and high-order rogue wave. However, as the number of breather and order of rogue wave in the hybrid solutions increase, the demands on the computer performance become higher. The generalized long-wave limit method of constructing hybrid solutions of breather and rogue wave, which has been developed in this paper, is also applicable to other 1+1 dimensional nonlinear wave equations possessing N-breather solution in Grammian form.

Compared to the breather solutions obtained through the Darboux transformation method in Ref. [46], we provide an explicit and compact Gram determinant form for the N-breather solution, from which hybrid solutions of breather and rogue wave can be derived conveniently through the long-wave limit. The difference between the hybrid solutions in this paper and the ones in Ref. [46] is that each hybrid solution in this paper contains all free parameters from the individual rogue waves and breathers involved in this solution and reveals a rich variety of interaction phenomena between breathers and rogue waves.

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