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Uncertainties of the standard quantum teleportation channel

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From the perspective of state-channel interaction, standard quantum teleportation can be viewed as a communication process characterized by both input and output, functioning as a quantum depolarizing channel. To achieve a precise quantification of the quantumness introduced by this channel, we examine its uncertainties, which encompass both state-dependent and state-independent uncertainties. Specifically, for qudit systems, we provide general formulas for these uncertainties. We analyze the uncertainties associated with standard quantum teleportation when induced by isotropic states, Werner states, and X-states, and we elucidate the correlation between these uncertainties and the parameters of the specific mixed states. Our findings demonstrate the validity of quantifying these uncertainties.

Keywords: uncertainty, standard quantum teleportation channel, state-channel interaction

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1. Introduction

Quantum teleportation, proposed over 30 years ago by Bennett *et al.*,[1] is one of the most important and classic applications of entanglement. It encapsulates the unique allure of quantum mechanics, and has been investigated both theoretically and experimentally.[2–9] It serves as the main information transmission method in quantum internet.[10] Thus, it is important to explore quantum teleportation from different angles. In the context of communication, quantum teleportation can be viewed as a communication process with three steps: input, channel encoding and decoding, and output. As such, it can be investigated from the perspective of state-channel interaction. Notably, standard quantum teleportation (SQT for short), which solely involves Bell diagonal measurements and Pauli rotations, functions as a depolarizing quantum channel.[11]

The characteristics of quantum channels have been extensively studied, including both classical and quantum capability,[12–14] the resource content of quantum channel,[15–18] cohering and decohering power of quantum channels.[19–25] Recent research has delved into the capacity of quantum channels to generate uncertainty, particularly in terms of variance. Given a quantum state ρ and an observable H, the variance of H in ρ quantifies the total uncertainty of H in ρ. It is defined as[26]

$$V(\rho,H) := \mathrm{Tr}\rho H^2 - (\mathrm{Tr}\rho H)^2 = \mathrm{Tr}\rho H_0^2 \tag{1}$$

with $H_0 = H - \mathrm{Tr}(\rho H)$. The well-known Heisenberg uncertainty relation as follows describes a counterintuitive fact: the total uncertainty of two observables cannot precisely measured simultaneously[26,27]

$$V(\rho,X)V(\rho,Y) \geqslant \left|\frac{1}{2}\mathrm{Tr}\rho[X,Y]\right|^2, \tag{2}$$

where $[X,Y] = XY - YX$ is the commutator of observables X and Y. An improved version of Heisenberg uncertainty relation is by introducing the anti-commutator. It is known as Schrödinger uncertainty relation[28]

$$\begin{aligned} &V(\rho,X)V(\rho,Y) \\ &\geqslant \left|\frac{1}{2}\mathrm{Tr}\rho[X,Y]\right|^2 + \left|\frac{1}{2}\mathrm{Tr}\rho\{X,Y\} - (\mathrm{Tr}\rho X)(\mathrm{Tr}\rho Y)\right|^2 \\ &= \left|\frac{1}{2}\mathrm{Tr}\rho[X,Y]\right|^2 + \left|\frac{1}{2}\mathrm{Tr}\rho\{X_0,Y_0\}\right|^2, \end{aligned} \tag{3}$$

where $\{X,Y\} = XY + YX$ is the anti-commutator of X and Y.

To achieve a more precise quantification of uncertainty and establish tighter uncertainty relation inequalities, researchers have endeavored to identify a quantity less than the variance. Wigner–Yanase skew information is a good option. It is defined as the noncommutative part of an observable and a state, and is also perceived as the information content of the state extracted by the observable.[29] The mathematical expression is as follows:

$$Q(\rho,H) := -\frac{1}{2}\mathrm{Tr}[\sqrt{\rho},H]^2 = -\frac{1}{2}\mathrm{Tr}[\sqrt{\rho},H_0]^2. \tag{4}$$

Note that

$$Q(\rho,H) = V(\rho,H) - C(\rho,H) \tag{5}$$

with

$$C(\rho,H) := \mathrm{Tr}\sqrt{\rho}H_0\sqrt{\rho}H_0$$

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$$= \mathrm{Tr}(\rho^{1/4}H_0\rho^{1/4})(\rho^{1/4}H_0\rho^{1/4})^{\dagger} \geqslant 0.$$

Therefore, Wigner–Yanase skew information is smaller than the variance. However, it does not satisfy the uncertainty relation as follows:

$$Q(\rho,X)Q(\rho,Y) \geq \left|\frac{1}{2}\mathrm{Tr}\rho[X,Y]\right|^2. \tag{6}$$

Based on these observations and some nice properties of $C(\rho,H)$ (such as concavity in states), Luo proposed to define $Q(\rho,H)$ as quantum uncertainty of H in ρ, and $C(\rho,H)$ as classical uncertainty of H in ρ.[30] As a result, the variance acting as total uncertainty is divided into classical uncertainty and quantum uncertainty.[30] Moreover, note that $V(\rho,H)$ is the arithmetic mean of $Q(\rho,H)$ and $J(\rho,H)$, where $J(\rho,H)$ is defined as

$$J(\rho,H) := V(\rho,H)+C(\rho,H) = \frac{1}{2}\mathrm{Tr}\{\sqrt{\rho},H_0\}^2. \tag{7}$$

Considering the geometric mean is smaller, which is denoted as

$$\begin{aligned} U(\rho,H) :&= \sqrt{V^2(\rho,H)-[V(\rho,H)-Q(\rho,H)]^2} \\ &= \sqrt{Q(\rho,H)J(\rho,H)}, \end{aligned}$$

Luo presented an improved uncertainty relation using geometric mean as follows:[30]

$$U(\rho,X)U(\rho,Y) \geqslant \frac{1}{4}\left|\mathrm{Tr}\rho[X,Y]\right|^2. \tag{8}$$

There are extensive researches on other uncertainty relations based on Wigner–Yanase skew information and its generalizations.[31–35]

However, when the system state is one of the eigenstates of one observable, the lower bound of inequalities (2) and (3) will vanish. In this case, measuring uncertainty with the standard deviation becomes meaningless. Therefore, Deutsch[36] first proposed to use the Shannon entropy of the measurement results of the observable, and constructed an entropic uncertainty relation. Subsequently, Maassen and Uffink improved Deutsch's results.[37] Their results were generalized to multipartite system with multi-observable.[38–42]

But the above researches only focused on the uncertainty of observables in states and their relations, in which both observables and states are Hermitian. By extending the observables to non-Hermitian operators, researchers generalized the uncertainty relations to non-Hermitian operators.[43,44] Recently, the uncertainty of a quantum channel was defined and studied from two aspects: one is to follow the generalization to non-Hermitian operators and define the channel uncertainty as the sum of variances of each Kraus operator in the state,[45–49] the second is to define the channel uncertainty as the sum of noncommutativity between Kraus operators and operator basis.[50]

To explore the details of correlation conversion in quantum tasks, Fu *et al*. studied the DQC1[51] model from the perspective of state-channel interaction.[52] The DQC1 model was decomposed into three phases as coherence generating, information writing and information reading. The process was simplified as a quantum channel. The uncertainties and coherence in the DQC1 model were investigated.[52] Motivated by the ideas in Ref. [52], we try to characterize the state-dependent[46] and independent uncertainties[50] for the standard quantum teleportation[11] in this paper. We obtain concrete quantification and clear delineation of uncertainties in the standard quantum teleportation.

The rest of the paper is arranged as follows. In Section 2, we review the operator-sum expression of the standard quantum teleportation channel. In Section 3, we present the definitions of state dependent and independent uncertainties of channel. In Section 4, we give general formulas for the SQT channel uncertainties and present some specific examples. In Section 5, we conclude our paper.

2. Standard quantum teleportation as a channel

The standard quantum teleportation is a teleportation protocol that only involves operations of Bell diagonal measurements and Pauli rotations. Suppose the sender Alice and the receiver Bob share an entangled bipartite state χ, where particle 2 is distributed to Alice and 3 to Bob. Alice possesses another particle 1 in a unknown state ρ. She wants to help Bob obtain the target state ρ without sending any physical particles, only using local operation and classical communication (LOCC). To complete the task, Alice performs Bell diagonal measurements on particles 1 and 2, and sends her measurement results to Bob by classical channel, Bob preforms a recovery operation on his particle 3. The schematic diagram of standard quantum teleportation for qubit systems and its simplified version is shown in Fig. 1. In Fig. 1, BSM is simplified for Bell state measurement and P for Pauli operations.

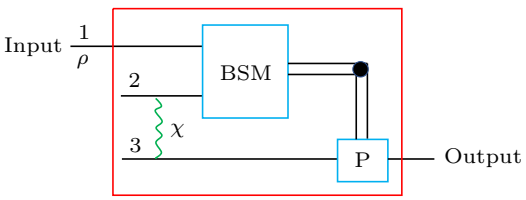


Fig. 1. Standard quantum teleportation.

From the perspective of state-channel interaction, the preparation of shared resources and LOCC operations by Alice and Bob can be viewed as channel manipulation. Consequently, the standard quantum teleportation process can be decomposed into a three-step communication framework: information input, interaction with the environment, and information output. The processes enclosed within the red dashed line in Fig. 1 can be regarded as a channel.

The mathematical description of the standard quantum teleportation is as follows. Assume that the system state of each particle is a d-dimensional Hilbert space $\mathcal{H}^d$. The state of ρ is a $d\times d$ positive matrix operated on $\mathcal{H}^d$ with $\mathrm{Tr}\rho=1$, i.e. a d-dimensional density matrix called qudit. The shared entangled state χ is a bipartite $d^2\times d^2$ density matrix operated on $\mathcal{H}^d\otimes\mathcal{H}^d$. Alice's measurement on qudits 1 and 2 is a generalized Bell measurement with measurement operators $\{E^{n,m}=(U^{n,m}\otimes I)|\Psi\rangle\langle\Psi|(U^{n,m}\otimes I)^\dagger\}$, which are defined as

$$|\Psi\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle, \tag{9}$$

$$U^{n,m}=\sum_{j=0}^{d-1}\exp\left(\frac{2\pi\mathrm{i}}{d}nj\right)|j\rangle\langle(j+m)\bmod d|. \tag{10}$$

Here $\{|i\rangle,i=0,1,\ldots,d-1\}$ is the orthonormal basis of $\mathcal{H}^d$ and $|i\rangle$ with one entry 1 on the i-th position and other entries 0. The Kraus operator-sum form of the standard quantum teleportation channel is

$$\mathcal{E}(\rho)=\sum_{n,m}p_{nm}U^{n,-m}\rho U^{n,-m^\dagger}, \tag{11}$$

where $p_{nm}=\mathrm{Tr}(\chi E^{n,m})$ is the probability of obtaining result n,m. It is a generalized depolarization channel. Especially, it is a noiseless channel when the shared resource is a Bell state.[11] Based on the operator-sum form in Eq. (11), it is obvious that the Kraus operators of the standard quantum teleportation channel is $K_{nm}=\sqrt{p_{nm}}U^{n,-m}$, satisfying $\sum_{mn}K_{mn}^\dagger K_{mn}=I$. When the shared state χ (not necessarily the maximally entangled state) and the input state (or target state) ρ are known, the theoretical output state can be obtained according to Eq. (11). This is useful when evaluating the capacity of teleportation in practice.

3. Uncertainties of quantum channel

In this section, we will present the state dependent uncertainty[46] and the state independent uncertainty.[50]

3.1. State-dependent uncertainties of quantum channel

In this subsection, we will discuss the uncertainties of SQT in terms of variance proposed by Sun and Li in 2021.[46]

For a quantum state ρ and a quantum channel $\mathcal{E}$ with the Kraus operators $\{K_i\}$ satisfying the completeness condition $\sum_i K_i^\dagger K_i=I$, the total uncertainty of $\mathcal{E}$ in ρ is defined as

$$V(\rho,\mathcal{E}):=\sum_i V(\rho,K_i), \tag{12}$$

where the effect of the channel on quantum state is $\mathcal{E}(\rho)=\sum_i K_i\rho K_i^\dagger$ and the generalized variance of the Kraus operator K_i in state ρ is

$$V(\rho,K_i):=\mathrm{Tr}\rho\frac{K_{i0}K_{i0}^\dagger+K_{i0}^\dagger K_{i0}}{2} \tag{13}$$

with $K_{i0}=K_i-\mathrm{Tr}(\rho K_i)$. By direct calculation, the total state-dependent uncertainty can be expressed as

$$V(\rho,\mathcal{E})=\frac{\mathrm{Tr}(\rho\mathcal{E}(I))+1}{2}-\sum_i|\mathrm{Tr}(\rho K_i)|^2 \tag{14}$$

with $\mathcal{E}(I)=\sum_i K_iK_i^\dagger$.

The quantum uncertainty of $\mathcal{E}$ in ρ (Ref. [46]) is defined in terms of the sum of the Wigner–Yanase skew information[29]

$$Q(\rho,\mathcal{E}):=\sum_i Q(\rho,K_i) \tag{15}$$

with

$$\begin{aligned}Q(\rho,K_i):&=\frac{1}{2}\mathrm{Tr}[\sqrt{\rho},K_{i0}][\sqrt{\rho},K_{i0}]^\dagger\\&=\frac{1}{2}\mathrm{Tr}[\sqrt{\rho},K_i][\sqrt{\rho},K_i]^\dagger.\end{aligned} \tag{16}$$

The classical uncertainty of $\mathcal{E}$ in ρ is defined as the difference between the total and quantum uncertainty as

$$C(\rho,\mathcal{E}):=V(\rho,\mathcal{E})-Q(\rho,\mathcal{E}), \tag{17}$$

or in terms of the sum of the classical uncertainty of K_i in ρ as follows:

$$C(\rho,\mathcal{E}):=\sum_i C(\rho,K_i)=\sum_i\mathrm{Tr}(\sqrt{\rho}K_{i0}\sqrt{\rho}K_{i0}^\dagger). \tag{18}$$

To eliminate the dependence on the input states, the average uncertainty of $\mathcal{E}$ is defined by taking the average over the pure state space as follows:

$$\overline{V}(\mathcal{E}):=\int V(|\phi\rangle\langle\phi|,\mathcal{E})\,\mathrm{d}\phi=\frac{d}{d+1}-\frac{\sum_i|\mathrm{Tr}K_i|^2}{d(d+1)}, \tag{19}$$

where $|\phi\rangle$ is the input pure state, $\mathrm{d}\phi$ is the normalized Haar measure over all pure states.

3.2. State–independent uncertainties of quantum channel

Recently, Luo *et al*. examined the total, classical, and quantum uncertainties generated by channels, focusing on eliminating the dependence on individual quantum states.[50] They exploited the action of a channel on an orthonormal basis of the space of observables. The total uncertainty quantifier is defined as the sum of noncommutativity between the Kraus operators of the channel and an orthonormal basis as follows:

$$\begin{aligned}T(\mathcal{E}):&=\frac{1}{d^2}\sum_{i,\alpha}\mathrm{Tr}[X_i,K_\alpha]^\dagger[X_i,K_\alpha]\\&=\frac{1}{d^2}\sum_{i,\alpha}\|[X_i,K_\alpha]\|^2,\end{aligned} \tag{20}$$

where $X_i,i=0,\ldots,d^2-1$ are Hermitian and form an orthonormal basis of the space of observables, the operator-sum form of the channel is $\mathcal{E}(\rho)=\sum_\alpha K_\alpha\rho K_\alpha{}^\dagger$. By the properties that

$\sum_i X_i^2 = dI$ and $\sum_i X_i A X_i = (\mathrm{Tr}A)I$ for any operator A (see Ref. [50] for proof), the total uncertainty can be reformed as

$$T(\mathcal{E}) = 2 - \frac{2}{d^2}\sum_i \mathrm{Tr} X_i \mathcal{E}(X_i) = 2 - \frac{2}{d^2}\sum_\alpha |\mathrm{Tr} K_\alpha|^2, \quad (21)$$

where $\mathcal{E}(X) = \sum_\alpha K_\alpha X K_\alpha^\dagger$ has the meaning that the channel operates on any input X (rather than a state ρ).

The state-independent classical uncertainty of a channel is defined as the linear entropy (Tsallis 2-entropy) of its Jamiołkowski–Choi state as follows:

$$C(\mathcal{E}) := S_2(J_\mathcal{E}) = 1 - \mathrm{Tr} J_\mathcal{E}^2, \quad (22)$$

where $J_\mathcal{E} = \mathcal{I} \otimes \mathcal{E}(|\Psi\rangle\langle\Psi|)$ is the Jamiołkowski–Choi state of $\mathcal{E}$. By direct calculation the classical uncertainty can be expressed as

$$C(\mathcal{E}) = 1 - \frac{1}{d^2}\sum_{\alpha,\beta} \left|\mathrm{Tr} K_\alpha^\dagger K_\beta\right|^2 = 1 - \frac{1}{d^2}\sum_i \|\mathcal{E}(X_i)\|^2. \quad (23)$$

The state-independent quantum uncertainty is quantified by the deviation from the identity channel as

$$Q(\mathcal{E}) = \frac{1}{d^2}\sum_i \|\mathcal{E}(X_i) - X_i\|^2. \quad (24)$$

This quantifier can also be represented only by Kraus operators as follows:

$$Q(\mathcal{E}) = 1 + \frac{1}{d^2}\sum_{\alpha,\beta} \left|\mathrm{Tr} K_\alpha^\dagger K_\beta\right|^2 - \frac{2}{d^2}\sum_\alpha |\mathrm{Tr} K_\alpha|^2. \quad (25)$$

It is obvious that the total uncertainty can be decomposed into the sum of classical and quantum uncertainty as follows:

$$T(\mathcal{E}) = C(\mathcal{E}) + Q(\mathcal{E}). \quad (26)$$

From the expressions in Eqs. (21), (23), and (25), it is apparent that the state-independent uncertainties are independent of the choice of the orthonormal basis of the observable space.

We must mention that all of the uncertainties of the channel are independent of the selection of Kraus operator sets. Take the state-dependent total uncertainty $V(\rho,\mathcal{E})$ as an example. Assume that $\{F_j\}$ is another Kraus operator set of $\mathcal{E}$. Then there exists a unitary matrix $U = (u_{ij})$ such that $K_i = \sum_j u_{ij} F_j$. As a result, the following equations hold

$$\begin{aligned}\sum_i |\mathrm{Tr}(\rho K_i)|^2 &= \sum_i \left|\mathrm{Tr}\left(\rho \sum_j u_{ij} F_j\right)\right|^2 \\ &= \sum_i \sum_j u_{ij}\overline{u_{ij}}\,\mathrm{Tr}(\rho F_j)\,\overline{\mathrm{Tr}(\rho F_j)} \\ &= \sum_j \left(\sum_i u_{ij}\overline{u_{ij}}\right)|\mathrm{Tr}(\rho F_j)|^2 = \sum_j |\mathrm{Tr}(\rho F_j)|^2,\end{aligned}$$

where $\overline{a}$ means the conjugate of complex number a and the last equation holds because U is unitary satisfying $\sum_i u_{ij}\overline{u_{ik}} = \delta_{jk}$. And there also exists

$$\begin{aligned}\mathcal{E}(I) &= \sum_i K_i K_i^\dagger = \sum_i \sum_j u_{ij}\overline{u_{ij}} F_j F_j^\dagger \\ &= \sum_j \left(\sum_i u_{ij}\overline{u_{ij}}\right) F_j F_j^\dagger = \sum_j F_j F_j^\dagger.\end{aligned}$$

From the above equations and Eq. (14), we have

$$V(\rho,\mathcal{E}) = \sum_i V(\rho,K_i) = \sum_j V(\rho,F_j).$$

Similarly, other uncertainties are also independent of the choice of Kraus operator sets.

Note that the state-dependent uncertainties of a quantum channel quantify the uncertainty of a channel in a state. It is determined by both the quantum channel and the state, which focuses on the interaction of state and channel. The state-independent uncertainties focus on eliminating the influence of individual quantum states. However, they also have the following relations:

(i) The state independent total uncertainty can be interpreted as the average uncertainty of all pure states in the channel $\mathcal{E}$ as follows:[50]

$$T(\mathcal{E}) = \frac{2(d+1)}{d}\overline{V}(\mathcal{E}) \quad (27)$$

with $\overline{V}(\mathcal{E})$ defined in Eq. (19).

(ii) For the maximally mixed state $\rho = I/d$, from Eq. (14) and (21), there exists

$$V(I/d,\mathcal{E}) = 1 - \frac{1}{d^2}\sum_\alpha |\mathrm{Tr} K_\alpha|^2 = T(\mathcal{E}) - 1. \quad (28)$$

Especially, the state-dependent quantum uncertainty for the maximally mixed state vanishes, i.e., $Q(I/d,\mathcal{E}) = 0$, but the state-independent quantum uncertainty in Eq. (25) remains unchanged.

4. Uncertainties of SQT channel

In this section, we first give the general formulas of the uncertainties of the SQT channel. Then we choose some special mixed states as the shared channel, and investigate the corresponding induced standard quantum teleportation channel and its uncertainties.

4.1. General formulae of uncertainties of SQT channel

Property 1 Assume that the shared state is χ in SQT and the induced channel is $\mathcal{E}$, the state to be teleportated expressed in the computational basis is $\rho = \sum_{i,j=0}^{d-1} \rho_{ij}|i\rangle\langle j|$, then the state dependent total uncertainty is

$$\begin{aligned}V(\rho,\mathcal{E}) = 1 - \sum_{n,m} p_{nm} \Bigg| &\sum_{k=0}^{d-1} \\ &\times \exp\left(\frac{2\pi \mathrm{i}}{d} kn\right) \rho_{k,(m+k) \bmod d}\Bigg|^2, \end{aligned} \quad (29)$$

where $p_{nm} = \mathrm{Tr}(\chi E^{n,m})$. If the spectral decomposition is $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, then the state-dependent quantum uncertainty is

$$Q(\rho,\mathcal{E}) = 1 - \sum_{n,m,i,i'} p_{nm}\sqrt{\lambda_i\lambda_{i'}}\left|\langle\psi_i|U^{n,-m}|\psi_{i'}\rangle\right|^2. \quad (30)$$

The average uncertainty is

$$\overline{V}(\mathcal{E}) = \frac{d}{d+1}(1-p_{00}). \quad (31)$$

Proof According to the definition of state-dependent total uncertainty in Eqs. (12)–(14), we have

$$\begin{aligned} V(\rho,\mathcal{E}) &= \frac{1}{2}\left[1+\mathrm{Tr}(\rho\textstyle\sum_{n,m}K_{nm}K_{nm}^\dagger)\right] \\ &\quad -\textstyle\sum_{n,m}|\mathrm{Tr}(\rho K_{nm})|^2, \end{aligned} \quad (32)$$

where $K_{nm} = \sqrt{p_{nm}}U^{n,-m}$. Note that the SQT channel is unital because of the following equations:

$$\begin{aligned} &\sum_{n,m}K_{nm}K_{nm}^\dagger = \sum_{n,m}p_{nm}U^{n,-m}U^{n,-m\dagger} \\ &= \sum_{n,m}p_{nm}\sum_k \exp\left(\frac{2\pi\mathrm{i}}{d}kn\right)|k\rangle\langle(k-m) \bmod d| \\ &\quad \times\sum_{\tilde{k}}\exp\left(\frac{2\pi\mathrm{i}}{d}(-\tilde{k})n\right)|(\tilde{k}-m) \bmod d\rangle\langle\tilde{k}| \\ &= \sum_{n,m}p_{nm}\sum_{k,\tilde{k}}\exp\left(\frac{2\pi\mathrm{i}}{d}(k-\tilde{k})n\right)|k\rangle\langle(k-m) \bmod d| \\ &\quad \times|(\tilde{k}-m) \bmod d\rangle\langle\tilde{k}| \\ &= \sum_{n,m}p_{nm}\sum_k|k\rangle\langle k| = I. \end{aligned} \quad (33)$$

Therefore, the total uncertainty is

$$V(\rho,\mathcal{E}) = 1 - \sum_{n,m}|\mathrm{Tr}(\rho K_{nm})|^2. \quad (34)$$

When we consider the expression of the state ρ, we get

$$\begin{aligned} &\left|\mathrm{Tr}(\rho U^{n,-m})\right|^2 \\ &= \left|\mathrm{Tr}\left(\sum_{i,j}\rho_{ij}|i\rangle\langle j|\sum_k\exp\left(\frac{2\pi\mathrm{i}}{d}kn\right)|k\rangle\langle(k-m) \bmod d|\right)\right|^2 \\ &= \left|\sum_k\exp\left(\frac{2\pi\mathrm{i}}{d}kn\right)\rho_{k,(m+k) \bmod d}\right|^2. \end{aligned} \quad (35)$$

As a result, the formula of the total uncertainty in Eq. (29) is obtained.

If the spectral decomposition of the input state is $\rho = \sum_i \lambda_i|\psi_i\rangle\langle\psi_i|$, then the state-dependent quantum uncertainty is

$$\begin{aligned} Q(\rho,\mathcal{E}) &= 1-\sum_{n,m}p_{nm}\mathrm{Tr}\left[U^{n,-m}\sqrt{\rho}U^{n,-m\dagger}\sqrt{\rho}\right] \\ &= 1-\sum_{n,m,i,i'}p_{nm}\sqrt{\lambda_i\lambda_{i'}}\left|\langle\psi_i|U^{n,-m}|\psi_{i'}\rangle\right|^2. \end{aligned}$$

The traces of the Kraus operators of the SQT channel are

$$\begin{aligned} |\mathrm{Tr}K_{nm}| &= \left|\mathrm{Tr}(\sqrt{p_{nm}}U^{n,-m})\right| \\ &= \sqrt{p_{nm}}\left|\sum_{k=0}^{d-1}\exp\left(\frac{2\pi\mathrm{i}}{d}kn\right)\langle(k-m) \bmod d|k\rangle\right| \\ &= \begin{cases}\sqrt{p_{nm}}d, & n=0,m=0,\\ 0, & \text{otherwise.}\end{cases} \end{aligned} \quad (36)$$

As a result, the total average uncertainty is

$$\overline{V}(\mathcal{E}) = \frac{d}{d+1} - \frac{\sum_i|\mathrm{Tr}K_i|^2}{d(d+1)} = \frac{d}{d+1}(1-p_{00}).$$

Property 2 Assuming that the shared state in the standard quantum teleportation is χ and the induced channel is $\mathcal{E}$, then the state-independent total uncertainty is

$$T(\mathcal{E}) = 2(1-p_{00}), \quad (37)$$

where $p_{nm} = \mathrm{Tr}(\chi E^{n,m})$. And the state-independent classical uncertainty is

$$C(\mathcal{E}) = 1 - \sum_{n,m=0}^{d-1}p_{nm}^2. \quad (38)$$

Proof According to the operator-sum expression of the induced channel in Eqs. (11) and (36), we obtain the total uncertainty

$$T(\mathcal{E}) = 2 - \frac{2}{d^2}\sum_{n,m}|\mathrm{Tr}K_{nm}|^2 = 2-2p_{00}.$$

Remember that $\mathrm{Tr}(U^{n,-m\dagger}U^{\tilde{n},-\tilde{m}}) = d\delta_{n\tilde{n},m\tilde{m}}$, therefore by direct calculations, we obtain

$$\begin{aligned} \mathrm{Tr}\left(K_{nm}^\dagger K_{\tilde{n}\tilde{m}}\right) &= \sqrt{p_{nm}}\sqrt{p_{\tilde{n}\tilde{m}}}\,\mathrm{Tr}\left(U^{n,-m\dagger}U^{\tilde{n},-\tilde{m}}\right) \\ &= \begin{cases}dp_{nm}, & n=\tilde{n},m=\tilde{m},\\ 0, & \text{otherwise.}\end{cases} \end{aligned} \quad (39)$$

By Eq. (23), the classical uncertainty of the standard quantum teleportation channel is

$$C(\mathcal{E}) = 1 - \frac{1}{d^2}\sum_{n,m}\left|\mathrm{Tr}K_{nm}^\dagger K_{nm}\right|^2 = 1-\sum_{n,m}p_{nm}^2.$$

Note If the density matrix of the shared state expressed in the computational basis is $\chi = \sum_{ij,st}\chi_{ij,st}|ij\rangle\langle st|$, then the probability is

$$\begin{aligned} p_{nm} &= \mathrm{Tr}(\chi E^{n,m}) = \frac{1}{d}\sum_{k,\tilde{k}}\exp\left(\frac{2\pi\,\mathrm{i}}{d}n(\tilde{k}-k)\right) \\ &\quad \times\chi_{(k,(k+m) \bmod d),(\tilde{k},(\tilde{k}+m) \bmod d)}. \end{aligned} \quad (40)$$

Combine Eqs. (37), (38), and (40) together, one can get the total and classical uncertainty directly.

4.2. Uncertainties of the SQT channel induced by specific states

In this subsection, we consider the uncertainties of SQT channels induced by specific states, such as the isotropic states, Werner states, and X-states.

4.2.1. Isotropic state

The isotropic two-qudit state is a convex combination of the maximally entangled state and the maximally mixed state defined as follows:[53–56]

$$\chi_\alpha = \alpha|\Psi\rangle\langle\Psi| + \frac{1-\alpha}{d^2} I\otimes I,$$
$$\alpha\in\mathbb{R}, \quad -\frac{1}{d^2-1} \leqslant \alpha \leqslant 1, \tag{41}$$

we have $p_{00} = \alpha + (1-\alpha)/d^2$ and $p_{nm} = (1-\alpha)/d^2, (n,m) \neq (0,0)$. Therefore, the corresponding Kraus operator-sum form of the induced SQT channel is

$$\mathcal{E}_{\chi_\alpha}(\rho) = \left(\alpha + \frac{1-\alpha}{d^2}\right)\rho + \frac{1-\alpha}{d^2} \times \sum_{(n,m)\neq(0,0)} U^{n,-m}\rho U^{n,-m\dagger}. \tag{42}$$

Based on the above definitions, we obtain the uncertainties of the standard quantum channel as follows: the state-dependent total uncertainty is

$$V(\rho,\mathcal{E}_{\chi_\alpha}) = (1-\alpha)\left[1 - \frac{\mathrm{Tr}(\rho^2)}{d}\right], \tag{43}$$

the state-dependent quantum uncertainty is

$$Q(\rho,\mathcal{E}_{\chi_\alpha}) = (1-\alpha)\left(1 - \frac{(\mathrm{Tr}(\sqrt{\rho}))^2}{d}\right), \tag{44}$$

and the classical uncertainty is

$$C(\rho,\mathcal{E}_{\chi_\alpha}) = (1-\alpha)\left(\frac{(\mathrm{Tr}(\sqrt{\rho}))^2}{d} - \frac{\mathrm{Tr}(\rho^2)}{d}\right). \tag{45}$$

We can see that the total uncertainty attains its minimum when the input state is a pure state. At the same time, the classical uncertainty vanishes. Because there is no classical mixing process but only quantum uncertainties. Therefore the quantum uncertainty attains its maximum. All the uncertainties are decreasing with α, which is an indicator of the degree of superposition of the maximum entangled state and the maximum mixed state.

From Eq. (43), it is easy to find that the state-dependent total uncertainty vanishes if and only if the induced isotropic state is a pure state, i.e., $\alpha = 1$. The state-dependent quantum uncertainty vanishes when $\alpha = 1$, or when the input state ρ is the maximally mixed state I/d for any α.

It is noteworthy that the state-dependent uncertainties are solely related to the teleportated state itself and the parameter α, which indicates the degree of mixing between the maximally entangled state and the mixed state. In Ref. [57], for arbitrary d-dimensional quantum state ρ, the normalized linear entropy mixedness $M_\mathrm{l}(\rho)$ and geometric mixedness $M_\mathrm{g}(\rho)$ are defined as follows:

$$M_\mathrm{l}(\rho) = \frac{d}{d-1}(1-\mathrm{Tr}\rho^2),$$
$$M_\mathrm{g}(\rho) = \frac{1}{d}(\mathrm{Tr}\sqrt{\rho})^2.$$

Thus, the state-dependent total and quantum uncertainty can be also expressed in terms of mixedness as follows:

$$V(\rho,\mathcal{E}_{\chi_\alpha}) = (1-\alpha)\frac{d-1}{d}\left[1 + \frac{M_\mathrm{l}(\rho)}{d}\right],$$
$$Q(\rho,\mathcal{E}_{\chi_\alpha}) = (1-\alpha)\left[1 - M_\mathrm{g}(\rho)\right].$$

The maximum coherence of the l_2 norm and the Wigner–Yanase skew information are respectively defined as[58,59]

$$C_{l_2}^{\max}(\rho) = \left(\mathrm{Tr}\rho^2 - \frac{1}{d}\right)^{1/2},$$
$$C_{sk}^{\max}(\rho) = 1 - \frac{1}{d}\left(\mathrm{Tr}\sqrt{\rho}\right)^2.$$

As a result, the state-dependent total and quantum uncertainties in terms of maximum coherence are as follows:

$$V(\rho,\mathcal{E}_{\chi_\alpha}) = (1-\alpha)\left[1 - \frac{1}{d^2} - \frac{(C_{l_2}^{\max}(\rho))^2}{d}\right],$$
$$Q(\rho,\mathcal{E}_{\chi_\alpha}) = (1-\alpha)C_{sk}^{\max}(\rho).$$

Similarly, the state-dependent uncertainties can also be represented in terms of the average coherence based on the l_2 norm or the Wigner–Yanase skew information, see the trade-off relations in Ref. [60].

The state-independent total uncertainty is

$$T(\mathcal{E}_{\chi_\alpha}) = 2(1-p_{00}) = 2(1-\alpha)\left(1 - \frac{1}{d^2}\right), \tag{46}$$

the state-independent classical uncertainty is

$$C(\mathcal{E}_{\chi_\alpha}) = 1 - \sum_{n,m} p_{nm}^2 = (1-\alpha^2)\left(1 - \frac{1}{d^2}\right) \tag{47}$$

and the quantum uncertainty is

$$Q(\mathcal{E}_{\chi_\alpha}) = T(\mathcal{E}_{\chi_\alpha}) - C(\mathcal{E}_{\chi_\alpha}) = 1 - 2p_{00} + \sum_{n,m} p_{nm}^2 = (1-\alpha)^2\left(1 - \frac{1}{d^2}\right). \tag{48}$$

Note that for a fixed dimension, the state-independent total uncertainty and classical uncertainty are decreasing with the parameter α, indicating that the shared channel is closer to the maximum entangled state and has less mixedness. However, the state-independent quantum uncertainty is increasing with α. The state-independent total uncertainty vanishes if and only if the state-independent quantum uncertainty vanishes, i.e., $T(\mathcal{E}_{\chi_\alpha}) = 0$ iff $Q(\mathcal{E}_{\chi_\alpha}) = 0$.

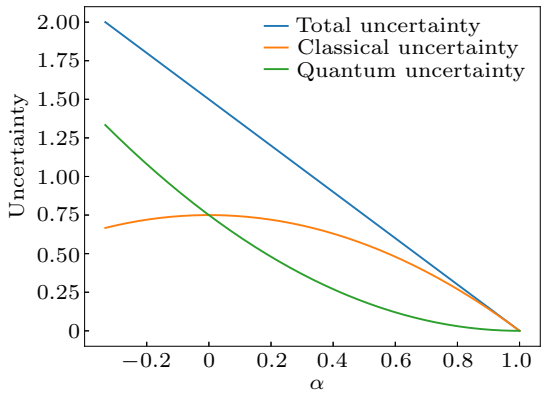


Fig. 2. Uncertainties of the SQT channel induced by isotropic state with 2×2 dimension.

The relation between the uncertainties and mixedness parameter α is shown in Fig. 2 when $d=2$. The detailed calculations of Eqs. (42)–(48) are presented in Appendix A.

4.2.2. Werner state

The Werner state with $d\times d$ dimension can be represented as[61]

$$\rho_w=\frac{1}{d^3-d}\left[(d-\phi)I\otimes I+(d\phi-1)V\right], \tag{49}$$

where $V=\sum_{i,j}|ij\rangle\langle ji|$ is the swap operator, i.e., $V|\varphi\psi\rangle=|\psi\varphi\rangle$ and $-1\le\phi\le 1$. The corresponding probabilities induced by the Werner state in the SQT channel are

$$p_{nm}=\begin{cases}\dfrac{d-\phi}{d^3-d}, & 2m\not\equiv 0 \pmod d,\\ \dfrac{(d-\phi)+(-1)^{2mn/d}(d\phi-1)}{d^3-d}, & 2m\equiv 0 \pmod d.\end{cases} \tag{50}$$

The state-dependent total uncertainty of the SQT channel is

$$V(\rho,\mathcal{E}_{\rho_w})=\begin{cases}1-\dfrac{(d-\phi)\,\mathrm{Tr}\rho^2}{d^2-1}-\dfrac{(d\phi-1)\sum_j\langle j|\rho|j\rangle^2}{d^2-1}, & d \text{ is odd},\\ 1-\dfrac{(d-\phi)\,\mathrm{Tr}\rho^2}{d^2-1}-\dfrac{(d\phi-1)\sum_j[\langle j|\rho|j\rangle^2+\langle (j+d/2) \bmod d|\rho|j\rangle^2]}{d^2-1}, & d \text{ is even}.\end{cases} \tag{51}$$

The state-dependent quantum uncertainty of the SQT channel is

$$Q(\rho,\mathcal{E}_{\rho_w})=\begin{cases}1-\dfrac{(d-\phi)(\mathrm{Tr}\sqrt{\rho})^2}{d^2-1}-\dfrac{(d\phi-1)\sum_j\langle j|\sqrt{\rho}|j\rangle^2}{d^2-1}, & d \text{ is odd},\\ 1-\dfrac{(d-\phi)(\mathrm{Tr}\sqrt{\rho})^2}{d^2-1}-\dfrac{(d\phi-1)\sum_j[\langle j|\sqrt{\rho}|j\rangle^2+\langle (j+d/2) \bmod d|\sqrt{\rho}|j\rangle^2]}{d^2-1}, & d \text{ is even}.\end{cases} \tag{52}$$

The state-independent total uncertainty of the SQT channel is

$$T(\mathcal{E}_{\rho_w})=2-2\frac{\phi+1}{d^2+d}. \tag{53}$$

The state-independent classical uncertainty of the SQT channel is

$$C(\mathcal{E}_{\rho_w})=\begin{cases}1-\dfrac{(d-\phi)^2}{d(d-1)(d+1)^2}-\dfrac{(\phi+1)^2}{d(d+1)^2}, & d \text{ is odd},\\ 1-\dfrac{(d-\phi)^2}{d(d-1)(d+1)^2}-\dfrac{(\phi+1)^2}{d(d+1)^2}-\dfrac{(d\phi-1)^2}{d(d-1)^2(d+1)^2}, & d \text{ is even}.\end{cases} \tag{54}$$

It is obvious that for fixed dimension, the state-independent total uncertainty decreases with ϕ, and it is greater than $4/3$ for any dimension. But the state-independent classical uncertainty increases when $\phi<1/d$ and decreases when $\phi>1/d$. When $\phi=1/d$, the classical uncertainty attains its maximum $1-1/d^2$, which corresponds to the channel induced by the maximally mixed state.

The relations between ϕ and uncertainties for (2×2)- and (3×3)-dimensional Werner states are displayed in Figs. 3 and 4 respectively. The detailed calculations of Eqs. (50)–(52) are presented in Appendix B.

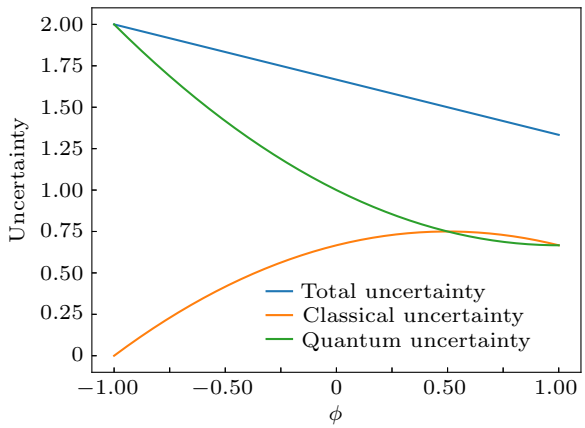


Fig. 3. Uncertainties of the SQT channel induced by Werner state with 2×2 dimension.

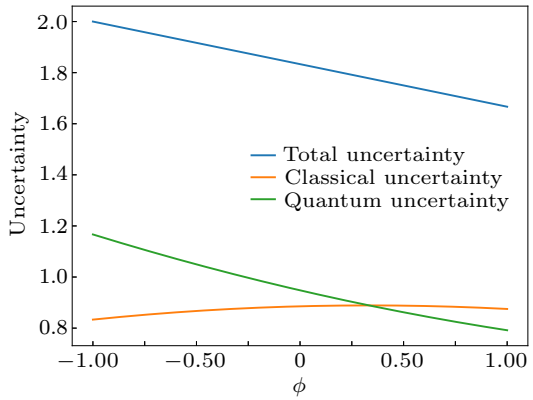


Fig. 4. Uncertainties of the SQT channel induced by Werner state with 3×3 dimension.

4.2.3. X-state

The X-state is a kind of mixed state with a density matrix in X shape. We consider the bipartite X-state with $d^2\times d^2$ dimension. When d is even, the shared X-state is

$$\rho_X = \sum_{i,j=0}^{d-1} X_{ij,ij}|ij\rangle\langle ij| + \sum_{i,j=0}^{d-1} X_{ij,(d-1-i,d-1-j)}|ij\rangle\langle d-1-i,d-1-j|. \quad (55)$$

When d is odd, the shared X-state is

$$\rho_X = \sum_{i,j=0}^{d-1} X_{ij,ij}|ij\rangle\langle ij| + \sum_{\substack{i,j=0\\(i,j)\neq(\frac{d-1}{2},\frac{d-1}{2})}}^{d-1} X_{ij,(d-1-i,d-1-j)}|ij\rangle\langle d-1-i,d-1-j|. \quad (56)$$

Therefore, when d is even, the corresponding probabilities induced by the X-state in the standard quantum teleportation channel are as follows:

(i) when $2m\not\equiv 0\ (\mathrm{mod}\, d)$ holds,

$$dp_{nm} = \sum_k X_{(k,(k+m)\bmod d),(k,(k+m)\bmod d)}; \quad (57)$$

(ii) when $2m\equiv 0\ (\mathrm{mod}\, d)$ holds,

$$dp_{nm} = \sum_k X_{(k,(k+m)\bmod d),(k,(k+m)\bmod d)} + \sum_k \exp\left(-\frac{2\pi \mathrm{i}}{d}(1+2k)n\right) \times X_{(k,(k+m)\bmod d),(\bar{k},\overline{(k+m)\bmod d})}, \quad (58)$$

where we introduce the notation that $\bar{k}=d-1-k$.

When d is odd, we have

$$dp_{n0} = \sum_k X_{kk,kk} + \sum_{k\neq\frac{d-1}{2}} \exp\left(-\frac{2\pi \mathrm{i}}{d}(1+2k)n\right) X_{kk,(d-1-k,d-1-k)}, \quad (59)$$

and for $m\neq 0$

$$dp_{nm} = \sum_k X_{(k,(k+m)\bmod d),(k,(k+m)\bmod d)}. \quad (60)$$

It is easy to note that the probability distribution has the property that

$$\sum_{n=0}^{d-1} p_{nk} = \sum_{n=0}^{d-1} X_{(n,n\oplus k)(n,n\oplus k)}. \quad (61)$$

The state-independent total uncertainty is

$$T(\mathcal{E}_{\rho_X}) = \begin{cases} 2\left[1-\frac{1}{d}\left(\sum_k X_{kk,kk} + \sum_{k\neq\frac{d-1}{2}} X_{kk,(d-1-k,d-1-k)}\right)\right], & d \text{ is odd},\\ 2\left[1-\frac{1}{d}\left(\sum_k X_{kk,kk} + \sum_k X_{kk,(d-1-k,d-1-k)}\right)\right], & d \text{ is even}. \end{cases} \quad (62)$$

By the fact that $\sum_{n=0}^{d-1}\exp\left[\left(\frac{2\pi \mathrm{i}}{d}\right)kn\right]=0$ for $k\neq 0$, we get the state-independent classical uncertainty as follows:

(i) when d is even,

$$C(\mathcal{E}_{\rho_X}) = 1-\frac{1}{d}\sum_m\left(\sum_k X_{(k,(k+m)\bmod d),(k,(k+m)\bmod d)}\right)^2 - \frac{1}{d}\sum_{m=0,d/2}\left(\sum_{\substack{k+\tilde{k}=\frac{d}{2}-1\\ k+\tilde{k}=d-1\\ k+\tilde{k}=\frac{3d}{2}-1}} X_{(k,(k+m)\bmod d),(\bar{k},\overline{(k+m)\bmod d})} X_{(\tilde{k},(\tilde{k}+m)\bmod d),(\bar{\tilde{k}},\overline{(\tilde{k}+m)\bmod d})}\right), \quad (63)$$

(ii) when d is odd,

$$C(\mathcal{E}_{\rho_X}) = 1-\frac{1}{d}\sum_m\left(\sum_k X_{(k,(k+m)\bmod d),(k,(k+m)\bmod d)}\right)^2 - \frac{1}{d}\sum_{\substack{k+\tilde{k}=d-1\\ k\neq\tilde{k}}} X_{(k,k),(\bar{k},\bar{k})} X_{(\tilde{k},\tilde{k}),(\bar{\tilde{k}},\bar{\tilde{k}})}. \quad (64)$$

Assuming that the input state is $\rho=\sum_{ij}\rho_{ij}|i\rangle\langle j|$, we exhibit two simplest cases for the state-dependent total uncertainty. Specifically, for the $d=2$ case, the probabilities are

$$\begin{aligned}
p_{00}&=\frac{1}{2}(X_{00,00}+X_{00,11}+X_{11,00}+X_{11,11}),\\
p_{01}&=\frac{1}{2}(X_{10,10}+X_{10,01}+X_{01,10}+X_{01,01}),\\
p_{10}&=\frac{1}{2}(X_{00,00}-X_{00,11}-X_{11,00}+X_{11,11}),\\
p_{11}&=\frac{1}{2}(X_{10,10}-X_{10,01}-X_{01,10}+X_{01,01}).
\end{aligned}\tag{65}$$

The state-dependent total uncertainty is

$$\begin{aligned}
V(\rho,\mathcal{E}_{\rho_X}) =\ & 1-\big[(X_{00,00}+X_{11,11})(|\rho_{00}|^2+|\rho_{11}|^2)\\
&+(X_{01,01}+X_{10,10})(|\rho_{01}|^2+|\rho_{10}|^2)\\
&+(X_{01,10}+X_{10,01})(\rho_{01}\rho_{10}^*+\rho_{10}\rho_{01}^*)\\
&+2(X_{00,11}+X_{11,00})\rho_{00}\rho_{11}\big].
\end{aligned}\tag{66}$$

When $d=3$, the probabilities are

$$\begin{aligned}
p_{00} &= \frac{1}{3}(X_{00,00}+X_{00,22}+X_{11,11}+X_{22,00}+X_{22,22}),\\
p_{01} &= p_{11}=p_{21}=\frac{1}{3}(X_{01,01}+X_{12,12}+X_{20,20}),\\
p_{02} &= p_{12}=p_{22}=\frac{1}{3}(X_{02,02}+X_{10,10}+X_{21,21}),\\
p_{10} &= \frac{1}{3}(X_{00,00}+\exp\left(\frac{4\pi\mathrm{i}}{3}\right)X_{00,22}+X_{11,11}\\
&\quad+\exp\left(\frac{2\pi\mathrm{i}}{3}\right)X_{22,00}+X_{22,22}),\\
p_{20} &= \frac{1}{3}(X_{00,00}+\exp\left(\frac{2\pi\mathrm{i}}{3}\right)X_{00,22}+X_{11,11}\\
&\quad+\exp\left(\frac{4\pi\mathrm{i}}{3}\right)X_{22,00}+X_{22,22}).
\end{aligned}\tag{67}$$

And the state-dependent total uncertainty is as follows:

$$\begin{aligned}
&V(\rho,\mathcal{E}_{\rho_X})\\
=\ & 1-\big[(X_{00,00}+X_{11,11}+X_{22,22})(|\rho_{00}|^2+|\rho_{11}|^2+|\rho_{22}|^2)\\
&+(X_{01,01}+X_{12,12}+X_{20,20})(|\rho_{01}|^2+|\rho_{12}|^2+|\rho_{20}|^2)\\
&+(X_{02,02}+X_{10,10}+X_{21,21})(|\rho_{02}|^2+|\rho_{10}|^2+|\rho_{21}|^2)\\
&+(X_{00,22}+X_{22,00})(\rho_{00}\rho_{11}+\rho_{11}\rho_{22}+\rho_{22}\rho_{00})\big].
\end{aligned}\tag{68}$$

5. Discussion and conclusion

Considering the importance of the standard quantum teleportation channel, we investigate it from the perspective of state-channel interaction. The main contributions of this paper are as follows:

(i) The uncertainty formulae of standard quantum teleportation channel in an arbitrary finite-dimensional system are given, including state-dependent and state-independent total uncertainties and quantum uncertainties.

(ii) The uncertainties of some special but important standard quantum teleportation channels induced by two-body mixed states are studied, including isotropic state, Werner state, and X-state. Furthermore, the relationship between the state-dependent uncertainties and quantum coherence is discussed.

The work in this paper provides a more detailed and comprehensive study of standard quantum teleportation channels. It provides a way to interpret the uncertainty in quantum teleportation process from the perspective of information theory, and helps to further understand the resource dynamics in quantum communication process.

Appendix A. Uncertainties of the isotropic state induced SQT channel

Lemma 1 Prove that for any basis $\{X_i\}$ of $B(\mathcal{H}^d)$ (not necessarily Hermitian) satisfying normalization condition $\mathrm{Tr}(X_i^\dagger X_j)=d\delta_{ij}$ and any operator $A\in B(\mathcal{H}^d)$, there exists

$$\sum_i X_i^\dagger A X_i=\sum_i X_i A X_i^\dagger=d\,\mathrm{Tr}(A)I.\tag{A1}$$

Proof Firstly, we need to prove that for any two bases $\{X_i\}$ and $\{Y_j\}$, there holds

$$\sum_i X_i A X_i^\dagger=\sum_j Y_j A Y_j^\dagger.\tag{A2}$$

Because $\{X_i\}$ and $\{Y_j\}$ are two bases of $B(\mathcal{H}^d)$, there exists a unitary matrix $U=(u_{ij})_{d^2\times d^2}$ with $\sum_k u_{ik}\overline{u_{jk}}=\delta_{ij}$ and $\sum_k \overline{u_{ki}}u_{kj}=\delta_{ij}$ such that $X_i=\sum_j u_{ij}Y_j$. As a result, we have the following equations

$$\begin{aligned}
&\sum_i X_i A X_i^\dagger=\sum_i\left(\sum_j u_{ij}Y_j\right)A\left(\sum_k \overline{u_{ik}}{Y_k}^\dagger\right)\\
=&\sum_{j,k}\left(\sum_i u_{ij}\overline{u_{ik}}\right)Y_j A {Y_k}^\dagger=\sum_{j,k}\delta_{jk}Y_j A {Y_k}^\dagger=\sum_j Y_j A {Y_j}^\dagger.
\end{aligned}$$

Now, for any matrix A, assume that its expression in the standard computational basis is $A=\sum_{r,s=0}^{d-1}a_{rs}|r\rangle\langle s|$, where a_{rs} are complex numbers. The trace of A is $\mathrm{Tr}A=\sum_{r=0}^{d-1}a_{rr}$. Considering the property in Eq. (A2), we can choose the basis $\{U^{n,m}\}_{n,m=0}^{d-1}$ in the calculation as follows:

$$\begin{aligned}
&\sum_{n,m=0}^{d-1}U^{n,m}AU^{n,m\dagger}=\sum_{n,m=0}^{d-1}\sum_{r,s=0}^{d-1}a_{rs}U^{n,m}|r\rangle\langle s|U^{n,m\dagger}\\
=&\sum_{n,m=0}^{d-1}\sum_{r,s=0}^{d-1}a_{rs}\sum_{k,\tilde{k}=0}^{d-1}\exp\left(\frac{2\pi\mathrm{i}}{d}n(k-\tilde{k})\right)|k\rangle\langle(k+m)\bmod d|r\rangle\langle s|(\tilde{k}+m)\bmod d\rangle\langle\tilde{k}|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n,m=0}^{d-1}\sum_{r,s=0}^{d-1} a_{rs}\exp\left(\frac{2\pi \mathrm{i}}{d}n(r-s)\right)|(r-m) \bmod d\rangle\langle(s-m) \bmod d| \\
&= \sum_{r=s=0}^{d-1} a_{rr}\sum_{n=0}^{d-1}\exp\left(\frac{2\pi \mathrm{i}}{d}0\right)\sum_{m=0}^{d-1}|(r-m) \bmod d\rangle\langle(r-m) \bmod d| \\
&\quad + \sum_{r\neq s} a_{rs}\sum_{n=0}^{d-1}\exp\left(\frac{2\pi \mathrm{i}}{d}n(r-s)\right)\sum_{m=0}^{d-1}|(r-m) \bmod d\rangle\langle(s-m) \bmod d| = d\,\mathrm{Tr}(A)I. \qquad \text{(A3)}
\end{aligned}
$$

Note that here we use the fact that $\langle(k+m) \bmod d|r\rangle\langle s|(\tilde{k}+m) \bmod d\rangle$ is nonzero if and only if $k=(r-m) \bmod d$ and $\tilde{k}=(s-m) \bmod d$ hold. In the last equation of Eq. (A3), we use the property that

$$
\sum_{k=0}^{d-1}\exp\left(\frac{2\pi \mathrm{i}}{d}nk\right)=\begin{cases} d, & n=0, \\ 0, & n\neq 0, \end{cases} \quad n\in \mathrm{Z}.
$$

Similar tricks can be used to prove that $\sum_i X_i^{\dagger}AX_i = d\,\mathrm{Tr}(A)I$, therefore the proof is completed.

Property A1 The probabilities corresponding to the isotropic state-induced channel are as follows:

$$
p_{00}=\mathrm{Tr}(\chi_\alpha E^{0,0})=\alpha+\frac{1-\alpha}{d^2},
$$

$$
p_{nm}=\mathrm{Tr}(\chi_\alpha E^{n,m})=\frac{1-\alpha}{d^2}, (n,m)\neq(0,0).
$$

Proof According to the definition of probability, we have

$$
\begin{aligned}
\mathrm{Tr}\,(E^{n,m}\chi_\alpha) &= \langle\Psi|(U^{n,m\dagger}\otimes I)(\alpha|\Psi\rangle\langle\Psi| \\
&\quad + \frac{1-\alpha}{d^2}I\otimes I)(U^{n,m}\otimes I)|\Psi\rangle \\
&= \alpha\langle\Psi|(U^{n,m\dagger}\otimes I)(|\Psi\rangle\langle\Psi|)(U^{n,m}\otimes I)|\Psi\rangle \\
&\quad + \frac{1-\alpha}{d^2}\langle\Psi|\Psi\rangle.
\end{aligned}
$$

For the first term, we have

$$
\begin{aligned}
\langle\Psi|(U^{n,m}\otimes I)|\Psi\rangle &= \frac{1}{d}\left(\sum_{j=0}^{d-1}\langle j|\langle j|\right)\left(\sum_{k=0}^{d-1}\exp\left(\frac{2\pi \mathrm{i}}{d}nk\right)|k\rangle\langle(k+m) \bmod d|\otimes I\right)\left(\sum_{l=0}^{d-1}|l\rangle|l\rangle\right) \\
&= \frac{1}{d}\sum_{j,k,l=0}^{d-1}\exp\left(\frac{2\pi \mathrm{i}}{d}nk\right)\langle j|k\rangle\langle j|l\rangle\langle(k+m) \bmod d|l\rangle \\
&\overset{j=k=l}{=\!=\!=} \frac{1}{d}\sum_{k=0}^{d-1}\exp\left(\frac{2\pi \mathrm{i}}{d}nk\right)\langle(k+m) \bmod d|k\rangle \overset{m=0}{=\!=} \frac{1}{d}\sum_{k=0}^{d-1}\exp\left(\frac{2\pi \mathrm{i}}{d}nk\right)=\begin{cases} 1, & n=0, \\ 0, & n\neq 0. \end{cases} \qquad \text{(A4)}
\end{aligned}
$$

Property A2 The state-dependent total uncertainty of the isotropic state-induced SQT channel is

$$
V(\rho,\mathcal{E}_{\chi_\alpha})=(1-\alpha)\left[1-\frac{\mathrm{Tr}(\rho^2)}{d}\right].
$$

Proof According to Eq. (34), we have

$$
V(\rho,\mathcal{E}_{\chi_\alpha})=1-\sum_{n,m}p_{nm}|\mathrm{Tr}(\rho U^{n,-m})|^2. \qquad \text{(A5)}
$$

It is easy to prove that $\{U^{n,-m}\}$ forms an orthonormal basis of the Hilbert–Schmidt inner product space with $\mathrm{Tr}\,U^{n,-m\dagger}U^{l,-j}=d\delta_{nl}\delta_{mj}$. Assume that the representation of ρ in the basis $\{U^{n,-m}\}$ is $\rho=\sum_{r,s=0}^{d-1}c_{rs}U^{r,-s}$. Because of the relation $\mathrm{Tr}\,\rho=\sum_{r,s=0}^{d-1}c_{rs}\,\mathrm{Tr}\,U^{r,-s}=dc_{00}=1$, there holds $c_{00}=1/d$. On the other hand, the following equations exist

$$
\begin{aligned}
\mathrm{Tr}(\rho U^{n,-m}) &= \mathrm{Tr}(\rho^{\dagger}U^{n,-m}) \\
&= \mathrm{Tr}\left[\sum_{r,s=0}^{d-1}\overline{c_{rs}}U^{r,-s\dagger}U^{n,-m}\right] \\
&= \sum_{r,s=0}^{d-1}\overline{c_{rs}}\,\mathrm{Tr}[U^{r,-s\dagger}U^{n,-m}] \\
&= d\sum_{r,s=0}^{d-1}\overline{c_{rs}}\delta_{rn}\delta_{sm}=d\overline{c_{nm}}. \qquad \text{(A6)}
\end{aligned}
$$

Therefore, the total uncertainty changes to

$$
\begin{aligned}
V(\rho,\mathcal{E}_{\chi_\alpha}) &= 1-d^2\sum_{n,m}p_{nm}|c_{nm}|^2 \\
&= 1-\left[\left(\alpha+\frac{1-\alpha}{d^2}\right)+(1-\alpha)\sum_{(n,m)\neq(0,0)}|c_{nm}|^2\right] \\
&= 1-\left[\alpha+(1-\alpha)\sum_{n,m=0}^{d-1}|c_{nm}|^2\right] \\
&= (1-\alpha)\left[1-\frac{1}{d}\mathrm{Tr}(\rho^2)\right]. \qquad \text{(A7)}
\end{aligned}
$$

Property A3 The state-dependent quantum uncertainty of the isotropic SQT channel is

$$
Q(\rho,\mathcal{E}_{\chi_\alpha})=(1-\alpha)\left(1-\frac{(\mathrm{Tr}(\sqrt{\rho}))^2}{d}\right).
$$

Proof According to the definition of state-dependent quantum uncertainty, the quantum uncertainty of the SQT

channel is

$$Q(\rho,\mathcal{E}_{\chi_\alpha}) = 1-\sum_{n,m}\mathrm{Tr}(\sqrt{\rho}K_{nm}\sqrt{\rho}{K_{nm}}^\dagger),$$

where $K_{nm}=\sqrt{p_{nm}}U^{n,-m}$ are Kraus operators of the SQT channel with $p_{00}=\alpha+(1-\alpha)/d^2$ and $p_{nm}=(1-\alpha)/d^2$, $(n,m)\neq(0,0)$. Subsequently, using Lemma 1, we get

$$\begin{aligned}
&\sum_{n,m}\mathrm{Tr}(\sqrt{\rho}K_{nm}\sqrt{\rho}{K_{nm}}^\dagger)\\
=\;&\alpha\,\mathrm{Tr}(\sqrt{\rho}U^{0,0}\sqrt{\rho}{U^{0,0}}^\dagger)\\
&+\frac{1-\alpha}{d^2}\mathrm{Tr}\left[\sqrt{\rho}\sum_{n,m=0}^{d-1}U^{n,-m}\sqrt{\rho}{U^{n,-m}}^\dagger\right]\\
=\;&\alpha+\frac{1-\alpha}{d^2}\mathrm{Tr}[\sqrt{\rho}\,\mathrm{Tr}(\sqrt{\rho})dI]\\
=\;&\alpha+\frac{1-\alpha}{d}(\mathrm{Tr}(\sqrt{\rho}))^2.
\end{aligned}$$

Therefore, the state-dependent quantum uncertainty of the isotropic SQT channel is

$$Q(\rho,\mathcal{E}_{\chi_\alpha}) = (1-\alpha)\left(1-\frac{(\mathrm{Tr}(\sqrt{\rho}))^2}{d}\right).$$

Appendix B. Uncertainties of the Werner state-induced SQT channel

Property B1 Prove that for the SQT channel induced by Werner state, the corresponding probabilities are

$$p_{nm}=\begin{cases}\dfrac{d-\phi}{d^3-d}, & 2m\not\equiv 0\ (\mathrm{mod}\,d),\\ \dfrac{(d-\phi)+(-1)^{2mn/d}(d\phi-1)}{d^3-d}, & 2m\equiv 0\ (\mathrm{mod}\,d).\end{cases}$$

Proof

$$\begin{aligned}
p_{nm}&=\mathrm{Tr}(E^{n,m}\rho_w)\\
&=\frac{1}{d^3-d}\langle\Psi|(U^{n,m\dagger}\otimes I)[(d-\phi)I\otimes I+(d\phi-1)V](U^{n,m}\otimes I)|\Psi\rangle\\
&=\frac{1}{d^3-d}\left[(d-\phi)+(d\phi-1)\langle\Psi|(U^{n,m\dagger}\otimes I)V(U^{n,m}\otimes I)|\Psi\rangle\right]\\
&=\frac{1}{d^3-d}\left[(d-\phi)+(d\phi-1)\langle\Psi|(U^{n,m\dagger}\otimes I)\sum_{i,j}|i\rangle\langle j|\otimes|j\rangle\langle i|(U^{n,m}\otimes I)|\Psi\rangle\right]\\
&=\frac{1}{d^3-d}\left[(d-\phi)+(d\phi-1)\langle\Psi|\sum_{i,j}U^{n,m\dagger}|i\rangle\langle j|U^{n,m}\otimes|j\rangle\langle i|\Psi\rangle\right]\\
&=\frac{1}{d^3-d}\left[(d-\phi)+(d\phi-1)\langle\Psi|\sum_{k,j}\exp\left(\frac{2\pi\mathrm{i}n(j-k)}{d}\right)|(k+m)\bmod d\rangle\langle(j+m)\bmod d|\otimes|j\rangle\langle k|\cdot|\Psi\rangle\right]\\
&=\frac{1}{d^3-d}\left[(d-\phi)+\frac{d\phi-1}{d}\sum_{k,j}\exp\left(\frac{2\pi\mathrm{i}n(j-k)}{d}\right)\langle j|(k+m)\bmod d\rangle\cdot\langle(j+m)\bmod d|k\rangle\right]\\
&\quad\begin{cases}\underline{\underline{j\neq(k+m)\bmod d\ \ \text{or}\ \ k\neq(j+m)\bmod d}}\ \dfrac{d-\phi}{d^3-d}, & 2m\not\equiv 0\ (\mathrm{mod}\,d),\\ \underline{\underline{j=(k+m)\bmod d,\ \ k=(j+m)\bmod d}}\ \dfrac{(d-\phi)+(-1)^{2mn/d}(d\phi-1)}{d^3-d}, & 2m\equiv 0\ (\mathrm{mod}\,d).\end{cases}
\end{aligned}\tag{B1}$$

Property B2 Prove that for the SQT channel induced by Werner state, the state-dependent total uncertainty is

$$V(\rho,\mathcal{E}_{\rho_w})=\begin{cases}1-\dfrac{(d-\phi)\,\mathrm{Tr}\rho^2}{d^2-1}-\dfrac{(d\phi-1)\sum_j\langle j|\rho|j\rangle^2}{d^2-1}, & d\text{ is odd},\\ 1-\dfrac{(d-\phi)\,\mathrm{Tr}\rho^2}{d^2-1}-\dfrac{(d\phi-1)\sum_j[\langle j|\rho|j\rangle^2+\langle(j+d/2)\bmod d|\rho|j\rangle^2]}{d^2-1}, & d\text{ is even}.\end{cases}$$

Proof According to Property B1, we have

$$\begin{aligned}
V(\rho,\mathcal{E}_{\rho_w})&=1-\sum_{n,m}p_{nm}|\,\mathrm{Tr}(\rho U^{n,-m})|^2\\
&=1-\frac{d-\phi}{d^3-d}\sum_{n,m}|\,\mathrm{Tr}(\rho U^{n,-m})|^2-\frac{(d\phi-1)}{d^3-d}\sum_{n,2m\equiv 0(\bmod d)}(-1)^{2mn/d}|\,\mathrm{Tr}(\rho U^{n,-m})|^2.
\end{aligned}\tag{B2}$$

Because $\{U^{n,-m}: n,m=0,1,\ldots,d-1\}$ is an orthogonal basis satisfying $\mathrm{Tr}(U^{r,-s\dagger}U^{n,-m})=d\delta_{rn}\delta_{sm}$, we can let $\rho=\sum_{r,s}c_{rs}U^{r,-s}$. As a result, we have

$$\sum_{n,m}|\mathrm{Tr}(\rho U^{n,-m})|^2=\sum_{n,m}|\mathrm{Tr}(\rho^\dagger U^{n,-m})|^2=\sum_{n,m}\left|\mathrm{Tr}\left(\sum_{rs}\overline{c_{rs}}U^{r,-s\dagger}U^{n,-m}\right)\right|^2=d^2\sum_{n,m}|c_{nm}|^2=d\,\mathrm{Tr}\rho^2. \tag{B3}$$

Therefore, we get

$$V(\rho,\mathcal{E}_{\rho_w})=1-\frac{d-\phi}{d^2-1}\mathrm{Tr}\rho^2-\frac{(d\phi-1)}{d^3-d}\sum_{2m\equiv 0(\bmod d)}(-1)^{2mn/d}|\mathrm{Tr}(\rho U^{n,-m})|^2. \tag{B4}$$

When the dimension d is odd, we have

$$V(\rho,\mathcal{E}_{\rho_w})=1-\frac{d-\phi}{d^2-1}\mathrm{Tr}\rho^2-\frac{(d\phi-1)}{d^3-d}\sum_n|\mathrm{Tr}(\rho U^{n,0})|^2=1-\frac{d-\phi}{d^2-1}\mathrm{Tr}\rho^2-\frac{(d\phi-1)}{d^2-1}\sum_j\langle j|\rho|j\rangle^2, \tag{B5}$$

where the last equation can be obtained by the following equations:

$$\begin{aligned}
&\sum_n|\mathrm{Tr}(\rho U^{n,0})|^2=\sum_n\mathrm{Tr}(\rho U^{n,0})\,\mathrm{Tr}(\rho U^{n,0^\dagger})=\sum_n\mathrm{Tr}[(\rho\otimes\rho)(U^{n,0}\otimes U^{n,0^\dagger})]\\
&=\mathrm{Tr}(\rho\otimes\rho)\left[\sum_n\left(\sum_j\exp\left(\frac{2\pi\mathrm{i}nj}{d}\right)|j\rangle\langle j|\right)\otimes\left(\sum_k\exp\left(-\frac{2\pi\mathrm{i}nk}{d}\right)|k\rangle\langle k|\right)\right]\\
&=\mathrm{Tr}(\rho\otimes\rho)\left[\sum_{j,k}\left(\sum_n\exp\left(\frac{2\pi\mathrm{i}n(j-k)}{d}\right)\right)|j\rangle\langle j|\otimes|k\rangle\langle k|\right]\\
&=\mathrm{Tr}(\rho\otimes\rho)\left[\sum_j d|j\rangle\langle j|\otimes|j\rangle\langle j|\right]=d\sum_j\mathrm{Tr}\left[\rho|j\rangle\langle j|\otimes\rho|j\rangle\langle j|\right]=d\sum_j(\mathrm{Tr}\rho|j\rangle\langle j|)^2=d\sum_j\langle j|\rho|j\rangle^2.
\end{aligned} \tag{B6}$$

When the dimension d is even, we have

$$\begin{aligned}
V(\rho,\mathcal{E}_{\rho_w})&=1-\frac{d-\phi}{d^2-1}\mathrm{Tr}\rho^2-\frac{(d\phi-1)}{d^3-d}\sum_n\left[|\mathrm{Tr}(\rho U^{n,0})|^2+(-1)^n|\mathrm{Tr}(\rho U^{n,-d/2})|^2\right]\\
&=1-\frac{d-\phi}{d^2-1}\mathrm{Tr}\rho^2-\frac{(d\phi-1)}{d^2-1}\sum_j\left[\langle j|\rho|j\rangle^2+\langle(j+\frac{d}{2})\bmod d|\rho|j\rangle^2\right].
\end{aligned} \tag{B7}$$

The last equation can be obtained by similar method in Eq. (B6).

Property B3 Prove that for the SQT channel induced by Werner state, the state-dependent quantum uncertainty is

$$Q(\rho,\mathcal{E}_{\rho_w})=\begin{cases}1-\dfrac{(d-\phi)(\mathrm{Tr}\sqrt{\rho})^2}{d^2-1}-\dfrac{(d\phi-1)\sum_j\langle j|\sqrt{\rho}|j\rangle^2}{d^2-1}, & d\text{ is odd},\\ 1-\dfrac{(d-\phi)(\mathrm{Tr}\sqrt{\rho})^2}{d^2-1}-\dfrac{(d\phi-1)\sum_j[\langle j|\sqrt{\rho}|j\rangle^2+\langle(j+d/2)\bmod d|\sqrt{\rho}|j\rangle^2]}{d^2-1}, & d\text{ is even}.\end{cases} \tag{B8}$$

Proof As the SQT channel is unital, the expression of the state-dependent quantum uncertainty is

$$Q(\rho,\mathcal{E}_{\rho_w})=1-\sum_{n,m}p_{nm}\mathrm{Tr}(\sqrt{\rho}U^{n,-m}\sqrt{\rho}U^{n,-m^\dagger}). \tag{B9}$$

Using the fact that $\mathrm{Tr}(U^{r,-s\dagger}U^{n,-m})=d\delta_{rn}\delta_{sm}$ and $\sum_{n,m}U^{n,-m}\sqrt{\rho}U^{n,-m^\dagger}=d(\mathrm{Tr}\sqrt{\rho})I$, we get

$$\begin{aligned}
Q(\rho,\mathcal{E}_{\rho_w})&=1-\frac{d-\phi}{d^3-d}\sum_{n,m}\mathrm{Tr}\left[\sqrt{\rho}U^{n,-m}\sqrt{\rho}U^{n,-m^\dagger}\right]-\frac{d\phi-1}{d^3-d}\sum_{2m\equiv 0(\bmod d)}(-1)^{2mn/d}\,\mathrm{Tr}\left[\sqrt{\rho}U^{n,-m}\sqrt{\rho}U^{n,-m^\dagger}\right]\\
&=1-\frac{d-\phi}{d^2-1}(\mathrm{Tr}\sqrt{\rho})^2-\frac{d\phi-1}{d^3-d}\sum_{2m\equiv 0(\bmod d)}(-1)^{2mn/d}\,\mathrm{Tr}\left[\sqrt{\rho}U^{n,-m}\sqrt{\rho}U^{n,-m^\dagger}\right].
\end{aligned} \tag{B10}$$

Note that when $m=0$, there exists

$$\sum_n U^{n,0}\sqrt{\rho}U^{n,0^\dagger}=\sum_n\left(\sum_{j,k}\exp\left(\frac{2\pi\mathrm{i}n(j-k)}{d}\right)|j\rangle\langle j|\sqrt{\rho}|k\rangle\langle k|\right)=d\sum_j|j\rangle\langle j|\sqrt{\rho}|j\rangle\langle j|. \tag{B11}$$

Similarly, when $2m = d$, we have

$$\begin{aligned}\sum_n (-1)^n U^{n,-d/2}\sqrt{\rho}(U^{n,-d/2})^\dagger &= \sum_n (-1)^n \left(\sum_{j,k} \exp\left(\frac{2\pi i n(j-k)}{d}\right)|j\rangle\left\langle\left(j-\frac{d}{2}\right) \bmod d\right|\sqrt{\rho}\left|\left(k-\frac{d}{2}\right) \bmod d\right\rangle\langle k|\right) \\ &= \sum_{j,k}\left(\sum_n \exp\left(\frac{2\pi i n(j+d/2-k)}{d}\right)\right)|j\rangle\left\langle\left(j-\frac{d}{2}\right) \bmod d\right|\sqrt{\rho}\left|\left(k-\frac{d}{2}\right) \bmod d\right\rangle\langle k| \\ &= d\sum_j |j\rangle\left\langle\left(j+\frac{d}{2}\right) \bmod d\right|\sqrt{\rho}|j\rangle\left\langle\left(j+\frac{d}{2}\right) \bmod d\right|. \end{aligned} \tag{B12}$$

As a result, when d is odd, we get

$$Q(\rho,\mathcal{E}_{\rho_w}) = 1-\frac{d-\phi}{d^2-1}(\mathrm{Tr}\sqrt{\rho})^2 - \frac{d\phi-1}{d^3-d}\mathrm{Tr}\left[\sqrt{\rho}(\sum_n U^{n,0}\sqrt{\rho}U^{n,0^\dagger})\right] = 1-\frac{d-\phi}{d^2-1}(\mathrm{Tr}\sqrt{\rho})^2 - \frac{d\phi-1}{d^2-1}\sum_j \langle j|\sqrt{\rho}|j\rangle^2, \tag{B13}$$

when d is even, we have

$$\begin{aligned} Q(\rho,\mathcal{E}_{\rho_w}) &= 1-\frac{d-\phi}{d^2-1}(\mathrm{Tr}\sqrt{\rho})^2 - \frac{d\phi-1}{d^3-d}\mathrm{Tr}\left[\sqrt{\rho}\sum_n\left(U^{n,0}\sqrt{\rho}\,U^{n,0^\dagger} + (-1)^n U^{n,-d/2}\sqrt{\rho}(U^{n,-d/2})^\dagger\right)\right] \\ &= 1-\frac{d-\phi}{d^2-1}(\mathrm{Tr}\sqrt{\rho})^2 - \frac{d\phi-1}{d^2-1}\sum_j\left[\langle j|\sqrt{\rho}|j\rangle^2 + \left\langle\left(j+\frac{d}{2}\right) \bmod d\right|\sqrt{\rho}|j\rangle^2\right]. \end{aligned} \tag{B14}$$

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References

[1] Bennett C H, Brassard G, Crépeau C, *et al*. 1993 *Phys. Rev. Lett.* **70** 016201
[2] Bouwmeester D, Pan J W, Mattle K, *et al*. 1997 *Nature* **390** 575
[3] Pirandola S, Eisert J, Weedbrook C, *et al*. 2015 *Nat. Photon.* **9** 641
[4] Lu C Y, Cao Y, Peng C Z, *et al*. 2022 *Rev. Mod. Phys* **94** 035001
[5] Chen X, Zhang J H and Zhang F L 2022 *Chin. Phys. B* **31** 030302
[6] Li H Y, Wei Y Z, *et al*. 2022 *Chin. Phys. B* **31** 020302
[7] Liu X F, Li D F, *et al*. 2022 *Chin. Phys. B* **31** 050301
[8] Hu X M, Guo Y, Liu B H, *et al*. 2023 *Nat. Rev. Phys.* **5** 339
[9] Liu Z D, Siltanen O, Kuusela T, *et al*. 2024 *Sci. Adv.* **10** eadj3435
[10] Wehner S, Elkouss D and Hanson R 2018 *Science* **362** eaam9288
[11] Bowen G and Bose S 2001 *Phys. Rev. Lett.* **87** 267901
[12] Nielsen M A and Chuang I L 2001 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[13] Holevo A S and Giovannetti V 2012 *Rep. Prog. Phys.* **75** 046001
[14] Caruso F, Giovannetti V, Lupo C, *et al*. 2014 *Rev. Mod. Phys.* **86** 1203
[15] Li L, Bu K F and Liu Z W 2020 *Phys. Rev. A* **101** 022335
[16] Liu Y and Yuan X 2020 *Phys. Rev. Res.* **2** 012035
[17] Zhang J, He K, Zhang H, *et al*. 2023 *Chin. Phys. B* **32** 090301
[18] Jebli L, Amzioug M, Ennadifi S E, *et al*. 2020 *Chin. Phys. B* **29** 110301
[19] Luo S L, Fu S S and Li N 2010 *Phys. Rev. A* **82** 052122
[20] Mani A and Karimipour V 2015 *Phys. Rev. A* **92** 032331
[21] Bu K F, Kumar A, Zhang L, *et al*. 2017 *Phys. Lett. A* **381** 1670
[22] Zanardi P, Styliaris G and Campos Venuti L 2017 *Phys. Rev. A* **95** 052306
[23] Zhang L, Ma Z H, Chen Z H, *et al*. 2018 *Quantum Inf. Process* **17** 1
[24] Styliaris G, Campos Venuti L and Zanardi P 2018 *Phys. Rev. A* **97** 032304
[25] Yang S R and Yu C S 2023 *Chin. Phys. B* **32** 040305
[26] Heisenberg W 1927 *Zeitschrift für Physik* **43** 172
[27] Robertson H P 1929 *Phys. Rev.* **34** 163
[28] Schröinger E 1930 *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse* **14** 296
[29] Wigner E P and Yanase M M 1963 *Proc. Natl. Acad. Sci.* **49** 910
[30] Luo S L 2005 *Phys. Rev. A* **72** 042110
[31] Zhang F G and Li Y M 2018 *Chin. Phys. B* **27** 090301
[32] Huang X F, Zhang T G and Jing N H 2020 *Commun. Theor. Phys.* **72** 075101
[33] Wu Z Q, Huang H J, Fei S M, *et al*. 2020 *Commun. Theor. Phys.* **72** 105102
[34] Zhang L, Huang J P, Wang J M, *et al*. 2022 *Commun. Theor. Phys.* **74** 075102
[35] Xu C, Wu Z Q and Fei S M 2024 *Commun. Theor. Phys.* **76** 035102
[36] Deutsch D 1983 *Phys. Rev. Lett.* **50** 631
[37] Maassen H and Uffink J B M 1988 *Phys. Rev. Lett.* **60** 1103
[38] Ming F, Wang D, Fan X G, *et al*. 2020 *Phys. Rev. A* **102** 012206
[39] Xie B F, Ming F, Wang D, *et al*. 2021 *Phys. Rev. A* **104** 062204
[40] Wu L, Ye L and Wang D 2022 *Phys. Rev. A* **102** 062219
[41] Li L J, Ming F, Song X K, *et al*. 2022 *Acta Phys. Sin.* **71** 070302 (in Chinese)
[42] Wang T Y and Wang D 2024 *Phys. Lett. B* **855** 138876
[43] Dou Y N and Du H K 2013 *J. Math. Phys.* **54** 103508
[44] Dou Y N and Du H K 2014 *Int. J. Theor. Phys.* **53** 952
[45] Luo S L and Sun Y 2019 *Commun. Theor. Phys.* **71** 1443
[46] Sun Y and Li N 2021 *Quantum Inf. Process* **20** 25
[47] Zhang L M, Gao T and Yan F L 2021 *Phys. Lett. A* **387** 127029
[48] Zhang J, Wei J N, Duan Z B, *et al*. 2021 *Commun. Theor. Phys.* **74** 015101
[49] Li H, Gao T and Yan F L 2023 *Phys. Scr.* **98** 015024
[50] Liu Y Z, Luo S L and Sun Y 2022 *Theor. Math. Phys.* **213** 1613
[51] Knill E and Laflamme R 1998 *Phys. Rev. Lett.* **81** 5672
[52] Fu S S, He J Y, Li X H, *et al*. 2023 *Phys. Scr.* **98** 045114
[53] Horodecki M and Horodecki P 1999 *Phys. Rev. A* **59** 4206
[54] Rains E M 1999 *Phys. Rev. A* **60** 179
[55] Rains E M 2000 *Phys. Rev. A* **63** 019902
[56] Bertlmann R A and Krammer P 2008 *J. Phys. A Math. Theor.* **41** 235303
[57] Peters N A, Wei T C and Kwiat P G 2004 *Phys. Rev. A* **70** 052309
[58] Yu C S, Yang S R and Guo B Q 2016 *Quantum Inf. Process* **15** 3773
[59] Luo S L and Sun Y 2019 *Phys. Lett. A* **383** 2869
[60] Zhang Q H and Fei S M 2024 *J. Phys. A: Math. Theor.* **57** 235301
[61] Werner R F 1989 *Phys. Rev. A* **40** 4277