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New interaction solutions in Mel'nikov equation obtained by modulating the phase shift

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The degradation and nonlinear interactions of a two-breather solution of the Mel'nikov equation are analyzed. By modulating the phase shift and limit method, we prove that in different regions near the non-singular boundaries, there are four kinds of solutions with repulsive interaction or attractive interaction in addition to the two-breather solution. They are the interaction solution between soliton and breather, the two-soliton solution, and the two-breather solution with small amplitude, which all exhibit repulsive interactions; and the two-breather solution with small amplitude, which exhibits attractive interaction. Interestingly, a new breather acts as a messenger to transfer energy during the interaction between two breather solutions with small amplitude.

Keywords: Mel'nikov equation, breather solution, phase shift

PACS: 02.30.IK, 05.45.Yv, 02.30.Jr

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1. Introduction

Breathers, also known as periodic solitons, are one of the most important soliton waves in the soliton family. They are nonlinear waves with periodic structures in time or space, and such local waves are ubiquitous phenomena in many physical systems, whether discrete or continuous,[1–5] and they are applied to a lot of partial differential equations.[6–9] In the late 1970s, Kuznetsov and Ma found a local wave packet named the Kuznetsov–Ma breather which is different from the usual bright soliton.[10] This wave is localized in space and periodic in time. After that, Akhmediev found nonlinear local waves with different structures from Kuznetsov–Ma breathers, named Akhmediev breathers.[11] The so-called intrinsic discrete breathers in Fermi–Pasta–Ulam lattices were reported in late 1980s.[12] The presence of modulation instability (MI) supports the generation of rogue waves (RWs).[13–16]

In recent years, nonlinear interactions of breathers have also become a hot topic among researchers. Miles found that the multi-soliton solutions of Kadomtsev–Petviashvili (KP) equation and Korteweg–de Vries (KdV) equation in some parameter regions are singular and they exhibit resonant interactions.[17] Tajiri and Murakami obtained two types of interactions between two y-periodic solitons:[18,19] attractive interaction and long-range interaction or extremely repulsive interaction. Lin and Kruglov studied molecule structures[20] and a new family[9] of breathers respectively. In particular, nonlinear interactions of breathers play an important role in the long- and short-wave systems. Although the frequencies and wave numbers of long-wave and short-wave differ greatly, their phase and group velocities are so close that there can be nonlinear interactions between the two waves. Recently, He studied resonant collisions among multi-breathers in the Mel'nikov system.[21] Liu and Li studied the dynamics of breathers on a periodic waves background in the nonlocal Mel'nikov equation.[22]

The fundamental model for the coupling system is provided by Zakharov system, which includes the system composed of short-wave linear Schrödinger equation nonlinearly coupled to long-wave Boussinesq equation.[23] The explicit form of coupled Schrödinger–Boussinesq system

$$u_{xx}+u_{xxxx}-3u_{xx}^2-3u_{tt}+\kappa|\Phi|_{xx}=0,$$
$$\mathrm{i}\Phi_t=u\Phi+\Phi_{xx}$$

can be extended to (2+1)-dimensional Mel'nikov equation[24]

$$3u_{yy}-u_{xt}-\left[3u^2+u_{xx}+\kappa|\Phi|\right]_{xx}=0,\ \mathrm{i}\Phi_y=u\Phi+\Phi_{xx},\quad(1)$$

by a special transformation

$$\Phi(x,y,x)=\Phi(x,y,t),\quad u(x,y,x)=u(x,y,t),$$

where κ is a constant, u and Φ are functions of x, y and t. Here u stands for long-wave component and Φ stands for short-wave component.

Mel'nikov equation (1) was proposed by Mel'nikov, which describes the interaction of long- and short-wave packets in the (x,y) space and it is a generalization of complex scalar KP equation or an extension of real scalar NLS

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equation.[24,25] Soliton solutions of Mel'nikov equation (1) have been reported by Hase using Hirota's bilinear method.[26] The exponentially localized solutions and N-rogue waves of Mel'nikov equation (1) have been studied.[27,28] Zhang has investigated Hybrid solutions of Eq. (1), he analyzed the interactions of dark soliton and anti-dark soliton, lump and line rogue wave, and two-breather solution, respectively.[29] Rao and Xu have investigated, respectively, dynamics of interaction among lumps, solitons and breathers of Mel'nikov equation (1).[30,31] By introducing the transformation

$$u = 2(\ln f)_{xx}, \quad \Phi = \sqrt{2}\frac{g}{f},$$

where f and g are functions of x, y and t, Eq. (1) can be changed into bilinear forms[26]

$$\begin{aligned}&(D_x^2 - \mathrm{i}D_y)g\cdot f = 0,\\ &(D_x^4 + D_xD_t - 3D_y^2)f\cdot f = 2\kappa[f^2 - gg^*].\end{aligned}$$

The standard identities for the Hirota D-operators[32]

$$\begin{aligned}&D_x^m D_y^n D_t^k f(x,y,t)\cdot g(x,y,t)\\ =\ &(\partial_x - \partial_{x'})^m(\partial_y - \partial_{y'})^n(\partial_t - \partial_{t'})^k\\ &\times f(x,y,t)g(x',y',t')|_{x'=x,y'=y,t'=t}.\end{aligned}$$

Using the Hirota direct method,[32] the multi-solitons of Eq. (1) can be derived as the following exponential forms:

$$u_N = 2(\ln f_N)_{xx}, \quad \Phi_N = \sqrt{2}\frac{g_N}{f_N}, \tag{2}$$

$$f_N = \sum_{\sigma=0,1} \mathrm{e}^{\left(\sum_{j=1}^{2N}\sigma_j\Psi_j + \sum_{1\le j<s}^{2N}\sigma_j\sigma_s A_{js}\right)},$$

$$g_N = \sum_{\sigma=0,1} \mathrm{e}^{\left(\sum_{j=1}^{2N}\sigma_j(\Psi_j+\ln\varpi_j) + \sum_{1\le j<s}^{2N}(\sigma_j+\ln\varpi_j)(\sigma_s+\ln\varpi_s)A_{js}\right)}, \tag{3}$$

with

$$\begin{aligned}&\Psi_j = k_j x + \omega_j t + p_j y + \Psi_j^0,\\ &\varpi_j = \frac{\mathrm{i}p_j + {k_j}^2}{\mathrm{i}p_j - {k_j}^2}, \quad \omega_j = \frac{3{p_j}^2}{k_j} - {k_j}^3 + \frac{4\kappa {k_j}^3}{{k_j}^4 + {p_j}^2},\\ &\mathrm{e}^{(A_{js})} = \frac{{k_j}^2{k_s}^2(k_j-k_s)^2 + (k_jp_s - k_sp_j)^2}{{k_j}^2{k_s}^2(k_j+k_s)^2 + (k_jp_s - k_sp_j)^2}.\end{aligned}$$

When parameters

$$k_{2j-1} = k_{2j}^*, \quad p_{2j-1} = p_{2j}^*, \quad \Psi_{2j-1}^0 = \Psi_{2j}^{0\,*}, \quad j = 1,2,\ldots, \tag{4}$$

and the above complex relationships lead to $\omega_{2j-1} = \omega_{2j}^*$ and $\Psi_{2j-1} = \Psi_{2j}^*$, then the multi-solitons (2) degenerate into the multi-breather solutions of Eq. (1).

In this work, we discuss waveforms and nonlinear interactions of the two-breather solution of Eq. (1) in the parameter regions near the singular boundaries of phase shift. In Section 2, we analyze the phase shift of the breather solution and obtain the non-singular boundaries of the two-breather solution. In Section 3, by modulating the phase shift, it is proved that in the regions near the non-singular boundaries of the phase shift, in addition to the two-breather solution, there are three new kinds of solutions. In Section 4, some conclusions and discussion are given.

2. Two-breather solution

2.1. One-breather solution

When we substitute $N=2$, $k_1 = k_2^* = \alpha_1 + \mathrm{i}\beta_1, w_1 = w_2^* = \omega_{\mathrm{r}1} + \mathrm{i}\omega_{\mathrm{i}1}, p_1 = p_2^* = \rho_1 + \mathrm{i}\sigma_1, \phi_1 = \phi_2^* = \xi_{01} - \ln(\sqrt{D}) + \mathrm{i}\eta_{01}, \varpi_1/\lambda_1 = \varpi_2\lambda_1 = \mathrm{e}^{\mathrm{i}\emptyset_1}$ into solutions (3) and then breather solutions of Mel'nikov equation are as follows:

$$u_1 = \frac{2\left(2\frac{\alpha_1\beta_1\sinh\xi\sin\eta}{\sqrt{D}} - \frac{({\beta_1}^2 - {\alpha_1}^2)\cos\eta\cosh\xi}{\sqrt{D}} + {\alpha_1}^2 - \frac{{\beta_1}^2}{D}\right)}{\left(\frac{\cos\eta}{\sqrt{D}} + \cosh\xi\right)^2},$$

$$\Phi_1 = \frac{\sqrt{2}\mathrm{e}^{\mathrm{i}\emptyset_1}\left(\cosh(\xi + \mathrm{i}\emptyset_1) + \frac{\cosh(\ln\lambda_1 + \mathrm{i}\eta))}{\sqrt{A_{12}}}\right)}{\cosh\xi + \frac{\cosh\eta}{\sqrt{A_{12}}}}, \tag{5}$$

with

$$\begin{aligned}&\xi = \alpha_1 x + \rho_1 y + \omega_{\mathrm{r}1}t + \xi_{01},\\ &\eta = \beta_1 x + \sigma_1 y + \omega_{i1}t + \eta_{01},\\ &D = \frac{(\alpha_1\sigma_1 - \beta_1\rho_1)^2 + {\beta_1}^2\left({\alpha_1}^2 + {\beta_1}^2\right)^2}{(\alpha_1\sigma_1 - \beta_1\rho_1)^2 - {\alpha_1}^2\left({\alpha_1}^2 + {\beta_1}^2\right)^2}.\end{aligned}$$

The existence condition for the non-singular solutions (5) is

$$\left(\frac{\alpha_1\sigma_1 - \beta_1\rho_1}{{\alpha_1}^2 + {\beta_1}^2}\right)^2 > \alpha_1^2.$$

As $\left(\frac{\alpha_1\sigma_1 - \beta_1\rho_1}{{\alpha_1}^2+{\beta_1}^2}\right)^2 \to \alpha_1^2$, $D\to\infty$ and

$$2\omega_{\mathrm{r}1} \to \omega_L = -\frac{1}{2}\left((2\alpha_1)^3 - 3\frac{(2\rho_1)^2}{2\alpha_1}\right),$$

the breather solutions (5) reduce to solitons

$$u_{\mathrm{sol}} = 2{\alpha_1}^2\mathrm{sech}^2\xi, \quad \Phi_{\mathrm{sol}} = \frac{\sqrt{2}\mathrm{e}^{\mathrm{i}\emptyset_1}\cosh(\xi + \mathrm{i}\emptyset_1)}{\cosh\xi}, \tag{6}$$

with the wave numbers $(2\alpha_1, 2\rho_1)$.

It can be easily proved that when $D \gg 1$, the breather solutions (5) degenerate to the soliton solutions (6) of Mel'nikov equation (1). It can be seen from Fig. 1 that when $D \gg 1$, there is little difference between breather and soliton. But the interaction of breather solution may be very different from the soliton, even if the amplitude of breather solution is very small.

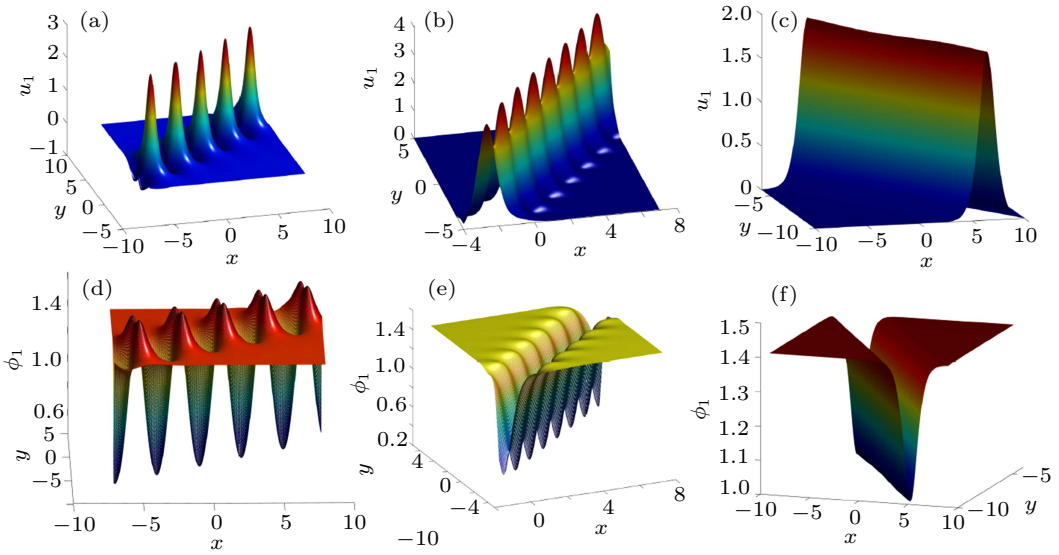


Fig. 1. The breather solution (5) with $D \gg 1$. (a) $(\alpha_1,\beta_1,\sigma_1,\rho_1) = (0.7,-1.6,-1.2,-0.8)$, $D = 18.82$, (b) $(\alpha_1,\beta_1,\sigma_1,\rho_1) = (1.29,-1.2,-1.7,-3)$, $D = 2.29\times10^2$, (c) $(\alpha_1,\beta_1,\sigma_1,\rho_1) = (0.98,0.5,1.96,-0.21045)$, $D = 1.526\times10^4$, (d) $(\alpha_1,\beta_1,\sigma_1,\rho_1) = (0.7,-1.6,-1.2,-0.8)$, $D = 18.82$, (e) $(\alpha_1,\beta_1,\sigma_1,\rho_1) = (1.29,-1.2,-1.7,-3)$, $D = 2.29\times10^2$ d, (f) $(\alpha_1,\beta_1,\sigma_1,\rho_1) = (0.98,0.5,1.96,-0.21045)$, $D = 1.526\times10^4$.

2.2. Two-breather solution and its phase shift

When $\kappa = 1$, two-breather solutions can be obtained based on the multi-soliton solutions (2) of Mel'nikov equation (1),

$$u_2 = 2(\ln f_2)_x, \quad \Phi_2 = \sqrt{2}\frac{g_2}{f_2}, \tag{7}$$

with

$$\begin{aligned} f_2 \;=\; & 1+2\,\mathrm{e}^{\xi_2}\cos(\eta_2)+2\mathrm{e}^{\xi_1}\cos(\eta_1) \\ & +2L_2\,\mathrm{e}^{\xi_1+\xi_2}\cos(\eta_2-\eta_1-\theta_2) \\ & +2L_1\,\mathrm{e}^{\xi_1+\xi_2}\cos(\eta_2+\eta_1+\theta_1)+D_1\,\mathrm{e}^{2\xi_1} \\ & +2D_1L_1L_2\,\mathrm{e}^{2\xi_1+\xi_2}\cos(\eta_2+\theta_1-\theta_2) \\ & +2D_2L_1L_2\,\mathrm{e}^{2\xi_2+\xi_1}\cos(\theta_2+\theta_1+\eta_1) \\ & +D_2\,\mathrm{e}^{2\xi_2}+{L_1}^2{L_2}^2D_1D_2\,\mathrm{e}^{2(\xi_1+\xi_2)}, \end{aligned} \tag{8}$$

$$\begin{aligned} g_2 \;=\; & 1+\mathrm{e}^{2(\mathrm{i}(\phi_1+\phi_2)+\xi_1+\xi_2)}D_1D_2{L_1}^2{L_2}^2 \\ & +\mathrm{e}^{2\mathrm{i}\phi_2+\xi_2}D_2+\mathrm{e}^{2\mathrm{i}\phi_1+2\xi_1}D_1 \\ & +2\mathrm{e}^{\mathrm{i}\phi_2+\xi_2}\cosh(\mathrm{i}\eta_2+\ln\lambda_2) \\ & +2\mathrm{e}^{\mathrm{i}\phi_1+\xi_1}\cosh(\mathrm{i}\eta_1+\ln\lambda_1) \\ & +2\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)+\xi_1+\xi_2}L_2 \\ & \times\cosh\left(\mathrm{i}(\eta_2-\theta_2-\eta_1)+\ln\left(\frac{\lambda_2}{\lambda_1}\right)\right) \\ & +2\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)+\xi_1+\xi_2}L_1 \\ & \times\cosh(\mathrm{i}(\theta_1+\eta_1+\eta_2)+\ln(\lambda_1\lambda_2)) \\ & +2\mathrm{e}^{\mathrm{i}(\phi_1+2\phi_2)+\xi_1+2\xi_2}L_1L_2D_2 \\ & \times\cosh(\mathrm{i}(\theta_1+\theta_2+\eta_1)+\ln\lambda_1) \\ & +2\mathrm{e}^{\mathrm{i}(2\phi_1+\phi_2)+2\xi_1+\xi_2}D_1L_1L_2 \\ & \times\cosh(\mathrm{i}(\theta_1-\theta_2+\eta_2)+\ln\lambda_2), \end{aligned} \tag{9}$$

where

$$\begin{aligned} & \xi_j = \alpha_j x+\rho_j y+\omega_{\mathrm{r}j}t+\xi_{0j}, \\ & \eta_j = \gamma_j x+\sigma_j y+\omega_{ij}t+\eta_{0j}, \\ & {\Upsilon_j}^2 = \left(\frac{\gamma_j\rho_j-\alpha_j\sigma_j}{{\alpha_j}^2+{\gamma_j}^2}\right)^2, \\ & D_j = \frac{{\Upsilon_j}^2+{\gamma_j}^2}{{\Upsilon_j}^2-{\alpha_j}^2} \quad (j=1,2), \\ & \varpi_j = \omega_{\mathrm{r}j}+\mathrm{i}\omega_{ij}, \quad \frac{\varpi_1}{\lambda_1} = \varpi_2\lambda_1 = \mathrm{e}^{\mathrm{i}\phi_1}, \\ & \frac{\varpi_3}{\lambda_2} = \varpi_4\lambda_2 = \mathrm{e}^{\mathrm{i}\phi_2}, \end{aligned}$$

and

$$\begin{aligned} L_1\,\mathrm{e}^{\mathrm{i}\theta_1} &= \frac{(\alpha_1+\mathrm{i}\gamma_1-\alpha_2-\mathrm{i}\gamma_2)^2+\left(\frac{\rho_2+\mathrm{i}\sigma_2}{\alpha_2+\mathrm{i}\gamma_2}-\frac{\rho_1+\mathrm{i}\sigma_1}{\alpha_1+\mathrm{i}\gamma_1}\right)^2}{(\alpha_1+\mathrm{i}\gamma_1+\alpha_2+\mathrm{i}\gamma_2)^2+\left(\frac{\rho_2+\mathrm{i}\sigma_2}{\alpha_2+\mathrm{i}\gamma_2}-\frac{\rho_1+\mathrm{i}\sigma_1}{\alpha_1+\mathrm{i}\gamma_1}\right)^2}, \\ L_2\,\mathrm{e}^{\mathrm{i}\theta_2} &= \frac{(\alpha_1+\mathrm{i}\gamma_1-\alpha_2+\mathrm{i}\gamma_2)^2+\left(\frac{\rho_2-\mathrm{i}\sigma_2}{\alpha_2-\mathrm{i}\gamma_2}-\frac{\rho_1+\mathrm{i}\sigma_1}{\alpha_1+\mathrm{i}\gamma_1}\right)^2}{(\alpha_1+\mathrm{i}\gamma_1+\alpha_2-\mathrm{i}\gamma_2)^2+\left(\frac{\rho_2-\mathrm{i}\sigma_2}{\alpha_2-\mathrm{i}\gamma_2}-\frac{\rho_1+\mathrm{i}\sigma_1}{\alpha_1+\mathrm{i}\gamma_1}\right)^2}. \end{aligned} \tag{10}$$

From Eqs. (8) and (9), it is clear that one breather moves τ after interaction and the other moves τ in the opposite direction.

Theorem 1 When two solitons move in the same direction on the x-axis, the high wave jumps backward and the low wave jumps forward to form an attraction phenomenon, it is called attractive interaction and phase shift $\tau > 0$.

Theorem 2 When two solitons move in the same direction on the x-axis, the high wave jumps forward and the low wave jumps backward to form a repulsive phenomenon, this interaction is called repulsive interaction and phase shift $\tau < 0$.

In this paper, we mainly consider the degeneracy of two-breather solution (8) and they show repulsive interaction. From Eq. (10), we can obtain

$$L_1^2L_2^2 = \frac{\left((K_1+\Upsilon_1-\Upsilon_2)^2+(\chi+E_1)^2\right)\left((K_1-\Upsilon_1+\Upsilon_2)^2+(\chi-E_1)^2\right)}{\left((K_2+\Upsilon_1-\Upsilon_2)^2+(\chi+E_2)^2\right)\left((K_2-\Upsilon_1+\Upsilon_2)^2+(\chi-E_2)^2\right)} \times \frac{\left((K_1-\Upsilon_1-\Upsilon_2)^2+(\chi-E_2)^2\right)\left((K_1+\Upsilon_1+\Upsilon_2)^2+(\chi+E_2)^2\right)}{\left((K_2-\Upsilon_1-\Upsilon_2)^2+(\chi-E_1)^2\right)\left((K_2+\Upsilon_1+\Upsilon_2)^2+(\chi+E_1)^2\right)}, \tag{11}$$

where $\varsigma_j = \frac{\alpha_j\rho_j+\gamma_j\sigma_j}{\alpha_j^2+\gamma_j^2}$ $(j=1,2)$, $K_1=\alpha_1-\alpha_2$, $K_2=\alpha_1+\alpha_2$, $E_1=\gamma_1-\gamma_2$, $E_2=\gamma_1+\gamma_2$, $\chi=\varsigma_2-\varsigma_1$.

From Theorem 2, we know that when $|L_1L_2|<1$ or $|L_1L_2|\to 0$, the two-breather solutions (7) have repulsive interaction or long-range interaction and when $|L_1L_2|>1$ or $|L_1L_2|\to\infty$, the solutions have attractive interaction or resonant interaction. We call these parameter relations as singular boundaries of the solutions (7) at $L_1^2L_2^2=0$ or ∞. Based on Eq. (11), we can have the conditions of singular interaction:

$$\Upsilon_2 = \Upsilon_1 \pm (\alpha_1+\alpha_2), \tag{12a}$$
$$\varsigma_2 = \varsigma_1 \mp (\gamma_1+\gamma_2), \tag{12b}$$

$$\Upsilon_2 = -\Upsilon_1 \pm (\alpha_1+\alpha_2), \tag{13a}$$
$$\varsigma_2 = \varsigma_1 \pm (\gamma_1-\gamma_2), \tag{13b}$$

or

$$\Upsilon_2 = \Upsilon_1 \pm (\alpha_1-\alpha_2), \tag{14a}$$
$$\varsigma_2 = \varsigma_1 \mp (\gamma_1-\gamma_2), \tag{14b}$$

$$\Upsilon_2 = -\Upsilon_1 \pm (\alpha_1-\alpha_2), \tag{15a}$$
$$\varsigma_2 = \varsigma_1 \pm (\gamma_1+\gamma_2). \tag{15b}$$

We define the first conditions as Eqs. (12a), (13a), (14a) and (15a), and likewise define Eqs. (12b), (13b), (14b) and (15b) as the second condition corresponding to the first condition respectively. The first condition and the second condition are singular boundaries of the two-breather solution. We label Eqs. (12a) and (13a) as lines b_1, b_2 and b_3, b_4, respectively, and label Eqs. (14a) and (15b) as lines l_1, l_2 and l_3, l_4, respectively. In each coupling equation, the upper and lower indices ($\pm$ or $\mp$) in the first condition correspond to the upper and lower indices ($\pm$ or $\mp$) in its second condition, respectively. According to the above boundary conditions, we establish the parametric coordinate systems about (Υ_1,Υ_2) and $(\varsigma_1,\varsigma_2)$ respectively in Fig. 2. The parameter space of Υi $(i=1,2)$ in Fig. 2(a) has four symmetric regions, so we assume parameters $\alpha_1>0$ and $\alpha_2>0$, and we only need to consider the behaviors of the two-breather solution at $\Upsilon_2>0$ and $\Upsilon_1<0$. More precisely, in this paper we only consider the repulsive interactions of the two-breather solutions in Fig. 2(a).

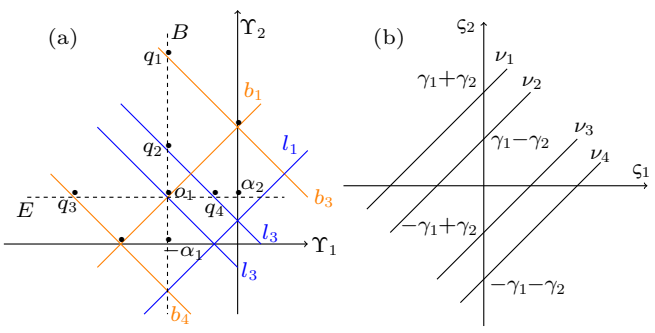


Fig. 2. A cross section of parameter space of the two-breather solution (8) and the two-breather solution (9).

2.3. Two-breather solution degenerating into breather-soliton solution

Based on the two coordinate spaces in Fig. 2, we choose

$$\begin{cases} \Upsilon_1 = -\alpha_1+\varepsilon, \\ \varsigma_1 = -\gamma_1+\dfrac{\rho_1}{\alpha_1}+\dfrac{\varepsilon\gamma_1}{\alpha_1}, \end{cases} \tag{16}$$

or

$$\begin{cases} \Upsilon_2 = \alpha_2+\varepsilon, \\ \varsigma_2 = \gamma_2+\dfrac{\rho_2}{\alpha_2}+\dfrac{\varepsilon\gamma_2}{\alpha_2}, \end{cases} \tag{17}$$

where ε is a small parameter. From Eqs. (16) and (17), we know that the solutions are in the the parameter regions near line B and line E respectively. We analyze the two-breather solution (8) and the two-breather solution (9) by condition (17), and then it is easily to know that $D_2\sim O(1/\varepsilon)$. We label $\frac{\sqrt{D_2}}{2}e^{\xi_2}=e^{\widetilde{\xi_2}}$ and choose a parameter ξ_{02} satisfing $\widetilde{\xi_2}\sim O(1)$. Then we neglect the terms that contain $1/\sqrt{D_2}$ in the two-breather solution (8) and the two-breather solution (9):

$$\begin{aligned} f_2 =\ & 1+2e^{\xi_1}\cos(\eta_1)+L_2L_1\,e^{2\widetilde{\xi_2}+\xi_1}\cos(\theta_2+\theta_1+\eta_1) \\ & +\frac{1}{2}e^{2\widetilde{\xi_2}}+D_1e^{2\xi_1}+\frac{1}{2}{L_1}^2{L_2}^2D_1e^{2(\xi_1+\widetilde{\xi_2})}, \end{aligned} \tag{18}$$

$$\begin{aligned} g_2 =\ & 1+e^{i(\emptyset_1+2\emptyset_2)+\xi_1+2\widetilde{\xi_2}}L_1L_2\cosh(i(\theta_1+\theta_2+\eta_1)+\ln\lambda_1) \\ & +\frac{D_1{L_1}^2{L_2}^2}{2}e^{2i(\emptyset_1+\emptyset_2)+2(\xi_1+\widetilde{\xi_2})}+e^{2(i\emptyset_1+\xi_1)}D_1 \\ & +\frac{1}{2}e^{2(i\emptyset_2+\widetilde{\xi_2})}. \end{aligned} \tag{19}$$

We label $2\widetilde{\xi_2}=\widetilde{\xi_L}$ and then consider the phase shift of the solution (18) and the solution (19). The two separated solitons before and after the collision are as follows:

$$\begin{cases} f_{\mathrm{L}} = D_1e^{2\xi_1}\left(1+\dfrac{1}{2}{L_2}^2{L_1}^2e^{\widetilde{\xi_L}}\right), \\ f_{\mathrm{B}} = 1+2e^{\xi_1}\cos(\eta_1)+D_1e^{2\xi_1}, \end{cases} \tag{20}$$

$$\begin{cases} \Phi_{\mathrm{L}} = \sqrt{2}\dfrac{e^{2i\emptyset_1}(2+{L_2}^2{L_1}^2e^{2i\emptyset_2+\widetilde{\xi_L}})}{2+{L_2}^2{L_1}^2e^{\widetilde{\xi_L}}}, \\ \Phi_{\mathrm{B}} = \sqrt{2}\dfrac{1+2e^{2i\emptyset_1+2\xi_1}D_1}{1+2e^{\xi_1}\cos(\eta_1)+D_1e^{2\xi_1}}, \end{cases} \tag{21}$$

and

$$\begin{cases} f_{\rm L} = 1+\dfrac{1}{2}\mathrm{e}^{\widetilde{\xi}_L}, \\ f_{\rm B} = \dfrac{1}{2}\mathrm{e}^{\widetilde{\xi}_L}(1+2L_1L_2\mathrm{e}^{\xi_1}\cos(\eta_1)+{L_2}^2{L_1}^2D_1\mathrm{e}^{2\xi_1}), \end{cases} \tag{22}$$

$$\begin{cases} \Phi_{\rm L} = \sqrt{2}\dfrac{2+\mathrm{e}^{2\mathrm{i}\phi_2+\widetilde{\xi}_L}}{2+\mathrm{e}^{\widetilde{\xi}_L}}, \\ \Phi_{\rm B} = \sqrt{2}\dfrac{\mathrm{e}^{2\mathrm{i}\phi_2}(1+2L_1L_2\mathrm{e}^{\mathrm{i}\phi_1+\xi_1}\cosh(\mathrm{i}\,(\theta_1+\theta_2+\eta_1))+{L_2}^2{L_1}^2D_1\mathrm{e}^{2\mathrm{i}\phi_1+2\xi_1})}{1+2L_1L_2\mathrm{e}^{\xi_1}\cos(\eta_1)+{L_2}^2{L_1}^2D_1\mathrm{e}^{2\xi_1}}. \end{cases} \tag{23}$$

From Eqs. (20)–(23), the waves $f_{\rm L}$ and $\Phi_{\rm L}$ are line solitons and waves $f_{\rm B}$ and $\Phi_{\rm B}$ are breather solutions. That is, the two-breather solutions in the parameter region near the line B and line E in Fig. 2(a) degenerate into the breather-line solutions.

The interaction of the long-wave solution (8) is the same as that of the short-wave solution (9), so we will only discuss the interaction of the long-wave solution (8).

3. Four kinds of solutions

3.1. Three kinds of solutions with repulsive interaction

3.1.1. Repulsive interaction between breather and soliton

We consider the nonlinear interaction of the two-breather solution (8) in the following parameter regions:

$$\begin{cases} \Upsilon_1 = \alpha_1-2\alpha_2-\varepsilon_1-\varepsilon_2, \\ \Upsilon_2 = \alpha_2+\varepsilon_2, \\ \dfrac{\rho_1}{\alpha_1}-\dfrac{\rho_2}{\alpha_2}+2\gamma_1\left(1-\dfrac{\alpha_2}{\alpha_1}\right) = \dfrac{\varepsilon_1\gamma_1}{\alpha_1}+\varepsilon_2\left(\dfrac{\gamma_1}{\alpha_1}+\dfrac{\gamma_2}{\alpha_2}\right)+\varepsilon_3, \end{cases} \tag{24}$$

or

$$\begin{cases} \Upsilon_1 = -\alpha_1+\varepsilon_1, \\ \Upsilon_2 = 2\alpha_1+\alpha_2-\varepsilon_1+\varepsilon_2, \\ \dfrac{\rho_1}{\alpha_1}-\dfrac{\rho_2}{\alpha_2}-2\gamma_2\left(\dfrac{\alpha_1}{\alpha_2}+1\right) = -\dfrac{\varepsilon_1\gamma_1}{\alpha_1}-\dfrac{(\varepsilon_1-\varepsilon_2)\gamma_2}{\alpha_2}+\varepsilon_3, \end{cases} \tag{25}$$

where $0<\varepsilon_j\ll 1$ $(j=1,2,3)$, $O(\varepsilon_j)\sim O(\varepsilon_1)$ and $|\alpha_1-\alpha_2|\gg\varepsilon_j$.

We first consider the dynamical behavior of the two-breather solution (8) in the parameter region (24). Because $\varepsilon_j\ll 1$ $(j=1,2,3)$, $L_1^2L_2^2\ll 1$. In other words, the phase shift $\tau=\ln|L_1L_2|\ll 0$ and $D_1=\frac{{\Upsilon_1}^2+{\gamma_1}^2}{{\Upsilon_1}^2-{\alpha_1}^2}\sim O(1/\varepsilon_1)\gg 1$. When D_1 is large enough, the two-breather solution degenerates into a two-line soliton. So the two-breather solution (8) exhibits the interaction between a soliton and a breather solution.

Figure 3 shows the two-breather solution (8) in the parameter region (24). Figure 3(a) is a three-dimensional map of Eq. (7), and Fig. 3(b) is a density map of Fig. 3(a). We can see from Fig. 3(b) that the breather c_3 and soliton s_3 move on the x-axis direction and the two waves approach each other. When they meet each other, the higher wave gets lower, the lower wave gets higher, and they converge into one wave which continues to move. And then, the breather c_3 with higher amplitude jumps to c_4 position in the positive direction of the x-axis, and the soliton s_3 with lower amplitude jumps to s_4 position in the negative direction of the x-axis, forming a repulsive phenomenon. According to Theorem 2, this interaction is a repulsive interaction.

From the above discussion, it can be seen that the two-breather solution in the region near point q_4 and line ν_1 in Fig. 2 shows repulsive interaction between the breather and the soliton.

Similarly, from Eq. (25), as $\varepsilon_j\ll 1$ $(j=1,2,3)$, the phase shift $\tau=\ln|L_1L_2|\ll 0$ and $D_2=\frac{{\Upsilon_2}^2+{\gamma_2}^2}{{\Upsilon_2}^2-{\alpha_2}^2}\sim O(1/\varepsilon_2)\gg 1$. The two-breather solution in the region near the point q_1 and line ν_3 degenerates to a interaction solution between a breather solution and a soliton solution with repulsive interaction.

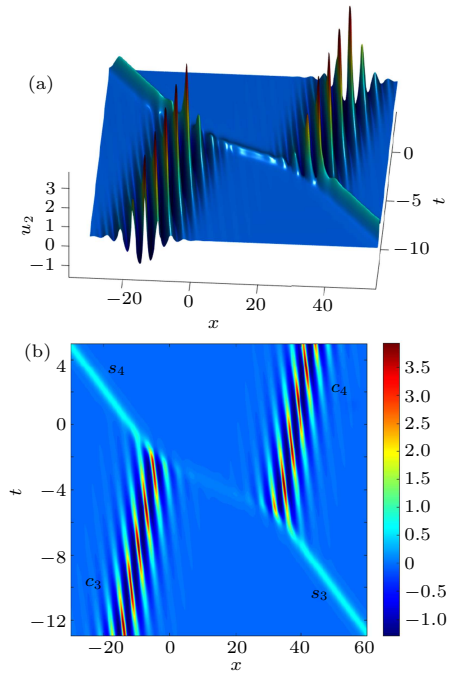


Fig. 3. The two-breather solution (8) degenerating into a combination of breather and soliton. (a) $\alpha_1=0.36$, $\gamma_1=-1.6$, $\rho_1=-1.2$, $\sigma_1=-0.8$, $\alpha_2=0.5904689585$, $\gamma_2=1.152633524$, $\rho_2=-0.7585834019$ and $\sigma_2=0.196416705$. (b) The contour plots of (a).

3.1.2. Repulsive interaction of two-breather solution with small amplitude

In the two coordinate spaces in Fig. 2, we choose

$$\begin{cases} \Upsilon_1 = -\alpha_1-\varepsilon_1, \\ \Upsilon_2 = \alpha_2+\varepsilon_2, \\ \left(\dfrac{\rho_1}{\alpha_1}-\dfrac{\rho_2}{\alpha_2}\right)-2(\gamma_1+\gamma_2) = \varepsilon_3, \end{cases} \tag{26}$$

where $|\varepsilon_j| \ll 1$ and $O(\varepsilon_j) \sim O(\varepsilon_1)$ $(j = 1,2,3)$. According to Eq. (26), it is easy to know that $D_j \gg 1$ $(j = 1,2)$. Then the amplitude of the two-breather solution (8) is proportional to $1/D_1 \sim O(\varepsilon_1)$ and $1/D_2 \sim O(\varepsilon_2)$. The wavelengths of the two-breather solution (8) are $2\pi/\gamma_1$ and $2\pi/\gamma_2$. $2\pi/\gamma_i > 1/D_i$ $(i = 1,2)$, so the waveform of the two-breather solution (8) is approximate to a two-soliton solution and it is called a small amplitude wave.

We substitute Eq. (26) into phase shift parameter (10), after calculation we obtain

$$L_1^2 L_2^2 \approx \frac{(2\alpha_1 - 2\alpha_2)^2 + \left(\frac{\rho_2}{\alpha_2} - \frac{\rho_1}{\alpha_1}\right)^2}{(2\alpha_1 + 2\alpha_2)^2 + \left(\frac{\rho_2}{\alpha_2} - \frac{\rho_1}{\alpha_1}\right)^2} \times \Delta + O(\Delta),$$

where

$$\Delta = \frac{(\varepsilon_1 - \varepsilon_2)^2 + \left(\frac{\varepsilon_2\gamma_2}{\alpha_2} + \frac{\varepsilon_1\gamma_1}{\alpha_1} - \varepsilon_3\right)^2}{(\varepsilon_1 + \varepsilon_2)^2 + \left(\frac{\varepsilon_2\gamma_2}{\alpha_2} + \frac{\varepsilon_1\gamma_1}{\alpha_1} - \varepsilon_3\right)^2}.$$

When $|\varepsilon_1 - \varepsilon_2| \ll |\varepsilon_1 + \varepsilon_2|$ and $|\frac{\varepsilon_2\gamma_2}{\alpha_2} + \frac{\varepsilon_1\gamma_1}{\alpha_1} - \varepsilon_3| \ll |\varepsilon_1 + \varepsilon_2|$, $L_1^2 L_2^2 \ll 1$, then the interaction of the two-breather solution is repulsive interaction.

We analyze the repulsive interaction of two parallel breather solutions with small amplitudes (see Fig. 4). Figures 4(a)–4(d) depict two parallel breathers with small amplitudes at different moments. As the higher wave approaches the lower wave, the higher wave is perturbed and emits a breather solution as a messenger soliton. The messenger soliton becomes larger in amplitude with time and is eventually absorbed by the low waves, finally forming an equilibrium state. We can also consider that the higher wave and lower wave exchange positions through the interaction. Figure 4(e) is the density map of the two-breather solution in x–y plane, and from Theorem 2 the two-breather solution presents repulsive interaction. Figure 4(f) shows the tangent plots of the two-breather solution at different moments in the y-axis, which shows that the two-breather solution has symmetry in time.

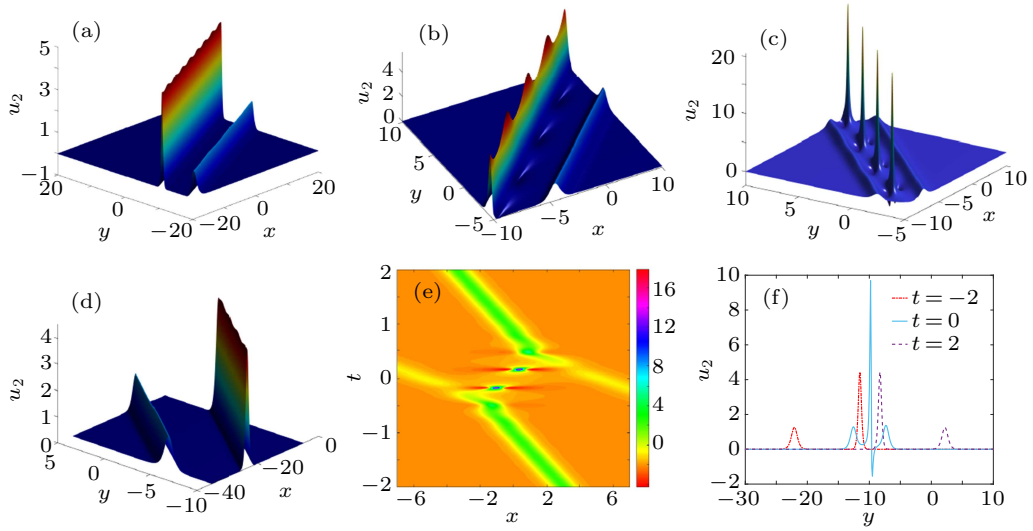


Fig. 4. The repulsive interaction of the two-breather solution (8) with small amplitude, the parameters $(\alpha_1, \gamma_1, \rho_1, \sigma_1) = (1.5, -0.5000146495, -3, -1.500151852)$, $(\alpha_2, \gamma_2, \rho_2, \sigma_2) = (0.8, 0.5, -1.6, -0.10988875)$. (a) $t = -6$, (b) $t = -0.6$, (c) $t = -0.04$, (d) $t = 2$. (e) The two-breather solution in x–t plane, (f) Cross sections of the two-breather solution in the y-axis.

So by the above illustration, we prove that the two-breather solution (8) in the parameter region near point o_1 and the line ν_1 in Fig. 2 is a two-breather solution with small amplitude.

3.1.3. Repulsive interaction of two-soliton

Based on the two coordinate spaces in Fig. 2, we choose

$$\begin{cases} \Upsilon_1 = -\alpha_1 + \varepsilon_1, \quad \Upsilon_2 = \alpha_2 + \varepsilon_2, \\ \varsigma_1 = -\gamma_1 + \dfrac{\rho_1}{\alpha_1} + \dfrac{\varepsilon\gamma_1}{\alpha_1}, \quad \varsigma_2 = \gamma_2 + \dfrac{\rho_2}{\alpha_2} + \dfrac{\varepsilon\gamma_2}{\alpha_2}, \end{cases} \tag{27}$$

where $0 < \varepsilon_j \sim O(\varepsilon) \ll 1$ and $(j = 1,2)$. Ensure $|\frac{\rho_1}{\alpha_1} - \frac{\rho_2}{\alpha_2} - 2(\gamma_1 + \gamma_2)| \neq 0$ and it is not small, then $L_1^2 L_2^2 \neq \frac{0}{0}$. The phase shift parameter relationship (10) is considered under the parameter condition Eq. (27). Ignore $O(\varepsilon_j)$, we have

$$L_1^2 L_2^2 = \frac{(2\alpha_1 - 2\alpha_2)^2 + \left(\frac{\rho_2}{\alpha_2} - \frac{\rho_1}{\alpha_1}\right)^2}{(2\alpha_1 + 2\alpha_2)^2 + \left(\frac{\rho_2}{\alpha_2} - \frac{\rho_1}{\alpha_1}\right)^2} < 1. \tag{28}$$

We label $\frac{\sqrt{D_j}}{2} e^{\xi_j} = e^{\tilde{\xi}_j}$ and then choose the parameter ξ_{0j} that satisfies $\tilde{\xi}_j \sim O(1)$ $(j = 1,2)$. The two-breather solution (8) is changed into

$$\begin{aligned} f = {} & 1 + \frac{4\, e^{\tilde{\xi}_2} \cos(\eta_2)}{\sqrt{D_2}} + \frac{4\, e^{\tilde{\xi}_1} \cos(\eta_1)}{\sqrt{D_1}} \\ & + \frac{8 L_2\, e^{\tilde{\xi}_1 + \tilde{\xi}_2} \cos(\eta_1 - \eta_2 + \theta_2)}{\sqrt{D_1}\sqrt{D_2}} \\ & + \frac{8 L_1\, e^{\tilde{\xi}_1 + \tilde{\xi}_2} \cos(\eta_2 + \eta_1 + \theta_1)}{\sqrt{D_1}\sqrt{D_2}} + 4\, e^{2\tilde{\xi}_2} \end{aligned}$$

$$+\frac{16L_1L_2\mathrm{e}^{2\tilde{\xi}_1+\tilde{\xi}_2}\cos(\eta_2+\theta_1-\theta_2)}{\sqrt{D_2}}$$
$$+\frac{16L_1L_2\mathrm{e}^{\tilde{\xi}_1+2\tilde{\xi}_2}\cos(\eta_1+\theta_2+\theta_1)}{\sqrt{D_1}}$$
$$+4\,\mathrm{e}^{2\tilde{\xi}_1}+16{L_2}^2{L_1}^2\mathrm{e}^{2\tilde{\xi}_1+2\tilde{\xi}_2}. \tag{29}$$

And then we neglect the terms containing $1/\sqrt{D_j}$ in Eq. (29)

$$f=1+4\mathrm{e}^{\xi_{L2}}+4\,\mathrm{e}^{\xi_{L1}}+16{L_2}^2{L_1}^2\mathrm{e}^{\xi_{L1}+\xi_{L2}}, \tag{30}$$

where $\xi_{L1}=2\tilde{\xi}_1,\xi_{L2}=2\tilde{\xi}_2$. Equation (2) is a two-soliton with the parameter $N=2$:

$$f_2=1+\mathrm{e}^{\Psi_1}+\mathrm{e}^{\Psi_2}+\mathrm{e}^{\Psi_1+\Psi_2+A_{12}}, \tag{31}$$

where $\Psi_1=k_1x+\omega_1t+p_1y+\Psi_{01}$, $\Psi_2=k_2x+\omega_2t+p_2y+\Psi_{02}$, and

$$A_{12}=\frac{(k_1-k_2)^2+\left(\frac{p_2}{k_2}-\frac{p_1}{k_1}\right)^2}{(k_1+k_2)^2+\left(\frac{p_2}{k_2}-\frac{p_1}{k_1}\right)^2}. \tag{32}$$

Mark $2\alpha_i=k_i,2\rho_i=p_i$ $(i=1,2)$, and then find that Eq. (32) is the phase shift parameter (28) of the two-breather solution. Figure 5 is a two-soliton, from which it can be seen that this two-soliton presents repulsive phenomenon and $L_1^2L_2^2<1$, so the two-soliton presents repulsive interaction.

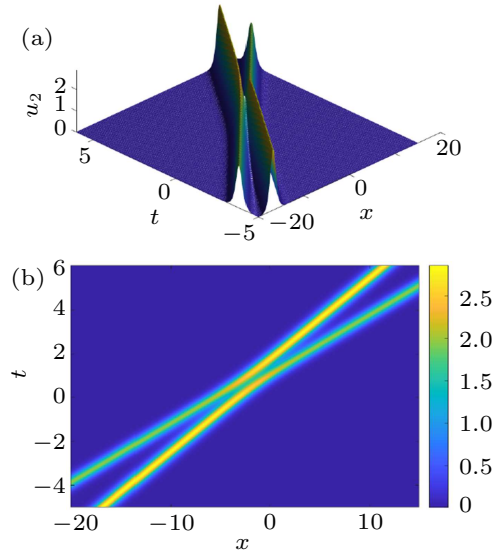


Fig. 5. The two-breather solution (8) degenerating into a combination of breather and soliton with repulsive interaction. (a) $\alpha_1=1.2$, $\alpha_2=1$ $\gamma_1=0.1716829487$, $\gamma_2=-0.1047281290$, $\gamma_1=1.2$, $\gamma_2=1/4$, $\sigma_1=-1.297801270$ and $\sigma_2=0.9847926623$, (b) The contour plots of (a).

By the above illustration, there are many two-breather solutions in the parameter region (27), where the two-breather solutions in the region near point o_1 and away from line ν_1 degenerate into two-soliton solutions with repulsive interaction and the wave numbers $(2\alpha_1,2\rho_1)$ and $(2\alpha_2,2\rho_2)$.

3.2. Attractive interaction between breather solution and soliton solution

We consider the nonlinear interactions of the two-breather solutions (8) in the following parameter regions:

$$\begin{cases}\Upsilon_1=-\alpha_1-2\alpha_2-\varepsilon_1-\varepsilon_2,\\ \Upsilon_2=\alpha_2+\varepsilon_2,\\ \dfrac{\rho_1}{\alpha_1}-\dfrac{\rho_2}{\alpha_2}-2\beta_1\left(1+\dfrac{\alpha_2}{\alpha_1}\right)=\dfrac{\varepsilon_1\gamma_1}{\alpha_1}+\varepsilon_2\left(\dfrac{\gamma_1}{\alpha_1}+\dfrac{\gamma_2}{\alpha_2}\right)+\varepsilon_3,\end{cases} \tag{33}$$

or

$$\begin{cases}\Upsilon_1=-\alpha_1+\varepsilon_1,\\ \Upsilon_2=2\alpha_1-\alpha_2-\varepsilon_1+\varepsilon_2,\\ \dfrac{\rho_1}{\alpha_1}-\dfrac{\rho_2}{\alpha_2}+2\beta_2\left(1-\dfrac{\alpha_1}{\alpha_2}\right)=-\dfrac{\varepsilon_1\beta_1}{\alpha_1}-\dfrac{(\varepsilon_1-\varepsilon_2)\beta_2}{\alpha_2}+\varepsilon_3,\end{cases} \tag{34}$$

where $0<|\varepsilon_j|\ll 1$, $O(\varepsilon_j)\sim O(\varepsilon_1)$ and $|\alpha_1-\alpha_2|\gg\varepsilon_j$ $(j=1,2,3)$.

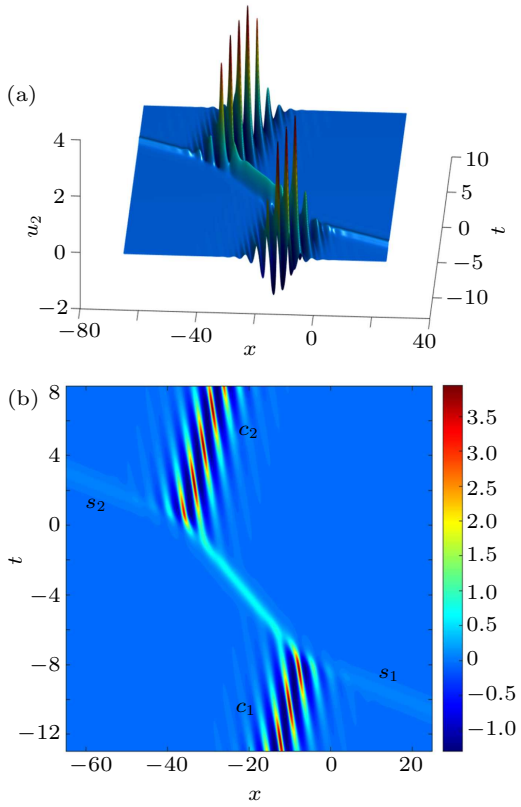


Fig. 6. The two-breather solution (8) degenerating into a combination of breather and soliton with attractive interaction. (a) $\alpha_1=0.36$, $\gamma_1=-1.6$, $\rho_1=-1.2$, $\sigma_1=-0.8$, $\alpha_2=0.2304699584$, $\gamma_2=-0.7071187644$, $\rho_2=-0.8012021662$, $\sigma_2=-0.8012021662$. (b) The contour plots of (a).

We consider the two-breather solutions in the parameter region (33). Because $\varepsilon_j\ll 1$, after calculation we know that $L_1^2L_2^2\gg 1$, the two-breather solution (8) exhibits attractive interaction between a soliton solution and a breather solution (see Fig. 6). Figure 6(a) is a three-dimensional map of Eq. (8), and Fig. 6(b) is a density map of Fig. 6(b). Figure 6 shows the attractive interaction between a soliton solution and a breather solution. When soliton s_1 and breather c_1 approach each other

over time on the x-axis, as the two waves meet each other, the wave c_1 with higher amplitude becomes lower, and the wave s_1 with lower amplitude becomes higher. After that, the synthesized wave starts to move for some time and then splits into breather c_2 and soliton s_2. During the interaction, the breather c_1 jumps to the negative direction of the x-axis and the soliton s_1 jumps to the positive direction of the x-axis, forming an attractive phenomenon. From Theorem 1, this interaction is attractive interaction. So when the two-breather solution is in the parameter region (33), that is, the solution in the region near point q_3 and line ν_3 in Fig. 2 shows the attractive interaction between a breather solution and a soliton solution.

Similarly, from Eq. (34)), we can show that the two-breather solution in the regions near the point q_2 and line ν_1 degenerates into a combination of a breather and a soliton. The solution presents an attractive interaction.

4. Summary

In this paper, some degenerate solutions of a two-breather solution of Mel'nikov equation are discussed. The non-singular boundary conditions of the two-breather solution are mapped into two coordinate spaces. We prove that in the regions near non-singular boundaries, in addition to the existence of two-breather solution, there are also four new kinds of solutions, namely, two-soliton solution, interaction solution between breather and soliton, and two-breather solution with small amplitude that all exhibit repulsive interactions, and the interaction solution between breather and soliton with attractive interaction. It needs to be pointed out that the two-breather solution with small amplitude, even though it has almost the same shape as a soliton, is very different. The two-breather solution with small amplitude can transmit energy through a new and growing breather during the interaction, while soliton does not have this characteristic.

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