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# Finite time hybrid synchronization of heterogeneous duplex complex networks via time-varying intermittent control

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This paper study the finite time internal synchronization and the external synchronization (hybrid synchronization) for duplex heterogeneous complex networks by time-varying intermittent control. There few study hybrid synchronization of heterogeneous duplex complex networks. Therefore, we study the finite time hybrid synchronization of heterogeneous duplex networks, which employs the time-varying intermittent control to drive the duplex heterogeneous complex networks to achieve hybrid synchronization in finite time. To be specific, the switch frequency of the controllers can be changed with time by devise Lyapunov function and boundary function, the internal synchronization and external synchronization are achieved simultaneously in finite time. Finally, numerical examples are presented to illustrate the validness of theoretical results.

**Keywords:** finite time synchronization, time-varying intermittent control, duplex heterogeneous networks, complex networks

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## 1. Introduction

In the past decades, the complex networks is an active field of scientific research, the complex networks are divided into many types, including public transport networks,[1] information networks,[2] financial network,[3] neural networks,[4] epidemic spreading networks[5,6] and so on. The synchronization is an important phenomenon of complex networks.[7–10] In sum, the synchronization are broadly divided into two types: the synchronization between network and nodes (the internal synchronization) and synchronization among some complex networks (the external synchronization). The controllers are added to networks for achieving synchronization.

Many control methods have been investigated for synchronization.[11–14] The intermittent control is one of discontinuous control method, the controller is not working continuously, the controller is actived in the work time, which is better than continuous control, the intermittent synchronization control have been developed. For example, in Ref. [15], the coupled network is synchronized by intermittent controllers. In Refs. [16,17], outer synchronization of complex networks intermittent control is investigated.

The time of synchronization can be divided into finite time and infinite time. In reality, the networks synchronization as quickly as possible, so finite time has better effect than infinite time. In order to quickly control the synchronization time, finite time control method is researched in Ref. [18]. Once the intermittent control is combined with finite time control, the time of synchronization can be predicted, which have been studied in Refs. [19–21]. The different parts of networks have different property, so the different parts of networks with different synchronization types.[22–24]

Duplex complex networks have a wide range of applications,[25,26] for example, in analyzing social networks and biological networks, duplex complex networks can be used to model the interactions between different levels, such as the relationship between individual and group behaviors.[27] Duplex complex networks can be used to describe the interactions between two different levels in gene regulatory networks. Due to the heterogeneity of the network, the study-of-heterogeneous duplex complex networks is of great significance for understanding complex systems in the real world.[28,29]

Intermittent control can control the behavior of dynamic systems by applying or withdrawing control actions at specific time intervals. This method of control is often used in systems that do not require continuous control inputs to maintain stability or desired behavior.[30]

Finite time time-varying intermittent control is a control strategy which the control input is applied intermittently and under certain conditions. The characteristics (such as magnitude, duration, and timing) vary over time. This method of control are suitable for systems that can maintain stability without needing to continuous control inputs in finite time.

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The purpose of finite time time-varying intermittent control is to save energy, reduce wear, and control complex dynamics, for ensuring the system attains the anticipated performance levels in finite time.

Due to diversity and heterogeneity of network, the different parts of the duplex network can achieve different types of synchronization. However, to the best of our knowledge, finite time synchronization of duplex networks focus on the same type for every layer network, every layer network should have different type for the duplex networks. There are few research on the finite time internal and the external synchronization for heterogeneous duplex complex networks by time-varying intermittent control. Motivated by the above discussions, this paper investigates the finite time internal and the external synchronization problem for heterogeneous duplex complex networks via time-varying intermittent control. The switch frequency of the controllers can be changed with time by devise time-varying Lyapunov function. The internal and the external synchronization can be achieved in finite time via time-varying intermittent control.

## 2. Network model and preliminaries

**Assumption 1**[7] The time-varying delay $\tau(t)$ is a function and satisfies $0 \le \tau(t) \le \mu$, which explain the time-varying delay signal is positive and bounded, maximum $\mu = 1$ in this paper. Derivative of the delay $\tau(t)$ exists, derivative of the delay $\tau(t)$ less than 1, which can also be written as $0 \le \dot{\tau}(t) \le \xi < 1$ $(\xi \ge 0)$, this assumption is true if $\tau(t)$ is constant.

Considering heterogeneous duplex complex networks (1)–(2) and a nonlinear dynamics mathematical model (3) are as follows:

$$\begin{aligned}\dot{x}_i(t) &= \tilde{A}x_i(t) + \tilde{B}f(x_i(t-\tau_1(t))) \\ &\quad + \sum_{j=1}^{N} c_{ij}^1 g_1(x_j(t)) + \sum_{j=1}^{N} d_{ij}^1 h_1(x_j(t-\tau_2(t))) \\ &\quad + \boldsymbol{u}_{\text{in}}(t) + \boldsymbol{u}_{\text{out}}(t), \qquad (1)\\ \dot{y}_i(t) &= \tilde{E}y_i(t) + \tilde{W}q(y_i(t-\tau_1(t))) \\ &\quad + \sum_{j=1}^{N} c_{ij}^2 g_2(y_j(t)) + \sum_{j=1}^{N} d_{ij}^2 h_2(x_j(t-\tau_2(t))), \qquad (2)\\ \dot{s}(t) &= Z(\boldsymbol{s}(t)), \qquad (3)\end{aligned}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{\text{in}}(t))^{\text{T}} \in R^n$ is the state variables of the $i$-th node for complex network (1).

$y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{\text{in}}(t))^{\text{T}} \in R^n$ is the state variables of the $i$-th node in complex network (2), $\boldsymbol{f} : R^n \to R^n$ and $\boldsymbol{q} : R^n \to R^n$ are the continuously vector functions which explain the dynamical equation of the $i$-th node, $\tau_1(t) > 0$ and $\tau_2(t) > 0$ are coupling time-varying delay, $f(z(t)) = g_2(y_j(t)) - g_1(x_j(t))$.

When $\tau_1(t) = \tau_2(t) = 0$, complex networks (1)–(2) without delay. $g_i : R^n \to R^n$ and $h_i : R^n \to R^n$ $(i = 1, 2)$ are couple functions. $s(t)$ is the state variables of the $i$-th node in system (3). $\tilde{\boldsymbol{A}} = \boldsymbol{A} + \Delta\boldsymbol{A}(t) \in R^{n\times n}$, $\tilde{\boldsymbol{B}} = \boldsymbol{B} + \Delta\boldsymbol{B}(t) \in R^{n\times n}$, $\tilde{\boldsymbol{E}} = \boldsymbol{E} + \Delta\boldsymbol{E}(t) \in R^{n\times n}$, and $\tilde{\boldsymbol{W}} = \boldsymbol{W} + \Delta\boldsymbol{W}(t) \in R^{n\times n}$ stand for the parametric uncertain, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are constant matrices, $\Delta\boldsymbol{A}$, $\Delta\boldsymbol{B}$, and $\Delta\boldsymbol{C}$ are parameter uncertainties. For $k = 1, 2$, $C^k \in R^{N\times N}$ and $D^k \in R^{N\times N}$ are the coupling configuration matrices which reflect the fundamental topology structure of network. If there is a connection between node $i$ and node $j (i \ne j)$, then $c_{ij}^k > 0$ and $d_{ij}^k > 0$; otherwise, $c_{ij}^k = 0$ and $d_{ij}^k = 0$. The diagonal elements of matrixs $C^k$ and $D^k$ are defined as $c_{ii}^k = -\sum_{j=1, j\ne i}^{N} c_{ij}^k$ and $d_{ii}^k = -\sum_{j=1, j\ne i}^{N} d_{ij}^k$. $\boldsymbol{s}(t) \in \boldsymbol{R}^n$ is equinoctial point, or chaotic attractors, or a periodic orbit. $\boldsymbol{u}_{\text{in}}(t) \in \boldsymbol{R}^n$ and $\boldsymbol{u}_{\text{out}}(t) \in \boldsymbol{R}^n$ are intermittent controllers which need be designed. The $\boldsymbol{u}_{\text{in}}(t)$ is the internal controller, which control the first node to the $d$-th node. The $\boldsymbol{u}_{\text{out}}(t)$ is the external controller, which control the $(d+1)$-th node to the $N$-th node.

**Definition 1** The hybrid synchronization (the internal and external synchronization) error is defined as follows:

$$\boldsymbol{e}(t) = \boldsymbol{e}_{\text{in}}(t) + \boldsymbol{e}_{\text{out}}(t),$$

$\boldsymbol{e}_{\text{in}}(t)$ is the internal synchronization error, which is defined as follows:

$$\boldsymbol{e}_{\text{in}}(t) = \boldsymbol{x}_i(t) - \boldsymbol{s}(t - \tau(t)), \quad i = 1, 2, \ldots, d,$$

where $\tau(t) > 0$ is time-varying delay, $\boldsymbol{x}_i(t) \in R^n$ is a state variable of complex networks (1), $\boldsymbol{s}(t)$ is state variable of system (3). The $\boldsymbol{e}_{\text{out}}(t)$ is the external synchronization error, which is defined

$$\boldsymbol{e}_{\text{out}}(t) = \boldsymbol{x}_j(t) - \boldsymbol{y}_j(t - \tau(t)), \quad j = d+1, d+2, \ldots, N.$$

Nodes 1 to $d$ are implemented internal synchronization control, and nodes $d+1$ to $N$ are implemented external synchronization control.

**Definition 1** There exists a setting time $T$ and meet the following conditions when hybrid synchronization is achieved,

$$\begin{aligned}\lim_{t\to T}\|\boldsymbol{e}(t)\| &= \lim_{t\to T}\|\boldsymbol{e}_{\text{in}}(t) + \boldsymbol{e}_{\text{out}}(t)\| \\ &= \lim_{t\to T}\|\boldsymbol{e}_{\text{in}}(t)\| + \lim_{t\to T}\|\boldsymbol{e}_{\text{out}}(t)\| \\ &= \lim_{t\to T}\|\boldsymbol{x}_i(t) - \boldsymbol{s}(t-\tau(t))\| \\ &\quad + \lim_{t\to T}\left\|\boldsymbol{x}_j(t) - \boldsymbol{y}_j(t-\tau(t))\right\|.\end{aligned}$$

**Lemma 1**[5] The $V(t)$ is a continuous positive-definite function and satisfies the following differential inequality,

$$\dot{V}(t) \le -\omega V^r(t), \quad \forall t \ge t_0, \quad V(t_0) \ge 0,$$

where $\omega > 0$, $0 < r < 1$, $V(t) \equiv 0$ for all $t \ge T(x_0)$ and the settling time is

$$T(x_0) \le \frac{V^{1-r}(x_0)}{\omega(1-r)}.$$

**Definition 2** For $\theta \in R^n$, $r>0$, the signal function $\mathrm{sign}(\cdot)$ and $|\cdot|^r$ are defined as

$$\mathrm{sign}(\theta) = \mathrm{diag}(\mathrm{sign}(\theta_1),\ \mathrm{sign}(\theta_2),\ldots,\mathrm{sign}(\theta_n)),$$
$$|\theta|^r = (|\theta_1|^r,|\theta_2|^r,\ldots,|\theta_n|^r)^{\mathrm{T}}.$$

**Assumption 2** The $\tau_1(t)$ and $\tau_2(t)$ are time-varying delay which bounded differential functions satisfy

$$0\le\tau_1(t)\le\tau_1^*,\quad 0\le\tau_2(t)\le\tau_2^*,$$
$$\dot{\tau}_1(t)\le\gamma<1,\quad \dot{\tau}_2(t)\le\gamma<1$$

for all $t$, where $\tau_1^*$, $\tau_2^*$, and $\gamma$ are some positive constants.

**Assumption 2** The functions $j(\cdot)$, $r(\cdot)$, and $h(\cdot)$ are three Lipschitz functions, *i.e.*, there exists positive constants $L_j$, $L_r$, $L_h$ such that for all $\mu$, $\nu\in R^n$,

$$\|\ j(\mu)-j(\nu)\ \|\le L_j\ \|\ \mu-\nu\ \|,$$
$$\|\ r(\mu)-r(\nu)\ \|\le L_r\ \|\ \mu-\nu\ \|,$$
$$\|\ h(\mu)-h(\nu)\ \|\le L_h\ \|\ \mu-\nu\ \|.$$

**Assumption 3** $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{B}$ are uncertain time-vary matrices, which are assumed to be

$$[\Delta\boldsymbol{A}(t),\Delta\boldsymbol{B}(t)] = \boldsymbol{M}\boldsymbol{F}(t)[\boldsymbol{N}_1,\boldsymbol{N}_2],$$

where $\boldsymbol{M}$, $\boldsymbol{N}_1$, $\boldsymbol{N}_2$ are known real matrices, $\boldsymbol{F}(t)$ is an unknown time-varying real matrix and satisfy $\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{F}(t)\le\boldsymbol{I}$.

**Lemma 2** [5] If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are real matrices, then the following inequality holds for any $\zeta>0$

$$\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{X}\le\zeta\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}+\frac{1}{\zeta}\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{Y}.$$

## 3. Main results

In this section, the finite time intermittent control scheme and the hybrid synchronization criterion are given. According to systems (1)–(3), the error state equations are written as follows:

$$\begin{aligned}\dot{e}(t) =\ & (I_d\otimes\tilde{A})e_{\mathrm{in}}(t)+(I_d\otimes\tilde{B})F(e_{\mathrm{in}}(\tau_1))\\ &+(I_d\otimes\tilde{E})e_{\mathrm{in}}(t)+(I_d\otimes\tilde{W})F(e_{\mathrm{in}}(\tau_1))\\ &+(C\otimes I_d)G(e_{\mathrm{in}}(t))+(D\otimes I_d)H(e_{\mathrm{in}}(\tau_2))\\ &+\boldsymbol{u}_{\mathrm{in}}(t)+\tilde{A}e_{\mathrm{out}}(t)+(I_{N-d}\otimes\tilde{B})F(e_{\mathrm{out}}(\tau_1))\\ &+\tilde{E}e_{\mathrm{out}}(t)+(I_{N-d}\otimes\tilde{E})F(e_{\mathrm{out}}(\tau_1))\\ &+\tilde{E}\tilde{W}F(e_{\mathrm{out}}(\tau_1)+I_{N-d}G(e_{\mathrm{out}}(t))\\ &+(D\otimes I_{N-d})H(e_{\mathrm{out}}(\tau_2))+\boldsymbol{u}_{\mathrm{out}}(t),\end{aligned}\tag{4}$$

where

$$\begin{aligned}\boldsymbol{e}(t) &= (e_1^{\mathrm{T}}(t),e_2^{\mathrm{T}}(t),\ldots,e_d^{\mathrm{T}}(t),e_{d+1}^{\mathrm{T}}(t),\ldots,e_N^{\mathrm{T}}(t))^{\mathrm{T}},\\ f(e_i(\tau_1)) &= f(y_i(t-\tau_1(t))-f(x_i(t-\tau_1(t))\\ &\quad+f(x_i(t-\tau_1(t))-f(s_i(t-\tau_1(t)),\\ F(e(\tau_1)) &= (f^{\mathrm{T}}(e_1(\tau_1)),f^{\mathrm{T}}(e_2(\tau_1)),\ldots,\\ &\quad f^{\mathrm{T}}(e_d(\tau_1)),f^{\mathrm{T}}(e_{d+1}(\tau_1)),\ldots,f^{\mathrm{T}}(e_N(\tau_1)))^{\mathrm{T}},\\ g(e_i(t)) &= g(y_i(t))-g(x_i(t))+g(x_i(t))-g(s_i(t)),\\ G(e(t)) &= (g^{\mathrm{T}}(e_1(t)),g^{\mathrm{T}}(e_2(t)),\ldots,\\ &\quad g^{\mathrm{T}}(e_d(t)),g^{\mathrm{T}}(e_{d+1}(t)),\ldots,g^{\mathrm{T}}(e_N(t)))^{\mathrm{T}},\\ h(e_i(\tau_2)) &= h(y_i(t-\tau_2(t))-h(x_i(t-\tau_2(t))\\ &\quad+h(x_i(t-\tau_2(t))-h(s_i(t-\tau_2(t)),\\ H(e(\tau_2)) &= (h^{\mathrm{T}}(e_1(\tau_2)),h^{\mathrm{T}}(e_2(\tau_2)),\ldots h^{\mathrm{T}}(e_d(\tau_2)),\\ &\quad h^{\mathrm{T}}(e_{d+1}(\tau_2)),\ldots,h^{\mathrm{T}}(e_N(\tau_2)))^{\mathrm{T}},\\ C &= C^2-C^1,\quad \boldsymbol{u}_{\mathrm{in}}(t)=(u_1^{\mathrm{T}}(t),u_2^{\mathrm{T}}(t),\ldots,u_d^{\mathrm{T}}(t))^{\mathrm{T}},\\ D &= D^2-D^1,\quad \boldsymbol{u}_{\mathrm{out}}(t)=(u_{d+1}^{\mathrm{T}}(t),u_{d+2}^{\mathrm{T}}(t),\ldots,u_N^{\mathrm{T}}(t))^{\mathrm{T}}.\end{aligned}$$

The developed intermittent controller is designed as

$$\boldsymbol{u}_{\mathrm{in}}(t)=\begin{cases}u_1(t), & \text{if } V_{\mathrm{in}}(t)\ge V_{\mathrm{in1}}(t),\\ u_1(t^-), & \text{if } V_{\mathrm{in2}}(t)<V_{\mathrm{in}}(t)<V_{\mathrm{in1}}(t),\\ 0, & \text{if } V_{\mathrm{in}}(t)\le V_{\mathrm{in2}}(t),\end{cases}$$

$$\boldsymbol{u}_{\mathrm{out}}(t)=\begin{cases}u_2(t), & \text{if } V_{\mathrm{out}}(t)\ge V_{\mathrm{out1}}(t),\\ u_2(t^-), & \text{if } V_{\mathrm{out2}}(t)<V_{\mathrm{out}}(t)<V_{\mathrm{out1}}(t),\\ 0, & \text{if } V_{\mathrm{out}}(t)\le V_{\mathrm{out2}}(t),\end{cases}$$

$$\begin{aligned}u_1(t) =\ & -\xi e_{\mathrm{in}}(t)-\frac{\alpha}{2}\mathrm{sign}(e_{\mathrm{in}}(t))\,|e_{\mathrm{in}}(t)|^{\sigma}\\ &-\frac{\alpha}{2}\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e_{\mathrm{in}}^{\mathrm{T}}(s)e_{\mathrm{in}}(s)\,\mathrm{d}s\right)^{(1+\sigma)/2}\frac{e_{\mathrm{in}}(t)}{e_{\mathrm{in}}(t)^2}\\ &-\frac{\alpha}{2}\left(\frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t}e_{\mathrm{in}}^{\mathrm{T}}(s)e_{\mathrm{in}}(s)\,\mathrm{d}s\right)^{(1+\sigma)/2}\frac{e_{\mathrm{in}}(t)}{e_{\mathrm{in}}(t)^2}\\ &+-\frac{\alpha}{2}\mathrm{sign}(e_{\mathrm{in}}(t))\,|e_{\mathrm{in}}(t)|^{\sigma},\\ u_2(t) =\ & -\frac{\lambda}{2}\left(\frac{\eta_3}{1-\varphi}\int_{t-\tau_1(t)}^{t}e_{\mathrm{out}}^{\mathrm{T}}(s)e_{\mathrm{out}}(s)\,\mathrm{d}s\right)^{(1+\mu)2}\frac{e_{\mathrm{out}}(t)}{e_{\mathrm{out}}(t)^2}\\ &-\frac{\lambda}{2}\left(\frac{\eta_4}{1-\varphi}\int_{t-\tau_2(t)}^{t}e_{\mathrm{out}}^{\mathrm{T}}(s)e_{\mathrm{out}}(s)\,\mathrm{d}s\right)^{(1+\mu)/2}\frac{e_{\mathrm{out}}(t)}{e_{\mathrm{out}}(t)^2}\\ &+\lambda\left(\frac{\eta_4}{1-\varphi}\int_{t-\tau_1(t)}^{t}e_{\mathrm{out}}^{\mathrm{T}}(s)e_{\mathrm{out}}(s)\,\mathrm{d}s\right)^{(1+\mu)/2}\\ &-\frac{\lambda}{2}\mathrm{sign}(e_{\mathrm{out}}(t))\,|e_{\mathrm{out}}(t)|^{\mu},\end{aligned}\tag{5}$$

where $\xi>0$, $\eta_1>0$, $\eta_2>0$, $\eta_3>0$, and $\eta_4>0$ are the control strengths, $\alpha>0$, $\lambda>0$, $0\le\sigma<1$, and $0\le\mu<1$ are tunable constants. $V_{\mathrm{in}}(t)$ and $V_{\mathrm{out}}(t)$ are two given Lyapunov functions, $V_{\mathrm{ini}}(t)$ and $V_{\mathrm{outj}}(t)$ are two nonnegative continuous functions. $t^-$ indicates the previous instant of time $t$. From expression (4), it is easy to see that the work time of controllers $u_1(t)$ and $u_2(t)$ are decided by the time-varying relation among

$V(t)$, $V_{\rm ini}(t)$, and $V_{\rm outj}(t)$. For brevity, take

$$\beta=\frac{1+\sigma}{2},\quad \chi=\frac{\mu+2}{3}.$$

$V_{\rm ini}(t)$ and $V_{\rm inj}(t)$ are given as

$$V_{\rm ini}(t)=\Big((kV(t_0))^{\beta}-\alpha_i(t)\beta(t-t_0)\Big)^{1/\beta},\quad i=1,2,$$
$$V_{\rm outj}(t)=\big((\vartheta V(t_0))^{1-\chi}-\lambda_j(t)\chi(t-t_0)\big)^{1/(\chi-1)},\quad j=1,2,$$

where $\alpha_i$, $\lambda_i$ , $k$, and $\vartheta$ are positive scalars, $\alpha>\alpha_1>\alpha_2$, $\lambda>\lambda_1>\lambda_2$, and $0<k<1$, $0<\vartheta<1$.

Based on the relation among $V_{\rm in}(t)$, $V_{\rm out}(t)$, $V_{\rm ini}(t)$, and $V_{\rm outj}(t)$, the intermittent control mechanism is described as:

(i) If $V_{\rm in}(t)\geq V_{\rm in1}(t)$ and $V_{\rm out}(t)\geq V_{\rm out1}(t)$, then the controllers $\boldsymbol{u}_{\rm in}(t)=u_1(t)$ and $\boldsymbol{u}_{\rm out}(t)=u_2(t)$ are actived at instant $t$ and continuously working.

(ii) If $V_{\rm in2}(t)\leq V_{\rm in}(t)\leq V_{\rm in1}(t)$ and $V_{\rm out2}(t)\leq V_{\rm out}(t)\leq V_{\rm out1}(t)$, the controllers $\boldsymbol{u}_{\rm in}(t)=u_1(t)$ and $\boldsymbol{u}_{\rm out}(t)=u_2(t)$ are working at the instant $t^-$, then the controllers $\boldsymbol{u}_{\rm in}(t)=u_1(t)$ and $\boldsymbol{u}_{\rm out}(t)=u_2(t)$ are continuously working after the time $t$.

(iii) If the following conditions are satisfied, $V_{\rm in2}(t)\leq V_{\rm in}(t)\leq V_{\rm in1}(t)$ and $V_{\rm out2}(t)\leq V_{\rm out}(t)\leq V_{\rm out1}(t)$, the controller $\boldsymbol{u}_{\rm in}(t)=0$ and $\boldsymbol{u}_{\rm out}(t)=0$ are working at the instant $t^-$, then the controller $\boldsymbol{u}_{\rm in}(t)=0$ and $\boldsymbol{u}_{\rm out}(t)=0$ are continuously working after the time $t$.

(iv) If $V_{\rm in}(t)\leq V_{\rm in2}(t)$ and $V_{\rm out}(t)\leq V_{\rm out2}(t)$, then the controller $\boldsymbol{u}_{\rm in}(t)=0$ and $\boldsymbol{u}_{\rm out}(t)=0$ are actived at instant $t$ and continuously working.

**Remark 1** The two controllers $\boldsymbol{u}_{\rm in}(t)$ and $\boldsymbol{u}_{\rm out}(t)$ can achieve finite time hybrid synchronization in the whole network, the controller $\boldsymbol{u}_{\rm in}(t)$ achieve finite time internal synchronization and controller $\boldsymbol{u}_{\rm out}(t)$ achieve finite time external synchronization. This paper focuses on finite time hybrid synchronization in the whole network. In this case, the controller may switch rapidly between $u(t)=u^*(t)$ and $u(t)=0$ with the trajectory of $V(t)$ touching the boundary function from $V_1(t)$ to $V_2(t)$. If the $\alpha_1(t)$, $\alpha_2(t)$, $\beta$, $\lambda_1(t)$, $\lambda_2(t)$, and $k$ are well chosen, the size of region $V_2(t)<V(t)<V_1(t)$ that corresponds to the switching frequency of the controller between $u(t)=u^*(t)$ and $u(t)=0$ will be changed with time. The switch frequency can be changed with time by the values of $\alpha_2(t)-\alpha_1(t)$ and $\lambda_2(t)-\lambda_1(t)$, which improve previous results.

The finite time synchronization criterion with the Lyapunov function $V_{\rm in}(t)$ and $V_{\rm out}(t)$ are designed as follows:

$$\begin{aligned}V_{\rm in}(t)&=e_{\rm in}^{\rm T}(t)e(t)+\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e_{\rm in}^{\rm T}(s)e_{\rm in}(s)\,{\rm d}s\\&\quad+\frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t}e_{\rm in}^{\rm T}(s)e(s)\,{\rm d}s,\\V_{\rm out}(t)&=\frac{\eta_3}{1-\varphi}\int_{t-\tau_1(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s\\&\quad+\frac{\eta_4}{1-\varphi}\int_{t-\tau_2(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s.\end{aligned}$$

Then the finite time synchronization criterion is given.

**Theorem 1** Suppose that Assumptions 1–3 hold, there exist tunable positive constants $\xi$, $\eta_1$, $\eta_2$, and any positive constants $s_1,\ldots,s_5$, such that

(i) $\Delta\leq 0$,

(ii) $s_2^{-1}L_j^2+s_3^{-1}L_j^2\lambda_{\max}(N_2^{\rm T}N_2)-s_1^{-1}L_j^2+\eta_1+\eta_3\leq 0$,

(iii) $s_5^{-1}L_h^2-\eta_2-\eta_2+\eta_4\leq 0$,

where

$$\begin{aligned}\Delta&=\lambda_{\max}(A^s)+s_1\lambda_{\max}(MM^{\rm T})+s_4^{-1}L_r^2+s_3L_r^2\\&\quad+s_1L_r^2-s_1^{-1}\lambda_{\max}(N_1^{\rm T}N_1)+s_2\lambda_{\max}(BB^{\rm T})\\&\quad+s_4\lambda_{\max}(CC^{\rm T})+s_5\lambda_{\max}(DD^{\rm T})+\frac{\eta_1+\eta_2+\eta_3}{1-\gamma}-\xi.\end{aligned}$$

Then the networks (1)–(3) with the intermittent control scheme (4) and (5) will be synchronized in a finite time $T$, and

$$T<T_1=\frac{(\vartheta V_{\rm out}(t_0))^{\chi}}{\lambda_2\chi}+t_0.$$

**Proof** Differentiating $V_{\rm in}(t)$ and $V_{\rm out}(t)$ along the solution of error system (3),

$$\begin{aligned}\dot V_{\rm in}(t)&\leq 2e_{\rm in}^{\rm T}(t)\dot e_{\rm in}(t)+\frac{\eta_1}{1-\gamma}e_{\rm in}^{\rm T}(t)e_{\rm in}^{\rm T}(t)\\&\quad-\eta_1e_{\rm in}^{\rm T}(\tau_1)e_{\rm in}(\tau_1)+\frac{\eta_2}{1-\gamma}e_{\rm in}^{\rm T}(t)e_{\rm in}(t)\\&\quad-\eta_2e_{\rm in}^{\rm T}(\tau_2)e_{\rm in}(\tau_2),\\\dot V_{\rm out}(t)&\leq\frac{\eta_3}{1-\varphi}e_{\rm out}^{\rm T}(t)e_{\rm out}^{\rm T}(t)\\&\quad-\eta_3e_{\rm out}^{\rm T}(\tau_1)e_{\rm out}(\tau_1)+\frac{\eta_4}{1-\varphi}e_{\rm out}^{\rm T}(t)e_{\rm out}(t)\\&\quad-\eta_4e_{\rm out}^{\rm T}(\tau_2)e_{\rm out}(\tau_2).\end{aligned}$$

Based on the error equation (3) and the control scheme (4) and (5), it is obtained that

$$\begin{aligned}&2e^{\rm T}(t)\dot e(t)\\&=2e_{\rm in}^{\rm T}(t)(I_N\otimes\tilde A)e_{\rm in}(t)+2e_{\rm in}^{\rm T}(t)(I_N\otimes\tilde B)F(e_{\rm in}(\tau_1))\\&\quad+2e_{\rm in}^{\rm T}(t)(C\otimes I_n)G(e_{\rm in}(t))\\&\quad+2e_{\rm in}^{\rm T}(t)(D\otimes I_n)H(e_{\rm in}(\tau_2))-2\xi e_{\rm in}^{\rm T}(t)e_{\rm in}(t)\\&\quad-\alpha e_{\rm in}^{\rm T}(t)\,{\rm sign}(e_{\rm in}(t))|e_{\rm in}(t)|^{\sigma}\\&\quad+2(I_N\otimes\tilde E)e_{\rm in}(t)+2e_{\rm in}^{\rm T}(t)(I_N\otimes\tilde W)F(e_{\rm in}(\tau_1))\\&\quad-\alpha\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e_{\rm in}^{\rm T}(s)e_{\rm in}(s)\,{\rm d}s\right)^{(1+\sigma)/2}\\&\quad-\alpha\left(\frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t}e_{\rm in}^{T}(s)e_{\rm in}(s)\,{\rm d}s\right)^{(1+\sigma)/2}\\&\quad-\lambda\left(\frac{\eta_3}{1-\gamma}\int_{t-\tau_1(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s\right)^{\mu/2}\\&\quad-\lambda(\frac{\eta_4}{1-\gamma}\int_{t-\tau_2(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s)^{\mu/2}.\end{aligned}$$

Furthermore, by Lemma 1, one has the following estimation:

$$\begin{aligned}
&2e_{\rm in}^{\rm T}(t)(I_N\otimes\tilde{A})e_{\rm in}(t)+2e_{\rm in}^{T}(t)(I_N\otimes\tilde{B})\\
&\times F(e_{\rm out}(\tau_1))+2e_{\rm in}^{\rm T}(t)(I_N\otimes\tilde{E})e_{\rm in}(t)\\
&+2e_{\rm in}^{\rm T}(t)(I_N\otimes\tilde{W})F(e_{\rm out}(\tau_1))\\
\le\ & 2e_{\rm in}^{\rm T}(t)(I_N\otimes\tilde{A})e_{\rm in}(t)+s_1e_{\rm in}^{\rm T}(t)\\
&\times(I_N\otimes MM^{\rm T})e_{\rm out}^{\rm T}(t)+2e_{\rm in}^{\rm T}(t)(I_N\otimes\tilde{E})\\
&\times e_{\rm in}(t)+s_1e_{\rm in}^{\rm T}(t)(I_N\otimes MM^{\rm T})e_{\rm out}^{\rm T}(t)\\
&+s_1^{-1}e_{\rm in}^{\rm T}(t)(I_N\otimes N_1^{\rm T}N_1)e_{\rm in}^{\rm T}(t)+s_2e_{\rm in}^{\rm T}(t)\\
&\times(I_N\otimes BB^{\rm T})e_{\rm in}^{\rm T}(t)+s_2^{-1}F^{\rm T}(e_{\rm out}(\tau_1))\\
&\times F(e_{\rm out}(\tau_1))+s_3e_{\rm in}^{\rm T}(t)(I_N\otimes MM^{\rm T})e_{\rm in}^{\rm T}(t)\\
&+s_3^{-1}F^{\rm T}(e_{\rm out}(\tau_1))(I_N\otimes N_2^{\rm T}N_2)F(e_{\rm out}(\tau_1))\\
\le\ & (\lambda_{\max}(A^s)+s_1\lambda_{\max}(MM^{\rm T})+s_2\lambda_{\max}(MM^{\rm T})\\
&+s_1^{-1}\lambda_{\max}(N_1^{\rm T}N_1)+s_2\lambda_{\max}(BB^{\rm T})+s_3\lambda_{\max}(MM^{\rm T}))\\
&\times e_{\rm in}^{\rm T}(t)e_{\rm in}(t)+\left(s_2^{-1}L_j^2+s_3^{-1}L_j^2\lambda_{\max}(N_2^{\rm T}N_2)\right)\\
&\times e_{\rm out}^{\rm T}(\tau_1)e_{\rm out}(\tau_1).
\end{aligned}$$

Similarly, we have the estimation for another term

$$\begin{aligned}
&2e_{\rm in}^{\rm T}(t)\,(C\otimes I_n)G(e(t))+2e_{\rm in}^{T}(t)(D\otimes I_n)\\
&\times H(e_{\rm in}^{\rm T}(\tau_2))+2e_{\rm out}^{\rm T}(t)(D\otimes I_n)H(e_{\rm in}^{\rm T}(\tau_2))\\
\le\ & s_4e_{\rm in}^{\rm T}(t)(CC^{\rm T}\otimes I_n)e_{\rm in}^{\rm T}(t)+s_4^{-1}G^{\rm T}(e_{\rm in}(t))G(e_{\rm in}(t))\\
&s_5e_{\rm in}^{\rm T}(t)(DD^{\rm T}\otimes I_n)e_{\rm in}(t)+s_5^{-1}H^{\rm T}(e_{\rm out}(\tau_2))\\
&\times H(e_{\rm out}(\tau_2))+s_4^{-1}G^{\rm T}(e_{\rm in}(t))G(e_{\rm in}(t))\\
\le\ & (s_4\lambda_{\max}(CC^{\rm T})+s_4^{-1}L_r^2+s_5\lambda_{\max}(DD^{\rm T}))e_{\rm in}^{\rm T}(t)e_{\rm in}(t)\\
&+s_5^{-1}L_h^2e_{\rm out}^{\rm T}(\tau_2)e_{\rm out}(\tau_2)+s_3^{-1}L_r^2+s_5^{-1}L_r^2\\
&+(s_4^{-1}L_r^2+s_5\lambda_{\max}(DD^{T}))e_{\rm in}^{\rm T}(t)e_{\rm in}(t).
\end{aligned}$$

It follows from Definition 2 and Lemma 2 that

$$\begin{aligned}
&-\alpha e^{\rm T}(t)\,{\rm sign}(e(t))|e(t)|^{\sigma}\\
=\ & -\alpha|e^{\rm T}(t)||e(t)|^{\sigma}\le-\alpha(e^{\rm T}(t)e(t))^{1+\sigma/2}.
\end{aligned}$$

Together with expressions (6) and (7), it results in

$$\begin{aligned}
\dot{V}_{\rm in}(t)\ \le\ & \Psi e_{\rm in}^{\rm T}(t)e_{\rm in}(t)+(s_5^{-1}L_h^2-\eta_2)e_{\rm in}^{\rm T}(\tau_2)e_{\rm in}(\tau_2)\\
&+\left(s_2^{-1}L_j^2+s_3^{-1}L_j^2\lambda_{\max}(N_2^{\rm T}N_2)-\eta_1\right)e_{\rm in}^{\rm T}(\tau_1)e_{\rm in}(\tau_1)\\
&+(L_j^2+s_3^{-1}L_j^2-\eta_1)e_{\rm in}^{\rm T}(\tau_1)e_{\rm in}(\tau_1)\\
&-\alpha\left(e_{\rm in}^{\rm T}(t)e_{\rm in}(t)\right)^{\sigma/2}\\
&-\alpha\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e_{\rm in}^{\rm T}(s)e_{\rm in}(s)\,{\rm d}s\right)^{\sigma/2}\\
&-\alpha\left(\frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t}e_{\rm in}^{\rm T}(s)e_{\rm in}(s)\,{\rm d}s\right)^{\sigma/2},\\
\dot{V}_{\rm out}(t)\ \le\ & (s_5^{-1}-\eta_3)e_{\rm out}^{\rm T}(\tau_2)e_{\rm out}(\tau_2)\\
&+\left(s_3^{-1}L_j^2\lambda_{\max}(N_2^{\rm T}N_2)-\eta_3\right)\\
&\times e_{\rm out}^{\rm T}(\tau_1)e_{\rm out}(\tau_1)+\eta_3e_{\rm out}^{\rm T}(\tau_1)e_{\rm out}(\tau_1)\\
&-\lambda\left(e_{\rm out}^{\rm T}(t)e_{\rm out}(t)\right)^{\mu/2}\\
&-\lambda\left(\frac{\eta_3}{1-\varphi}\int_{t-\tau_1(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s\right)^{(\mu+1)/2}\\
&-\lambda\left(\frac{\eta_4}{1-\varphi}\int_{t-\tau_2(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s\right)^{(\mu+1)/2}\\
&\times\left(s_3^{-1}L_j^2\lambda_{\max}(N_2^{\rm T}N_2)-\eta_3\right)L_j^2\eta_3e_{\rm out}^{\rm T}(\tau_1)e_{\rm out}(\tau_1).
\end{aligned}$$

Then, it follows from Lemma 2 that

$$\begin{aligned}
\dot{V}_{\rm in}(t)\ \le\ & -\alpha\left(e_{\rm in}^{\rm T}(t)e_{\rm in}(t)\right)^{\sigma/2}\\
&-\alpha\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e_{\rm in}^{\rm T}(s)e_{\rm in}(s)\,{\rm d}s\right)^{\sigma/2}\\
&\times\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e_{\rm in}^{\rm T}(s)e_{\rm in}(s)\,{\rm d}s\right)^{\sigma/2}\\
&-\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e_{\rm in}^{\rm T}(s)e_{\rm in}(s)\,{\rm d}s\right)^{\sigma/2}\\
&-\alpha\left(\frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t}e_{\rm in}^{\rm T}(s)e_{\rm in}(s)\,{\rm d}s\right)^{\sigma/2}\\
\le\ & -\alpha V_{\rm in}^{\beta}(t), \qquad (6)\\
\dot{V}_{\rm out}(t)\ \le\ & (s_5^{-1}-\eta_3)e_{\rm out}^{\rm T}(\tau_2)e_{\rm out}(\tau_2)\\
&+\left(s_3^{-1}L_j^2\lambda_{\max}(N_2^{\rm T}N_2)-\eta_3\right)e_{\rm out}^{\rm T}(\tau_1)e_{\rm out}(\tau_1)\\
&-\lambda(e_{\rm out}^{\rm T}(t)e_{\rm out}(t))^{\mu/2}\\
&-\lambda\left(\frac{\eta_3}{1-\varphi}\int_{t-\tau_1(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s\right)^{\mu/2}\\
&-\lambda\left(\frac{\eta_4}{1-\varphi}\int_{t-\tau_2(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s\right)^{\mu/2}\\
&-\left(\frac{\eta_3}{1-\varphi}\int_{t-\tau_1(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s\right)^{\mu/2}\\
\le\ & -\lambda\left(e_{\rm out}^{\rm T}(t)e_{\rm out}(t)\right)^{\mu/2}\\
&-\lambda\left(\frac{\eta_4}{1-\varphi}\int_{t-\tau_2(t)}^{t}e_{\rm out}^{\rm T}(s)e_{\rm out}(s)\,{\rm d}s\right)^{\mu/2}\\
\le\ & -\lambda V_{\rm out}^{\mu}(t). \qquad (7)
\end{aligned}$$

Integrating expression (6) from $t_0$ to $t$, one gets

$$\begin{aligned}
V(t)\ =\ & V_{\rm in}(t)+V_{\rm out}(t)\\
\le\ & \left((kV(t_0))^{1-\beta}-\alpha(1-\beta)(t-t_0)\right)^{1/(1-\beta)},
\end{aligned}$$

where $0<k<1$, $V(t_0)>V_{\rm in1}(t_0)$, $V(t_0)>V_{\rm out1}(t_0)$. The controller $\boldsymbol{u}_{\rm in}(t)=u_1(t)$ and $\boldsymbol{u}_{\rm out}(t)=u_2(t)$ are activated at $t_0$.

Since $\alpha>\alpha_1$, $\alpha>\lambda_1$, the convergence rate of $V_{\rm in}(t)$ is faster than that of the boundary function $V_{\rm in1}(t)$, $V_{\rm out}(t)$ is faster than that of the boundary function $V_{\rm out1}(t)$, there exists an instant $V_{\rm in}(t)$ when the trajectory of $V_{\rm in}(t)$ and touches $V_{\rm in1}(t)$, the trajectory of $V_{\rm out}(t)$ and touches $V_{\rm out1}(t)$. Based on the proposed control mechanism, the controllers $u_{\rm in}(t)=u_1(t)$ and $u_{\rm out}(t)=u_2(t)$ are still active after $V_{\rm in}(t)$, the trajectory of $V_{\rm in}(t)$ and $V_{\rm out}(t)$ will enter the region $V_{\rm in2}(t)<V_{\rm in}(t)<V_{\rm in1}(t)$ and $V_{\rm out2}(t)<V_{\rm out}(t)<V_{\rm out1}(t)$, respectively.

Further, due to $\alpha > \alpha_2$, $\alpha > \lambda_2$, there is an instant $V_{\text{in}}(t)$ such that the trajectory of $V_{\text{in}}(t)$ touches the boundary $V_{\text{in2}}(t)$, the trajectory of $V_{\text{out}}(t)$ touches the boundary $V_{\text{out2}}(t)$, as well the controller $u_{\text{in}}(t) = 0$ and $u_{\text{out}}(t) = 0$ are activated at $V_{\text{in}}(t)$. If there exists another instant $V_{\text{in}}(t)$ such that $V_{\text{in}}(t)$ touches the boundary $V_{\text{in1}}(t)$, the controller $u_{\text{in}}(t) = u_1(t)$ and $u_{\text{out}}(t) = u_2(t)$ are activated again, the trajectory of $V_{\text{in}}(t)$ and $V_{\text{out}}(t)$ touches the boundary $V_{\text{in2}}(t)$ and $V_{\text{out2}}(t)$, activating the controllers $V_{\text{in}}(t) = 0$ and $V_{\text{out}}(t) = 0$ once again separately.

Based on the above discussion, there is an instant

$$t_r > T_2 = t_0 + \frac{(kV(t_0))^{1-\beta}}{\alpha_2(1-\beta)}$$

such that the trajectory of $V_{\text{in}}(t)$ and $V_{\text{out}}(t)$ touch the boundary $V_{\text{in1}}(t)$ and $V_{\text{out2}}(t)$, respectively, $V_{\text{in}}(t_r) = V_{\text{in1}}(t_r)$ and $V_{\text{out}}(t_r) = V_{\text{out1}}(t_r)$. It yields directly

$$||e(t)||^2 \le V_{\text{in}}(t) + V_{\text{out}}(t) \le V_{\text{in1}}(t) + V_{\text{out1}}(t), \quad t \ge t_1.$$

Therefore, by Lemma 1, one obtains that the error system (4) is finite time stable with the setting time $\tau_1(t)$ and $\tau_2(t)$, thus finite time hybrid synchronization is achieved.

**Remark 2** If networks without delay, the delays $\tau_1(t)$ and $\tau_2(t)$ in the above formulas are equal to 0.

## 4. Numerical example

Assuming the network (1)–(2) consists of twenty nodes, respectively. The function $f(\cdot)$ in both of the subnetworks is chosen as

$$f(z(t)) = (|z_1(t)+2| - |z_1(t)-2|, 1, 0, 1)^{\mathrm{T}}.$$

The nonlinear inner connecting functions and the time-varying delays are taken as

$$g(\cdot) = h(\cdot) = \tanh(\cdot), \quad \tau_1(t) = \tau_2(t) = \cos t,$$
$$L_j = 1.4, \quad L_h = L_r = 1.2, \quad \gamma = 1.3, \quad \varphi = 1.1.$$

$S(t)$ is set as $\begin{cases} \dot{x} = \varphi(y-x) + \vartheta yz, \\ \dot{y} = cx - dxz - yg + w, \\ \dot{z} = xy - bz, \\ \dot{w} = -fy - hw. \end{cases}$

The initial conditions are set as

$$\varphi = 30, \quad b = 4.2, \quad c = 24, \quad d = 5.3,$$
$$\vartheta = 35, \quad f = 102, \quad h = 1, \quad g = 2,$$
$$x_i(t_0) = (0.11i, \quad 0.12i, \quad 0.13i, \quad 0.14i)^{\mathrm{T}},$$
$$y_i(t_0) = (-1-0.1i, \quad -1-0.2i, \quad -2i, -i)^{\mathrm{T}},$$
$$i = 1, 2, \ldots, 20;$$
$$\boldsymbol{M}_{20\times 20} = \boldsymbol{N}_1 = \boldsymbol{N}_2 = \boldsymbol{I},$$
$$F(t) = \mathrm{diag}(0.3, \sin t, \cos t, 0.5, 0.3, \sin t, \cos t, 0.5, 0.3, \sin t, \cos t, 0.5, 0.3, \sin t, \cos t, 0.5, \sin t, \cos t, 0, 1),$$
$$s_i = 2 (i = 1, \ldots, 1), \quad \xi = 1, \quad \eta_1 = 2, \quad \eta_2 = 3, \quad \sigma = 4,$$
$$\alpha = 3, \quad \alpha_1(t) = 2\sin t,$$
$$\alpha_2(t) = \cos t, \quad k = 0.3, \beta = 0.1, \quad \eta_3 = 0.2,$$
$$\eta_4 = 0.2, \quad \lambda = 0.1, \quad \chi = 0.4.$$

$$\boldsymbol{A} = \begin{bmatrix}
0.2 & 0.2 & 1 & 1.1 & 0 & 1.3 & 1 & 0 & 1.5 & 0 & 0 & 0.2 & 0 & -0.3 & 0 & 0 & -1.2 & -0.3 & -0.2 & 0.56 \\
0.3 & 0 & -0.3 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & -0.3 & 0 & 0 & 0 & 0.14 & 0 & 0.2 & 0 & -1.2 & 1.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.11 & 0.50 & -0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0.32 & 0.2 & -1.2 & 0 & 0 \\
0.12 & 0 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0.56 & 0 & 0.1 & 0 & 0 & -1.2 & -0.2 & 0.56 \\
0 & 0.65 & 0 & -0.3 & 0 & 0.2 & 0 & 0.2 & 0 & -2.2 & 0 & 0 & -1.12 & 1 & 0 & 0 & 0 & 0.2 & 0.2 & -1.2 \\
0 & 0 & 0.8 & 0 & 0.1 & 0 & 0 & 1.03 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0.1 & 0.2 & 0.2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.12 & 0.1 & 0 & 0.35 & -0.2 & 0 & 0 & -1.2 & 0 & 0 & 0.1 & -0.2 & 0.8 & 0 \\
0 & 0.8 & -0.2 & 0.1 & 0 & -0.11 & 0 & 0 & 0.4 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.34 & 0 & 0 & 0 & 0.56 \\
0.12 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0.23 & 0 & 0 & 0.2 & 0 & 0 & 0.1 & 0 & 0 & -0.2 & -1.2 & 0 \\
0.11 & 0 & 0 & 1.1 & 0 & 0.21 & 0 & 0.22 & 0 & 0.54 & 0.21 & 0.2 & 0.2 & 0 & -0.12 & 0 & 0.21 & 0 & 0 & -1.4 \\
0 & -0.5 & 0 & 0 & 0 & 0 & 0.32 & 0 & 2.1 & 0 & 0 & 0 & 0 & -3.2 & 0 & 0 & 0.2 & 0 & -0.2 & 0.56 \\
0.23 & 0 & 0.32 & 0 & 0.2 & 0 & 2.1 & 0 & 0 & 0 & 0.1 & 0 & 0.2 & 0 & 0 & 0.32 & 0 & 0.2 & 0 & 0 \\
0.25 & 0 & 0 & 2.1 & 0 & 1 & 0.2 & -0.32 & 0 & 0.32 & 0 & 0.2 & 0 & 1 & -0.54 & 0 & 0.23 & 0.12 & 0 & -1.2 \\
1 & 0 & 0.32 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0.2 & 0.2 & 0.8 & 0.26 \\
0 & 2.1 & 0 & 0 & 0 & 2.3 & 1.8 & 1 & 0 & 0.25 & 0 & 1.2 & 2.1 & 0 & 0 & 0.2 & 0 & 0.22 & 0 & 1 \\
-0.2 & 0 & -0.12 & 1 & 0.8 & 0 & 0 & 0.2 & 0 & -0.2 & 0 & 0.12 & 0 & 0.2 & 0.2 & 0 & 0.8 & 0.32 & -0.2 & 0.16 \\
0 & 0 & 0 & 0 & 0 & 0.1 & 1 & 0 & 1.2 & 0 & 0.12 & 0 & 0 & 0 & 0 & 0 & 0.12 & 0.92 & 0.2 & 0.3 \\
1 & 1 & -0.45 & 0 & 0 & 1 & 2.3 & 1.2 & 0 & 1.3 & 1.3 & 3.2 & 0.35 & 0 & 0.16 & 0.2 & 0.2 & 2.1 & 0 & 0.16 \\
0.12 & 0 & 0 & -0.3 & 0 & 0 & 0 & 0 & 1.2 & 0 & 0 & 1.2 & 0 & -0.2 & 0 & 1 & 0.13 & 0.2 & -0.2 & 1 \\
2 & 1 & 1 & 0 & 0 & 1.8 & 0.21 & 0 & 0 & 0.52 & 0 & 0 & -1.12 & 0 & 0 & 0.23 & 0 & 0.12 & 1 & 0.3
\end{bmatrix},$$

$$\boldsymbol{B}=\begin{bmatrix}
2.5 & 0 & 2.5 & 0 & 2.7 & 0 & 0 & 2.2 & 0 & 2.2 & 0 & 2.5 & 2.5 & 0 & -2.3 & 2.5 & 0 & 2.7 & 0 & -2.9 \\
0 & -2.4 & 0 & 2.5 & 0 & -2.9 & 2.5 & 2.2 & 2.2 & 2.5 & -2.4 & 0 & 0 & 2.8 & 0 & 2.2 & 0 & 2.2 & -2.9 & 0 \\
-2.9 & 0 & -2.9 & 0 & -2.9 & 0 & 0 & 0 & 2.8 & 0 & 0 & -2.9 & 2.2 & 0 & 2.6 & -2.3 & -2.3 & -2.9 & 0 & 0 \\
0 & 2.2 & 2.5 & -2.1 & 0 & -2.4 & 0 & 2.8 & 0 & 0 & 2.2 & 0 & 2.2 & 0 & 2.2 & 0 & 2.2 & 0 & -2.3 & -2.9 \\
-2.1 & 0 & 0 & 0 & 2.5 & 0 & -2.1 & 0 & -2.9 & 2.2 & -2.3 & 0 & 2.5 & 2.8 & 0 & 2.6 & 2.2 & -2.9 & 0 & 0 \\
-2.9 & 0 & 0 & 2.4 & 0 & 0 & 0 & -2.4 & 0 & 2.6 & 0 & 2.4 & -2.3 & -2.9 & 2.2 & 0 & -2.9 & 0 & -2.9 & 0 \\
0 & 2.3 & 2.2 & 0 & 0 & 2.6 & -2.9 & 0 & 2.2 & 0 & -2.9 & 2.2 & 0 & 0 & 0 & 2.2 & 0 & 2.6 & 2.2 & 0 \\
2.6 & 0 & 0 & -2.4 & 2.6 & 0 & 0 & 2.6 & 0 & 2.1 & 0 & 0 & -2.9 & 0 & 2.6 & 0 & 2.5 & 0 & 2.6 & -2.9 \\
0 & -2.9 & 2.2 & 0 & 0 & 0 & 2.2 & 0 & -2.1 & 0 & 2.2 & 2.6 & 0 & 2.6 & 0 & 2.6 & 0 & -2.4 & 0 & 0 \\
-2.9 & 0 & 0 & -2.3 & 2.5 & 2.3 & 0 & 0 & 0 & 2.5 & 2.2 & 0 & -2.3 & 0 & 2.6 & 0 & 0 & 0 & 2.6 & 2.6 \\
0 & 2.5 & -2.7 & 2.6 & 0 & 0 & -2.3 & 0 & 2.8 & 0 & 0 & 2.6 & -2.1 & 2.5 & 0 & 0 & -2.9 & 2.6 & 0 & 0 \\
-2.9 & 0 & 0 & 0 & 2.8 & 2.5 & 0 & 2.7 & 0 & 0 & -2.4 & 0 & 0 & 0 & 2.5 & -2.4 & 0 & 0 & -2.9 & 0 \\
0 & 0 & 2.5 & 2.7 & 0 & 0 & 2.7 & 0 & 2.2 & 2.8 & 0 & 0 & 2.5 & 2.2 & 0 & 0 & 0 & -2.4 & 0 & 2.6 \\
-2.4 & 2.7 & 0 & 0 & 2.6 & 2.8 & 0 & 2.1 & -2.9 & 0 & 0 & 2.8 & 2.2 & 0 & 2.2 & 2.5 & 2.5 & 0 & 2.6 & 0 \\
0 & 0 & 2.7 & 2.2 & 0 & 0 & -2.4 & 0 & 0 & 2.2 & 2.6 & 0 & -2.4 & 0 & -2.9 & 0 & 0 & -2.3 & 0 & -2.3 \\
2.7 & 2.5 & 0 & 2.7 & 2.2 & 0 & 2.2 & 2.8 & -2.4 & 0 & 0 & 0 & 0 & 2.5 & 0 & 2.9 & -2.3 & 0 & 0 & 0 \\
0 & 0 & 2.2 & 0 & 0 & 2.8 & 0 & 0 & 0 & 2.8 & 0 & 2.8 & 2.2 & 0 & 2.1 & 0 & 0 & 0 & 2.4 & 2.2 \\
2.6 & 2.7 & 0 & 2.2 & 2.1 & 0 & -2.4 & 2.5 & 0 & 0 & -2.4 & 0 & 0 & 2.6 & 0 & 2.1 & -2.1 & -2.3 & 0 & 0 \\
2.5 & 0 & 0 & 2.7 & 0 & 2.5 & 0 & 0 & 2.8 & 0 & 0 & 0 & -2.4 & 0 & 2.5 & 0 & 0 & 0 & -2.3 & 2.2 \\
0 & 0 & 2.1 & 0 & 2.1 & 0 & 2.5 & 0 & 0 & 2 & 0 & 2.1 & 0 & 2.1 & 0 & 2.1 & 0 & 2.2 & 0 & 2.8
\end{bmatrix},$$

$$\boldsymbol{E}=\begin{bmatrix}
0 & -2.8 & -2.9 & 0 & -2.2 & 1 & 0 & 2 & -2.1 & 0 & -2.3 & 2.3 & 1 & 0 & -2.6 & -2.5 & 0 & 1 & -2.6 & 0 \\
-2.8 & 0 & -2 & -2.7 & 0 & 0 & -2 & -2.4 & -2 & 0 & 2.2 & 2 & 0 & 2.5 & 0 & -2.5 & -2.3 & 0 & -1.5 & 0.9 \\
-1.9 & -1.9 & 0 & 0 & 3 & 2.4 & 0 & 2.3 & 0 & 0 & 2 & 0 & 2.5 & 0 & -3 & 0 & -3 & -1.6 & -2 & -2.6 \\
-0.4 & 0 & -2.1 & 2.4 & 0 & -2.4 & 0 & -2 & 2.4 & 3 & 0 & 2.2 & 1 & 2.6 & 2.6 & -3 & 1 & 0 & -2.3 & 2.6 \\
0 & 0 & -2 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & -3 & 2.2 & 0 & -2.3 & 2.7 & 0 & 2 & 2 \\
1.4 & 0 & 0 & 2.2 & 0 & -3 & 0 & 2.7 & 1 & 0 & 2.5 & 2.4 & 1 & 0 & 1.1 & 1.5 & 1.2 & 2.2 & 2.9 & 2.6 \\
0 & 0 & -2.4 & -2.3 & 2.3 & 2.2 & 2.9 & 0 & 2.1 & 2.5 & -3 & 0 & 0 & 0 & 1.8 & 0 & 3 & 0 & 3 & 2.8 \\
-2.6 & 1.1 & 0 & 1.1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2.3 & 1.7 & 1.4 & 0 & 2.6 & 1.9 & 2.6 & 1.2 \\
-3 & 2.1 & -3 & 2 & 0 & 2.1 & 2.7 & 2 & 0 & 2.5 & 2 & -2.5 & 0 & -2 & 0 & 2.6 & 2.1 & 0 & -2 & 0 \\
0 & 0 & 2.1 & 0 & 2.3 & 0 & -2.1 & 0 & -3 & 0 & -2.1 & 0 & 3 & 1.1 & 2.5 & 2.5 & 0 & 2.6 & 3 & 2 \\
-2.4 & 2.2 & 0 & 1 & 0 & -2.5 & 1 & 0 & 2.5 & 1 & 0 & -2.2 & 0 & 0 & 1.4 & 0 & 2.6 & 1 & -3 & -2.4 \\
1 & -2.4 & 0 & 0 & -3 & 0 & 0 & 0 & -3 & -2.1 & 3 & 0 & 0 & 2.7 & 0 & 2.1 & 2 & 0 & -2.3 & 0 \\
0 & -2 & 0 & 3 & 2.8 & -2 & -2.5 & 3 & 0 & 2.6 & 2.1 & 2.5 & 1 & 2.3 & 2.2 & -2.1 & -2.2 & 1 & 2.1 & 2.3 \\
-2.5 & 0 & 2.6 & 2.5 & 0 & 0 & 2.3 & 0 & -2.1 & 2.7 & 0 & 0 & 2.4 & 0 & 1.9 & -3 & 0 & 0 & -2.2 & 0 \\
2.6 & 2.7 & 0 & 2.3 & 2.7 & 0 & 2.1 & -2.2 & 2.1 & 2 & -2 & 0 & 0 & 3 & 0 & 0 & -2.1 & -3 & 0 & -2.1 \\
2 & 0 & 2.5 & 1 & -3 & 2.8 & 2.5 & 1 & 0 & 2.2 & 2.8 & 2.7 & 1 & 2.4 & -3 & 1.7 & 2.1 & 0 & 1 & 3 \\
0 & 2 & 2.1 & 2.5 & 3 & 2.2 & 2 & 0 & 2.4 & 2.9 & 2.2 & 0 & 2.3 & 2.1 & -2.1 & 2.7 & -3 & 3 & 0 & 0 \\
2.6 & -1.5 & -2.4 & 0 & -2.2 & 0 & 2.9 & -2 & 0 & 3 & 0 & 2.3 & 1 & -2 & 2.6 & 0 & 1.4 & 2.5 & -3 & -3 \\
0 & 0 & 0 & -2.1 & 0 & 1 & 2.3 & 0 & 1 & 2.6 & 1.2 & 3 & 0 & 1.5 & 1 & 1.6 & 0 & -2 & 0 & 2 \\
-2.4 & -2.1 & 2.3 & 1 & 2.8 & 0 & 2.4 & 0 & 2.9 & 2.1 & 0 & 2.6 & 1 & 2.1 & 2.7 & 2.2 & 0 & 2.4 & 2.4 & 2
\end{bmatrix},$$

$$\boldsymbol{W}=\begin{bmatrix}
0 & 2.1 & -2.7 & 2.9 & 0 & 1 & -3 & -2.9 & 0 & 2.2 & 0 & 0 & -2.8 & 2 & -2.9 & 2.2 & -2.7 & -2.8 & -2.7 & -2.8 \\
-1.9 & -3 & 0 & -2.9 & -2.8 & -2.5 & 3 & -2.7 & 1 & -2.5 & 2.7 & -2.7 & 2.1 & 2.5 & -2.8 & 2.2 & 3 & 2.3 & 0 & 3 \\
2.1 & -2.4 & -2.7 & 0 & -2.6 & -2.3 & -2.7 & 2.2 & -2.5 & 0 & -2.5 & 0 & 2.8 & 2 & -2.4 & 3 & -2.8 & 0 & -2.1 & -2.7 \\
-1.9 & -2.5 & 0 & 2.9 & 2.1 & 0 & 0 & -2.3 & 1 & 1 & 0 & 2.4 & 2.3 & 0 & 2.6 & 0 & 3 & 2.8 & -2.8 & 2.8 \\
2.1 & 0 & -2.6 & -2.5 & -1.4 & 2.5 & 0 & 1.2 & 0 & 2.9 & 2.5 & 0 & 2.8 & 0 & 2.8 & -2.2 & 0 & 2.4 & 3 & 3 \\
0 & -2.6 & 0 & -2.6 & 0 & -2.6 & 0 & -2.1 & 2.7 & -2.4 & -2 & 1 & 0 & 2.6 & -2.6 & 0 & -2.7 & 0 & 2 & 1.1 \\
3 & 0 & -2.5 & -2.2 & 2.1 & 2.5 & 0 & -2.7 & -2.4 & 0 & 0 & 2.3 & 1 & 1.7 & 0 & -2.5 & 3 & 2.5 & -1.3 & 3 \\
-2.8 & -2.5 & -2.3 & 0 & -2.1 & -2.5 & 2.7 & 0 & 1 & 2.3 & 0 & 2.9 & 1 & 0 & -2.2 & 2.8 & 0 & 2 & 0 & 2.5 \\
0 & -2 & 2.2 & 2.7 & 2 & 2.7 & -2.4 & 2 & 2.1 & 2.5 & 2.6 & 2.6 & 3 & 2.8 & -2.3 & 0 & -2 & 0 & 2.4 & 2.9 \\
-2.7 & 0 & 0 & 0 & 0 & 2.9 & 0 & -2.2 & 0 & 0 & 0 & 1 & -2.2 & 0 & 2.8 & -2.5 & 0 & 2.2 & 0 & 0 \\
0 & -2.7 & 2.7 & -2.1 & 2.3 & 0 & 2 & -2.4 & 0 & 2 & 2.6 & 2 & 0 & 2.6 & 0 & 2.5 & 2 & 2.4 & 2.5 & 2.6 \\
3 & -2.1 & -2.2 & 2 & 0 & 0 & -2 & 0 & 2.3 & 2.3 & 0 & 0 & -2.1 & 2.6 & 2.1 & 0 & -2.3 & 0 & 2.9 & 0 \\
-2.8 & 1 & 2.3 & 2.3 & 0 & 2.2 & 2.4 & 1.8 & 0 & -2 & 0 & -1.3 & 3 & 0 & 2.4 & 1.4 & 2 & -2.2 & 2.7 & 2.8 \\
0 & -2 & 2.7 & 3 & 0 & 0 & -2.4 & 0 & 0 & 0 & 2.7 & 1 & -2.2 & 2.7 & 2.9 & -2 & 1 & 2.8 & -2.1 & 1 \\
3 & 3 & -2.4 & -2.3 & 0 & -2.2 & 1 & 2.4 & 2.4 & 2.8 & 1 & 2.1 & 2.5 & 2.3 & 2 & 3 & 0 & 2.9 & 2.4 & 0 \\
0 & 1 & -2.4 & 1.4 & -2.3 & 0 & -2.2 & 2.6 & 3 & -2 & -2.3 & 0 & 0 & 2.7 & 2.3 & 0 & 0 & 2.4 & 0 & 0 \\
-2.6 & 2.9 & 2 & -2.4 & 2.4 & -2 & 1 & 0 & 0 & 0 & 0 & 2.5 & 0 & 2 & 2.4 & 2.3 & 1 & 0 & 2 & 2.5 \\
3 & 0 & -2.2 & 0 & 2.9 & 2 & 2.1 & 2.9 & 2 & 2.2 & 0 & 2.8 & 2.9 & -2.1 & 0 & 2.9 & 3 & -2 & 1 & -2.1 \\
0 & 2.2 & -2.6 & -2.1 & -2.3 & -2.1 & 1 & -2.4 & 0 & 2.5 & -2.2 & 2.6 & -2.2 & 2.1 & -2.3 & 1 & -2.4 & 2.2 & 0 & 2.7 \\
0 & 0 & -2.1 & 2.2 & 2.5 & 0 & 2.5 & 0 & 2.1 & -2.3 & 2.6 & 0 & -2.1 & 2.6 & 1 & 0 & -1.3 & 2.1 & 2.6 & -1.2
\end{bmatrix},$$

$$\boldsymbol{C}^1=\begin{bmatrix}
0.1 & 0.3 & 0 & 0.1 & 0 & 0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.2 & 0 & -0.3 & 0 & 0 & -1.2 & -0.3 & -0.2 & 0.56\\
0 & 0 & -0.3 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & -0.3 & 0 & 0 & 0 & 0.14 & 0 & 0.2 & 0 & -1.2 & 0.2 & 0\\
0 & 0 & 0 & 0 & 0 & 0.11 & 0.50 & -0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0.32 & 0.2 & -1.2 & 0 & 0\\
0.12 & 0 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0.56 & 0 & 0.1 & 0 & 0 & -1.2 & -0.2 & 0.56\\
0 & 0.65 & 0 & -0.3 & 0 & 0.2 & 0 & 0.2 & 0 & -2.2 & 0 & 0 & -1.12 & 0 & 0 & 0 & 0 & 0.2 & 0.2 & -1.2\\
0 & 0 & 0.8 & 0 & 0.1 & 0 & 0 & 1.03 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0.1 & 0.2 & 0.2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0.12 & 0.1 & 0 & 0.35 & -0.2 & 0 & 0 & -1.2 & 0 & 0 & 0.1 & -0.2 & 0.8 & 0\\
0 & 0.8 & -0.2 & 0.1 & 0 & -0.11 & 0 & 0 & 0.4 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.34 & 0 & 0 & 0 & 0.56\\
0.12 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0.23 & 0 & 0 & 0.2 & 0 & 0 & 0.1 & 0 & 0 & -0.2 & -1.2 & 0\\
0.11 & 0 & 0 & 1.1 & 0 & 0.21 & 0 & 0.22 & 0 & 0.54 & 0.21 & 0.2 & 0.2 & 0 & -0.12 & 0 & 0.21 & 0 & 0 & -1.4\\
0 & -0.5 & 0 & 0 & 0 & 0 & 0.32 & 0 & 2.1 & 0 & 0 & 0 & 0 & -3.2 & 0 & 0 & 0.2 & 0 & -0.2 & 0.56\\
0.23 & 0 & 0.32 & 0 & 0.2 & 0 & 2.1 & 0 & 0 & 0 & 0.1 & 0 & 0.2 & 0 & 0 & 0.32 & 0 & 0.2 & 0 & 0\\
0.25 & 0 & 0 & 2.1 & 0 & 0 & 0.2 & -0.32 & 0 & 0.32 & 0 & 0.2 & 0 & 0 & -0.54 & 0 & 0.23 & 0.12 & 0 & -1.2\\
0 & 0 & 0.32 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0.2 & 0.2 & 0.8 & 0.56\\
0 & 2.1 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0.25 & 0 & 0 & 2.1 & 0 & 0 & 0.2 & 0 & 0.22 & 0 & 0\\
-0.2 & 0 & -0.12 & 0 & 0.8 & 0 & 0 & 0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 & 0.2 & 0 & 0.8 & 0.32 & -0.2 & 0.56\\
0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0.12 & 0.92 & 0 & 0\\
0 & 0 & -0.45 & 0 & 0 & 0 & 0 & 1.2 & 0 & 0 & 0 & 0 & 0.35 & 0 & 0.36 & 0.2 & 0.2 & 2.1 & 0 & 0.56\\
0.32 & 0 & 0 & -0.3 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0.2 & 0 & -0.2 & 0 & 0 & 0.23 & 0.2 & -0.2 & 0\\
0 & 0 & 0 & 0 & 0 & 0.8 & 0.11 & 0 & 0 & 0.52 & 0 & 0 & -1.12 & 0 & 0 & 0.23 & 0 & 0.32 & 0 & 0
\end{bmatrix},$$

$$\boldsymbol{C}^2=\begin{bmatrix}
0 & 0.4 & 1.2 & 1.1 & 3.3 & 0 & 0 & 1 & 2.5 & 0 & 0.2 & 0 & -0.3 & -2.3 & 0 & -1.2 & 0 & 1.3 & -0.2 & 0.1\\
0.3 & -0.3 & 0 & 0.23 & 0 & 0.2 & 1.3 & 0 & 0.1 & -2.3 & 0 & 1.2 & 0 & 0.24 & 0.2 & 3.3 & -1.2 & 0 & 1.2 & 0\\
1 & 0 & 0 & 0.2 & 0 & 0.11 & 0 & -1.3 & 0 & 0.32 & 0.5 & 0 & 0.4 & 0 & 1 & 0 & 1.2 & -0.2 & 0 & 0.3\\
2.3 & 1.3 & 0.25 & 0 & 2.32 & 0 & 2.3 & 0 & 0.1 & 0 & 0.3 & 0.23 & 0 & 0.36 & 0 & 0.1 & -1.2 & 0 & -2.2 & 1.5\\
0.3 & 0 & 0.15 & -0.3 & 0 & 0.2 & 0 & 0.32 & 0 & -2.2 & 0 & 0 & 1.2 & 0.12 & 0 & 1 & 0 & 0.2 & 1.2 & -0.2\\
0 & 2.3 & 0.18 & 0 & 0.11 & 0 & 2 & 0 & 0 & 2.3 & 0 & 1.2 & 0.2 & 0 & 0 & 0.1 & 1.2 & 0 & 0.3 & 0\\
0.22 & 0 & 0 & 0 & 0 & 0 & 1.2 & 0.1 & 0 & 0.3 & -0.2 & 0 & 1.04 & -1.2 & 1 & 0 & 0.1 & -0.2 & 0.32 & 1\\
0 & 0.7 & -0.2 & 0.3 & 0 & -0.11 & 0 & 0 & 0.4 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.34 & 0 & 0 & 0 & 0.5\\
0.13 & 0 & 2.3 & 0 & 0.1 & 0 & 0 & 1.2 & 0.2 & 0 & 1.3 & 0.12 & 0 & 1 & 0.1 & 2.3 & 0 & -1.2 & -1.2 & 1\\
1 & 0 & 1 & 1.1 & 0 & 1.21 & 1 & 0.12 & 1 & 0.1 & 0 & 1.2 & 0 & 2.3 & 0 & 1 & 2.3 & 0 & 1.2 & -1.4\\
0 & 2.5 & 0.32 & 1 & 1 & 1.3 & 0 & 0.32 & 0 & 1.1 & -0.2 & 0 & 0 & -3.2 & 0 & 0 & 0.2 & 0 & -0.2 & 0.56\\
0.21 & 0 & 1.32 & 1 & 1.2 & 0 & 1.1 & 0 & 1 & 0 & 1.1 & 1 & 1.2 & 1 & 0 & 1.32 & 0 & 0.2 & 1 & 0\\
0.15 & 0 & 0 & 1.1 & 0 & 0 & 0.1 & -0.31 & 0 & 0.12 & 0 & 1.2 & 1 & 1 & -0.54 & 0 & 0.23 & 1.12 & 1 & -1.1\\
1 & 1 & 1.32 & 0 & 1 & 1.1 & 0 & 1 & 1.2 & 0 & 0 & 1 & 0 & 1.8 & 0 & 1 & 0.2 & 0.2 & 1.1 & 1.2\\
0 & 1.1 & 0 & 1 & 0 & 1 & 0.8 & 1 & 0 & 1.25 & 0 & 0 & 1.1 & 1 & 0 & 1.2 & 1 & 0.12 & 0 & 0.1\\
-1.2 & 0 & -1.12 & 0 & 1.8 & 0 & 1 & 1.2 & 0 & -1.2 & 0 & 1.2 & 0 & 0.1 & 0.1 & 0 & 1.8 & 0.31 & 0 & 1.51\\
1 & 0 & 1 & 0 & 0 & 0.11 & 0 & 0 & 0.21 & 0 & 0.12 & 0 & 1 & 0 & 1 & 0 & 0.11 & 0.12 & 0 & 1\\
0 & 1 & -1.45 & 0 & 0 & 1 & 0 & 1.1 & 0 & 0 & 1 & 0 & 0.15 & 0 & 0.31 & 1.2 & 0.1 & 1.1 & 0 & 1.51\\
0.12 & 0 & 0 & -0.13 & 0 & 0 & 1 & 0 & 0.12 & 0 & 0 & 0.12 & 0 & -10.2 & 0 & 0 & 0.13 & 0.12 & -1.2 & 1\\
1 & 0 & 1 & 0 & 1 & 0.8 & 0.11 & 0 & 1 & 0.11 & 0 & 0 & -1.1 & 0 & 0 & 1.32 & 0 & 0.31 & 0 & 1.01
\end{bmatrix},$$

$$\boldsymbol{D}^1=\begin{bmatrix}
0.2 & 0.2 & 1 & 1.1 & 0 & 1.3 & 1 & 0 & 1.5 & 0 & 0 & 0.2 & 0 & -0.3 & 0 & 0 & -1.2 & -0.3 & -0.2 & 0.56\\
0.3 & 0 & -0.3 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & -0.3 & 0 & 0 & 0 & 0.14 & 0 & 0.2 & 0 & -1.2 & 1.2 & 0\\
0 & 0 & 0 & 0 & 0 & 0.11 & 0.50 & -0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0 & 0.32 & 0.2 & -1.2 & 0 & 0\\
0.12 & 0 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0.56 & 0 & 0.1 & 0 & 0 & -1.2 & -0.2 & 0.56\\
0 & 0.65 & 0 & -0.3 & 0 & 0.2 & 0 & 0.2 & 0 & -2.2 & 0 & 0 & -1.12 & 1 & 0 & 0 & 0 & 0.2 & 0.2 & -1.2\\
0 & 0 & 0.8 & 0 & 0.1 & 0 & 0 & 1.03 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0.1 & 0.2 & 0.2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0.12 & 0.1 & 0 & 0.35 & -0.2 & 0 & 0 & -1.2 & 0 & 0 & 0.1 & -0.2 & 0.8 & 0\\
0 & 0.8 & -0.2 & 0.1 & 0 & -0.11 & 0 & 0 & 0.4 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.34 & 0 & 0 & 0 & 0.56\\
0.12 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0.23 & 0 & 0 & 0.2 & 0 & 0 & 0.1 & 0 & 0 & -0.2 & -1.2 & 0\\
0.11 & 0 & 0 & 1.1 & 0 & 0.21 & 0 & 0.22 & 0 & 0.54 & 0.21 & 0.2 & 0.2 & 0 & -0.12 & 0 & 0.21 & 0 & 0 & -1.4\\
0 & -0.5 & 0 & 0 & 0 & 0 & 0.32 & 0 & 2.1 & 0 & 0 & 0 & 0 & -3.2 & 0 & 0 & 0.2 & 0 & -0.2 & 0.56\\
0.23 & 0 & 0.32 & 0 & 0.2 & 0 & 2.1 & 0 & 0 & 0 & 0.1 & 0 & 0.2 & 0 & 0 & 0.32 & 0 & 0.2 & 0 & 0\\
0.25 & 0 & 0 & 2.1 & 0 & 1 & 0.2 & -0.32 & 0 & 0.32 & 0 & 0.2 & 0 & 1 & -0.54 & 0 & 0.23 & 0.12 & 0 & -1.2\\
1 & 0 & 0.32 & 0 & 0 & 0.1 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0.2 & 0.2 & 0.8 & 0.26\\
0 & 2.1 & 0 & 0 & 0 & 2.3 & 1.8 & 1 & 0 & 0.25 & 0 & 1.2 & 2.1 & 0 & 0 & 0.2 & 0 & 0.22 & 0 & 1\\
-0.2 & 0 & -0.12 & 1 & 0.8 & 0 & 0 & 0.2 & 0 & -0.2 & 0 & 0.12 & 0 & 0.2 & 0.2 & 0 & 0.8 & 0.32 & -0.2 & 0.16\\
0 & 0 & 0 & 0 & 0 & 0.1 & 1 & 0 & 1.2 & 0 & 0.12 & 0 & 0 & 0 & 0 & 0 & 0.12 & 0.92 & 0.2 & 0.3\\
1 & 1 & -0.45 & 0 & 0 & 1 & 2.3 & 1.2 & 0 & 1.3 & 1.3 & 3.2 & 0.35 & 0 & 0.16 & 0.2 & 0.2 & 2.1 & 0 & 0.16\\
0.12 & 0 & 0 & -0.3 & 0 & 0 & 0 & 0 & 1.2 & 0 & 0 & 1.2 & 0 & -0.2 & 0 & 1 & 0.13 & 0.2 & -0.2 & 1\\
2 & 1 & 1 & 0 & 0 & 1.8 & 0.21 & 0 & 0 & 0.52 & 0 & 0 & -1.12 & 0 & 0 & 0.23 & 0 & 0.12 & 1 & 0.3
\end{bmatrix},$$

$$\boldsymbol{D}^2 = \begin{bmatrix}
2.5 & 0 & 2.5 & 0 & 2.7 & 0 & 0 & 2.2 & 0 & 2.2 & 0 & 2.5 & 2.5 & 0 & -2.3 & 2.5 & 0 & 2.7 & 0 & -2.9 \\
0 & -2.4 & 0 & 2.5 & 0 & -2.9 & 2.5 & 2.2 & 2.2 & 2.5 & -2.4 & 0 & 0 & 2.8 & 0 & 2.2 & 0 & 2.2 & -2.9 & 0 \\
-2.9 & 0 & -2.9 & 0 & -2.9 & 0 & 0 & 0 & 2.8 & 0 & 0 & -2.9 & 2.2 & 0 & 2.6 & -2.3 & -2.3 & -2.9 & 0 & 0 \\
0 & 2.2 & 2.5 & -2.1 & 0 & -2.4 & 0 & 2.8 & 0 & 0 & 2.2 & 0 & 2.2 & 0 & 2.2 & 0 & 2.2 & 0 & -2.3 & -2.9 \\
-2.1 & 0 & 0 & 0 & 2.5 & 0 & -2.1 & 0 & -2.9 & 2.2 & -2.3 & 0 & 2.5 & 2.8 & 0 & 2.6 & 2.2 & -2.9 & 0 & 0 \\
-2.9 & 0 & 0 & 2.4 & 0 & 0 & 0 & -2.4 & 0 & 2.6 & 0 & 2.4 & -2.3 & -2.9 & 2.2 & 0 & -2.9 & 0 & -2.9 & 0 \\
0 & 2.3 & 2.2 & 0 & 0 & 2.6 & -2.9 & 0 & 2.2 & 0 & -2.9 & 2.2 & 0 & 0 & 0 & 2.2 & 0 & 2.6 & 2.2 & 0 \\
2.6 & 0 & 0 & -2.4 & 2.6 & 0 & 0 & 2.6 & 0 & 2.1 & 0 & 0 & -2.9 & 0 & 2.6 & 0 & 2.5 & 0 & 2.6 & -2.9 \\
0 & -2.9 & 2.2 & 0 & 0 & 0 & 2.2 & 0 & -2.1 & 0 & 2.2 & 2.6 & 0 & 2.6 & 0 & 2.6 & 0 & -2.4 & 0 & 0 \\
-2.9 & 0 & 0 & -2.3 & 2.5 & 2.3 & 0 & 0 & 0 & 2.5 & 2.2 & 0 & -2.3 & 0 & 2.6 & 0 & 0 & 0 & 2.6 & 2.6 \\
0 & 2.5 & -2.7 & 2.6 & 0 & 0 & -2.3 & 0 & 2.8 & 0 & 0 & 2.6 & -2.1 & 2.5 & 0 & 0 & -2.9 & 2.6 & 0 & 0 \\
-2.9 & 0 & 0 & 0 & 2.8 & 2.5 & 0 & 2.7 & 0 & 0 & -2.4 & 0 & 0 & 0 & 2.5 & -2.4 & 0 & 0 & -2.9 & 0 \\
0 & 0 & 2.5 & 2.7 & 0 & 0 & 2.7 & 0 & 2.2 & 2.8 & 0 & 0 & 2.5 & 2.2 & 0 & 0 & 0 & -2.4 & 0 & 2.6 \\
-2.4 & 2.7 & 0 & 0 & 2.6 & 2.8 & 0 & 2.1 & -2.9 & 0 & 0 & 2.8 & 2.2 & 0 & 2.2 & 2.5 & 2.5 & 0 & 2.6 & 0 \\
0 & 0 & 2.7 & 2.2 & 0 & 0 & -2.4 & 0 & 0 & 2.2 & 2.6 & 0 & -2.4 & 0 & -2.9 & 0 & 0 & -2.3 & 0 & -2.3 \\
2.7 & 2.5 & 0 & 2.7 & 2.2 & 0 & 2.2 & 2.8 & -2.4 & 0 & 0 & 0 & 0 & 2.5 & 0 & 2.9 & -2.3 & 0 & 0 & 0 \\
0 & 0 & 2.2 & 0 & 0 & 2.8 & 0 & 0 & 0 & 2.8 & 0 & 2.8 & 2.2 & 0 & 2.1 & 0 & 0 & 0 & 2.4 & 2.2 \\
2.6 & 2.7 & 0 & 2.2 & 2.1 & 0 & -2.4 & 2.5 & 0 & 0 & -2.4 & 0 & 0 & 2.6 & 0 & 2.1 & -2.1 & -2.3 & 0 & 0 \\
2.5 & 0 & 0 & 2.7 & 0 & 2.5 & 0 & 0 & 2.8 & 0 & 0 & 0 & -2.4 & 0 & 2.5 & 0 & 0 & 0 & -2.3 & 2.2 \\
0 & 0 & 2.1 & 0 & 2.1 & 0 & 2.5 & 0 & 0 & 2 & 0 & 2.1 & 0 & 2.1 & 0 & 2.1 & 0 & 2.2 & 0 & 2.8
\end{bmatrix}.$$

Figure 1 displays the uncontrolled synchronization error $e(t)$ when the network with delay. Figure 2 displays the controlled synchronization error $e(t)$ in finite time when network with delay. Figure 3 displays the uncontrolled synchronization error $e(t)$ when the network without delay. Figure 4 displays the controlled synchronization error $e(t)$ in finite time when network without delay.

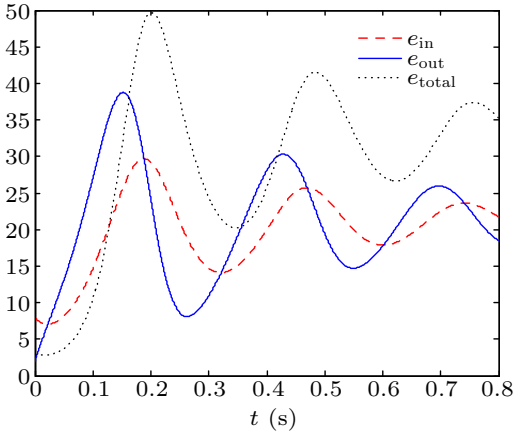



**Fig. 1.** Uncontrolled synchronization error with delay.

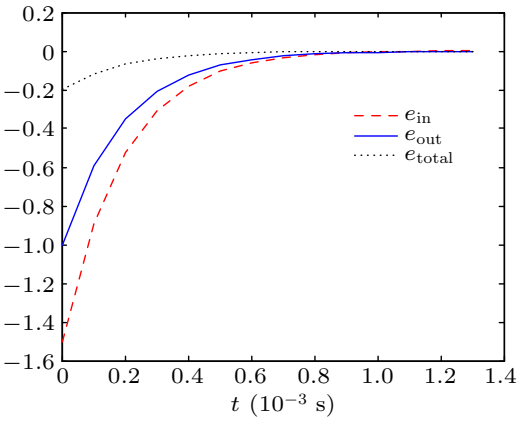



**Fig. 2.** Controlled synchronization error with delay.

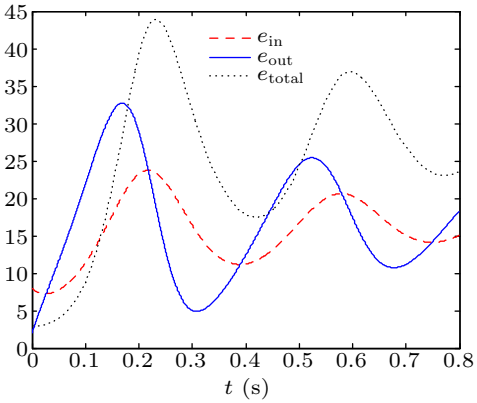



**Fig. 3.** Uncontrolled synchronization error without delay.

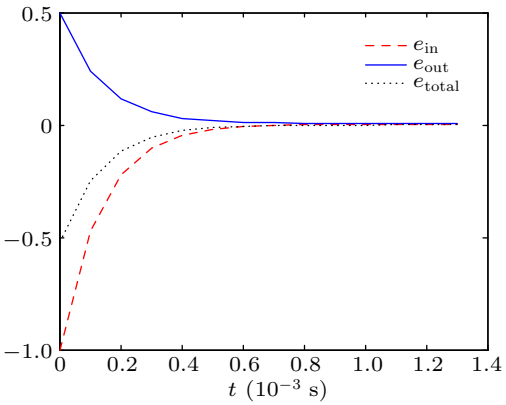



**Fig. 4.** Controlled synchronization error without delay.

## 5. Conclusions

Finite time hybrid synchronization of duplex heterogeneous complex networks via time-varying intermittent control has been investigated in this paper. The internal synchronization and the external synchronization all be achieved synchronization in finite time by time-varying intermittent control. The switch frequency of the controllers determined by the de-

signed time-varying Lyapunov function. Time-varying intermittent control combined with finite time control have been proposed to guarantee the synchronization within a finite time. Numerical example show that the finite time control scheme is feasible and effective.

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