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Riemann–Hilbert approach to the higher-order Kaup–Newell equation on the half line

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The higher-order Kaup–Newell equation is examined by applying the Fokas unified method on the half-line. We demonstrate that the solution can be expressed in relation to the resolution of the Riemann–Hilbert problem. The jump matrix for this problem is derived from the spectral matrix, which is calculated based on both the initial conditions and the boundary conditions. The jump matrix is explicitly dependent and expressed through the spectral functions, which are derived from the initial and boundary information, respectively. These spectral functions are interdependent and adhere to a so-called global relationship.

Keywords: higher-order Kaup–Newell equation, Fokas unified method, Riemann–Hilbert problem

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1. Introduction

In Ref. [1], Abhinav *et al.* gave coupled equations

$$\begin{aligned} q_t &= \mathrm{i}q_{xx} - (4\beta+1)q^2 r_x - 4\beta q q_x r \\ &\quad + \frac{\mathrm{i}}{2}(1+2\beta)(4\beta+1)q^3 r^2, \end{aligned} \tag{1}$$

$$\begin{aligned} r_t &= -\mathrm{i}r_{xx} - (4\beta+1)r^2 q_x - 4\beta r r_x q \\ &\quad - \frac{\mathrm{i}}{2}(1+2\beta)(4\beta+1)q^2 r^3, \end{aligned} \tag{2}$$

when $\beta = -1/2$ and $r = -q^*$ the system (1), (2) reduce to discrete nonlinear Schrödinger equation (DNLS) I

$$\mathrm{i}q_t + q_{xx} - \mathrm{i}(q^2 q^*)_x = 0, \tag{3}$$

the symbol '*' is used to indicate the complex conjugate, and the subscript notation for x (or t) is a way to denote the partial derivative taken with respect to x (or t). Equation (3) is alternatively referred to as the Kaup–Newell (KN) equation, as cited in Ref. [2]. In recent times, extensive studies have been conducted on the KN equation, focusing on aspects such as spectral issues, exact solutions, Hamiltonian structures, Painléve properties, and other characteristics, as detailed in Refs. [2–8]. Equation (3) is a classic dispersion relation, originating from the magnetohydrodynamic equation that includes the Hall effect. It specifically characterizes the nonlinear Alfvén waves in the field of plasma physics, as discussed in Refs. [9–11].

While setting $\beta = 0$ and $r = -q^*$, the system consisting of Eqs. (1) and (2) simplifies to the DNLS II form

$$\mathrm{i}q_t + q_{xx} - \mathrm{i}qq^* q_x = 0. \tag{4}$$

If $\beta = 1/4$ and $r = -q^*$, the system composed of Eqs. (1) and (2) is transformed into DNLS III

$$\mathrm{i}q_t + q_{xx} - \mathrm{i}q^2 q_x^* + \frac{1}{2}q^3 q^{*2} = 0. \tag{5}$$

Fan[3] provided a higher-order generalization of Eqs. (1) and (2)

$$\begin{aligned} &q_t - \frac{1}{4}[2q_{xxx} - 6(2\beta-1)rq_x^2 - 6(4\beta-1)qq_x r_x \\ &-6(2\beta-1)qrq_{xx} + 6(2\beta-1)(4\beta-1)q^3 r r_x \\ &+3(8\beta^2 - 12\beta + 3)q^2 r^2 q_x \\ &+4\beta(2\beta-1)(4\beta-1)q^4 r^3] = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &r_t - \frac{1}{4}[2r_{xxx} - 6(2\beta-1)qr_x^2 - 6(4\beta-1)rq_x r_x \\ &-6(2\beta-1)qrr_{xx} + 6(2\beta-1)(4\beta-1)qr^3 q_x \\ &+3(8\beta^2 - 12\beta + 3)q^2 r^2 r_x \\ &+4\beta(2\beta-1)(4\beta-1)q^3 r^4] = 0. \end{aligned} \tag{7}$$

The systems (6) and (7) serve to illustrate the complex nonlinear impacts in the realm of nonlinear optics and similar areas of study. Equations (6) and (7) also have three important Schrödinger-type reductions.

Initially, with $\beta = 0$, under the transformations $x \to \mathrm{i}x$, $t \to \mathrm{i}t$, and $r = -q^*$, the systems of Eqs. (6) and (7) are transformed into

$$q_t = -\frac{1}{2}q_{xxx} + \left(\frac{3\mathrm{i}}{2}|q|^2 q_x\right)_x + \left(\frac{3}{4}|q|^4 q\right)_x. \tag{8}$$

This can be regarded as the higher-order DNLS I or the higher-order Kaup–Newell equation.

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Given that $n=3$ and the parameters are suitably chosen, Eq. (8) can also be obtained from the generalized KN hierarchy.

Next, with $\beta=1/4$, as x approaches $\mathrm{i}x$, t approaches $\mathrm{i}t$, and r is set to the negative of the complex conjugate of q, Eqs. (6) and (7) transform into

$$q_t=-\frac{1}{2}q_{xxx}-\frac{3\mathrm{i}}{4}|q|^2q_{xx}-\frac{3\mathrm{i}}{4}q^*q_x^2+\frac{3}{8}|q|^4q_x. \tag{9}$$

The equation can be regarded as the higher-order DNLS II or higher-order Chen–Lee–Liu (CLL) equation.

Third, when $\beta=1/2$ and $x\to\mathrm{i}x$, $t\to\mathrm{i}t$ and $r=-q^*$, the systems (6) and (7) become

$$q_t=-\frac{1}{2}q_{xxx}+\frac{3\mathrm{i}}{2}qq_xq_x^*-\frac{3}{4}|q|^4q_x. \tag{10}$$

The equation can be considered as the higher-order DNLS III or higher-order Gerdjikov–Ivanov (GI) equation.

In this article, we employ the Fokas unified approach to investigate the Riemann–Hilbert (R-H) problem for the higher-order KN equation on the half-line. Lenells[14–17] analyzed the existing literature on the initial boundary value problems and R-H problems on the half-line. In this research, we apply the Fokas method to broaden Lenells' examination of the initial boundary value problem for the finite interval $(-\infty,0]$. The structure of this paper is summarized as follows. In Section 2, we present findings pertaining to the higher-order K-N equation and decompose the characteristic function to examine the interplay between its respective components. In Section 3, we consider the jump matrix $G(x,t)$ with explicit dependencies, utilizing the spectral functions $f(\xi),z(\xi)$ and $F(\xi),Z(\xi)$, with the initial data $q_0(x)=q(x,0)$, $q_0^*(x)=q^*(x,0)$ and boundary data $h_0(t),h_1(t),h_2(t)$ and $g_0(t),g_1(t),g_2(t)$. This issue involves crossing the boundary $\mathrm{Im}\xi^6=0$. The spectral functions show a relationship of interdependence and are tied together by a uniform condition known as the global relation.

2. Fundamental Riemann–Hilbert problem

In this section, we introduce the Lax pairs for the equations and the segments situated in the complex plane, as well as the matrix that facilitates the transition between these separate segments.

2.1. Formulas and symbols

(i) The Pauli matrix can be expressed as σ_3, $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\sigma_+=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\sigma_-=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $\sigma_+^*=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, and $\sigma_-^*=\begin{pmatrix}0&0\\0&1\end{pmatrix}$.

(ii) The matrices A and T are both 2×2 in size, with A denoting the commutator of T; this means $[A,T]=AT-TA$.

(iii) The matrix commutator is given by $\hat{\sigma}_3$, and in this paper, we compute $\mathrm{e}^{\hat{\sigma}_3}$ as $\mathrm{e}^{\hat{\sigma}_3}S=\mathrm{e}^{\sigma_3}S\mathrm{e}^{-\sigma_3}$.

(iv) The complex conjugation of a function is denoted by a line above the function symbol, such as, $\overline{s(\cdot)}$.

2.2. Lax pairs

Lax pairs for the higher-order KN equation have been obtained in Ref. [3]. The higher-order KN equation is required to adhere to the Lax pair criteria for generic parameters, $\xi\in\mathbb{C}$, as follows:

$$\begin{aligned}\partial_xJ(x,t;\xi)&=M(x,t;\xi)Y(x,t;\xi),\\ \partial_tJ(x,t;\xi)&=N(x,t;\xi)Y(x,t;\xi),\end{aligned} \tag{11}$$

where

$$Q=\begin{pmatrix}0&q\\-q^*&0\end{pmatrix},\quad Q_x=\begin{pmatrix}0&q_x\\-q_x^*&0\end{pmatrix}. \tag{12}$$

Assuming

$$Y=J\mathrm{e}^{\mathrm{i}(\xi^2x+2\xi^6t)\sigma_3},\quad -\infty<x\le0,\ \ 0\le t\le T, \tag{13}$$

we obtain the corresponding Lax pair

$$J_x+\mathrm{i}\xi^2[\sigma_3,J]=\xi QJ, \tag{14}$$

$$J_t+2\mathrm{i}\xi^6[\sigma_3,J]=\xi PJ. \tag{15}$$

The Lax pair can be represented as

$$\mathrm{d}\left(\mathrm{e}^{\mathrm{i}(\xi^2x+2\xi^6t)\hat{\sigma}_3}J(x,t;\xi)\right)=\mathrm{e}^{\mathrm{i}(\xi^2x+2\xi^6t)\hat{\sigma}_3}U(x,t;\xi)J, \tag{16}$$

where $-\infty<x\le0,\ \ 0\le t\le T$,

$$\begin{aligned}U(x,t;\xi)&=U_1(x,t;\xi)\mathrm{d}x+U_2(x,t;\xi)\mathrm{d}t,\\ U_1(x,t;\xi)&=\xi Q,\ \ U_2(x,t;\xi)=N_1,\end{aligned}$$

where

$$\begin{aligned}P=&\ 2Q\xi^4-\mathrm{i}Q^2\sigma_3\xi^3+\mathrm{i}\sigma_3Q_x\xi^2+Q^3\xi^2\\ &-\frac{1}{2}(QQ_x-Q_xQ)\xi-\frac{3}{4}\mathrm{i}Q^4\sigma_3\xi\\ &-\frac{1}{2}Q_{xx}+\frac{3}{2}\mathrm{i}\sigma_3Q^2Q_x+\frac{3}{4}Q^5,\\ N_1=&\ 2Q\xi^5-\mathrm{i}Q^2\sigma_3\xi^4+\mathrm{i}\sigma_3Q_x\xi^3+Q^3\xi^3\\ &-\frac{1}{2}(QQx-QxQ)\xi^2-\frac{3}{4}\mathrm{i}Q^4\sigma_3\xi^2\\ &-\frac{1}{2}Q_{xx}\xi+\frac{3}{2}\mathrm{i}\sigma_3Q^2Q_x\xi+\frac{3}{4}Q^5\xi,\end{aligned}$$

and also

$$\begin{aligned}N_1=&\ 2(q\sigma_+-q^*\sigma_-)\xi^5+\mathrm{i}(qq^*\sigma_+^*-qq^*\sigma_-^*)\xi^4\\ &+(\mathrm{i}q_x\sigma_++\mathrm{i}q_x^*\sigma_-)\xi^3\\ &+\left[\frac{1}{2}(qq_x^*-q^*q_x)-\frac{3}{4}\mathrm{i}q^2q^{*2}\right]\sigma_+^*\xi^2\\ &+\left[\frac{1}{2}(q_xq^*-q_x^*q)+\frac{3}{4}\mathrm{i}q^2q^{*2}\right]\sigma_-^*\xi^2\end{aligned}$$

$$+\left(-\frac{1}{2}q_{xx}-\frac{3\mathrm{i}}{2}qq^{*}q_{x}+\frac{3}{4}q^{3}q^{*2}\right)\sigma_{-}\xi$$
$$+\left(\frac{1}{2}q^{*}_{xx}-\frac{3\mathrm{i}}{2}qq^{*}q^{*}_{x}-\frac{3}{4}q^{2}q^{*3}\right)\sigma_{-}\xi.$$

In tackling the inverse spectral problem using the R-H method, our goal is to derive a spectral solution that converges to the identity matrix as $\xi\to\infty$. As the result does not conform to the attribute, we apply the Lenells method to Eq. (16) in order to derive the expected asymptotic behavior.[14–17]

2.3. Analysis of spectra and limiting behavior analysis

Suppose that the following represents the structure of a solution for Eq. (16):

$$J(x,t;\xi)=D_0+\frac{D_1}{\xi}+\frac{D_2}{\xi^2}+\frac{D_3}{\xi^3}+\frac{D_4}{\xi^4}+\frac{D_5}{\xi^5}+\mathscr{O}(\frac{1}{\xi^6}),$$
$$\xi\to\infty,$$

where D_0,D_1,D_2,D_3,D_4,D_5 are not affected by ξ. Comparison coefficients, we obtain

$$O(\xi^2):\ \mathrm{i}[\sigma_3,D_0]=0,$$
$$O(\xi):\ \mathrm{i}[\sigma_3,D_1]=QD_0,$$
$$O(1):\ D_{0x}+\mathrm{i}[\sigma_3,D_2]=QD_1,$$

where D_0 denotes a matrix composed solely of diagonal entries.

Let $D_0=\begin{pmatrix}D_0^{11}&0\\0&D_0^{22}\end{pmatrix}$. Utilizing $O(\xi)$, we obtain the expression

$$D_1^{(\mathrm{o})}=\begin{pmatrix}0&\frac{\mathrm{i}}{2}qD_0^{22}\\\frac{\mathrm{i}}{2}q^{*}D_0^{11}&0\end{pmatrix},$$

where $D_1^{(\mathrm{o})}$ is the off-diagonal part of D_1. Utilizing $O(1)$, we obtain

$$D_{0x}=-\frac{1}{2\mathrm{i}}\sigma_3Q^2D_0=\frac{1}{2\mathrm{i}}qq^{*}(\sigma_{+}^{*}-\sigma_{-}^{*})D_0. \tag{17}$$

Using Eq. (15), the coefficient of comparison, we arrive at

$$O(\xi^6):\ 2\mathrm{i}[\sigma_3,D_0]=0,$$
$$O(\xi^5):\ 2\mathrm{i}[\sigma_3,D_1]=2QD_0,$$
$$O(\xi^4):\ 2\mathrm{i}[\sigma_3,D_2]=2QD_1-\mathrm{i}Q^2\sigma_3D_0,$$
$$O(\xi^3):\ 2\mathrm{i}[\sigma_3,D_3]=\mathrm{i}\sigma_3Q_xD_0+Q^3D_0-\mathrm{i}Q^2\sigma_3D_1+2QD_2,$$
$$O(\xi^2):\ 2\mathrm{i}[\sigma_3,D_4]=-\frac{1}{2}(QQ_x-Q_xQ)D_0-\frac{3}{4}\mathrm{i}Q^4\sigma_3D_0+\mathrm{i}\sigma_3Q_xD_1+Q^3D_1-\mathrm{i}Q^2\sigma_3D_2+2QD_3,$$
$$O(\xi^1):\ 2\mathrm{i}[\sigma_3,D_5]=-\frac{1}{2}Q_{xx}D_0+\frac{3}{2}\mathrm{i}\sigma_3Q^2Q_xD_0+\frac{3}{4}Q^5D_0-\frac{1}{2}(QQ_x-Q_xQ)D_1-\frac{3}{4}\mathrm{i}Q^4\sigma_3D_1+\mathrm{i}\sigma_3Q_xD_2+Q^3D_2-\mathrm{i}Q^2\sigma_3D_3+2QD_4,$$
$$O(1):\ D_{0t}=2QD_5-\mathrm{i}Q^2\sigma_3D_4+\mathrm{i}\sigma_3Q_xD_3+Q^3D_3-\frac{1}{2}(QQ_x-Q_xQ)D_2-\frac{3}{4}\mathrm{i}Q^4\sigma_3D_2-\frac{1}{2}Q_{xx}D_1+\frac{3}{2}\mathrm{i}\sigma_3Q^2Q^xD_1+\frac{3}{4}Q^5D_1.$$

The following relationship can be derived from $O(\xi^1)$:

$$\begin{aligned}D_5^{(\mathrm{o})}=&-\frac{1}{8\mathrm{i}}\sigma_3Q_{xx}D_0+\frac{3}{8}Q^2Q_xD_0+\frac{3}{16\mathrm{i}}\sigma_3Q^5D_0\\&-\frac{1}{8\mathrm{i}}\sigma_3(QQ_x-Q_xQ)D_1^{(\mathrm{o})}-\frac{3}{16}Q^4D_1^{(\mathrm{o})}+\frac{1}{4}Q_xD_2^{(\mathrm{d})}\\&+\frac{1}{4\mathrm{i}}\sigma_3Q^3D_2^{(\mathrm{d})}-\frac{1}{4}Q^2D_3^{(\mathrm{o})}+\frac{1}{2\mathrm{i}}\sigma_3QD_4^{(\mathrm{d})},\end{aligned} \tag{18}$$

the off-diagonal element of D_3 is represented as $D_3^{(\mathrm{o})}$, while the diagonal element of D_2 is indicated by $D_2^{(\mathrm{d})}$. By utilizing Eq. (18) and adopting a computational complexity of $O(1)$, we can obtain the intended outcome

$$\begin{aligned}D_{0t}=&-\frac{1}{4\mathrm{i}}\sigma_3QQ_{xx}D_0+\frac{1}{4\mathrm{i}}\sigma_3Q_{xx}QD_0+\frac{7}{8}Q^3Q_xD_0\\&+\frac{1}{8}\sigma_3Q_xQ^3D_0-\frac{11}{16\mathrm{i}}\sigma_3Q^6D_0+\frac{1}{16}Q^6D_0\\&-\frac{5}{8}Q^2Q_xQD_0-\frac{1}{8}QQ_xQ^2D_0+\frac{\mathrm{i}}{4}\sigma_3Q_x^2D_0.\end{aligned} \tag{19}$$

Using Eqs. (17) and (19), we arrive at the following definition:

$$D_0(x,t)=\exp\left(\mathrm{i}\int_{(x_0,t_0)}^{(x,t)}\Delta\sigma_3\right), \tag{20}$$

where

$$\Delta(x,t)=\Delta_1(x,t)\,\mathrm{d}x+\Delta_2(x,t)\,\mathrm{d}t,\ \ \Delta_1(x,t)=-\frac{1}{2\mathrm{i}}Q^2, \tag{21}$$

$$\begin{aligned}\Delta_2(x,t)=&-\frac{1}{4}QQ_{xx}-\frac{1}{4}Q_{xx}Q+\frac{7}{8\mathrm{i}}Q^3Q_x\sigma_3+\frac{1}{8\mathrm{i}}Q_xQ^3\\&+\frac{11}{16}Q^6-\frac{1}{16}Q^6\sigma_3-\frac{5}{8\mathrm{i}}Q^2Q_xQ\sigma_3\\&-\frac{1}{8\mathrm{i}}QQ_xQ^2\sigma_3+\frac{1}{4}Q^2,\end{aligned} \tag{22}$$

and $(x_0,t_0)\in D$ simultaneously. In order to simplify calculations, we select $(x_0,t_0)=(0,0)$. Since the integral in Eq. (20) is invariant to the path taken, we may define a new function $\lambda(x,t;\xi)$ as follows:

$$J(x,t;\xi)=\mathrm{e}^{\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta\hat{\sigma}_3}\lambda(x,t;\xi)D_0(x,t),$$
$$-\infty<x\le 0,\ \ 0\le t\le T. \tag{23}$$

Then,

$$\lambda(x,t;\xi)=1+\mathscr{O}(\frac{1}{\xi}),\ \ \xi\to\infty. \tag{24}$$

Following a direct calculation, we arrive at the Lax pair as given in Eq. (16)

$$\mathrm{d}[\mathrm{e}^{\mathrm{i}(\xi^2x+2\xi^6t)\hat{\sigma}_3}\lambda(x,t;\xi)]=W(x,t;\xi),\ \ \xi\in\mathbb{C}, \tag{25}$$

where

$$\begin{aligned}W(x,t;\xi)&=\mathrm{e}^{\mathrm{i}(\xi^2x+2\xi^6t)\hat{\sigma}_3}H(x,t;\xi)\lambda(x,t;\xi),\\ H(x,t;\xi)&=H_1(x,t;\xi)\mathrm{d}x+H_2(x,t;\xi)\mathrm{d}t\\ &=\mathrm{e}^{-\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta\hat{\sigma}_3}[U(x,t;\xi)-\mathrm{i}\Delta\sigma_3].\end{aligned}$$

With the provided definitions of $U(x,t;\xi)$ and Δ, it is feasible to deduce both H_1 and H_2:

$$H_1(x,t;\xi)=\begin{pmatrix}\frac{\mathrm{i}}{2}qq^* & q\xi\,\mathrm{e}^{-2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta}\\ -q^*\xi\,\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta} & -\frac{\mathrm{i}}{2}qq^*\end{pmatrix},$$

$$H_2(x,t;\xi)=\begin{pmatrix}H_2^{11} & H_2^{12}\\ H_2^{21} & H_2^{22}\end{pmatrix},$$

where

$$\begin{aligned}H_2^{11}&=\mathrm{i}qq^*\xi^4+\frac{1}{2}(qq_x^*-q^*q_x)\xi^2-\frac{3\mathrm{i}}{4}q^2q^{*2}\xi^2-\frac{\mathrm{i}}{4}q_{xx}^*q\\ &\quad-\frac{\mathrm{i}}{4}q_{xx}q^*-\frac{7}{8}q^2q^*q_x^*-\frac{1}{8}qq^{*2}q_x+\frac{11\mathrm{i}}{16}q^3q^{*3}\\ &\quad-\frac{\mathrm{i}}{16}q^3q^{*3}+\frac{5}{8}qq^{*2}q_x+\frac{1}{8}q^2q_x^*q^*+\frac{\mathrm{i}}{4}qq^*,\\ H_2^{12}&=\mathrm{e}^{-2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta}\Big(2q\xi^5+\mathrm{i}q_x\xi^3-q^2q^*\xi^3-\frac{1}{2}q_{xx}\xi\\ &\quad-\frac{3\mathrm{i}}{2}qq^*q_x\xi+\frac{3}{4}q^3q^{*2}\xi\Big),\\ H_2^{21}&=\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta}\Big(-2q^*\xi^5+\mathrm{i}q_x^*\xi^3+qq^{*2}\xi^3+\frac{1}{2}q_{xx}^*\xi\\ &\quad-\frac{3\mathrm{i}}{2}qq^*q_x^*\xi-\frac{3}{4}q^2q^{*3}\xi\Big),\\ H_2^{22}&=-\mathrm{i}qq^*\xi^4+\frac{1}{2}(q_xq^*-q_x^*q)\xi^2+\frac{3\mathrm{i}}{4}q^2q^{*2}\xi^2+\frac{\mathrm{i}}{4}q_{xx}q^*\\ &\quad+\frac{\mathrm{i}}{4}q_{xx}^*q-\frac{7}{8}qq^{*2}q_x+\frac{1}{8}q^2q^*q_x^*-\frac{11\mathrm{i}}{16}q^3q^{*3}\\ &\quad-\frac{\mathrm{i}}{16}q^3q^{*3}+\frac{5}{8}q^2q^*q_x^*+\frac{1}{8}qq^{*2}q_x-\frac{\mathrm{i}}{4}qq^*.\end{aligned}$$

Then Eq. (25) for $\lambda(x,t;\xi)$ can be written as

$$\lambda_x+\mathrm{i}\xi^2[\sigma_3,\lambda]=H_1\lambda,\ \ \lambda_t+2\mathrm{i}\xi^6[\sigma_3,\lambda]=H_2\lambda, \tag{26}$$

where $-\infty<x\le0$, $0\le t\le T$, $\xi\in\mathbb{C}$.

2.4. Characteristics functions and their interconnections

We investigate the scenario in which the domain D is specified as the region $\{-\infty<x\le0, 0\le t\le T\}$, where T denotes a constant positive quantity. In accordance with Refs. [12,13], we delineate three particular solutions to Eq. (25). The functions $\lambda_j(x,t,\xi)$ are characterized as 2×2 matrix-valued and are established by

$$\lambda_j(x,t;\xi)=I+\int_{(x_j,t_j)}^{(x,t)}\mathrm{e}^{-\mathrm{i}(\xi^2x+2\xi^6t)\hat{\sigma}_3}W(\tau,\eta,\xi), \tag{27}$$

where $-\infty<x<0$, $0<t<T$ and I is the identity matrix. As depicted in Fig. 1, the integral corresponds to a continuous line that links the points (x_j,t_j) and (x,t), demonstrating the subsequent values: $(x_1,t_1)\to(-\infty,t)$, $(x_2,t_2)=(0,0)$, $(x_3,t_3)=(0,T)$.

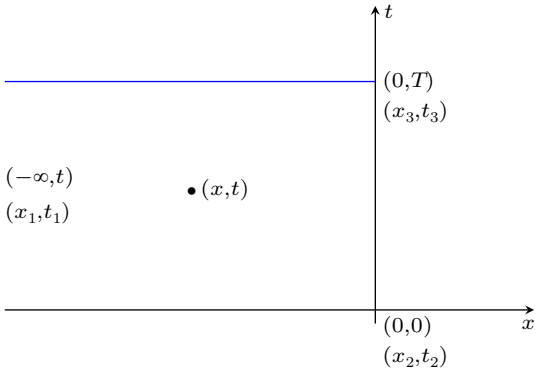


Fig. 1. Three distinct locations in the (x,t) coordinate space.

The value of the integral remains unaffected by the chosen integration path, according to the principles of mathematical analysis, while λ_1, λ_2, and λ_3 are mutually independent. The functions λ_1, λ_2, and λ_3 are established through the analysis of diverse paths within the complex ξ-plane. Returning to the idea presented in Ref. [6], we select three separate paths that converge to the point (x,t), as illustrated in Fig. 2.

Based on the above analysis, we obtain

$$\begin{aligned}\lambda_1(x,t;\xi)&=I+\int_{-\infty}^{x}\mathrm{e}^{\mathrm{i}\xi^2(\tau-x)\hat{\sigma}_3}(H_1\lambda_1)(\tau,t,\xi)\mathrm{d}\tau,\\ \lambda_2(x,t;\xi)&=I-\int_{x}^{0}\mathrm{e}^{\mathrm{i}\xi^2(\tau-x)\hat{\sigma}_3}(H_1\lambda_2)(\tau,t,\xi)\mathrm{d}\tau\\ &\quad+\mathrm{e}^{-\mathrm{i}\xi^2x\hat{\sigma}_3}\int_0^t\mathrm{e}^{2\mathrm{i}\xi^6(\eta-t)\hat{\sigma}_3}(H_2\lambda_2)(0,\eta,\xi)\mathrm{d}\eta,\\ \lambda_3(x,t;\xi)&=I-\int_{x}^{0}\mathrm{e}^{\mathrm{i}\xi^2(\tau-x)\hat{\sigma}_3}(H_1\lambda_3)(\tau,t,\xi)\mathrm{d}\tau\\ &\quad-\mathrm{e}^{-\mathrm{i}\xi^2x\hat{\sigma}_3}\int_t^T\mathrm{e}^{2\mathrm{i}\xi^6(\eta-t)\hat{\sigma}_3}(H_2\lambda_3)(0,\eta,\xi)\mathrm{d}\eta.\end{aligned} \tag{28}$$

The selection of the path yields the following inequality:

$$\begin{aligned}&(x_1,t_1)\to(x,t):-\infty<\tau<x,\\ &(x_2,t_2)\to(x,t):x<\tau<0,\ 0<\eta<t,\\ &(x_3,t_3)\to(x,t):x<\tau<0,\ t<\eta<T.\end{aligned}$$

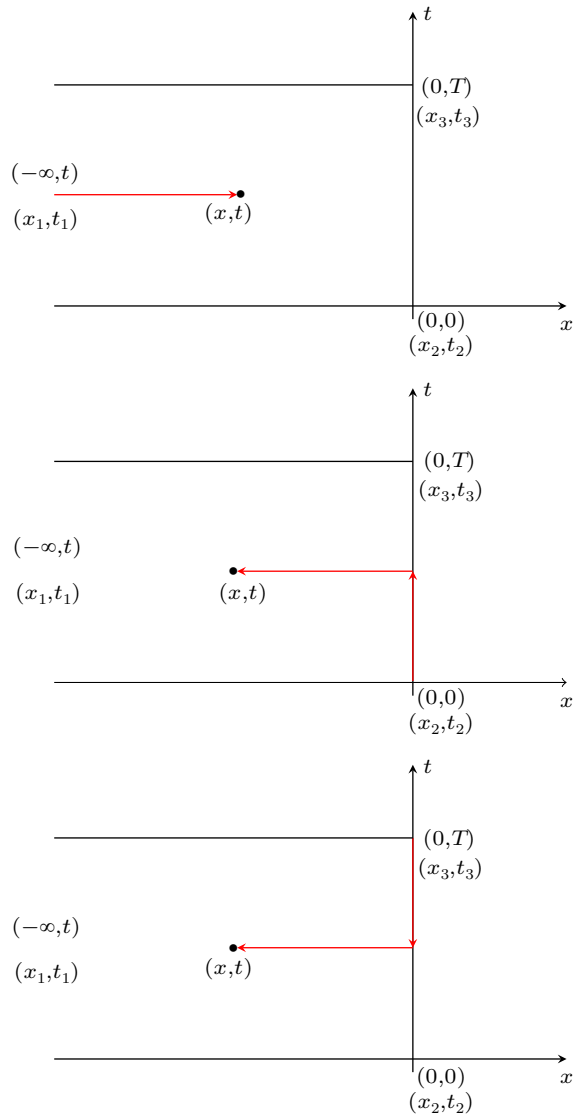


Fig. 2. The λ_1, λ_2, and λ_3 in the (x,t)-domain.

It has been found that $\mathrm{e}^{-2\mathrm{i}\xi^2(\tau-x)}$ and $\mathrm{e}^{-4\mathrm{i}\xi^6(\eta-t)}$ are elements of the first column in the matrix represented by Eq. (27). By employing the previously mentioned inequality, we can note that these expressions are constrained. As a result, we can partition the complex ξ-plane into six separate sectors:

$$\begin{aligned}
&\lambda_1^{(1)}(x,t;\xi):(x_1,t_1)\to(x,t):\{\mathrm{Im}\xi^2\geq 0\},\\
&\lambda_2^{(1)}(x,t;\xi):(x_2,t_2)\to(x,t):\{\mathrm{Im}\xi^2\leq 0\}\bigcap\{\mathrm{Im}\xi^6\geq 0\},\\
&\lambda_3^{(1)}(x,t;\xi):(x_3,t_3)\to(x,t):\{\mathrm{Im}\xi^2\leq 0\}\bigcap\{\mathrm{Im}\xi^6\leq 0\}.
\end{aligned}$$

The second column is the inverse of the first column as indicated in Eq. (27). Consequently, the result of the second column is the negation of the first column, which is constrained within

$$\begin{aligned}
&\lambda_1^{(2)}(x,t;\xi):(x_1,t_1)\to(x,t):\{\mathrm{Im}\xi^2\leq 0\},\\
&\lambda_2^{(2)}(x,t;\xi):(x_2,t_2)\to(x,t):\{\mathrm{Im}\xi^2\geq 0\}\bigcap\{\mathrm{Im}\xi^6\leq 0\},\\
&\lambda_3^{(2)}(x,t;\xi):(x_3,t_3)\to(x,t):\{\mathrm{Im}\xi^2\geq 0\}\bigcap\{\mathrm{Im}\xi^6\geq 0\}.
\end{aligned}$$

We provide D_j $(j=1,2,3,4,5,6)$ in the complex plane of ξ as $D_j=\varepsilon_j\cup(-\varepsilon_j)$, $\varepsilon_j=\{\xi\in\mathbb{C}|k\pi+\frac{j-1}{6}\pi<\mathrm{Arg}\xi<k\pi+\frac{j}{6}\pi\}$, $-\varepsilon_j=\{-\xi\in\mathbb{C}|\xi\in\varepsilon_j\}$, $j=1,2,3,4,5,6$, $k=0,\pm1,\pm2,\ldots$, (see Fig. 3).

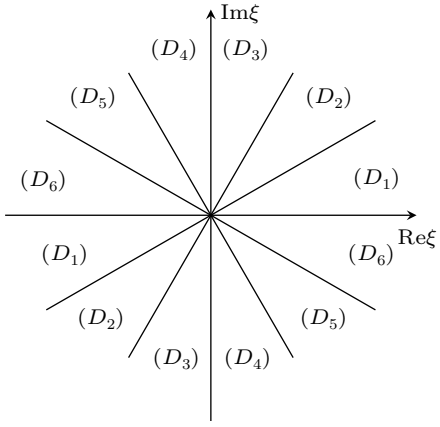


Fig. 3. The ξ-plane is partitioned into regions D_j, $j=1,2,3,4,5,6$.

Next, we obtain

$$\begin{aligned}
&\lambda_1(x,t;\xi)=(\lambda_1^{D_1\cup D_2\cup D_3}(x,t;\xi),\lambda_1^{D_4\cup D_5\cup D_6}(x,t;\xi)),\\
&\lambda_2(x,t;\xi)=(\lambda_2^{D_5}(x,t;\xi),\lambda_2^{D_2}(x,t;\xi)),\\
&\lambda_3(x,t;\xi)=(\lambda_3^{D_4\cup D_6}(x,t;\xi),\lambda_3^{D_1\cup D_3}(x,t;\xi)).
\end{aligned}\tag{29}$$

The λ_j functions serve as the critical eigenfunctions required for constructing a R-H problem within the complex ξ-plane. Nevertheless, the functions $\lambda_2(0,t;\xi)$, $\lambda_3(0,t;\xi)$, $\lambda_2(x,0;\xi)$, $\lambda_3(x,T;\xi)$, $\lambda_2(0,T;\xi)$, and $\lambda_3(0,0;\xi)$ remain bounded within more extensive regions:

$$\begin{aligned}
&\lambda_2(0,t;\xi)=(\lambda_2^{D_1\cup D_3\cup D_5}(0,t;\xi),\ \lambda_2^{D_2\cup D_4\cup D_6}(0,t;\xi)),\\
&\lambda_3(0,t;\xi)=(\lambda_3^{D_2\cup D_4\cup D_6}(0,t;\xi),\ \lambda_3^{D_1\cup D_3\cup D_5}(0,t;\xi)),\\
&\lambda_2(x,0;\xi)=(\lambda_2^{D_4\cup D_5\cup D_6}(x,0;\xi),\ \lambda_2^{D_1\cup D_2\cup D_3}(x,0;\xi)),\\
&\lambda_3(x,T;\xi)=(\lambda_3^{D_4\cup D_5\cup D_6}(x,T;\xi),\ \lambda_3^{D_1\cup D_2\cup D_3}(x,T;\xi)),\\
&\lambda_2(0,T;\xi)=(\lambda_2^{D_1\cup D_3\cup D_5}(0,T;\xi),\ \lambda_2^{D_2\cup D_4\cup D_6}(0,T;\xi)),\\
&\lambda_3(0,0;\xi)=(\lambda_3^{D_2\cup D_4\cup D_6}(0,0;\xi),\ \lambda_3^{D_1\cup D_3\cup D_5}(0,0;\xi)).
\end{aligned}\tag{30}$$

The λ_j satisfies the condition that $\lambda_j(x,t;\xi)=I+\mathscr{O}(\frac{1}{\xi})$ as $\xi\to\infty$, $j=1,2,3$.

2.5. The spectral functions and their propositions

By examining the individual D_j $(j=1,2,3,4,5,6)$ domains and computing the jump matrices that link these regions, one can formulate the R-H problem. The results suggest that the correlation jump matrix can be specified in terms of two spectral functions $v(\xi)$ and $V(\xi)$, both of which are 2×2 matrices. The composition of these matrices is delineated as follows:

$$\lambda_1(x,t;\xi)=\lambda_2(x,t;\xi)\,\mathrm{e}^{-\mathrm{i}(\xi^2x+2\xi^6t)\hat{\sigma}_3}v(\xi),\tag{31}$$

$$\lambda_3(x,t;\xi)=\lambda_2(x,t;\xi)\,\mathrm{e}^{-\mathrm{i}(\xi^2x+2\xi^6t)\hat{\sigma}_3}V(\xi). \tag{32}$$

Calculating Eq. (31) at $(x,t)=(0,0)$, one obtains

$$v(\xi)=\lambda_1(0,0;\xi). \tag{33}$$

When Eq. (31) is evaluated at $(x,t)=(0,T)$, it is indicated that

$$\begin{aligned} V(\xi)&=(\mathrm{e}^{2\mathrm{i}\xi^6T\hat{\sigma}_3}\lambda_2(0,T;\xi))^{-1},\\ V(\xi)^{-1}&=(\mathrm{e}^{2\mathrm{i}\xi^6T\hat{\sigma}_3}\lambda_2(0,T;\xi)). \end{aligned} \tag{34}$$

Furthermore, Eqs. (31) and (32) imply

$$\lambda_3(x,t;\xi)=\lambda_1(x,t;\xi)\,\mathrm{e}^{-\mathrm{i}(\xi^2x+2\xi^6t)\hat{\sigma}_3}(v(\xi))^{-1}V(\xi). \tag{35}$$

This will lead to the establishment of the overall relationship. Subsequently, by assessing the function $\lambda_1(x,0,\xi)$ at the point $x=0$ and the function $\lambda_2(0,t,\xi)$ at the instant $t=T$, we can derive the functions $v(\xi)$ and $V(\xi)$. Additionally, these functions, denoted as $\lambda_j(x,t;\xi)$ (where $j=1,2,3$), comply with the following formulas:

$$\begin{aligned} \lambda_1(x,0;\xi)&=I+\int_{-\infty}^{x}\mathrm{e}^{\mathrm{i}\xi^2(\tau-x)\hat{\sigma}_3}(H_1\lambda_1)(\tau,0,\xi)\,\mathrm{d}\tau,\\ \lambda_2(0,t;\xi)&=I+\int_{0}^{t}\mathrm{e}^{\mathrm{i}\xi^6(\eta-t)\hat{\sigma}_3}(H_2\lambda_2)(0,\eta,\xi)\,\mathrm{d}\eta,\\ \lambda_2(x,0;\xi)&=I-\int_{x}^{0}\mathrm{e}^{\mathrm{i}\xi^2(\tau-x)\hat{\sigma}_3}(H_1\lambda_2)(\tau,0,\xi)\,\mathrm{d}\tau,\\ \lambda_3(0,t;\xi)&=I-\int_{t}^{T}\mathrm{e}^{2\mathrm{i}\xi^6(\eta-t)\hat{\sigma}_3}(H_2\lambda_3)(0,\eta,\xi)\,\mathrm{d}\eta. \end{aligned} \tag{36}$$

We calculate the result of the equation H_1 at $t=0$, and consider $q_0(x)=q(x,0)$ and $q_0^*(x)=q^*(x,0)$ as the initial values for $q(x,t)$ and $q^*(x,t)$, respectively. Define $h_0(t)=q(0,t)$, $g_0(t)=q^*(0,t)$, $h_1(t)=q_x(0,t)$, and $g_1(t)=q_x^*(0,t)$, $h_2(t)=q_{xx}(0,t)$, $g_2(t)=q_{xx}^*(0,t)$ as the boundary values for $q(x,t)$ and $q^*(x,t)$, and establish the value of the calculation when $x=0$ as H_2. Taking these findings into account, we can acquire

$$H_1(x,0;\xi)=\begin{pmatrix} \frac{\mathrm{i}}{2}q_0^*(x)q_0(x) & q_0(x)\xi\,\mathrm{e}^{-2\mathrm{i}\int_0^x\Delta_1}\\ -q_0^*(x)\xi\,\mathrm{e}^{2\mathrm{i}\int_0^x\Delta_1} & -\frac{\mathrm{i}}{2}q_0(x)q_0^*(x)\end{pmatrix},$$

$$H_2(0,t;\xi)=\begin{pmatrix} H_2^{11} & H_2^{12}\\ H_2^{21} & H_2^{22}\end{pmatrix},$$

where

$$\begin{aligned} H_2^{11}&=\mathrm{i}h_0(t)g_0(t)\xi^4-\frac{3\mathrm{i}}{4}h_0(t)^2g_0(t)^2\xi^2+\frac{11\mathrm{i}}{16}h_0(t)^3g_0(t)^3\\ &\quad-\frac{\mathrm{i}}{16}h_0(t)^3g_0(t)^3+\frac{\mathrm{i}}{4}h_0(t)g_0(t),\\ H_2^{12}&=\mathrm{e}^{-2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}\Big[2h_0(t)\xi^5-h_0(t)^2g_0(t)\xi^3\\ &\quad+\frac{3}{4}h_0(t)^3g_0(t)^2\xi\Big],\\ H_2^{21}&=\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}\Big[-2g_0(t)\xi^5+h_0(t)g_0(t)^2\xi^3\\ &\quad-\frac{3}{4}h_0(t)^2g_0(t)^3\xi\Big],\\ H_2^{22}&=-\mathrm{i}h_0(t)g_0(t)\xi^4+\frac{3\mathrm{i}}{4}h_0(t)^2g_0(t)^2\xi^2\\ &\quad-\frac{11\mathrm{i}}{16}h_0(t)^3g_0(t)^3-\frac{\mathrm{i}}{16}h_0(t)^3g_0(t)^3-\frac{\mathrm{i}}{4}h_0(t)g_0(t). \end{aligned}$$

The calculation of $H_1(x,0;\xi)$ and $H_2(0,t;\xi)$ depends solely on the functions $q_0(x)$, $q_0^*(x)$, $h_0(t)$, $h_1(t)$, $g_0(t)$, and $g_1(t)$, $h_2(t)$, $g_2(t)$. Consequently, the initial conditions specify the integral in Eq. (23) that computes $v(\xi)$, and the boundary conditions are employed to establish $h_0(t)$, $h_1(t)$, $g_0(t)$, $g_1(t)$ as well as $h_2(t)$, $g_2(t)$ via the integral in Eq. (23), which in turn determines $V(\xi)$. The following declaration describes the analytical properties of the matrices $\lambda_j(x,t;\xi)$ $(j=1,2,3)$ derived from Eq. (23).

Proposition 1 The matrix functions

$$\begin{aligned} \lambda_j(x,t;\xi)&=(\lambda_j^{(1)}(x,t;\xi),\\ \lambda_j^{(2)}(x,t;\xi))&=\begin{pmatrix}\lambda_j^{11} & \lambda_j^{12}\\ \lambda_j^{21} & \lambda_j^{22}\end{pmatrix},\quad j=1,2,3 \end{aligned} \tag{37}$$

present the following analytical attributes:

(i) $\det\lambda_j(x,t;\xi)=1$, $j=1,2,3$.

(ii) The analytic characteristics of the function $\lambda_1^{(1)}(x,t;\xi)$ are evident, and $\lim\limits_{\xi\to\infty}\lambda_1^{(1)}(x,t;\xi)=(1,0)^{\mathrm{T}}$, $\xi\in\{\mathrm{Im}\xi^2\geq 0\}$.

(iii) The analytic characteristics of the function $\lambda_1^{(2)}(x,t;\xi)$ are evident, and $\lim\limits_{\xi\to\infty}\lambda_1^{(2)}(x,t;\xi)=(0,1)^{\mathrm{T}}$, $\xi\in\{\mathrm{Im}\xi^2\leq 0\}$.

(iv) The analytic characteristics of the function $\lambda_2^{(1)}(x,t;\xi)$ are evident, and $\lim\limits_{\xi\to\infty}\lambda_2^{(1)}(x,t;\xi)=(1,0)^{\mathrm{T}}$, $\xi\in\{\mathrm{Im}\xi^2\leq 0\}\bigcap\{\mathrm{Im}\xi^6\geq 0\}$.

(v) The analytic characteristics of the function $\lambda_2^{(2)}(x,t;\xi)$ are evident, and $\lim\limits_{\xi\to\infty}\lambda_2^{(2)}(x,t;\xi)=(0,1)^{\mathrm{T}}$, $\xi\in\{\mathrm{Im}\xi^2\geq 0\}\bigcap\{\mathrm{Im}\xi^6\leq 0\}$.

(vi) The analytic characteristics of the function $\lambda_3^{(1)}(x,t;\xi)$ are evident, and $\lim\limits_{\xi\to\infty}\lambda_3^{(1)}(x,t;\xi)=(1,0)^{\mathrm{T}}$, $\xi\in\{\mathrm{Im}\xi^2\leq 0\}\bigcap\{\mathrm{Im}\xi^6\leq 0\}$.

(vii) The analytic characteristics of the function $\lambda_3^{(2)}(x,t;\xi)$ are evident, and $\lim\limits_{\xi\to\infty}\lambda_3^{(2)}(x,t;\xi)=(0,1)^{\mathrm{T}}$, $\xi\in\{\mathrm{Im}\xi^2\geq 0\}\bigcap\{\mathrm{Im}\xi^6\geq 0\}$.

Given that $\lambda(x,t;\xi)=1+\mathscr{O}(\frac{1}{\xi})$ as ξ approaches infinity, it is understood that $\lambda(x,t;\xi)$ converges to the identity matrix as $\xi\to\infty$. Consequently, the first row of the matrix converges to $(1,0)$, while the second row converges to $(0,1)$.

For $j=1,2,3$, $\lambda(x,t,\xi)=\lambda_j(x,t,\xi)$ satisfy the symmetric relation:

Proposition 2

$$\lambda^{11}(x,t;\xi)=\overline{\lambda^{22}(x,t;\bar{\xi})},\quad \lambda^{12}(x,t;\xi)=\overline{\lambda^{21}(x,t;\bar{\xi})}, \tag{38}$$

as well as

$$\lambda^{11}(x,t;-\xi)=\lambda^{11}(x,t;\xi),\lambda^{12}(x,t;-\xi)=-\lambda^{12}(x,t;\xi),$$
$$\lambda^{21}(x,t;-\xi)=-\lambda^{21}(x,t;\xi),\lambda^{22}(x,t;-\xi)=\lambda^{22}(x,t;\xi). \tag{39}$$

We have the ability to create matrix functions $v(\xi)$ and $V(\xi)$:

Proposition 3

$$v(\xi)=\begin{pmatrix}\overline{f(\bar{\xi})} & z(\xi)\\ \overline{z(\bar{\xi})} & f(\xi)\end{pmatrix},\ V(\xi)=\begin{pmatrix}\overline{F(\bar{\xi})} & Z(\xi)\\ \overline{Z(\bar{\xi})} & F(\xi)\end{pmatrix}. \tag{40}$$

The definitions of λ_1 and λ_2 have implications that encompass both $v(\xi)$ and $V(\xi)$,

$$v(\xi)=\lambda_1(0,0;\xi)=I+\int_{-\infty}^{0}\mathrm{e}^{\mathrm{i}\xi^2\tau\hat{\sigma}_3}(H_1\lambda_1)(\tau,0;\xi)\mathrm{d}\tau,$$
$$V^{-1}(\xi)=\mathrm{e}^{-2\mathrm{i}\xi^6T\hat{\sigma}_3}\lambda_2(0,T;\xi)$$
$$=I+\int_0^T\mathrm{e}^{2\mathrm{i}\xi^6\eta\hat{\sigma}_3}(H_2\lambda_2)(0,\eta;\xi)\mathrm{d}\eta. \tag{41}$$

The subsequent attributes can be deduced based on the conditions of the determinant

$$\begin{pmatrix}\overline{f(\bar{\xi})}\\ \overline{z(\bar{\xi})}\end{pmatrix}=\lambda_1^{(1)}(0,0;\xi)=\begin{pmatrix}\lambda_1^{11}(0,0;\xi)\\ \lambda_1^{21}(0,0;\xi)\end{pmatrix},$$
$$\begin{pmatrix}z(\xi)\\ f(\xi)\end{pmatrix}=\lambda_1^{(2)}(0,0;\xi)=\begin{pmatrix}\lambda_1^{12}(0,0;\xi)\\ \lambda_1^{22}(0,0;\xi)\end{pmatrix},$$
$$\begin{pmatrix}F(\xi)\\ -\overline{Z(\bar{\xi})}\mathrm{e}^{4\mathrm{i}\xi^6T}\end{pmatrix}=\lambda_2^{(1)}(0,T;\xi)=\begin{pmatrix}\lambda_2^{11}(0,T;\xi)\\ \lambda_2^{21}(0,T;\xi)\end{pmatrix},$$
$$\begin{pmatrix}-\mathrm{e}^{-4\mathrm{i}\xi^6T}Z(\xi)\\ \overline{F(\bar{\xi})}\end{pmatrix}=\lambda_2^{(2)}(0,T;\xi)=\begin{pmatrix}\lambda_2^{12}(0,T;\xi)\\ \lambda_2^{22}(0,T;\xi)\end{pmatrix};$$

$$\partial_x\lambda_1^{(2)}(x,0;\xi)+2\mathrm{i}\xi^2\sigma\lambda_1^{(2)}(x,0;\xi)=H_1(x,0;\xi)\lambda_1^{(2)}(x,0;\xi),$$
$$\xi\in D_4\bigcup D_5\bigcup D_6,\ -\infty<x<0;$$
$$\partial_t\lambda_2^{(2)}(0,t;\xi)+4\mathrm{i}\xi^6\sigma\lambda_2^{(2)}(0,t;\xi)=H_2(0,t;\xi)\lambda_2^{(2)}(0,t;\xi),$$
$$\xi\in D_2\bigcup D_4\bigcup D_6,\ 0<t<T, \tag{42}$$

where $\sigma=\begin{pmatrix}1&0\\0&0\end{pmatrix}$,

$$\det v(\xi)=\det V(\xi)=1,$$
$$f(-\xi)=f(\xi),\ z(-\xi)=-z(\xi),$$
$$F(-\xi)=F(\xi),\ Z(-\xi)=-Z(\xi),$$
$$f(\xi)\overline{f(\bar{\xi})}-z(\xi)\overline{z(\bar{\xi})}=1,$$
$$F(\xi)\overline{F(\bar{\xi})}-Z(\xi)\overline{Z(\bar{\xi})}=1,$$
$$f(\xi)=1+\mathscr{O}(\frac{1}{\xi}),\ z(\xi)=\mathscr{O}(\frac{1}{\xi}),\ \xi\to\infty,\ \mathrm{Im}\xi^2\geq 0,$$
$$F(\xi)=1+\mathscr{O}(\frac{1}{\xi}),\ Z(\xi)=\mathscr{O}(\frac{1}{\xi}),\ \xi\to\infty,\ \mathrm{Im}\xi^6\geq 0.$$

These characteristics can be inferred from the analytic and boundedness of $\lambda_1(x,0;\xi)$ and $\lambda_2(0,t;\xi)$, in conjunction with the necessity for a determinant of one and the limiting behavior of these eigenfunctions as ξ approaches certain values.

2.6. Jump matrix

We show that the spectral functions are interconnected in a holistic manner, rather than operating independently. The R-H problem pertaining to the higher KN equations can be inferred from Refs. [18–28]. Equations (31), (32), and (35) must be expressed in the form of jump matrices.

For the purpose of simplicity, let us suppose that

$$\theta(\xi)=\xi^2x+2\xi^6t,$$
$$\rho(\xi)=\overline{f(\bar{\xi})}F(\xi)-\overline{z(\bar{\xi})}Z(\xi),$$
$$\beta(\xi)=f(\xi)Z(\xi)-F(\xi)z(\xi).$$

Consider the function $L(x,t;\xi)$ defined as follows:

$$L_+^1(x,t,\xi)=\left(\lambda_2^{D_5}(x,t,\xi),\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,t,\xi)}{f(\xi)}\right),$$
$$\xi\in D_5,$$
$$L_-^1(x,t,\xi)=\left(\lambda_3^{D_4\cup D_6}(x,t,\xi),\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,t,\xi)}{\overline{\rho(\bar{\xi})}}\right),$$
$$\xi\in D_4\cup D_6,$$
$$L_+^2(x,t,\xi)=\left(\frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,t,\xi)}{\rho(\xi)},\lambda_3^{D_1\cup D_3}(x,t,\xi)\right),$$
$$\xi\in D_1\cup D_3,$$
$$L_-^2(x,t,\xi)=\left(\frac{\lambda_1^{D_1\cup D_2D_3}(x,t,\xi)}{\overline{f(\bar{\xi})}},\lambda_3^{D_2}(x,t,\xi)\right),$$
$$\xi\in D_2. \tag{43}$$

These definitions show that

$$\det L(x,t;\xi)=1,\ L(x,t;\xi)=I+\mathscr{O}(\frac{1}{\xi}),\ \xi\to\infty.$$

Theorem 1 Assume that $q(x,t;\xi)$ is a smooth function and $L(x,t;\xi)$ is given by Eq. (43), which complies with the conditions of the jump setting

$$L_+(x,t,\xi)=L_-(x,t,\xi)G(x,t,\xi),\ \mathrm{Im}\xi^6=0, \tag{44}$$

where

$$G(x,t,\xi)=\begin{cases}G_1(x,t,\xi),\ \mathrm{Arg}\xi=k\pi\text{ or }k\pi+\frac{\pi}{2},\\ G_2(x,t,\xi),\ \mathrm{Arg}\xi=k\pi+\frac{\pi}{6}\text{ or }k\pi+\frac{\pi}{3},\\ G_3(x,t,\xi),\ \mathrm{Arg}\xi=k\pi+\frac{2\pi}{3}\text{ or }k\pi+\frac{5\pi}{6},\end{cases} \tag{45}$$

with

$$G_1(x,t,\xi)=\begin{pmatrix} 1 & \frac{z(\xi)}{f(\xi)}\mathrm{e}^{-2\mathrm{i}\theta(\xi)} \\ -\frac{\overline{z(\bar{\xi})}}{\overline{f(\bar{\xi})}}\mathrm{e}^{2\mathrm{i}\theta(\xi)} & \frac{1}{f(\xi)\overline{f(\bar{\xi})}} \end{pmatrix},$$

$$G_2(x,t,\xi)=\begin{pmatrix} \frac{\overline{f(\bar{\xi})}}{\rho(\xi)} & \mathrm{e}^{-2\mathrm{i}\theta(\xi)}Z(\xi) \\ 0 & \overline{\frac{f(\bar{\xi})}{\rho(\xi)}} \end{pmatrix},$$

$$G_3(x,t,\xi)=\begin{pmatrix} \frac{f(\xi)}{\overline{\rho(\bar{\xi})}} & 0 \\ \mathrm{e}^{-2\mathrm{i}\theta(\xi)}\overline{Z(\bar{\xi})} & \overline{\frac{\rho(\bar{\xi})}{f(\xi)}} \end{pmatrix}.$$

Proof We formulate Eqs. (31), (32), (35), and (44) as follows, thereby delineating their interconnections and deducing the conditions for the jump environment:

$$\begin{cases} \lambda_1^{D_1\cup D_2\cup D_3}=\overline{f(\bar{\xi})}\lambda_2^{D_5}+\overline{z(\bar{\xi})}\mathrm{e}^{2\mathrm{i}\theta(\xi)}\lambda_2^{D_2}, \\ \lambda_1^{D_4\cup D_5\cup D_6}=z(\xi)\mathrm{e}^{-2\mathrm{i}\theta(\xi)}\lambda_2^{D_5}+f(\xi)\lambda_2^{D_2}, \end{cases} \tag{46}$$

$$\begin{cases} \lambda_3^{D_4\cup D_6}=\overline{F(\bar{\xi})}\lambda_2^{D_5}+\overline{Z(\bar{\xi})}\mathrm{e}^{2\mathrm{i}\theta(\xi)}\lambda_2^{D_2}, \\ \lambda_3^{D_1\cup D_3}=Z(\xi)\mathrm{e}^{-2\mathrm{i}\theta(\xi)}\lambda_2^{D_5}+F(\xi)\lambda_2^{D_2}, \end{cases} \tag{47}$$

$$\begin{cases} \lambda_3^{D_4\cup D_6}=\overline{\rho(\bar{\xi})}\lambda_1^{D_1\cup D_2\cup D_3}+\mathrm{e}^{2\mathrm{i}\theta(\xi)}\overline{\beta(\bar{\xi})}\lambda_1^{D_4\cup D_5\cup D_6}, \\ \lambda_3^{D_1\cup D_3}=\mathrm{e}^{-2\mathrm{i}\theta(\xi)}\beta(\xi)\lambda_1^{D_1\cup D_2\cup D_3}+\rho(\xi)\lambda_1^{D_4\cup D_5\cup D_6}. \end{cases} \tag{48}$$

By employing Eqs. (46)–(48) in association with the interconnections between various boundaries, it is possible to infer the jump matrices $G_i(x,t;\xi)$ for $i=1,2,3$:

$$\begin{aligned} &\left[\lambda_2^{D_5}(x,t,\xi),\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,t,\xi)}{f(\xi)}\right] \\ =&\left[\frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,t,\xi)}{\overline{f(\bar{\xi})}},\lambda_2^{D_2}(x,t,\xi)\right]G_1(x,t;\xi), \\ &\left[\frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,t,\xi)}{\rho(\xi)},\lambda_1^{D_1\cup D_3}(x,t,\xi)\right] \\ =&\left[\frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,t,\xi)}{\overline{f(\bar{\xi})}},\lambda_2^{D_2}(x,t,\xi)\right]G_2(x,t;\xi), \\ &\left[\lambda_2^{D_5}(x,t,\xi),\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,t,\xi)}{f(\xi)}\right] \\ =&\left[\lambda_2^{D_4\cup D_6}(x,t,\xi),\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,t,\xi)}{\overline{\rho(\bar{\xi})}}\right]G_3(x,t;\xi). \end{aligned} \tag{49}$$

This completes the proof.

Hypothesis 1 Regarding the functions $f(\xi)$ and $\rho(\xi)$, we postulate the following.

(i) $f(\xi)$ contains $2s$ simple zeros $\{\zeta_j\}_{j=1}^{2s}$ ($2s=2s_1+2s_2$). We assume that ζ_j ($j=1,2,\ldots,2s_1$) pertains to $D_4\cup D_5\cup D_6$, and $\bar{\zeta}_j$ ($j=2s_1+1,2s_1+2,\ldots,2s$) pertains to $D_1\cup D_2\cup D_3$.

(ii) $\rho(\xi)$ contains $2t$ simple zeros $\{\varepsilon_j\}_{j=1}^{2t}$ ($2t=2t_1+2t_2$). We assume that ε_j ($j=1,2,\ldots,2t_1$) pertains to $D_1\cup D_2\cup D_3$, and $\bar{\varepsilon}_j$ ($j=2t_1+1,2t_1+2,\ldots,2t$) pertains to $D_4\cup D_5\cup D_6$.

(iii) Differences exist between the elementary zeros of $f(\xi)$ and those of $\rho(\xi)$.

The solution represented by $[L(x,t;\xi)]_1$ is applied to address the R-H problem for the initial column, and $[L(x,t;\xi)]_2$ is employed for the purposes of the second column. The residue at the point associated with the function $L(x,t;\xi)$ can be computed using Eqs. (31), (32), and (35). We can write $\dot{f}(\xi)=\mathrm{d}f/\mathrm{d}\xi$ and $\dot{\rho}(\xi)=\mathrm{d}\rho/\mathrm{d}\xi$.

We can then derive the following proposition:

Proposition 4

(i) $\mathrm{Res}\{[L(x,t;\xi)]_2,\zeta_j\}=\frac{\mathrm{e}^{-2\mathrm{i}\theta(\zeta_j)}z(\zeta_j)}{\dot{f}(\zeta_j)}[L(x,t;\zeta_j)]_1$, $j=1,2,\ldots,2s_1$.

(ii) $\mathrm{Res}\{[L(x,t;\xi)]_1,\ \bar{\zeta}_j\}=\frac{\mathrm{e}^{2\mathrm{i}\theta(\bar{\zeta}_j)\overline{z(\zeta_j)}}}{\overline{\dot{f}(\zeta_j)}}[L(x,t;\bar{\zeta}_j)]_2$, $j=2s_1+1,2s_1+2,\ldots,2s$.

(iii) $\mathrm{Res}\{[L(x,t;\xi)]_2,\bar{\varepsilon}_j\}=-\frac{\mathrm{e}^{-2\mathrm{i}\theta(\bar{\varepsilon}_j)}\beta(\bar{\varepsilon}_j)}{\overline{\dot{\rho}(\varepsilon_j)}}[L(x,t;\bar{\varepsilon}_j)]_1$, $j=2t_1+1,2t_1+2,\ldots,2t$.

(iv) $\mathrm{Res}\{[L(x,t;\xi)]_1,\varepsilon_j\}=-\frac{\mathrm{e}^{2\mathrm{i}\theta(\varepsilon_j)}\overline{\beta(\bar{\varepsilon}_j)}}{\dot{\rho}(\varepsilon_j)}[L(x,t;\varepsilon_j)]_2$, $j=1,2,\ldots,2t_1$.

Proof Consider the pair

$$\left[\lambda_2^{D_5}(x,t,\xi),\rho^{-1}(\xi)\lambda_1^{D_4\cup D_5\cup D_6}(x,t,\xi)\right],$$

noting that the zeros ζ_j ($j=1,2,\ldots,2s_1$) of $f(\xi)$ correspond to the poles of $\rho^{-1}(\xi)\lambda_1^{D_4\cup D_5\cup D_6}(x,t,\xi)$.

Then, we consider

$$\begin{aligned} &\mathrm{Res}\left\{\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,t;\xi)}{f(\xi)},\zeta_j\right\} \\ =&\lim_{\xi\to\zeta_j}(\xi-\zeta_j)\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,t;\xi)}{f(\xi)} \\ =&\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,t;\xi)}{\dot{f}(\zeta_j)}. \end{aligned}$$

Inputting $\xi=\zeta_j$ into the second part of Eq. (48) generates

$$\lambda_1^{D_4\cup D_5\cup D_6}(x,t;\zeta_j)=z(\zeta_j)\lambda_2^{D_5}(x,t;\zeta_j)\mathrm{e}^{-2\mathrm{i}\theta(\zeta_j)}.$$

Furthermore,

$$\mathrm{Res}\left\{\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,t;\xi)}{f(\xi)},\zeta_j\right\}=\frac{z(\zeta_j)\mathrm{e}^{-2\mathrm{i}\theta(\zeta_j)}\lambda_2^{D_5}(x,t;\zeta_j)}{\dot{f}(\zeta_j)}.$$

The same can be said for others.

2.7. The inverse problem

Equation (44) for the jump relation is identical to

$$L_+(x,t;\xi)-L_-(x,t;\xi)=L_-\tilde{G}(x,t;\xi), \tag{50}$$

where $\tilde{G}(x,t;\xi)=G(x,t;\xi)-I$. The function's asymptotic expansion is derived using Proposition 1, Proposition 2 and the asymptotic conditions stated in Eq. (28):

$$L(x,t;\xi)=I+\frac{\overline{L}(x,t;\xi)}{\xi}+\mathscr{O}(\frac{1}{\xi}),\ \ \xi\to\infty,\ \ \xi\in\mathbb{C}\setminus\Gamma, \quad (51)$$

where $\Gamma=\{\xi^6=\mathbb{R}\}$. Applying Eqs. (50) and (51), one can obtain the subsequent results

$$L(x,t;\xi)=I+\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{L_+(x,t;\xi')\tilde{G}(x,t;\xi')}{\xi-\xi'}\mathrm{d}\xi',\ \ \xi\in\mathbb{C}\setminus\Gamma, \quad (52)$$

then

$$\overline{L}(x,t;\xi)=-\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}L_+(x,t;\xi')\tilde{G}(x,t;\xi')\mathrm{d}\xi'. \quad (53)$$

The potential function $m(x,t)$ can be ascertained using the spectral functions λ_j, where j ranges from 1 to 3. We intend to reconstruct the potential function $m(x,t)$. As shown in Subsection 2.2, we provide proof that

$$D_1^{(\mathrm{o})}=\begin{pmatrix}0 & \frac{1}{2\mathrm{i}}qD_0^{22}\\ \frac{1}{2\mathrm{i}}q^*D_0^{11} & 0\end{pmatrix}.$$

As $\xi\to\infty$, the function $D(x,t;\xi)$ can be expressed as $D_0+\frac{D_1}{\xi}+\frac{D_2}{\xi^2}+\frac{D_3}{\xi^3}+\frac{D_4}{\xi^4}+\frac{D_5}{\xi^5}+\mathscr{O}(\frac{1}{\xi^6})$, satisfying Eq. (16). Consequently,

$$q(x,t)=2\mathrm{i}m(x,t)\,\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta}, \quad (54)$$

where

$$\lambda(x,t,\xi)=I+\frac{m_1}{\xi}+\frac{m_2}{\xi^2}+\frac{m_3}{\xi^3}++\frac{m_4}{\xi^4}+\frac{m_5}{\xi^5}+\mathscr{O}\left(\frac{1}{\xi^6}\right),\ \ (\xi\to\infty)$$

is the corresponding solution of Eq. (11), we obtain

$$q\overline{q}=4|m|^2,\ \ q_x\overline{q}-q\overline{q}_x=4(\overline{m}m_x-m\overline{m}_x)+16\mathrm{i}|m|^4.$$

Next, we can address the inverse problem in the following manner.

(i) Use any one of the three spectral functions λ_j $(j=1,2,3)$ to compute $m(x,t)$ according to

$$m(x,t)=\lim_{\xi\to\infty}(\xi\lambda_j(x,t;\xi))_{12}. \quad (55)$$

(ii) Determine $\Delta(x,t)$ from Eqs. (21) and (22).

(iii) Finally, $q(x,t)$ is given by Eq. (54).

3. Spectral functions and Riemann–Hilbert problem

In this part, we delve into the analysis of spectral functions and address the R-H issue. Here are its definitions and attributes.

3.1. Attributes of spectral functions

Definition 1 [Regarding $f(\xi)$ and $z(\xi)$] We suppose that $q_0(x)=q(x,0)$, and define the map

$$S:\{q_0(x)\}\to\{f(\xi),z(\xi)\},$$

where $f(\xi)$ and $z(\xi)$ are spectral functions

$$\begin{pmatrix}z(\xi)\\ f(\xi)\end{pmatrix}=\lambda_1^{(2)}(x,0;\xi)=\begin{pmatrix}\lambda_1^{12}(x,0;\xi)\\ \lambda_1^{22}(x,0;\xi)\end{pmatrix},\ \mathrm{Im}\xi^2\le 0,$$

with

$$\lambda_1(x,0;\xi)=I+\int_{-\infty}^{x}\mathrm{e}^{\mathrm{i}\xi^2(\tau-x)\hat{\sigma}_3}(H_1\lambda_1)(\tau,0;\xi)\mathrm{d}\tau.$$

$H_1(x,0;\xi)$ is represented in terms of $q(x,0;\xi)$

$$H_1(x,0;\xi)=\begin{pmatrix}\frac{\mathrm{i}}{2}q_0^*(x)q_0(x) & q_0(x)\xi\,\mathrm{e}^{-2\mathrm{i}\int_0^x\Delta_1}\\ -q_0^*(x)\xi\,\mathrm{e}^{2\mathrm{i}\int_0^x\Delta_1} & -\frac{\mathrm{i}}{2}q_0(x)q_0^*(x)\end{pmatrix}.$$

Proposition 5 The key characteristics of $f(\xi)$ and $z(\xi)$ are listed here:

(i) For $\mathrm{Im}\xi^2\le 0$, $f(\xi)$ and $z(\xi)$ are analytical.

(ii) $f(\xi)=1+\mathscr{O}(\frac{1}{\xi}),z(\xi)=\mathscr{O}(\frac{1}{\xi})$ as $\xi\to\infty$, $\mathrm{Im}\xi^2\le 0$.

(iii) As $Q:\{f(\xi),z(\xi)\}\to\{q_0(x)\}$ and $S^{-1}=Q$. The definition of the inverse mapping S for Q is given as

$$q_0(x)=2\mathrm{i}\lim_{\xi\to\infty}(\xi L^{(x)}(x,\xi))_{12},$$
$$m(x)=-2\mathrm{i}\lim_{\xi\to\infty}(\xi L^{(x)}(x,\xi))_{21},$$

where $L^{(x)}(x,\xi)$ meets the following R-H condition (see Theorem 2).

Theorem 2 Let

$$L_-^{(x)}(x,\xi)=\left(\lambda_2^{D_4\cup D_5\cup D_6}(x,\xi),\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,\xi)}{f(\xi)}\right),$$
$$\mathrm{Im}\xi^2\le 0,\ \ \xi\in D_4\cup D_5\cup D_6,$$
$$L_+^{(x)}(x,\xi)=\left(\frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,\xi)}{\overline{f(\bar{\xi})}},\lambda_2^{D_1\cup D_2\cup D_3}(x,\xi)\right),$$
$$\mathrm{Im}\xi^2\ge 0,\ \ \xi\in D_1\cup D_2\cup D_3. \quad (56)$$

Then, $L^{(x)}(x,\xi)$ satisfies the following R-H problem:

(i) $L^{(x)}(x,\xi)=\begin{cases}L_+^{(x)}(x,\xi),\ \mathrm{Im}\xi^2\ge 0\\ L_-^{(x)}(x,\xi),\ \mathrm{Im}\xi^2\le 0\end{cases}$ is a piecewise analytic function.

(ii) $L_+^{(x)}(x,\xi)=L_-^{(x)}(x,\xi)G^{(x)}(x,\xi)$, $\xi^2\in\mathbb{R}$, and

$$G^{(x)}(x,\xi)=\begin{pmatrix}1 & \frac{z(\xi)}{f(\xi)}\mathrm{e}^{-2\mathrm{i}\xi^2x}\\ -\frac{\overline{z(\bar{\xi})}}{f(\xi)}\mathrm{e}^{2\mathrm{i}\xi^2x} & \frac{1}{f(\xi)\overline{f(\bar{\xi})}}\end{pmatrix},\ \xi^6\in\mathbb{R}. \quad (57)$$

(iii) $L^{(x)}(T,\xi)=I+\mathscr{O}(\frac{1}{\xi}),\ \ \xi\to\infty$.

The $f(\xi)$ contains $2m$ simple zeros $\{\vartheta_j\}_1^{2m}$, $(2m = 2m_1 + 2m_2)$, and we assume that ϑ_j $(j = 1,2,\ldots,2m_1)$ is a part of $D_4 \cup D_5 \cup D_6$ and $\bar{\vartheta}_j$ $(j = 2m_1+1, 2m_1+2, \ldots, 2m)$ is a part of $D_1 \cup D_2 \cup D_3$.

$[L_+^{(x)}(x,\xi)]_1$ possesses simple poles at $\xi = \vartheta_j$, for j ranging from 1 to $2m_1$. The positions of the simple poles within $[L_-^{(x)}(x,\xi)]_2$ are located at $\xi = \bar{\vartheta}_j$, for j taking values from $2m_1+1$ to $2m$.

The residue requirements are as follows:

$$\begin{aligned}&\mathrm{Res}\{[L^{(x)}(x,\xi)]_2, \vartheta_j\} = \frac{z(\vartheta_j)\,\mathrm{e}^{-2\mathrm{i}\vartheta_j^2 x}}{\dot{f}(\vartheta_j)}[L^{(x)}(x,\vartheta_j)]_1,\\ &j = 1,2,\ldots,2m_1, \end{aligned} \tag{58}$$

$$\begin{aligned}&\mathrm{Res}\{[L^{(x)}(x,\xi)]_1, \bar{\vartheta}_j\} = \frac{\overline{z(\vartheta_j)}\,\mathrm{e}^{2\mathrm{i}\bar{\vartheta}_j^2 x}}{\overline{\dot{f}(\vartheta_j)}}[L^{(x)}(x,\bar{\vartheta}_j)]_2,\\ &j = 2m_1+1, 2m_1+2, \ldots, 2m. \end{aligned} \tag{59}$$

Proof While substituting $t = 0$ into Eq. (47), we obtain the following expression:

$$\begin{cases} \lambda_1^{D_1\cup D_2\cup D_3} = \overline{f(\bar{\xi})}\lambda_2^{D_4\cup D_5\cup D_6} + \overline{z(\bar{\xi})}\,\mathrm{e}^{2\mathrm{i}\xi^2 x}\lambda_2^{D_1\cup D_2\cup D_3}, \\ \lambda_1^{D_4\cup D_5\cup D_6} = z(\xi)\,\mathrm{e}^{-2\mathrm{i}\xi^2 x}\lambda_2^{D_4\cup D_5\cup D_6} + f(\xi)\lambda_2^{D_1\cup D_2\cup D_3}. \end{cases} \tag{60}$$

The jump matrix $L^{(x)}(x,\xi)$ satisfies the jump conditions as established by computational analysis. We analyze $[L_+^{(x)}(x,\xi)]_1$

$$\begin{aligned}\mathrm{Res}\left\{\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,\xi)}{f(\xi)}, \vartheta_j\right\} &= \lim_{\xi\to\vartheta_j}(\xi-\vartheta_j)\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,\xi)}{f(\xi)}\\ &= \frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,\xi)}{\dot{f}(\vartheta_j)}.\end{aligned}$$

Replacing $\xi = \vartheta_j$ with the second equation of Eq. (60), we deduce

$$\lambda_1^{D_4\cup D_5\cup D_6}(x,\vartheta_j) = \mathrm{e}^{-2\mathrm{i}\vartheta_j{}^2 x} z(\vartheta_j)\lambda_2^{D_4\cup D_5\cup D_6}(x,\vartheta_j).$$

Furthermore,

$$\begin{aligned}&\mathrm{Res}\left\{\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,\xi)}{f(\xi)}, \vartheta_j\right\}\\ &= \frac{\mathrm{e}^{-2\mathrm{i}\vartheta_j{}^2 x} z(\vartheta_j)}{\dot{f}(\vartheta_j)}\lambda_2^{D_4\cup D_5\cup D_6}(x,\vartheta_j),\\ &j = 1,2,\ldots,2m_1.\end{aligned}$$

This is equivalent to Eq. (58). The correctness of the formula in Eq. (59) can be proven by employing an analogous method.

Definition 2 [Regarding $F(\xi)$ and $Z(\xi)$] Take $h_0(t), h_1(t), h_2(t)$, and define the map

$$\tilde{S}: \{h_0(t), h_1(t), h_2(t)\} \to \{F(\xi), Z(\xi)\},$$

where

$$\begin{pmatrix} Z(\xi) \\ F(\xi) \end{pmatrix} = \lambda_3^{(2)}(0,\xi) = \begin{pmatrix} \lambda_3^{12}(0,\xi) \\ \lambda_3^{22}(0,\xi) \end{pmatrix}, \ \mathrm{Im}\xi^6 \geq 0,$$

with

$$\lambda_3(0,\xi) = I - \int_t^T \mathrm{e}^{2\mathrm{i}\xi^6(\eta-T)\hat{\sigma}_3}(H_2\lambda_3)(\eta,\xi)\,\mathrm{d}\eta.$$

$H_2(0,T;\xi)$ is given by

$$H_2(0,t;\xi) = \begin{pmatrix} H_2^{11}(0,t;\xi) & H_2^{12}(0,t;\xi) \\ H_2^{21}(0,t;\xi) & H_2^{22}(0,t;\xi) \end{pmatrix},$$

$$\begin{aligned}H_2^{11}(0,t;\xi) &= \mathrm{i}h_0(t)g_0(t)\xi^4 - \frac{3\mathrm{i}}{4}h_0(t)^2 g_0(t)^2\xi^2\\ &\quad + \frac{11\mathrm{i}}{16}h_0(t)^3 g_0(t)^3 - \frac{\mathrm{i}}{16}h_0(t)^3 g_0(t)^3\\ &\quad + \frac{\mathrm{i}}{4}h_0(t)g_0(t),\end{aligned}$$

$$\begin{aligned}H_2^{12}(0,t;\xi) &= \mathrm{e}^{-2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}\Big(2h_0(t)\xi^5 - h_0(t)^2 g_0(t)\xi^3\\ &\quad + \frac{3}{4}h_0(t)^3 g_0(t)^2\xi\Big),\end{aligned}$$

$$\begin{aligned}H_2^{21}(0,t;\xi) &= \mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}\Big[-2g_0(t)\xi^5 + h_0(t)g_0(t)^2\xi^3\\ &\quad - \frac{3}{4}h_0(t)^2 g_0(t)^3\xi\Big],\end{aligned}$$

$$\begin{aligned}H_2^{22}(0,t;\xi) &= -\mathrm{i}h_0(t)g_0(t)\xi^4 + \frac{3\mathrm{i}}{4}h_0(t)^2 g_0(t)^2\xi^2\\ &\quad - \frac{11\mathrm{i}}{16}h_0(t)^3 g_0(t)^3 - \frac{\mathrm{i}}{16}h_0(t)^3 g_0(t)^3\\ &\quad - \frac{\mathrm{i}}{4}h_0(t)g_0(t),\end{aligned}$$

where $\Delta_2(0,\eta) = \begin{pmatrix} -\frac{5}{8}q^3q^{*3} - \frac{1}{4}qq^* & 0 \\ 0 & -\frac{3}{4}q^3q^{*3} - \frac{1}{4}qq^* \end{pmatrix}$.

Proposition 6 The key characteristics of $F(\xi)$ and $Z(\xi)$ are as follows:

(i) For $\mathrm{Im}\xi^6 \geq 0$, $F(\xi)$ and $Z(\xi)$ are analytical.

(ii) $F(\xi) = 1 + \mathscr{O}(\frac{1}{\xi})$, $Z(\xi) = \mathscr{O}(\frac{1}{\xi})$ as $\xi \to \infty$, $\mathrm{Im}\xi^6 \geq 0$.

(iii) As $\tilde{Q}: \{F(\xi), Z(\xi)\} \to \{h_0(t), h_1(t), h_2(t)\}$ and $\tilde{S}^{-1} = \tilde{Q}$, the definition of the inverse map S for Q is as follows:

$$h_0(t) = 2\mathrm{i}m_{12}^{(1)}(t)\,\mathrm{e}^{2\mathrm{i}\int_0^t \Delta_2(\xi)\,\mathrm{d}\xi}, \tag{61}$$

$$\begin{aligned}h_1(t) &= 4\,\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}m_{12}^{(3)} - \mathrm{i}q^* h_0(t)^2\\ &\quad - q^*\,\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}m_{12}^{(1)}h_0(t) + 2\mathrm{i}m_{22}^{(2)}h_0(t),\end{aligned} \tag{62}$$

$$\begin{aligned}h_2(t) &= -3\mathrm{i}q^* h_0(t)h_1(t) + \frac{3}{2}q^{*2}h_0(t)^3\\ &\quad + [h_0(t)q_x^* - q^* h_1(t)]\,\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}m_{12}^{(1)}\\ &\quad - \frac{3\mathrm{i}}{2}h_0(t)^2 q^{*2}\,\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}m_{12}^{(1)}\\ &\quad + 2\mathrm{i}h_1(t)m_{22}^{(2)} - 2h_0(t)^2 q^* m_{22}^{(2)}\\ &\quad + 2\mathrm{i}h_0(t)q^*\,\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}m_{12}^{(3)} + 4h_0(t)m_{22}^{(4)}\end{aligned}$$

$$-8\mathrm{i}\,\mathrm{e}^{2\mathrm{i}\int_{(0,0)}^{(x,t)}\Delta_2}m_{12}^{(5)}. \tag{63}$$

$L^{(t)}(t,\xi)$ fulfills the subsequent R-H problem (see Theorem 3).

Theorem 3 Let

$$\begin{aligned}
&L_+^{(t)}(t,\xi)=\left(\lambda_2^{D_1\cup D_3}(t,\xi),\frac{\lambda_3^{D_1\cup D_3}(t,\xi)}{F(\xi)}\right),\\
&\mathrm{Im}\xi^6\geq 0,\ \ \xi\in D_1\cup D_3,\\
&L_-^{(t)}(t,\xi)=\left(\frac{\lambda_3^{D_4\cup D_6}(t,\xi)}{\overline{F(\bar{\xi})}},\lambda_2^{D_4\cup D_6}(t,\xi)\right),\\
&\mathrm{Im}\xi^6\leq 0,\ \ \xi\in D_4\cup D_6.
\end{aligned} \tag{64}$$

Then, $L^{(t)}(t,\xi)$ fulfills the subsequent R-H problem:

(i) $L^{(t)}(t,\xi)=\begin{cases}L_+^{(t)}(t,\xi),\ \mathrm{Im}\xi^6\geq 0\\ L_-^{(t)}(t,\xi),\ \mathrm{Im}\xi^6\leq 0\end{cases}$ is a piecewise analytic function.

(ii) $L_+^{(t)}(t,\xi)=L_-^{(t)}(t,\xi)G^{(t)}(t,\xi)$, $\xi^6\in\mathbb{R}$, and

$$G^{(t)}(t,\xi)=\begin{pmatrix}1 & \frac{Z(\xi)}{F(\xi)}\mathrm{e}^{-4\mathrm{i}\xi^6 t}\\ -\frac{\overline{Z(\bar{\xi})}}{\overline{F(\bar{\xi})}}\mathrm{e}^{4\mathrm{i}\xi^6 t} & \frac{1}{F(\xi)\overline{F(\bar{\xi})}}\end{pmatrix},\ \ \xi^6\in\mathbb{R}. \tag{65}$$

(iii) $L^{(t)}(T,\xi)=I+\mathscr{O}(\frac{1}{\xi}),\ \ \xi\to\infty$.

$F(\xi)$ contains $2b$ simple zeros $\{\gamma_j\}_1^{2b}$, $(2b=2b_1+2b_2)$, and we assume that γ_j $(j=1,2,\ldots,2b_1)$ is a part of $D_1\bigcup D_3$ and $\bar{\gamma}_j$ $(j=2b_1+1,2b_1+2,\ldots,2b)$ is a part of $D_4\bigcup D_6$.

$[L_+^{(t)}(t,\xi)]_2$ possesses simple poles at $\xi=\gamma_j$, for $j=1,2,\ldots,2b_1$. The positions of the simple poles within $[L_-^{(t)}(t,\xi)]_1$ are given by $\xi=\bar{\gamma}_j$, for j ranging from $2b_1+1$ to $2b$.

Then, the residue conditions are

$$\begin{aligned}
&\mathrm{Res}\{[L^{(t)}(t,\xi)]_2,\gamma_j\}=\frac{Z(\gamma_j)\mathrm{e}^{-4\mathrm{i}\gamma_j^6 t}}{\dot{F}(\gamma_j)}[L^{(t)}(t,\gamma_j)]_1,\\
&j=1,2,\ldots,2b_1,
\end{aligned} \tag{66}$$

$$\begin{aligned}
&\mathrm{Res}\{[L^{(t)}(t,\xi)]_1,\bar{\gamma}_j\}=\frac{\overline{Z(\gamma_j)}\mathrm{e}^{4\mathrm{i}\bar{\gamma}_j^6 t}}{\overline{\dot{F}(\gamma_j)}}[L^{(t)}(t,\bar{\gamma}_j)]_2,\\
&j=2b_1+1,2b_1+2,\ldots,2b.
\end{aligned} \tag{67}$$

Proof Having substituted $x=0$ into Eq. (47), we arrive at the following equation:

$$\begin{cases}\lambda_3^{D_4\cup D_6}=\overline{F(\bar{\xi})}\lambda_2^{D_1\cup D_3}+\overline{Z(\bar{\xi})}\mathrm{e}^{4\mathrm{i}\xi^6 t}\lambda_2^{D_4\cup D_6},\\ \lambda_3^{D_1\cup D_3}=Z(\xi)\mathrm{e}^{-4\mathrm{i}\xi^6 t}\lambda_2^{D_1\cup D_3}+F(\xi)\lambda_2^{D_4\cup D_6}.\end{cases} \tag{68}$$

The jump matrix $L^{(t)}(t,\xi)$ satisfies the jump conditions, as established by computational investigation. We analyze $[L_+^{(t)}(t,\xi)]_2$

$$\begin{aligned}
\mathrm{Res}\left\{\frac{\lambda_3^{D_1\cup D_3}(t,\xi)}{F(\xi)},\gamma_j\right\}&=\lim_{\xi\to\gamma_j}(\xi-\gamma_j)\frac{\lambda_3^{D_1\cup D_3}(t,\xi)}{F(\xi)}\\
&=\frac{\lambda_3^{D_1\cup D_3}(t,\gamma_j)}{\dot{F}(\gamma_j)}.
\end{aligned}$$

Replacing $\xi=\gamma_j$ with the second equation of Eq. (68), we deduce

$$\lambda_3^{D_1\cup D_3}(t,\gamma_j)=\mathrm{e}^{-4\mathrm{i}\gamma^6 t}Z(\gamma_j)\lambda_2^{D_1\cup D_3}(t,\gamma_j).$$

Furthermore,

$$\begin{aligned}
&\mathrm{Res}\left\{\frac{\lambda_3^{D_1\cup D_3}(t,\xi)}{F(\xi)},\gamma_j\right\}=\frac{Z(\gamma_j)\mathrm{e}^{-4\mathrm{i}\gamma_j^6 t}}{\dot{F}(\gamma_j)}[L^{(t)}(t,\gamma_j)]_1,\\
&j=1,2,\ldots,2b_1.
\end{aligned}$$

The equation is equivalent to Eq. (66). The correctness of Eq. (67) can be established through an analogous method.

Definition 3 [Regarding $\rho(\xi)$ and $\beta(\xi)$] We posit that $q_T(x)$ is equivalent to $q(x,T)$ and $q_T^*(x)$ is the same as $q^*(x,T)$. Furthermore, we take into account the spectral functions

$$\begin{aligned}
\rho(\xi)&=\overline{f(\bar{\xi})}F(\xi)-\overline{z(\bar{\xi})}Z(\xi),\\
\beta(\xi)&=f(\xi)Z(\xi)-z(\xi)F(\xi).
\end{aligned}$$

We establish the mapping

$$\tilde{\tilde{S}}:\{g_T(x)\}\to\{\rho(\xi),\beta(\xi)\},$$

where

$$\begin{aligned}
&\begin{pmatrix}\beta(\xi)\\ \rho(\xi)\end{pmatrix}=\lambda_3^{(2)}(0,\xi)=\begin{pmatrix}\lambda_3^{12}(0,\xi)\\ \lambda_3^{22}(0,\xi)\end{pmatrix},\ \mathrm{Im}\xi^2\geq 0,\\
&\lambda_3(x,T;\xi)=I+\int_0^x\mathrm{e}^{\mathrm{i}\xi^2(\tau-x)\hat{\sigma}_3}(H_1\lambda_3)(x,T;\xi)\mathrm{d}\xi.
\end{aligned}$$

$H_1(x,T;\xi)$ is given by

$$H_1(x,T;\xi)=\begin{pmatrix}\frac{\mathrm{i}}{2}q_T q_T^* & \xi q_T\mathrm{e}^{-2\mathrm{i}\int_0^x\Delta\hat{\sigma}_3}\\ -\xi q_T^*\mathrm{e}^{2\mathrm{i}\int_0^x\Delta\hat{\sigma}_3} & -\frac{\mathrm{i}}{2}q_T\ q_T^*\end{pmatrix}.$$

Proposition 7 The important attributes of $\rho(\xi)$ and $\beta(\xi)$ are listed here:

(i) For $\mathrm{Im}\xi^2\leq 0$, $\rho(\xi)$ and $\beta(\xi)$ are analytical;

(ii) $\rho(\xi)=1+\mathscr{O}(\frac{1}{\xi}),\beta(\xi)=\mathscr{O}(\frac{1}{\xi})$ as $\xi\to\infty$, $\mathrm{Im}\xi^2\leq 0$;

(iii) $\rho(\xi)\overline{\rho(\bar{\xi})}-\beta(\xi)\overline{\beta(\bar{\xi})}=1$, $\xi^2\in\mathbb{R}$;

(iv) $\rho(-\xi)=\rho(\xi),\beta(-\xi)=-\beta(\xi)$, $\mathrm{Im}\xi^2\leq 0$;

(v) As $\tilde{\tilde{Q}}:\{\rho(\xi),\beta(\xi)\}\to\{g_T(x)\}$ and $\tilde{\tilde{S}}^{-1}=\tilde{\tilde{Q}}$, the definition of the inverse map $\tilde{\tilde{S}}$ for $\tilde{\tilde{Q}}$ is as follows:

$$q_T(x)=2\mathrm{i}\lim_{\xi\to\infty}(\xi L^{(T)}(x,\xi))_{12}, \tag{69}$$

$$q_T^*(x)=-2\mathrm{i}\lim_{\xi\to\infty}(\xi L^{(T)}(x,\xi))_{21}, \tag{70}$$

where $L^{(T)}(x,\xi)$ fulfills the subsequent R-H problem.

Theorem 4 Let

$$L_-^{(T)}(x,\xi)=\left(\lambda_3^{D_4\cup D_5\cup D_6}(x,\xi),\frac{\lambda_1^{D_4\cup D_5\cup D_6}(x,\xi)}{\overline{\rho(\bar{\xi})}}\right),$$

$$\mathrm{Im}\xi^2 \le 0, \xi \in D_4 \cup D_5 \cup D_6,$$

$$L_+^{(T)}(x,\xi) = \left(\lambda_3^{D_1\cup D_2\cup D_3}(x,\xi), \frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,\xi)}{\rho(\xi)}\right),$$

$$\mathrm{Im}\xi^2 \ge 0, \xi \in D_1 \cup D_2 \cup D_3. \tag{71}$$

Then, $L^{(T)}(x,\xi)$ fulfills the subsequent R-H problem:

(i) $L^{(T)}(x,\xi) = \begin{cases} L_-^{(T)}(x,\xi), & \mathrm{Im}\xi^2 \le 0 \\ L_+^{(T)}(x,\xi), & \mathrm{Im}\xi^2 \le 0 \end{cases}$ is a piecewise analytic function.

(ii) $L_+^{(T)}(x,\xi) = L_-^{(T)}(x,\xi)G^{(T)}(x,\xi)$, $\xi^2 \in \mathbb{R}$, and

$$G^{(T)}(x,\xi) = \begin{pmatrix} \frac{1}{\rho(\xi)\overline{\rho(\bar{\xi})}} & \frac{\overline{\beta(\bar{\xi})}}{\rho(\xi)} \mathrm{e}^{-2\mathrm{i}(\xi^2 x + 2\xi^6 T)} \\ -\frac{\beta(\xi)}{\overline{\rho(\bar{\xi})}} \mathrm{e}^{2\mathrm{i}(\xi^2 x + 2\xi^6 T)} & 1 \end{pmatrix},$$

$$\xi^2 \in \mathbb{R}. \tag{72}$$

(iii) $L^{(T)}(x,\xi) = I + \mathscr{O}(\frac{1}{\xi}),\ \xi \to \infty$.

$\rho(\xi)$ contains $2a$ simple zeros $\{\mu_j\}_{j=1}^{2a}$ ($2a = 2a_1 + 2a_2$), we assume that μ_j ($j = 1,2,\ldots,2a_1$) is a part of $D_1 \bigcup D_2 \bigcup D_3$, and $\bar{\mu}_j$ ($j = 2a_1+1, 2a_1+2, \ldots, 2a$) is a part of $D_4 \bigcup D_5 \bigcup D_6$.

Simple poles are located in $[L_+^{(T)}(x,\xi)]_2$ at points where $\xi = \mu_j$, for j taking values from 1 to $2a_1$. The positions of the simple poles within $[L_-^{(T)}(x,\xi)]_1$ are at $\xi = \bar{\mu}_j$, for j ranging from $2a_1+1$ to $2a$.

The residue requirements are as follows:

$$\mathrm{Res}\{[L^{(T)}(x,\xi)]_1, \mu_j\} = \frac{\mathrm{e}^{2\mathrm{i}(\mu_j^2 x + 2\mu_j^6 t)}}{\dot{\rho}(\mu_j)\beta(\mu_j)}[L^{(T)}(x,\mu_j)]_2,$$

$$j = 1,2,\ldots,2a_1, \tag{73}$$

$$\mathrm{Res}\{[L^{(T)}(x,\xi)]_2, \bar{\mu}_j\} = \frac{\mathrm{e}^{-2\mathrm{i}(\bar{\mu_j}^2 x + 2\bar{\mu_j}^6 t)}}{\overline{\dot{\rho}(\mu_j)\beta(\mu_j)}}[L^{(T)}(x,\bar{\mu}_j)]_1,$$

$$j = 2a_1+1, 2a_1+2, \ldots, 2a. \tag{74}$$

Proof After inserting $t = T$ into Eq. (48), we derive the subsequent formula

$$\begin{cases} \lambda_3^{D_4\cup D_5\cup D_6} = \overline{\rho(\bar{\xi})}\lambda_1^{D_1\cup D_2\cup D_3} \\ \qquad\qquad + \mathrm{e}^{2\mathrm{i}(\xi^2 x + 2\xi^6 T)}\overline{\beta(\bar{\xi})}\lambda_1^{D_4\cup D_5\cup D_6}, \\ \lambda_3^{D_1\cup D_2\cup D_3} = \mathrm{e}^{-2\mathrm{i}(\xi^2 x + 2\xi^6 T)}\beta(\xi)\lambda_1^{D_1\cup D_2\cup D_3} \\ \qquad\qquad + \rho(\xi)\lambda_1^{D_4\cup D_5\cup D_6}. \end{cases} \tag{75}$$

The jump matrix $L^{(T)}(T,\xi)$ complies with the jump conditions, as established via computational analysis. We analyze $[L_+^{(T)}(T,\xi)]_1$

$$\mathrm{Res}\left\{\frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,\xi)}{\rho(\xi)}, \mu_j\right\} = \lim_{\xi\to\mu_j}(\xi-\mu_j)\frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,\xi)}{\rho(\xi)} = \frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,\mu_j)}{\dot{\rho}(\mu_j)}.$$

Replacing $\xi = \mu_j$ with the second equation of Eq. (75), we deduce

$$\lambda_1^{D_1\cup D_2\cup D_3}(t,\mu_j) = \frac{\mathrm{e}^{4\mathrm{i}\mu_j^6 T}\lambda_3^{D_1\cup D_2\cup D_3}(t,\mu_j)}{\beta(\mu_j)}.$$

Furthermore,

$$\mathrm{Res}\left\{\frac{\lambda_1^{D_1\cup D_2\cup D_3}(x,\xi)}{\rho(\xi)}, \mu_j\right\} = \frac{\mathrm{e}^{2\mathrm{i}(\mu_j^2 x + 2\mu_j^6 T)}}{\dot{\rho}(\mu_j)\beta(\mu_j)}[L^{(T)}(x,\mu_j)]_2,$$

$j = 1,2,\ldots,2a_1$.

This is equivalent to Eq. (73). The accuracy of Eq. (74) can be verified by following an identical procedure.

3.2. The fundamental Riemann–Hilbert problem

Theorem 5 The matrix functions $v(\xi)$ and $V(\xi)$ are determined by the pairs $f(\xi)$ and $z(\xi)$, as well as $F(\xi)$ and $Z(\xi)$, respectively. $f(\xi)$, $z(\xi)$, $F(\xi)$, and $Z(\xi)$ are represented by the spectral functions $q_0(x)$, $h_0(t)$, and $h_1(t)$ in Definitions 1 and 2. Assume that $f(\xi)$, $z(\xi)$, $F(\xi)$, and $Z(\xi)$ meet the global relation. Then,

$$f(\xi)Z(\xi) - z(\xi)F(\xi) = \mathrm{e}^{4\mathrm{i}\xi^6 T}c^+(\xi),\ \mathrm{Im}\xi^2 \ge 0,$$

where $c^+(\xi)$ is an entire function $v(\xi) = \lambda_1(0,0;\xi)$, $V(\xi) = V(T,\xi) = (\mathrm{e}^{2\mathrm{i}\xi^6 T}\lambda_2(0,T;\xi))^{-1}$. The global relation is transformed into $f(\xi)Z(\xi) - z(\xi)F(\xi) = 0$ when $\xi \to \infty$. $L(x,t,\xi)$ adheres to the R-H problem in the subsequent manner.

(i) The function $L(x,t;\xi)$ behaves as an analytic function across slices, consistently holding a determinant value of unity.

(ii) $L(x,t;\xi)$ meets the jump condition

$$L_+(x,t;\xi) = L_-(x,t;\xi)G(x,t;\xi),\ \xi^6 \in \mathbb{R}. \tag{76}$$

(iii) The simple poles of $[L(x,t;\xi)]_1$ are positioned at $\xi = \bar{\varepsilon}_j$ for j ranging from $2t_1+1$ to $2t$, and at $\xi = \bar{\zeta}_j$ for j varying from $2s_1+1$ to $2s$. Furthermore, $[L(x,t;\xi)]_2$ exhibits simple poles at $\xi = \varepsilon_j$ for j from 1 to $2t_1$, and at $\xi = \zeta_j$ for j from 1 to $2s_1$.

(iv) $L(x,t;\xi) = I + \mathscr{O}(\frac{1}{\xi}),\ \xi \to \infty$.

(v) $L(x,t;\xi)$ meets the residue condition as specified in Hypothesis 1.

The function $L(x,t;\xi)$ not only exists but is also the only one of its kind

$$q(x,t) = 2\mathrm{i}\lim_{\xi\to\infty}(\xi L(x,t;\xi))_{12},$$
$$q^*(x,t) = -2\mathrm{i}\lim_{\xi\to\infty}(\xi L(x,t;\xi))_{21}. \tag{77}$$

Additionally, the solutions of Eq. (8), represented by $q(x,t)$ and $q^*(x,t)$, respectively, are given through these equations.

Proof It can be inferred that the (2×2) matrix function $L(x,t;\xi)$ satisfies a non-singular Riemann–Hilbert problem,

given that $f(\xi)$ and $F(\xi)$ are devoid of zeros. By leveraging the relationship between the jump matrix $G(x,t,\xi)$ and the symmetry conditions, one can show that this problem possesses a solution that is valid universally. In the case where $f(\xi)$ and $F(\xi)$ possess only a finite number of zeros, an equivalent scenario without any zeros can be achieved by imposing an algebraic system of equations that is guaranteed to have a solution.

Lemma 1 [Vanishing lemma] According to Theorem 5, the R-H problem possesses exclusively trivial solutions when the boundary condition is imposed that $L(x,t;\xi)$ tends to zero as $\xi \to \infty$.

Proof Define

$$\begin{aligned}\chi_+(\xi) &= L_+(\xi)L_-^\dagger(-\bar{\xi}),\ \mathrm{Im}\xi^6 \geq 0,\\ \chi_-(\xi) &= L_-(\xi)L_+^\dagger(-\bar{\xi}),\ \mathrm{Im}\xi^6 \leq 0.\end{aligned} \tag{78}$$

Given that x and t are interdependent, through the utilization of symmetry properties, we can deduce that

$$\begin{aligned}&G_1^\dagger(-\bar{\xi}) = G_1(\xi),\ G_2^\dagger(-\bar{\xi}) = G_2(\xi),\\ &G_3^\dagger(-\bar{\xi}) = G_3(\xi).\end{aligned} \tag{79}$$

Then,

$$\begin{aligned}\chi_+(\xi) &= L_-(\xi)G(\xi)L_-^\dagger(-\bar{\xi}),\ \mathrm{Im}\xi^6 \in \mathbb{R},\\ \chi_-(\xi) &= L_-(\xi)G^\dagger(-\bar{\xi})L_-^\dagger(-\bar{\xi}),\ \mathrm{Im}\xi^6 \in \mathbb{R},\end{aligned} \tag{80}$$

for $\mathrm{Im}\xi^6 \in \mathbb{R}$, $\chi_+(\xi) = \chi_-(\xi)$, in accordance with Eqs. (78) and (79). Consequently, $\chi_+(\xi)$ and $\chi_-(\xi)$ establish a complete function, which is uniformly zero as it approaches infinity.

It is revealed that the matrix $G_1(\mathrm{i}\iota)$, where ι belongs to the set of real numbers, is a positively definite matrix. Considering that $\chi_-(\iota)$ vanishes in a similar manner for ι in the set $\mathrm{i}\mathbb{R}$, which means

$$L_+(\mathrm{i}\iota)G_1(\mathrm{i}\iota)L_+^\dagger(\mathrm{i}\iota) = 0,\ \iota \in \mathbb{R}. \tag{81}$$

When $\iota \in \mathbb{R}$, one can infer that $L_+(\mathrm{i}\iota) = 0$. Consequently, both $L_+(\xi)$ and $L_-(\xi)$ vanish in a consistent fashion.

Proof that $q(x,t)$ satisifies the higher KN equation.

4. Conclusion

In this paper, we thoroughly examined the higher KN equation on the half-line using the Fokas method. Initially, we divided the equation in the complex plane by utilizing spectral and asymptotic analytical methods. Following this, we conducted detailed computations to ascertain the jump matrix across the various partitioned regions, and finally, we arrived at a solution to the equation by applying the R-H approach.

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