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A kinetic description of the goods exchange market allowing transfer of agents

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In order to avoid the worsening of wealth inequality, it is necessary to explore the influencing factors of wealth distribution and discuss measures to reduce wealth inequality. We investigate the wealth distribution in the goods exchange market by using the kinetic theory of rarefied gas. The trading objects are two kinds of commodities (commodities A and B) and the trading subjects are agents of two groups (dealers and speculators). We deduce the interaction rules according to the principle of utility maximization and consider the transfer of agents in the Boltzmann equation. The steady solution of the Fokker–Planck equation for a special case is obtained and the effects of trading strategy and transfer frequency on the steady distribution are analyzed in numerical experiments. The conclusions illustrate that the transfer of agents is conducive to reducing the inequality of wealth distribution.

Keywords: kinetic model, goods exchange, Boltzmann equation, Fokker–Planck equation

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1. Introduction

Data from the World Inequality Report 2022 show that 76% of the global total wealth is owned by 10% of the population and the poorest 50% of the population only own 2% of the global total wealth, which is enough to state that the inequality of wealth distribution has become a serious problem.[1] To a certain extent, it has influenced the political, economic and social development of countries. Therefore, it is necessary to utilize appropriate methods to investigate wealth distribution, explore the corresponding adjustment strategies and prevent the inequality of wealth distribution from worsening.

In recent years, a large number of scholars have applied the methods of statistical physics to study economic and social phenomena. Cordier *et al.*[2] developed a Cordier–Pareschi–Toscani (CPT) model to explain the formation of a Pareto tail in wealth distribution, in which the deterministic exchange of wealth among agents and the influence of market random factors are considered. Cordier *et al.*[3] assumed that there is only one type of agent in the financial market who is free to choose whether to invest in a stock or a bond and described the formation mechanism of stock price through a kinetic model. Zhong *et al.*[4] embedded a suitable value function in the interaction rule of wealth and used a kinetic model to describe the distribution of wealth between risky and risk-free assets. Maldarella and Pareschi[5] developed a relatively simple kinetic model to describe the interaction between two different types of agents and investigated the asymptotic behavior of price formation and wealth distribution. A dynamical model was also used in Brugna and Toscani,[6] where the formation of the opinions for agents is discussed. Utilizing the kinetic model of wealth distribution, Bertotti[7] considered taxation and the process of redistribution in a closed society, aiming to analyze the evolution of the wealth density function caused by binary interactions. Meng *et al.*[8] added a non-Maxwellian collision kernel to the Boltzmann equation and studied the evolution of the agent's wealth distribution. The results show that taxation and redistribution have the property of changing the Pareto index. By adding two different value functions to the interaction rule and introducing a non-Maxwellian collision kernel into the dynamic model, Zhao *et al.*[9] obtained the steady-state distribution of call center service time in an exact form.

In the multi-agent market, agent behaviors, including the knowledge level, opinion, risk appetite and saving propensity have an important impact on wealth distribution. Pareschi and Toscani[10] investigated the influence of agents' knowledge on the inequality of wealth distribution. Toscani[11] considered the transmission of opinions and knowledge spillover among agents and established a kinetic model to depict the mechanism of opinion formation. Dimarco *et al.*[12] developed a dynamic model to characterize the impact of higher education on social stratification and found that universal education reduces socioeconomic inequality. However, if educational attainment is dependent on social status, education will hinder social equality. Boudin *et al.*[13] discussed the evolution of opinion formation in a closed market and considered the influence of both agent interaction and mass media. Assuming that the interaction between agents is driven by both compromise and self-thinking, Düring *et al.*[14] established a dynamic

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model of opinion formation and proposed an effective optimal control strategy to break consensus in the process of opinion formation. Furioli *et al.*[15] obtained a Fokker–Planck equation to discuss socio-economic phenomena such as wealth distribution, knowledge diffusion and opinion formation. Wang *et al.*[16] studied the influence of decision-making ability on wealth distribution in a multi-agent system and summarized that the imbalance of decision-making ability is one of the reasons for worsening wealth inequality. Hu and Chen[17] found that an agent's psychological factors have both favorable and unfavorable aspects in terms of investment decision-making, and the agent's investment ability helps the agent to increase returns and reduce losses. Paul *et al.*[18] studied the relationship between the inequality index and the saving propensity and came to the conclusion that with the increase of the saving propensity, the inequality index decreases, which indicates that the equality of social wealth distribution increases. Cao and Motsch[19] built kinetic models of wealth distribution under unbiased, poor-biased and rich-biased conditions, acquired the steady-state solutions for the three cases, and proved that the social Gini coefficient is close to 1 under the rich-biased condition.

Assuming that agents are motivated to trade to maximize their own utility levels and the binary interactions between agents follow the relevant rules of the Edgeworth box, Toscani *et al.*[20] combined the kinetic theory of rarefied gas with the theory of price in microeconomics to study the wealth distribution in a goods exchange market. Brugna and Toscani[21] constructed a system of kinetic equations to describe a goods trading market that contains two groups of agents, each with disparate selection of transaction strategy. Brugna and Toscani[21] described goods trading based on the price theory of microeconomics, specifically using the Cobb–Douglas utility function to rationalize agents' trading motives. Hu *et al.*[22] employed the constant elasticity of substitution utility function to describe the commodity preference of agents, redefine the transaction strategies of two agent groups in the market and discuss the impact of the trading strategies and commodity preference of agents on prices through numerical experiments. Düring and Toscani[23] discussed a dynamic model of commodity exchange across multiple countries and showed that wealth is generally distributed in a bimodal pattern.

Many scholars describe the transition of gases through Boltzmann equations. Giovangigli[24] derived a multi-component flow model by using the kinetic theory of gases and studied the symmetric hyperbolic parabolic structure of the model. Desvillettes *et al.*[25] developed a kinetic model to describe the chemical reaction of gas mixtures, which converges to the Euler equation of reactive multidirectional gas mixtures at the limiting level. Rossani and Spiga[26] employed the Boltzmann equation to analyze the interaction of gas mixtures and the equilibrium state under collision. Dimarco *et al.*[27] believe that in epidemic transmission, the susceptible population may be infected and the infected eventually recover or die. Namely, the agents' categories vary. Supposing that wealth exchange interacts with infectious disease epidemics, Bernadi *et al.*[28] applied dynamic models to assess economic trends during epidemics and demonstrated that vaccination helps reduce wealth inequality. Bisi[29] proposed a model to describe the international trade between agents of different countries, considered the possibility of agents transferring from one country to another and characterized the transfer by a Boltzmann operator. Bisi and Loy[30] established a general dynamic model with transfer probability, simulated the evolution of wealth distribution function under the unsteady interaction rate and proved that the Pareto index of the steady-state distribution depends on the transfer probability of the agent.

Research on wealth distribution in the existing literature has achieved significant results and laid a theoretical foundation for subsequent studies. However, there are still some problems that need to be further investigated. For example, the inter-group transfer of agents should be considered. Based on the transfer of agents between two groups, we deduce trading rules according to the utility theory in microeconomics and use different trading strategies to distinguish the two groups of agents (dealers and speculators). Compared with the works in Refs. [21,22,29], our contributions are as follows.

Firstly, Brugna and Toscani[21] and Hu *et al.*[22] studied goods exchange between two groups of agents with different trading strategies. However, their work does not consider the transfer between different groups of agents. In our paper, we allow agents to transfer from one group to another.

Secondly, different from the rules of wealth interaction in Ref. [29], we deduce the rules of binary interaction according to the principle of utility maximization based on the Cobb–Douglas utility function.

Thirdly, Bisi[29] assumed that there is only one commodity in the market and studied international trade between agents of two countries. We assume that there are two different commodities A and B in the market. The agent's trading tendencies for goods A and B are λ_a and λ_b, respectively. We allow $\lambda_a = \lambda_b$ and $\lambda_a \neq \lambda_b$.

This paper is organized as follows. In Section 2, we introduce the basic model of goods exchange and give the updating rules of quantities of each type of goods. In Section 3, we construct the Boltzmann operators including binary interaction and agent transfer. Section 4 aims to deduce the evolution equations for fractions of agents in each group and the average quantities of each type of goods. We use the "continuous trading limit" process to simplify the Boltzmann equation and obtain the corresponding Fokker–Planck equation in

Section 5. Particularly, we find a steady-state solution of the Fokker–Planck equation for a special case. In numerical experiments, we analyze the effects of trading strategy and transfer frequency on the wealth distribution. The conclusion is presented in Section 6.

2. Basic model

Based on price theory, this section introduces the theoretical framework of goods trading, which lays a theoretical foundation for the establishment of our model. Suppose that a market is composed of agents in two groups, with fixed numbers N_1 and N_2 of agents belonging to groups 1 and 2, respectively. All agents possess goods of two different types, denoted by A and B. At the beginning, the quantities of commodities A and B owned by an agent in group 1 are $a_k = a_k(0)$ and $b_k = b_k(0)$, respectively. Therefore, at time $t \geq 0$, the total amounts of commodities A and B traded by agents in group 1 are represented by

$$\sum_{k=1}^{N_1} a_k(t) = N_{1\mathrm{A}}(t), \quad \sum_{k=1}^{N_1} b_k(t) = N_{1\mathrm{B}}(t).$$

Similarly, the initial quantities of commodities A and B for the agents in group 2 are $\widetilde{a}_k = \widetilde{a}_k(0)$ and $\widetilde{b}_k = \widetilde{b}_k(0)$. At time $t \geq 0$, the total amounts of commodities A and B traded by agents in group 2 are represented by

$$\sum_{k=1}^{N_2} \widetilde{a}_k(t) = N_{2\mathrm{A}}(t), \quad \sum_{k=1}^{N_2} \widetilde{b}_k(t) = N_{2\mathrm{B}}(t).$$

Assuming that the market is closed, the total number of commodities A and B satisfies the following conservation law:

$$\begin{cases} N_{1\mathrm{A}}(t) + N_{2\mathrm{A}}(t) = N_{1\mathrm{A}} + N_{2\mathrm{A}} = I_{\mathrm{A}}, \\ N_{1\mathrm{B}}(t) + N_{2\mathrm{B}}(t) = N_{1\mathrm{B}} + N_{2\mathrm{B}} = I_{\mathrm{B}} \end{cases}$$

at any subsequent time $t \geq 0$.

Consider a situation where agents in two groups adopt different trading strategies. The agents in group 1 place all the commodities they own in trade without saving, while the agents in group 2 have a marked saving propensity and trade only part of their goods; that is, trade only $(\lambda_a \widetilde{a}_k(t_-), \lambda_b \widetilde{b}_k(t_-))$ when they own $(\widetilde{a}_k(t_-), \widetilde{b}_k(t_-))$, where $0 < \lambda_a, \lambda_b < 1$ are fixed constant, relating to the trading propensity of agents in goods A and B. We call the agents in group 1 dealers and the agents in group 2 speculators, respectively.

Assuming that the price of commodity A is equal to unit and the price of commodity B is $P(t) > 0$. At any subsequent time $t \geq \Delta t$, the wealth of the agents in the two groups reads as

$$\begin{cases} w_k(t) = a_k(t_-) + P(t) b_k(t_-), \\ \widetilde{w}_k(t) = \lambda_a \widetilde{a}_k(t_-) + P(t) \lambda_b \widetilde{b}_k(t_-), \end{cases} \tag{1}$$

where $t_- = t - \Delta t$ denotes the last exchange time.

In order to rationalize the motivation of agents for goods trading, we assume that the agent's behavior is driven by a utility function and the purpose of each agent is to maximize his utility. We use the Cobb–Douglas utility function

$$U(a) = a^{\alpha}(w_k(t) - a)^{\beta}, \quad \alpha + \beta = 1. \tag{2}$$

In Eq. (2), $\alpha \geq 0$ and $\beta \geq 0$ represent the preference of agents for goods A and B, respectively. $\alpha > \beta$ denotes that an agent is more likely to have goods A and vice versa when $\alpha < \beta$. The choice $\alpha = \beta = 1/2$ denotes that an agent possesses the same preference for goods A and B.

Assuming that there is a main counterparty for every transaction. The main counterparty randomly chooses the object of his transaction. In order to explore the evolution of wealth distribution of the two types of agents, we set the two types of agents as the main counterparty and give the corresponding updating rules of commodity quantities, respectively.

Now, we consider the cases of a dealer k_1 acting as the main counterparty. When the dealer k_1 deals with another dealer k_2, the maximization of Eq. (2) updates the quantities of goods in the rules

$$\begin{cases} a_{k_1}(t) = \alpha w_{k_1}(t), \ \ P(t) b_{k_1}(t) = \beta w_{k_1}(t), \\ a_{k_2}(t) = \alpha w_{k_2}(t), \ \ P(t) b_{k_2}(t) = \beta w_{k_2}(t). \end{cases} \tag{3}$$

The value of price $P(t)$ is derived from the law of conservation of the total number of goods A and B at time $t - \Delta t$. In this case, we have

$$\begin{cases} \alpha(w_{k_1}(t) + w_{k_2}(t)) = a_{k_1}(t_-) + a_{k_2}(t_-), \\ \beta(w_{k_1}(t) + w_{k_2}(t)) = P(t)(b_{k_1}(t_-) + b_{k_2}(t_-)). \end{cases} \tag{4}$$

Using Eq. (4) yields

$$P_{11}(t) = \frac{\beta}{\alpha} \frac{a_{k_1}(t_-) + a_{k_2}(t_-)}{b_{k_1}(t_-) + b_{k_2}(t_-)}, \tag{5}$$

where $P_{11}(t)$ represents the price of goods B when both parties of the transaction come from group 1. Substituting the price $P_{11}(t)$ into Eq. (3), we obtain the new quantities of goods for the dealer k_1

$$\begin{cases} a_{k_1}(t) = a_{k_1}(t_-) + \beta \left(\dfrac{a_{k_1}(t_-) + a_{k_2}(t_-)}{b_{k_1}(t_-) + b_{k_2}(t_-)} b_{k_1}(t_-) - a_{k_1}(t_-) \right), \\ b_{k_1}(t) = b_{k_1}(t_-) + \alpha \left(\dfrac{b_{k_1}(t_-) + b_{k_2}(t_-)}{a_{k_1}(t_-) + a_{k_2}(t_-)} a_{k_1}(t_-) - b_{k_1}(t_-) \right). \end{cases}$$

Similarly, when a dealer k_1 trades with a speculator k_2, the updating rules of quantities of goods of the dealer k_1 are written as

$$\begin{cases} a_{k_1}(t) = a_{k_1}(t_-) + \beta \left(\dfrac{a_{k_1}(t_-) + \lambda_a a_{k_2}(t_-)}{b_{k_1}(t_-) + \lambda_b b_{k_2}(t_-)} b_{k_1}(t_-) - a_{k_1}(t_-) \right), \\ b_{k_1}(t) = b_{k_1}(t_-) + \alpha \left(\dfrac{b_{k_1}(t_-) + \lambda_b b_{k_2}(t_-)}{a_{k_1}(t_-) + \lambda_a a_{k_2}(t_-)} a_{k_1}(t_-) - b_{k_1}(t_-) \right). \end{cases}$$

When $\lambda_a = \lambda_b = \lambda$, the above trading rules become

$$\begin{cases} a_{k_1}(t) = a_{k_1}(t_-) + \beta\left(\dfrac{a_{k_1}(t_-)+\lambda a_{k_2}(t_-)}{b_{k_1}(t_-)+\lambda b_{k_2}(t_-)} b_{k_1}(t_-) - a_{k_1}(t_-)\right), \\ b_{k_1}(t) = b_{k_1}(t_-) + \alpha\left(\dfrac{b_{k_1}(t_-)+\lambda b_{k_2}(t_-)}{a_{k_1}(t_-)+\lambda a_{k_2}(t_-)} a_{k_1}(t_-) - b_{k_1}(t_-)\right). \end{cases}$$

Then, we discuss the cases of a speculator k_1 acting as the main counterparty. When both parties in the transaction are speculators, the updating rules of quantities of commodity A and B are written as

$$\begin{cases} a_{k_1}(t) = a_{k_1}(t_-) + \beta\left(\dfrac{a_{k_1}(t_-)+a_{k_2}(t_-)}{b_{k_1}(t_-)+b_{k_2}(t_-)} b_{k_1}(t_-) - a_{k_1}(t_-)\right), \\ b_{k_1}(t) = b_{k_1}(t_-) + \alpha\left(\dfrac{b_{k_1}(t_-)+b_{k_2}(t_-)}{a_{k_1}(t_-)+a_{k_2}(t_-)} a_{k_1}(t_-) - b_{k_1}(t_-)\right). \end{cases}$$

When speculator k_1 trades with dealer k_2, we obtain the following trading rules:

$$\begin{cases} a_{k_1}(t) = a_{k_1}(t_-) + \dfrac{\beta}{\lambda_a}\left(\dfrac{\lambda_a a_{k_1}(t_-)+a_{k_2}(t_-)}{\lambda_b b_{k_1}(t_-)+b_{k_2}(t_-)}\lambda_b b_{k_1}(t_-) - \lambda_a a_{k_1}(t_-)\right), \\ b_{k_1}(t) = b_{k_1}(t_-) + \dfrac{\alpha}{\lambda_b}\left(\dfrac{\lambda_b b_{k_1}(t_-)+b_{k_2}(t_-)}{\lambda_a a_{k_1}(t_-)+a_{k_2}(t_-)}\lambda_a a_{k_1}(t_-) - \lambda_b b_{k_1}(t_-)\right). \end{cases}$$

If we let $\lambda_a = \lambda_b = \lambda$, the above updating rules become

$$\begin{cases} a_{k_1}(t) = a_{k_1}(t_-) + \beta\left(\dfrac{\lambda a_{k_1}(t_-)+a_{k_2}(t_-)}{\lambda b_{k_1}(t_-)+b_{k_2}(t_-)} b_{k_1}(t_-) - a_{k_1}(t_-)\right), \\ b_{k_1}(t) = b_{k_1}(t_-) + \alpha\left(\dfrac{\lambda b_{k_1}(t_-)+b_{k_2}(t_-)}{\lambda a_{k_1}(t_-)+a_{k_2}(t_-)} a_{k_1}(t_-) - b_{k_1}(t_-)\right). \end{cases}$$

To sum up, when both parties in the transaction are agents of the same kind (both parties are dealers or are speculators), the updating rules of goods of the main counterparty are the same. Similar expressions for updating rules of goods are also found in Refs. [21,22]. Using the above rules, we shall build a Boltzmann system.

3. Boltzmann equation for goods exchange and agent transfer

In this part, we construct a dynamical model for goods exchange for agents in two groups and investigate the possibility of agents' transfer from group 1 to group 2 or from group 2 to group 1. The binary interaction is modelled by the updating rule for the quantity of goods which is analogous to that presented in Ref. [22], and transfer is modelled by the Boltzmann type operator.

The objective of this paper is to study the evolution of the wealth distributions of agents in two groups. We use $f_i(a,b,t)$ $(i=1,2)$ to represent the wealth distribution of the agent in group i, which is determined by the quantities of goods $a,b \in \mathbb{R}_+$ and time $t \in \mathbb{R}_+$. Debt is not allowed in our model. Thus, we assume that no one has a negative quantity of goods. More precisely, in order to ensure the continuity of density function as $a,b \to 0$, we suppose that $f_i(0,0,t) = 0$ and $f_i(a,b,t) \geq 0$. The sth moments of wealth distribution functions are represented by

$$\mathcal{M}^s_{i\mathrm{A}}(t) = \int_{R^2_+} a^s f_i(a,b,t)\,\mathrm{d}a\mathrm{d}b,$$

$$\mathcal{M}^s_{i\mathrm{B}}(t) = \int_{R^2_+} b^s f_i(a,b,t)\,\mathrm{d}a\mathrm{d}b,$$

where $i=1,2$ and $s \geq 0$. The option $s=0$ represents the fractions of agents in each group in the total number of agents

$$\begin{cases} \rho_1(t) = \int_{R^2_+} f_1(a,b,t)\,\mathrm{d}a\mathrm{d}b, \\ \rho_2(t) = \int_{R^2_+} f_2(a,b,t)\,\mathrm{d}a\mathrm{d}b, \end{cases} \tag{6}$$

which changes with time due to the agents' transfer. The total wealth density function $f(a,b,t)$ satisfies

$$f(a,b,t) = f_1(a,b,t) + f_2(a,b,t), \quad \int_{R^2_+} f(a,b,t)\,\mathrm{d}a\mathrm{d}b = 1.$$

Thus, we have $\rho_1(t) + \rho_2(t) = 1$ at any time $t > 0$.

The option $s=1$ yields the total quantities of each goods of group i $(i=1,2)$

$$\begin{cases} M_{1\mathrm{A}}(t) = \int_{R^2_+} a f_1(a,b,t)\,\mathrm{d}a\mathrm{d}b, \\ M_{1\mathrm{B}}(t) = \int_{R^2_+} b f_1(a,b,t)\,\mathrm{d}a\mathrm{d}b, \end{cases} \tag{7}$$

$$\begin{cases} M_{2\mathrm{A}}(t) = \int_{R^2_+} a f_2(a,b,t)\,\mathrm{d}a\mathrm{d}b, \\ M_{2\mathrm{B}}(t) = \int_{R^2_+} b f_2(a,b,t)\,\mathrm{d}a\mathrm{d}b. \end{cases} \tag{8}$$

The time evolution equation for the wealth distribution function $f_i(a,b,t)$ is expressed as

$$\frac{\partial f_i(a,b,t)}{\partial t} = \sum_{j=1}^{2} Q_i(f_i,f_j)(a,b) + Q_i^{\mathrm{T}}(f_1,f_2)(a,b), \quad i=1,2. \tag{9}$$

In Eq. (9), $Q_i(f_i,f_j)(a,b)$ is a binary transaction operator, describing the effect of the transaction between two agents from group i and group j (including the case of $i=j$). The specific trading rules are detailed later. The operator $Q_i^{\mathrm{T}}(f_1,f_2)(a,b)$ denotes cases where dealers change their roles to speculators, or vice versa.

When two dealers have a binary interaction, the quantities of goods A and B they own before the transaction are (a,b) and (a_1,b_1), respectively. After the transaction, the amounts of goods A and B that the main counterparty owns become (a^*,b^*). The trading rules considering market risk are as follows:

$$\begin{cases} a^* = a + \beta\left(\dfrac{a+a_1}{b+b_1} b - a\right) + \mu_a a, \\ b^* = b + \alpha\left(\dfrac{b+b_1}{a+a_1} a - b\right) + \mu_b b, \end{cases} \tag{10}$$

where μ_a and μ_b are random variables satisfying $\langle \mu_a \rangle = \langle \mu_b \rangle = 0$ ($\langle\cdot\rangle$ denotes the mathematical expectation) and $\mathrm{Var}(\mu_a) = \mathrm{Var}(\mu_b) = \gamma_\mu$. γ_μ is a constant.

The operator $Q_1(f_1,f_1)(a,b)$ obeys a Boltzmann equation, which combining with any smooth function $\varphi(a,b)$ with supported set in R_+^2 reads as

$$\int_{R_+^2}\varphi(a,b)Q_1(f_1,f_1)(a,b)\mathrm{d}a\mathrm{d}b = \sigma\left\langle\int_{R_+^4}(\varphi(a^*,b^*)-\varphi(a,b))f_1(a,b,t)f_1(a_1,b_1,t)\mathrm{d}\pi\right\rangle,$$

where σ is the interaction probability independent of the quantities of goods and $\mathrm{d}\pi=\mathrm{d}a\mathrm{d}b\mathrm{d}a_1\mathrm{d}b_1$. When a dealer interacts with a speculator, we consider two cases. If $\lambda_a\neq\lambda_b$, the trading rules are written as

$$\begin{cases} a^*=a+\beta\left(\dfrac{a+\lambda_a a_1}{b+\lambda_b b_1}b-a\right)+\mu_a a,\\ b^*=b+\alpha\left(\dfrac{b+\lambda_b b_1}{a+\lambda_a a_1}a-b\right)+\mu_b b. \end{cases}\tag{11}$$

If $\lambda_a=\lambda_b=\lambda$, the trading rules become

$$\begin{cases} a^*=a+\beta\left(\dfrac{a+\lambda a_1}{b+\lambda b_1}b-a\right)+\mu_a a,\\ b^*=b+\alpha\left(\dfrac{b+\lambda b_1}{a+\lambda a_1}a-b\right)+\mu_b b. \end{cases}\tag{12}$$

The weak form of Boltzmann operator $Q_1(f_1,f_2)(a,b)$ reads as

$$\int_{R_+^2}\varphi(a,b)Q_1(f_1,f_2)(a,b)\mathrm{d}a\mathrm{d}b = \sigma\left\langle\int_{R_+^4}(\varphi(a^*,b^*)-\varphi(a,b))f_1(a,b,t)f_2(a_1,b_1,t)\mathrm{d}\pi\right\rangle.$$

When two speculators have a binary interaction, the quantities of goods A and B of the main counterparty follow rules (10). The weak form of Boltzmann operator $Q_2(f_2,f_2)(a,b)$ is written as

$$\int_{R_+^2}\varphi(a,b)Q_2(f_2,f_2)(a,b)\mathrm{d}a\mathrm{d}b = \sigma\left\langle\int_{R_+^4}(\varphi(a^*,b^*)-\varphi(a,b))f_2(a,b,t)f_2(a_1,b_1,t)\mathrm{d}\pi\right\rangle.$$

Assuming that a speculator interacts with a dealer, the trading rules of the main counterparty become

$$\begin{cases} a^*=a+\dfrac{\beta}{\lambda_a}\left(\dfrac{\lambda_a a+a_1}{\lambda_b b+b_1}\lambda_b b-\lambda_a a\right)+\mu_a a,\\ b^*=b+\dfrac{\alpha}{\lambda_b}\left(\dfrac{\lambda_b b+b_1}{\lambda_a a+a_1}\lambda_a a-\lambda_b b\right)+\mu_b b. \end{cases}\tag{13}$$

When $\lambda_a=\lambda_b=\lambda$, the trading rules are amended as

$$\begin{cases} a^*=a+\beta\left(\dfrac{\lambda a+a_1}{\lambda b+b_1}b-a\right)+\mu_a a,\\ b^*=b+\alpha\left(\dfrac{\lambda b+b_1}{\lambda a+a_1}a-b\right)+\mu_b b. \end{cases}\tag{14}$$

The weak form of Boltzmann operator $Q_2(f_2,f_1)(a,b)$ is expressed as

$$\int_{R_+^2}\varphi(a,b)Q_2(f_2,f_1)(a,b)\mathrm{d}a\mathrm{d}b = \sigma\left\langle\int_{R_+^4}(\varphi(a^*,b^*)-\varphi(a,b))f_2(a,b,t)f_1(a_1,b_1,t)\mathrm{d}\pi\right\rangle.$$

Now, we construct the operators $Q_i^{\mathrm{T}}(f_1,f_2)(a,b)$ $(i=1,2)$ that describe the transfer of agents. Assuming that the transfer occurs after a binary interaction, we have four concrete cases

$$\text{(i) } 1+1\to1+2,\quad \text{(ii) } 2+2\to1+2,$$
$$\text{(iii) } 1+2\to1+1,\quad \text{(iv) } 1+2\to2+2.$$

The cases (i) and (ii) indicate that the parties in the transaction change from the same group to different groups. The cases (iii) and (iv) represent that the parties in the transaction change from different groups to the same one.

The Boltzmann operators $Q_i^{\mathrm{T}}(f_1,f_2)(a,b)$ $(i=1,2)$ contain the above four cases simultaneously. Hence, we have

$$Q_i^{\mathrm{T}}(f_1,f_2)(a,b)=\sum_{l\in\{\mathrm{i,ii,iii,iv}\}}Q_i^{\mathrm{T}(l)}(f_1,f_2)(a,b).\tag{15}$$

We write the construction procedure of the operator $Q_1^{\mathrm{T}}(f_1,f_2)(a,b)$ in detail. The construction procedure corresponding to operator $Q_2^{\mathrm{T}}(f_1,f_2)(a,b)$ is similar.

The occurrence of the transition (i) results in a net loss for agents of group 1. Thus, the Boltzmann operator $Q_1^{\mathrm{T(i)}}(f_1,f_2)(a,b)$ is written as

$$Q_1^{\mathrm{T(i)}}(f_1,f_2)(a,b)=-\int_{R_+^2}Kf_1(a,b,t)f_1(a_1,b_1,t)\mathrm{d}a_1\mathrm{d}b_1,$$

where K denotes the transfer frequency between individuals with quantities of goods (a,b) and (a_1,b_1), respectively. We assume that K is a constant and $0\le K\le1$. The transition (ii) leads to a net gain for agents of group 1. We write the operator $Q_1^{\mathrm{T(ii)}}(f_1,f_2)(a,b)$ in the form

$$Q_1^{\mathrm{T(ii)}}(f_1,f_2)(a,b)=\int_{R_+^2}Kf_2(a,b,t)f_2(a_1,b_1,t)\mathrm{d}a_1\mathrm{d}b_1.$$

For the agents of group 1, we give the specific forms of the operators $Q_1^{\mathrm{T(iii)}}(f_1,f_2)(a,b)$ and $Q_1^{\mathrm{T(iv)}}(f_1,f_2)(a,b)$ in which transitions (iii) and (iv) occur

$$Q_1^{\mathrm{T(iii)}}(f_1,f_2)(a,b)=\int_{R_+^2}Kf_2(a,b,t)f_1(a_1,b_1,t)\mathrm{d}a_1\mathrm{d}b_1,$$

$$Q_1^{\mathrm{T(iv)}}(f_1,f_2)(a,b)=-\int_{R_+^2}Kf_1(a,b,t)f_2(a_1,b_1,t)\mathrm{d}a_1\mathrm{d}b_1.$$

For the sum of the four transfer operators, we obtain

$$Q_1^{\mathrm{T}}(f_1,f_2)(a,b)=\sum_{l\in\{\mathrm{i,ii,iii,iv}\}}Q_1^{\mathrm{T}(l)}(f_1,f_2)(a,b)$$

$$
\begin{aligned}
&= -\int_{R_+^2} K f_1(a,b,t) f_1(a_1,b_1,t)\,\mathrm{d}a_1\,\mathrm{d}b_1 \\
&\quad + \int_{R_+^2} K f_2(a,b,t) f_2(a_1,b_1,t)\,\mathrm{d}a_1\,\mathrm{d}b_1 \\
&\quad + \int_{R_+^2} K f_2(a,b,t) f_1(a_1,b_1,t)\,\mathrm{d}a_1\,\mathrm{d}b_1 \\
&\quad - \int_{R_+^2} K f_1(a,b,t) f_2(a_1,b_1,t)\,\mathrm{d}a_1\,\mathrm{d}b_1. \quad (16)
\end{aligned}
$$

By exchanging the roles of agents in groups 1 and 2 when transferring from (i) to (iv), we have the following concrete form of the operator:

$$
\begin{aligned}
Q_2^{\mathrm{T}}(f_1,f_2)(a,b) &= \sum_{l\in\{\mathrm{i,ii,iii,iv}\}} Q_2^{\mathrm{T}(l)}(f_1,f_2)(a,b) \\
&= \int_{R_+^2} K f_1(a,b,t) f_1(a_1,b_1,t)\,\mathrm{d}a_1\,\mathrm{d}b_1 \\
&\quad - \int_{R_+^2} K f_2(a,b,t) f_2(a_1,b_1,t)\,\mathrm{d}a_1\,\mathrm{d}b_1 \\
&\quad - \int_{R_+^2} K f_2(a,b,t) f_1(a_1,b_1,t)\,\mathrm{d}a_1\,\mathrm{d}b_1 \\
&\quad + \int_{R_+^2} K f_1(a,b,t) f_2(a_1,b_1,t)\,\mathrm{d}a_1\,\mathrm{d}b_1. \quad (17)
\end{aligned}
$$

In summary, we acquire the following Boltzmann system:

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t}\int_{R_+^2}\varphi(a,b) f_1(a,b,t)\,\mathrm{d}a\mathrm{d}b \\
= \int_{R_+^2}\varphi(a,b) Q_1(f_1,f_1)(a,b)\,\mathrm{d}a\mathrm{d}b \\
\quad + \int_{R_+^2}\varphi(a,b) Q_1(f_1,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b \\
\quad + \int_{R_+^2}\varphi(a,b) Q_1^{\mathrm{T}}(f_1,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b, \\
\dfrac{\mathrm{d}}{\mathrm{d}t}\int_{R_+^2}\varphi(a,b) f_2(a,b,t)\,\mathrm{d}a\mathrm{d}b \\
= \int_{R_+^2}\varphi(a,b) Q_2(f_2,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b \\
\quad + \int_{R_+^2}\varphi(a,b) Q_2(f_2,f_1)(a,b)\,\mathrm{d}a\mathrm{d}b \\
\quad + \int_{R_+^2}\varphi(a,b) Q_2^{\mathrm{T}}(f_1,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b.
\end{cases} \quad (18)
$$

4. Macroscopic equations for the number of agents and each good

In the dynamical model described in the preceding section, each transaction leads to the movement of goods and the transfer of people between the two groups of agents. In this part, we discuss the evolution of the density of agents in each group and the average number of goods they own.

We assume that the prices of goods A and B are equal in all cases; that is, the relative price $P_{ij}(t) = 1$ $(i,j = 1,2)$. Therefore, the original trading rules (10)–(12) should be amended accordingly. For the agents of group 1 (dealers), the trading rule when they deal with any object is simplified as

$$
\begin{cases}
a^* = a - \beta a + \alpha b + \mu_a a, \\
b^* = b - \alpha b + \beta a + \mu_b b,
\end{cases} \quad (19)
$$

which states that, for a dealer, the quantity of goods he owns after a single transaction is linearly dependent on the quantity before the transaction. In each interaction, the dealer sells α percentage of commodity B to purchase commodity A, and $\alpha + \mu_a$ is the relative loss (or gain) of commodity A. Similarly, the dealer sells β percentage of goods A in exchange for goods B, and $\beta + \mu_b$ is the relative loss (or gain) of goods B. For the agents of group 2 (speculators), the trading rule becomes

$$
\begin{cases}
a^* = a - \beta a + \alpha\dfrac{\lambda_b}{\lambda_a} b + \mu_a a, \\
b^* = b - \alpha b + \beta\dfrac{\lambda_a}{\lambda_b} a + \mu_b b.
\end{cases} \quad (20)
$$

Speculators only put commodity A with proportion λ_a and commodity B with proportion λ_b into the market; λ_b/λ_a is the speculators' trading strategy, which is represented by the proportion of speculators' investments in the two goods. At this point, the proportion of goods B sold by the agent in exchange for goods A in a single transaction becomes $\alpha\lambda_b/\lambda_a$, and the proportion of goods A sold by the agent in exchange for goods B becomes $\beta\lambda_a/\lambda_b$. Obviously, when $\lambda_a = \lambda_b = \lambda$, Eq. (20) is reduced to Eq. (19). At this time, the updating rules of the commodity quantities of the two groups of agents are the same.

By choosing the test function $\varphi(a,b) = 1$, the evolution equations for fractions of agents in each group are obtained from the weak form of Boltzmann equation (18). Obviously, the effect of a binary transaction vanishes at this time and the change of density of agents in each group depends only on the transfer operator $Q_i^{\mathrm{T}}(f_1,f_2)(a,b)$ with $i = 1,2$. Putting $\varphi(a,b) = 1$ in the corresponding Boltzmann equation, we obtain

$$
\begin{cases}
\dfrac{\mathrm{d}\rho_1}{\mathrm{d}t} = K(\rho_2 - \rho_1), \\
\dfrac{\mathrm{d}\rho_2}{\mathrm{d}t} = K(\rho_1 - \rho_2),
\end{cases} \quad (21)
$$

where ρ_1 and ρ_2 are the same as in formula (6). When $\mathrm{d}\rho_1/\mathrm{d}t = \mathrm{d}\rho_2/\mathrm{d}t = 0$, ρ_1 and ρ_2 reach a steady-state and no longer change with time. Combining with the condition $\rho_1 + \rho_2 = 1$, we obtain the unique equilibrium state

$$
(\rho_1)_\infty = (\rho_2)_\infty = \frac{1}{2}.
$$

That is, if agents in groups 1 and 2 transfer to each other with equal transfer frequency K, each group has the same number of agents when $t \to \infty$. This conclusion is consistent with that in Ref. [29].

Choosing the test functions $\varphi(a,b) = a$ and $\varphi(a,b) = b$, we explore the evolution of the average number of goods owned by agents of each group. Now, considering agents of group 1, the evolution of the number of commodities they own relies on both the binary transaction and the agent's transfer. Using the conditions $M_{1\mathrm{A}} + M_{2\mathrm{A}} = I_{\mathrm{A}}$, $M_{1\mathrm{B}} + M_{2\mathrm{B}} = I_{\mathrm{B}}$,

$I_{\rm A}+I_{\rm B}=I$ and $\rho_1+\rho_2=1$ ($M_{i{\rm A}}$, and $M_{i{\rm B}}$ ($i=1,2$) defined in formulas (7) and (8)), we have

$$\begin{aligned}
&\int_{R_+^2} aQ_1(f_1,f_1)(a,b)\,\mathrm{d}a\mathrm{d}b\\
&=\sigma\left\langle\int_{R_+^4}(a^*-a)f_1(a,b,t)f_1(a_1,b_1,t)\,\mathrm{d}\pi\right\rangle\\
&=\sigma\int_{R_+^4}(\alpha b-\beta a)f_1(a,b,t)f_1(a_1,b_1,t)\,\mathrm{d}\pi\\
&=\sigma\alpha M_{1{\rm B}}\rho_1-\sigma\beta M_{1{\rm A}}\rho_1,\\
&\int_{R_+^2} bQ_1(f_1,f_1)(a,b)\,\mathrm{d}a\mathrm{d}b\\
&=\sigma\left\langle\int_{R_+^4}(b^*-b)f_1(a,b,t)f_1(a_1,b_1,t)\,\mathrm{d}\pi\right\rangle\\
&=\sigma\int_{R_+^4}(\beta a-\alpha b)f_1(a,b,t)f_1(a_1,b_1,t)\,\mathrm{d}\pi\\
&=\sigma\beta M_{1{\rm A}}\rho_1-\sigma\alpha M_{1{\rm B}}\rho_1,\\
&\int_{R_+^2} aQ_1(f_1,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b\\
&=\sigma\left\langle\int_{R_+^4}(a^*-a)f_1(a,b,t)f_2(a_1,b_1,t)\,\mathrm{d}\pi\right\rangle\\
&=\sigma\int_{R_+^4}(\alpha b-\beta a)f_1(a,b,t)f_2(a_1,b_1,t)\,\mathrm{d}\pi\\
&=\sigma\alpha M_{1{\rm B}}\rho_2-\sigma\beta M_{1{\rm A}}\rho_2,\\
&\int_{R_+^2} bQ_1(f_1,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b\\
&=\sigma\left\langle\int_{R_+^4}(b^*-b)f_1(a,b,t)f_2(a_1,b_1,t)\,\mathrm{d}\pi\right\rangle\\
&=\sigma\int_{R_+^4}(\beta a-\alpha b)f_1(a,b,t)f_2(a_1,b_1,t)\,\mathrm{d}\pi\\
&=\sigma\beta M_{1{\rm A}}\rho_2-\sigma\alpha M_{1{\rm B}}\rho_2.
\end{aligned}$$

Due to the operator $Q_1^{\rm T}(f_1,f_2)(a,b)$, we obtain

$$\begin{aligned}
&\int_{R_+^2} aQ_1^{\rm T}(f_1,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b=K(I_{\rm A}-2M_{1{\rm A}}),\\
&\int_{R_+^2} bQ_1^{\rm T}(f_1,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b=K(I_{\rm B}-2M_{1{\rm B}}).
\end{aligned}$$

Therefore, the evolution equations for the average number of goods owned by agents in group 1 become the form

$$\begin{cases}
\dfrac{\mathrm{d}M_{1{\rm A}}}{\mathrm{d}t}=\sigma\alpha M_{1{\rm B}}-(\sigma\beta+2K)M_{1{\rm A}}+I_{\rm A}K,\\
\dfrac{\mathrm{d}M_{1{\rm B}}}{\mathrm{d}t}=\sigma\beta M_{1{\rm A}}-(\sigma\alpha+2K)M_{1{\rm B}}+I_{\rm B}K.
\end{cases}\tag{22}$$

We obtain the unique equilibrium state

$$(M_{1{\rm A}})_\infty=\frac{\sigma\alpha I+2I_{\rm A}K}{2\sigma+4K},\ \ (M_{1{\rm B}})_\infty=\frac{\sigma\beta I+2I_{\rm B}K}{2\sigma+4K}.$$

At steady state, the average amounts of goods A and B owned by agents in group 1 depend on the frequency of agents' transfer.

Similarly, for agents in group 2, the effects of binary interaction on the variation of goods' quantities are expressed as

$$\begin{aligned}
&\int_{R_+^2} aQ_2(f_2,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b\\
&=\sigma\left\langle\int_{R_+^4}(a^*-a)f_2(a,b,t)f_2(a_1,b_1,t)\,\mathrm{d}\pi\right\rangle\\
&=\sigma\int_{R_+^4}\left(\alpha\frac{\lambda_b}{\lambda_a}b-\beta a\right)f_2(a,b,t)f_2(a_1,b_1,t)\,\mathrm{d}\pi\\
&=\sigma\alpha\frac{\lambda_b}{\lambda_a}M_{2{\rm B}}\rho_2-\sigma\beta M_{2{\rm A}}\rho_2,\\
&\int_{R_+^2} bQ_2(f_2,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b\\
&=\sigma\left\langle\int_{R_+^4}(b^*-b)f_2(a,b,t)f_2(a_1,b_1,t)\,\mathrm{d}\pi\right\rangle\\
&=\sigma\int_{R_+^4}\left(\beta\frac{\lambda_a}{\lambda_b}a-\alpha b\right)f_2(a,b,t)f_2(a_1,b_1,t)\,\mathrm{d}\pi\\
&=\sigma\beta\frac{\lambda_a}{\lambda_b}M_{2{\rm A}}\rho_2-\sigma\alpha M_{2{\rm B}}\rho_2,\\
&\int_{R_+^2} aQ_2(f_2,f_1)(a,b)\,\mathrm{d}a\mathrm{d}b\\
&=\sigma\left\langle\int_{R_+^4}(a^*-a)f_2(a,b,t)f_1(a_1,b_1,t)\,\mathrm{d}\pi\right\rangle\\
&=\sigma\int_{R_+^4}\left(\alpha\frac{\lambda_b}{\lambda_a}b-\beta a\right)f_2(a,b,t)f_1(a_1,b_1,t)\,\mathrm{d}\pi\\
&=\sigma\alpha\frac{\lambda_b}{\lambda_a}M_{2{\rm B}}\rho_1-\sigma\beta M_{2{\rm A}}\rho_1,\\
&\int_{R_+^2} bQ_2(f_2,f_1)(a,b)\,\mathrm{d}a\mathrm{d}b\\
&=\sigma\left\langle\int_{R_+^4}(b^*-b)f_2(a,b,t)f_1(a_1,b_1,t)\,\mathrm{d}\pi\right\rangle\\
&=\sigma\int_{R_+^4}\left(\beta\frac{\lambda_a}{\lambda_b}a-\alpha b\right)f_2(a,b,t)f_1(a_1,b_1,t)\,\mathrm{d}\pi\\
&=\sigma\beta\frac{\lambda_a}{\lambda_b}M_{2{\rm A}}\rho_1-\sigma\alpha M_{2{\rm B}}\rho_1.
\end{aligned}$$

The effects of agents' transfer on the variation of quantities of commodities A and B are given as

$$\begin{aligned}
&\int_{R_+^2} aQ_2^{\rm T}(f_1,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b=K(I_{\rm A}-2M_{2{\rm A}}),\\
&\int_{R_+^2} bQ_2^{\rm T}(f_1,f_2)(a,b)\,\mathrm{d}a\mathrm{d}b=K(I_{\rm B}-2M_{2{\rm B}}).
\end{aligned}$$

Then, the equations for the average number of goods owned by agents in group 2 become

$$\begin{cases}
\dfrac{\mathrm{d}M_{2{\rm A}}}{\mathrm{d}t}=\sigma\alpha\dfrac{\lambda_b}{\lambda_a}M_{2{\rm B}}-(\sigma\beta+2K)M_{2{\rm A}}+I_{\rm A}K,\\
\dfrac{\mathrm{d}M_{2{\rm B}}}{\mathrm{d}t}=\sigma\beta\dfrac{\lambda_a}{\lambda_b}M_{2{\rm A}}-(\sigma\alpha+2K)M_{2{\rm B}}+I_{\rm B}K.
\end{cases}\tag{23}$$

The steady-state solutions of the quantities of goods owned by agents in group 2 are obtained by solving formula (23). Namely, we have

$$(M_{2\mathrm{A}})_\infty = \frac{\lambda_a I_\mathrm{A}\sigma\alpha + \lambda_b I_\mathrm{B}\sigma\alpha + 2\lambda_a I_\mathrm{A} K}{2\lambda_a\sigma + 4\lambda_a K},$$
$$(M_{2\mathrm{B}})_\infty = \frac{\lambda_b I_\mathrm{B}\sigma\beta + \lambda_a I_\mathrm{A}\sigma\beta + 2\lambda_b I_\mathrm{B} K}{2\lambda_b\sigma + 4\lambda_b K}.$$

The quantities of commodities A and B owned by agents in group 2 at steady state depend on both the speculators' trading strategies and the frequency of agent transfer. Consider the case where the agent has the same propensity to trade for goods A and B ($\lambda_a = \lambda_b = \lambda$). At this time, we have

$$(M_{2\mathrm{A}})_\infty = \frac{\sigma\alpha I + 2I_\mathrm{A} K}{2\sigma + 4K},\ \ (M_{2\mathrm{B}})_\infty = \frac{\sigma\beta I + 2I_\mathrm{B} K}{2\sigma + 4K}.$$

Obviously, both groups of agents have equal equilibrium state quantities of goods A and B when $\lambda_a = \lambda_b = \lambda$; that is, there are $(M_{1\mathrm{A}})_\infty = (M_{2\mathrm{A}})_\infty$ and $(M_{1\mathrm{B}})_\infty = (M_{2\mathrm{B}})_\infty$.

5. Continuous trading limit and Fokker–Planck approximation

5.1. The Fokker–Planck equations of the distribution function

According to Bisi,[29] it is difficult to derive the analytic expressions of the distribution function from the Boltzmann equation (18). The usual practice is to use the "continuous trading limit" methods to derive the Fokker–Planck equation. The principle of the "continuous trading limit" methods is that the amount of wealth exchanged each time may be very small; that is, the difference in the amount of wealth of the agent before and after the transaction is negligible. Thus, the entire market is viewed as consisting of an infinite number of small binary transactions and therefore is considered continuous. The "continuous trading limit" is achieved by introducing the scaling

$$\alpha \to \varepsilon\alpha,\ \ \beta \to \varepsilon\beta,\ \ \mu_a \to \sqrt{\varepsilon}\mu_a,\ \ \mu_b \to \sqrt{\varepsilon}\mu_b,$$

where ε is a small positive parameter. Now, we derive the evolution equation of the dealer's wealth distribution $f_1(a,b,t)$. From the trading rule (19), we acquire

$$\langle a^* - a\rangle = \varepsilon(\alpha b - \beta a),\ \ \langle b^* - b\rangle = \varepsilon(\beta a - \alpha b),$$
$$\langle (a^* - a)^2\rangle = \varepsilon^2(\alpha b - \beta a)^2 + \varepsilon\gamma_\mu a^2,$$
$$\langle (b^* - b)^2\rangle = \varepsilon^2(\beta a - \alpha b)^2 + \varepsilon\gamma_\mu b^2,$$
$$\langle (a^* - a)(b^* - b)\rangle = -\varepsilon^2(\alpha b - \beta a)^2.$$

The second-order Taylor expansion of the test function $\varphi(a^*, b^*)$ around (a,b) possesses the form

$$\begin{aligned}
&\langle \varphi(a^*,b^*) - \varphi(a,b)\rangle \\
&= \varepsilon\left[(\alpha b - \beta a)\frac{\partial\varphi}{\partial a} + (\beta a - \alpha b)\frac{\partial\varphi}{\partial b}\right. \\
&\quad \left. + \frac{\gamma_\mu a^2}{2}\frac{\partial^2\varphi}{\partial a^2} + \frac{\gamma_\mu b^2}{2}\frac{\partial^2\varphi}{\partial b^2}\right] + \langle R_\varepsilon(a,b)\rangle, \qquad (24)
\end{aligned}$$

where $R_\varepsilon(a,b)$ represents the remainder term including ε^n ($n \geq 2$).

Our goal is to study the long-term behavior of the distribution function. Thus, we scale the time such that $\tau = \varepsilon t$. The scaled distribution function $g_1(a,b,\tau) = f_1(a,b,t)$ satisfies the Boltzmann equation

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+^2} \varphi(a,b) g_1(a,b,\tau)\,\mathrm{d}a\mathrm{d}b \\
&= \frac{1}{\varepsilon}\int_{R_+^2} \varphi(a,b) Q_1(g_1,g_1)(a,b)\,\mathrm{d}a\mathrm{d}b \\
&\quad + \frac{1}{\varepsilon}\int_{R_+^2} \varphi(a,b) Q_1(g_1,g_2)(a,b)\,\mathrm{d}a\mathrm{d}b \\
&\quad + \int_{R_+^2} \varphi(a,b) Q_1^\mathrm{T}(g_1,g_2)(a,b)\,\mathrm{d}a\mathrm{d}b. \qquad (25)
\end{aligned}$$

We substitute formula (24) into the Boltzmann equation (25). The first and second items on the right-hand side of the formula (25) become

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[\int_{R_+^2} \varphi(a,b) Q_1(g_1,g_1)(a,b)\,\mathrm{d}a\mathrm{d}b\right. \\
&\quad \left. + \int_{R_+^2} \varphi(a,b) Q_1(g_1,g_2)(a,b)\,\mathrm{d}a\mathrm{d}b\right] \\
&= \sigma\int_{R_+^4}\left[(\alpha b - \beta a)\frac{\partial\varphi}{\partial a} + (\beta a - \alpha b)\frac{\partial\varphi}{\partial b}\right. \\
&\quad \left. + \frac{1}{2}\gamma_\mu a^2\frac{\partial^2\varphi}{\partial a^2} + \frac{1}{2}\gamma_\mu b^2\frac{\partial^2\varphi}{\partial b^2}\right] g_1(\tau) g_1(\tau)\,\mathrm{d}\pi \\
&\quad + \sigma\int_{R_+^4}\left[(\alpha b - \beta a)\frac{\partial\varphi}{\partial a} + (\beta a - \alpha b)\frac{\partial\varphi}{\partial b} + \frac{1}{2}\gamma_\mu a^2\frac{\partial^2\varphi}{\partial a^2}\right. \\
&\quad \left. + \frac{1}{2}\gamma_\mu b^2\frac{\partial^2\varphi}{\partial b^2}\right] g_1(\tau) g_2(\tau)\,\mathrm{d}\pi + \overline{R}_\varepsilon(\varphi), \qquad (26)
\end{aligned}$$

where

$$\begin{aligned}
\overline{R}_\varepsilon(\varphi) &= \frac{\sigma}{\varepsilon}\left[\int_{R_+^4} \langle R_\varepsilon(a,b)\rangle g_1(\tau) g_1(\tau)\,\mathrm{d}\pi\right. \\
&\quad \left. + \int_{R_+^4} \langle R_\varepsilon(a,b)\rangle g_1(\tau) g_2(\tau)\,\mathrm{d}\pi\right].
\end{aligned}$$

Since $R_\varepsilon(a,b)$ contains ε^n ($n \geq 2$), when $\varepsilon \to 0$, $\overline{R}_\varepsilon(\varphi) \to 0$. Meanwhile, the last item on the right-hand side of the formula (25) becomes

$$\begin{aligned}
&\int_{R_+^2} \varphi(a,b) Q_1^\mathrm{T}(g_1,g_2)(a,b)\,\mathrm{d}a\mathrm{d}b \\
&= -\int_{R_+^2} \varphi(a,b) K\rho_1 g_1(\tau)\,\mathrm{d}a\mathrm{d}b + \int_{R_+^2} \varphi(a,b) K\rho_2 g_2(\tau)\,\mathrm{d}a\mathrm{d}b \\
&\quad + \int_{R_+^2} \varphi(a,b) K\rho_1 g_2(\tau)\,\mathrm{d}a\mathrm{d}b - \int_{R_+^2} \varphi(a,b) K\rho_2 g_1(\tau)\,\mathrm{d}a\mathrm{d}b
\end{aligned}$$

$$= -\int_{R_+^2} \varphi(a,b) K g_1(\tau) \mathrm{d}a\mathrm{d}b + \int_{R_+^2} \varphi(a,b) K g_2(\tau) \mathrm{d}a\mathrm{d}b. \quad (27)$$

We substitute Eqs. (26) and (27) into Eq. (25), and pass the derivative of the test function to the distribution function by using integration by parts and acquire the evolution equation of $g_1(a,b,\tau)$

$$\begin{aligned}\frac{\partial g_1}{\partial \tau} &= \sigma\Bigg[(\beta M_{1\mathrm{A}} - \alpha M_{1\mathrm{B}})\frac{\partial g_1}{\partial a} + (\alpha M_{1\mathrm{B}} - \beta M_{1\mathrm{A}})\frac{\partial g_1}{\partial b} \\ &\quad + \frac{\rho_1\gamma_\mu}{2}\frac{\partial^2(a^2 g_1)}{\partial a^2} + \frac{\rho_1\gamma_\mu}{2}\frac{\partial^2(b^2 g_1)}{\partial b^2} \\ &\quad + (\beta M_{1\mathrm{A}} - \alpha M_{1\mathrm{B}})\frac{\partial g_2}{\partial a} + (\alpha M_{1\mathrm{B}} - \beta M_{1\mathrm{A}})\frac{\partial g_2}{\partial b} \\ &\quad + \frac{\rho_1\gamma_\mu}{2}\frac{\partial^2(a^2 g_2)}{\partial a^2} + \frac{\rho_1\gamma_\mu}{2}\frac{\partial^2(b^2 g_2)}{\partial b^2}\Bigg] + Kg_2 - Kg_1. \quad (28)\end{aligned}$$

Using the same method, we obtain the evolution equation of $g_2(a,b,\tau)$

$$\begin{aligned}\frac{\partial g_2}{\partial \tau} &= \sigma\Bigg[(\beta M_{2\mathrm{A}} - \frac{\lambda_b}{\lambda_a}\alpha M_{2\mathrm{B}})\frac{\partial g_2}{\partial a} + (\alpha M_{2\mathrm{B}} - \frac{\lambda_a}{\lambda_b}\beta M_{2\mathrm{A}})\frac{\partial g_2}{\partial b} \\ &\quad + \frac{\rho_2\gamma_\mu}{2}\frac{\partial^2(a^2 g_2)}{\partial a^2} + \frac{\rho_2\gamma_\mu}{2}\frac{\partial^2(b^2 g_2)}{\partial b^2} \\ &\quad + (\beta M_{2\mathrm{A}} - \frac{\lambda_b}{\lambda_a}\alpha M_{2\mathrm{B}})\frac{\partial g_1}{\partial a} + (\alpha M_{2\mathrm{B}} - \frac{\lambda_a}{\lambda_b}\beta M_{2\mathrm{A}})\frac{\partial g_1}{\partial b} \\ &\quad + \frac{\rho_2\gamma_\mu}{2}\frac{\partial^2(a^2 g_1)}{\partial a^2} + \frac{\rho_2\gamma_\mu}{2}\frac{\partial^2(b^2 g_1)}{\partial b^2}\Bigg] + Kg_1 - Kg_2. \quad (29)\end{aligned}$$

The second-order partial derivative terms on the right-hand side of the Fokker–Planck equations (28) and (29) measure the diffusion of the quantities of goods due to uncertainty and are called diffusion terms. The first-order partial derivative terms are called the drift terms, which cause the quantities of goods of agents in each group to move towards a suitable weighted average. According to the statements in Ref. [29], it is difficult to compute the steady solutions of Eqs. (28) and (29). Therefore, some restrictions need to be placed on the equations.

In the economic activities of a real society, people often think that putting all the commodities into the trade increases the risk. Thus, they have a certain tendency to save. Suppose that the government encourages agents to save; that is, only dealers are allowed to become speculators and no transfers are allowed in the opposite direction. In other words, unlike the case where there are two-way transfers of agents in the previous sections, only one-way transfers of agents are allowed here. Thus, we have $(\rho_1)_\infty = 0$, $(M_{1\mathrm{A}})_\infty = 0$, $(M_{1\mathrm{B}})_\infty = 0$, $(\rho_2)_\infty = 1$ and $(M_{2\mathrm{A}})_\infty + (M_{2\mathrm{B}})_\infty = I$. Therefore, the steady distribution $(g_1)_\infty$ vanishes when $(g_2)_\infty$ is given by the equation

$$\begin{aligned}\frac{\partial g_2}{\partial t} &= \sigma\Bigg[(\beta M_{2\mathrm{A}} - \frac{\lambda_b}{\lambda_a}\alpha M_{2\mathrm{B}})\frac{\partial g_2}{\partial a} + (\alpha M_{2\mathrm{B}} - \frac{\lambda_a}{\lambda_b}\beta M_{2\mathrm{A}})\frac{\partial g_2}{\partial b} \\ &\quad + \frac{\gamma_\mu}{2}\frac{\partial^2(a^2 g_2)}{\partial a^2} + \frac{\gamma_\mu}{2}\frac{\partial^2(b^2 g_2)}{\partial b^2}\Bigg] - Kg_2. \quad (30)\end{aligned}$$

When $\lambda_a = \lambda_b = \lambda$, the evolution equation (30) becomes

$$\begin{aligned}\frac{\partial g_2}{\partial t} &= \sigma\Bigg[(\beta M_{2\mathrm{A}} - \alpha M_{2\mathrm{B}})\frac{\partial g_2}{\partial a} + (\alpha M_{2\mathrm{B}} - \beta M_{2\mathrm{A}})\frac{\partial g_2}{\partial b} \\ &\quad + \frac{\gamma_\mu}{2}\frac{\partial^2(a^2 g_2)}{\partial a^2} + \frac{\gamma_\mu}{2}\frac{\partial^2(b^2 g_2)}{\partial b^2}\Bigg] - Kg_2.\end{aligned}$$

5.2. A solvable case

In what follows, we consider a case where there is no random influence ($\gamma_\mu = 0$). As a result, the diffusion term in Eq. (30) disappears and Eq. (30) is simplified to

$$\begin{aligned}\frac{\partial g_2}{\partial t} &= \sigma\Bigg[(\beta M_{2\mathrm{A}} - \frac{\lambda_b}{\lambda_a}\alpha M_{2\mathrm{B}})\frac{\partial g_2}{\partial a} \\ &\quad + (\alpha M_{2\mathrm{B}} - \frac{\lambda_a}{\lambda_b}\beta M_{2\mathrm{A}})\frac{\partial g_2}{\partial b}\Bigg] - Kg_2.\end{aligned}$$

When $t \to \infty$, we obtain the following partial differential equation:

$$\begin{aligned}&\sigma\Bigg[(\beta M_{2\mathrm{A}} - \frac{\lambda_b}{\lambda_a}\alpha M_{2\mathrm{B}})\frac{\partial g_2}{\partial a} \\ &+ \left(\alpha M_{2\mathrm{B}} - \frac{\lambda_a}{\lambda_b}\beta M_{2\mathrm{A}}\right)\frac{\partial g_2}{\partial b}\Bigg] - Kg_2 = 0. \quad (31)\end{aligned}$$

The two commodities are homogeneous except that the agent has different tendencies to trade them ($\lambda_a \neq \lambda_b$). Therefore, the steady wealth distribution has two symmetrical forms. According to Polyanin,[31] we solve Eq. (31) and acquire the steady solution

$$\begin{aligned}(g_2)_\infty &= \exp\left\{\frac{K}{\sigma\left(\beta M_{2\mathrm{A}} - \frac{\lambda_b}{\lambda_a}\alpha M_{2\mathrm{B}}\right)}a\right\}\Phi\left(A(a,b)\right) \\ &= \exp\left\{\frac{K}{\sigma\left(\alpha M_{2\mathrm{B}} - \frac{\lambda_a}{\lambda_b}\beta M_{2\mathrm{A}}\right)}b\right\}\Phi\left(A(a,b)\right),\end{aligned}$$

where $\Phi(A(a,b))$ is an arbitrary function of $A(a,b)$ and $A(a,b) = \sigma(\alpha M_{2\mathrm{B}} - \frac{\lambda_a}{\lambda_b}\beta M_{2\mathrm{A}})a - \sigma(\beta M_{2\mathrm{A}} - \frac{\lambda_b}{\lambda_a}\alpha M_{2\mathrm{B}})b$. Obviously, the steady-state wealth distribution of the agent in group 2 (speculator) depends on the trading strategies λ_b/λ_a (the ratio of the trading propensity between two goods) and transfer frequency K. When $\lambda_a = \lambda_b = \lambda$, the influence of the trading strategies λ_b/λ_a vanishes and the steady-state wealth distribution only depends on the transfer frequency K. In order to study the effects of trading strategies on the wealth distribution of agents, the following work mainly considers the case $\lambda_a \neq \lambda_b$ and the case $\lambda_a = \lambda_b = \lambda$ is easily derived.

To analyze the influence of agent transfer on the evolution of wealth distribution, we write the Fokker–Planck equation

that does not involve the agent's transfer; that is, formula (31) is reduced to

$$\sigma\left[\left(\beta M_{2\mathrm{A}}-\frac{\lambda_b}{\lambda_a}\alpha M_{2\mathrm{B}}\right)\frac{\partial\widetilde{g_2}}{\partial a}+\left(\alpha M_{2\mathrm{B}}-\frac{\lambda_a}{\lambda_b}\beta M_{2\mathrm{A}}\right)\frac{\partial\widetilde{g_2}}{\partial b}\right]=0. \tag{32}$$

According to Polyanin,[31] the steady solution of formula (32) is

$$(\widetilde{g_2})_\infty=\Psi\left(A(a,b)\right),$$

where $\Psi(A(a,b))$ is an arbitrary function of $A(a,b)$.

To compare $(g_2)_\infty$ and $(\widetilde{g_2})_\infty$, we take the same form for $\Phi(A(a,b))$ and $\Psi(A(a,b))$ and find that the influence of agent transfer on steady wealth distribution is described by $T_1(a)$ and $T_2(b)$, where $T_1(a)$ and $T_2(b)$ are defined by

$$T_1(a)=\frac{(g_2)_\infty}{(\widetilde{g_2})_\infty}=\exp\left\{\frac{K}{\sigma\left(\beta M_{2\mathrm{A}}-\frac{\lambda_b}{\lambda_a}\alpha M_{2\mathrm{B}}\right)}a\right\},$$

$$T_2(b)=\frac{(g_2)_\infty}{(\widetilde{g_2})_\infty}=\exp\left\{\frac{K}{\sigma\left(\alpha M_{2\mathrm{B}}-\frac{\lambda_a}{\lambda_b}\beta M_{2\mathrm{A}}\right)}b\right\}. \tag{33}$$

The specific effects will be analyzed in the numerical experiments in Subsection 5.3.

5.3. Numerical experiment

In this section, we perform two tests. Test 1 discusses the effects of trading strategies and transfer frequency on steady wealth distribution in the above special case (see Subsection 5.2). We compare the steady wealth distribution of the speculator by taking different values of λ_a, λ_b and K. In test 2, the solution of Eq. (30) at $t\to\infty$ is acquired through numerical experiment and the image of the steady wealth distribution obtained in this case is drawn.

5.3.1. Test 1: The influence of agent transfer on steady wealth distribution

In this test, we want to observe the change in steady wealth distribution due to the addition of the agent transfer operator in the Boltzmann equation. We consider the simplified case where $\sigma\beta M_{2\mathrm{A}}=\sigma\alpha M_{2\mathrm{B}}=1$ and $K=0.3$ in formula (33). Compared with the case where the agent's transfer is not considered, the effects of agent transfer on the steady wealth distribution are characterized by $T_1(a)$ and $T_2(b)$.

We let $\lambda_a<\lambda_b$ and plot the graphs of $T_1(a)$ and $T_2(b)$ in Fig. 1. The meaning of $\lambda_a<\lambda_b$ is that the agent prefers to trade commodity B and hold commodity A. In other words, agents believe that goods A have a higher value and they prefer to hold goods A than sell it. Therefore, when $\lambda_a<\lambda_b$, we focus on analyzing the influence of $T_1(a)$ on steady-state wealth distribution. From Fig. 1, we know that $T_1(a)\le 1$ and $T_1(a)$ decreases with the increase of the quantity of goods A. When the quantity $a\to+\infty$, $T_1(a)\to 0$. According to $(g_2)_\infty=T_1(a)\cdot(\widetilde{g_2})_\infty$, we conclude that the tail of steady-state wealth distribution with agent transfer becomes thinner than that of wealth distribution without agent transfer at the high wealth level. In other words, the transfer of individuals reduces the probability of the emergence of extremely rich people, which is conducive to the equality of wealth distribution. In addition, when the gap between λ_a and λ_b is different, the influence of agent transfer on steady wealth distribution is also different. The smaller the gap between λ_a and λ_b, the smaller the value of $T_1(a)$. In the commodity exchange market, allowing agents to transfer helps to improve the unequal distribution of wealth. Thus, it is required that the gap between agent's trading propensities for different commodities is small. When $\lambda_a>\lambda_b$, we can easily reach a similar conclusion by analyzing the influence of $T_2(b)$ on steady-state wealth distribution.

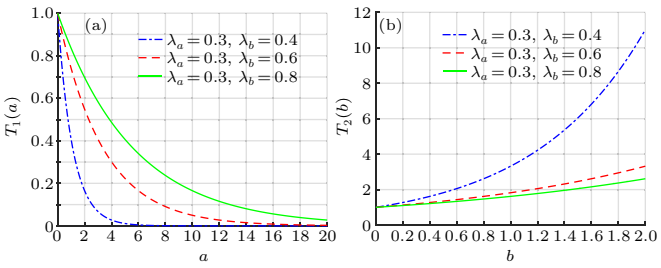


Fig. 1. Test 1: Comparison of the effects of agent transfer on wealth distribution under different values of λ_a and λ_b. (Blue) $\lambda_a=0.3$ and $\lambda_b=0.4$. (Red) $\lambda_a=0.3$ and $\lambda_b=0.6$. (Green) $\lambda_a=0.3$ and $\lambda_b=0.8$. In each case, we fix $\sigma\beta M_{2\mathrm{A}}=\sigma\alpha M_{2\mathrm{B}}=1$ and $K=0.3$.

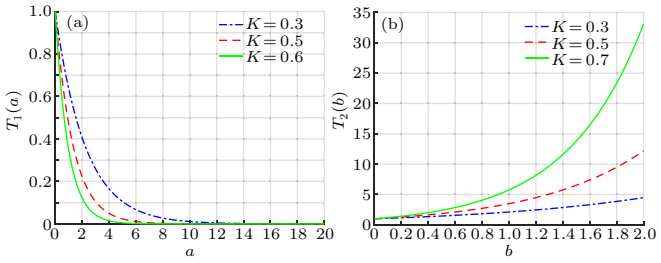


Fig. 2. Test 1: Comparison of the effects of agent transfer on wealth distribution under different values of K. (Blue) $K=0.3$. (Red) $K=0.5$. (Green) $K=0.7$. In each case, we fix $\sigma\beta M_{2\mathrm{A}}=\sigma\alpha M_{2\mathrm{B}}=1$, $\lambda_a=0.3$ and $\lambda_b=0.5$.

According to the expressions of $T_1(a)$ and $T_2(b)$, the frequency K of the agent's transfer also affects the steady-state wealth distribution. Therefore, holding $\lambda_a<\lambda_b$, we take different values of K to plot the corresponding images of $T_1(a)$ and $T_2(b)$ in Fig. 2. It can be seen from the trend of $T_1(a)$ that when the frequency of transfer increases, the speed of decay of $T_1(a)$ increases, meaning that the effect of individual transfer on wealth distribution is enhanced. In conclusion, a high transfer frequency is conducive to the equalization of wealth distribution.

5.3.2. Test 2: Steady wealth distribution in the general case

For Eq. (30), it is difficult to solve the steady solution when $t \to \infty$. Therefore, we find its numerical solution and draw a graph of $g_\infty(a,b)$. In this test, we fix $\gamma_\mu = 0.5$, $\sigma\beta M_{2A} = \sigma\alpha M_{2B} = 5$, $\lambda_a = 0.2$, $\lambda_b = 0.7$ and $K = 1$. From Fig. 3, we know that the numerical solution depends on both variables a and b. As the variables a and b increase, the wealth distribution $g_\infty(a,b)$ increases first and then decreases. Finally, the wealth distribution $g_\infty(a,b)$ approaches 0. The trend of the numerical solution indicates that a small number of people possess the most wealth. Employing actual data (source: the World Inequality Database), we plot Fig. 4, where the curves depict the change of social wealth held by the richest 10% of the population (blue) and the poorest 50% (red) in the world's major economies from 1995 to 2022. The distance between the blue and red curves in each image represents the degree of wealth inequality in the economy. It is worth noting that the gap between the rich and poor is large as shown in Fig. 4. The level of wealth inequality in China and India has increased over the past two decades. The actual case in Fig. 4 demonstrates that the trend of our numerical solution $g_\infty(a,b)$ fits the real economy.

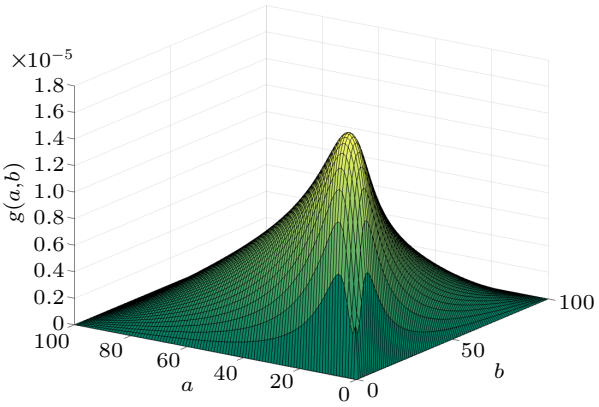


Fig. 3. Test 2: Steady wealth distribution in the general case where we fix $\gamma_\mu = 1$, $\sigma\beta M_{2A} = \sigma\alpha M_{2B} = 5$, $\lambda_a = 0.2$, $\lambda_b = 0.7$ and $K = 1$.

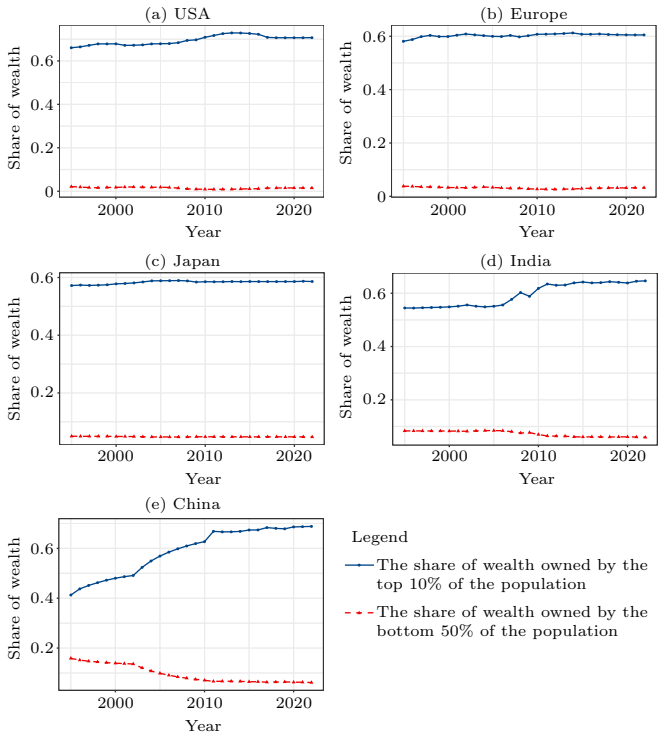


Fig. 4. Wealth inequality in the world's major economies (1995–2022).

6. Final remarks

In this work, we utilize the kinetic theory of rarefied gas to study the wealth distribution in a goods exchange market that allows agents to transfer between groups 1 and 2. The evolution of the wealth distribution function of the agent in each group depends on both binary interactions and agent transfer. Suppose that agents in the market have the same preference for two kinds of commodities, which is described by the Cobb–Douglas utility function. The evolution processes of the wealth distribution of agents in the two groups are described by different Boltzmann equations. In the Boltzmann system, agents are allowed to employ their transaction strategies to improve utilities. Specifically, we suppose that agents in group 1 (dealers) place all their commodities in the trade, while agents in group 2 (speculators) place only a portion of their commodities in the trade. In other words, speculators maximize their utilities by choosing the proportion of commodities they trade. In addition, the Boltzmann equations also contain the possible transfer of agents. Starting from the Boltzmann system, we derive macroscopic equations for the fractions of agents of each group and the average quantities of two goods they own. From the equilibrium solutions, it is seen that if there are two-way transfers of agents with equal transfer frequency K, the fractions of agents are the same; that is, we have $\rho_1 = \rho_2 = 1/2$ when $t \to \infty$. In addition, we acquire the average quantities of two goods in the two agent groups and find that the average quantities of dealers rely on transfer frequency, while the average quantities of speculators rely on trading strategies and transfer frequency. Finally, we derive equations of Fokker–Planck type by the "continuous trading limit" process and obtain a steady solution for a special case. In numerical experiments, we perform two tests. Test 1 discusses the effects of trading strategies and transfer frequency on steady wealth distribution. Through test 2, we plot the curves of the numerical solution of the Fokker–Planck equation (30) and find that the wealth distribution increases first and subsequently decreases with the increase of variables a and b. This paper draws conclusions that the transfer of agents (the adjustment of agent's trading strategy) reduces the inequality of wealth distribution. Narrowing the gap between agents' trading propensities for different goods and increasing the frequency of agent transfer would enhance the equalization of wealth distribution.

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