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Hengchun Hu(胡恒春)† and Xu Xu(徐旭)

Citation: Chin. Phys. B, 2025, 34 (**3**): 030501. DOI: 10.1088/1674-1056/ada43b
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Abundant invariant solutions of (3+1)-dimensional combined pKP-BKP equation

Hengchun Hu(胡恒春)[†] and Xu Xu(徐旭)

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, *China*

(Received 12 November 2024; accepted manuscript online 31 December 2024)

Lie symmetry analysis is applied to a (3+1)-dimensional combined potential Kadomtsev–Petviashvili equation with B-type Kadomtsev–Petviashvili equation (pKP-BKP equation) and the corresponding similarity reduction equations are obtained with the different infinitesimal generators. Invariant solutions with arbitrary functions and constants for the (3+1)-dimensional pKP-BKP equation, including the lump solution, the periodic-lump solution, the two-kink solution, the breather solution and the lump–two-kink solution, have been studied analytically and graphically.

Keywords: (3+1)-dimensional combined pKP-BKP equation, Lie symmetry, invariant solutions

PACS: 05.45.Yv, 02.30.Ik, 02.30.Jr

DOI: 10.1088/1674-1056/ada43b **CSTR:** 32038.14.CPB.ada43b

1. Introduction

Nonlinear evolution equations are widely used in the fields of fluid dynamics, optical fibers, plasma dynamics, etc. With the development of soliton theory, scientists have obtained methods to solve these nonlinear partial differential equations, such as Darboux transformation, Hirota direct method, Painlevé analysis method, classical and non-classical Lie group transformation methods, tanh expansion method and so on.[1–14] The Lie group method is one of the most significant methods in soliton theory to study the exact solution of nonlinear equations. The main idea behind the Lie group method is that when one parameter group of transformation acts on the space of the independent and dependent variables, it leaves the partial differential equations in the original state. In recent decades, the Lie group method has been applied to obtain exact closed-form solutions of many partial differential equations, such as the KdV equation, Burgers equation, KP equation and so on.[15–22]

The well-known KP equation

$$(u_t + 6uu_x + u_{xxx})_x + au_{yy} = 0, \tag{1}$$

describes the evolution of nonlinear, long waves of small amplitude with slow dependence on the transverse coordinate,[23] which is one of the famous integrable equations in soliton theory. Setting $u = u_x$ in Eq. (1) and integrating x once, one can have the potential KP equation

$$u_{xt} + 6u_xu_{xx} + u_{xxxx} + au_{yy} = 0. \tag{2}$$

Moreover, the (2+1)-dimensional KP equation associated with B-type infinite dimensional Lie algebra reads

$$\left(15(u_x)^3 + 15u_xu_{xxx} + u_{xxxxx}\right)_x + 5\left(u_{xxxy} + 3\left(u_xu_y\right)_x\right) + u_{xt} - u_{yy} = 0, \tag{3}$$

which describes the interactions between exponentially localized structures and Eq. (3) has been used as a model for the shallow water wave in fluids and the electrostatic wave potentials in plasmas.[24] In Ref. [25], a combined form of the pKP Eq. (2) and BKP Eq. (3) is called pKP-BKP equation and given in the form

$$\begin{aligned} &a_1\left(15u_x^3 + 15u_xu_{xxx} + u_{xxxxx}\right)_x + a_2\left(6u_xu_{xx} + u_{xxxx}\right) \\ &+ a_3\left(u_{xxxy} + 3\left(u_xu_y\right)_x\right) + a_4u_{xx} + a_5u_{xt} + a_6u_{yy} = 0, \end{aligned} \tag{4}$$

where a_i $(i = 1,\ldots,6)$ are arbitrary constants with $a_5 \neq 0$. This newly proposed Eq. (4) has been studied extensively to admit multiple soliton solutions, kink wave, lump solution, M-breather and periodic travelling wave solutions in Refs. [25–29].

Very recently, the author introduced a combined version of pKP-BKP equation in (3+1)-dimension

$$\begin{aligned} &u_{xt} + \alpha\left(15u_x^3 + 15u_xu_{xxx} + u_{xxxxx}\right)_x \\ &+ \beta\left(6u_xu_{xx} + u_{xxxx}\right) + \gamma(u_{xxxy} + 3(u_xu_y)_x) \\ &+ au_{xx} + bu_{xy} + cu_{xz} + \mu u_{yy} = 0, \end{aligned} \tag{5}$$

where some additional terms, namely, bu_{xy} and cu_{xz}, are added to the pKP-BKP Eq. (4) with $a_5 = 1$.[30] This equation shows the double effect of both the pKP equation and the BKP equation, which is used to reveal nonlinear fluctuation phenomena in nonlinear optics, plasma physics and hydrodynamics.[31,32] When $\mu = -\frac{\gamma^2}{5\alpha}$, $\alpha \neq 0$, it has been demonstrated that Eq. (5) is integrable in the Painlevé sense and the multiple soliton solutions, lump solutions have been derived by the simplified Hirota bilinear method.[30] Breathing wave solutions, multiple soliton fission and multiple wave interaction solutions,

[†]Corresponding author. E-mail: hhengchun@163.com

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and Y-type soliton solutions have been obtained.[31–34] Moreover, two special (3+1)-dimensional pKP-BKP equations are also studied in Ref. [30] by eliminating some elements from Eq. (5).

In this paper, we aim to find many different similarity solutions with arbitrary functions and constants of the special pKP-BKP equation by fixing the constants $\alpha = 1/5$, $\beta = \gamma = \mu = a = b = c = 1$ for simplicity

$$\begin{aligned} & u_{xt} + \left(3u_x^3 + 3u_x u_{xxx} + \frac{1}{5}u_{xxxxx}\right)_x \\ & + 6u_x u_{xx} + u_{xxxx} + u_{xxxy} \\ & + 3\left(u_x u_y\right)_x + u_{xx} + u_{xy} + u_{xz} + u_{yy} = 0. \end{aligned} \tag{6}$$

To the best of our knowledge, the Lie group analysis and abundant similarity solutions to Eq. (5) have not been reported in the literature. We will use the classical Lie group method to obtain various similarity solutions with arbitrary functions, including the lump solution, the lump–kink solution, the lump–two-kink solution, the two-kink solution, the interaction between solitary and periodic wave solution, the periodic lump, the periodic kink and the breather solution in different coordinate planes.

The paper is organized as follows. In Section 2, we derive the corresponding infinitesimal generators with arbitrary functions through Lie group analysis for the pKP-BKP Eq. (6). In Section 3, various types of reduction equations for Eq. (6) are obtained based on different vector fields. Some different similarity solutions with arbitrary constants and functions are derived by resolving these reduction equations. Some interesting similarity solutions are also exhibited graphically by selecting appropriate arbitrary functions. Finally, summary and discussion are provided in Section 4.

2. Lie symmetry analysis of Eq. (6)

In this section, the classical Lie symmetry method is applied to Eq. (6) and the corresponding infinitesimal generators of Eq. (6) are obtained directly. First of all, we make a one-parameter Lie group transformation for Eq. (6)

$$\begin{aligned} & x \to x + \varepsilon\xi, \quad y \to y + \varepsilon\eta, \quad z \to z + \varepsilon\omega, \\ & t \to t + \varepsilon\tau, \quad u \to u + \varepsilon\Phi, \end{aligned} \tag{7}$$

where ε is a small parameter. The infinitesimal generators $\{\xi, \eta, \omega, \tau, \Phi\}$ are undetermined functions depending on x, y, z, t, and u. The corresponding vector field

$$\boldsymbol{V} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial t} + \Phi\frac{\partial}{\partial u} \tag{8}$$

is a Lie point symmetry of Eq. (6) if and only if

$$\mathrm{Pr}^{(6)}\boldsymbol{V}(\Delta)|_{\Delta=0} = 0, \tag{9}$$

where $\mathrm{Pr}^{(6)}\boldsymbol{V}$ is the sixth prolongation of Eq. (8) as follows:

$$\begin{aligned} \mathrm{Pr}^{(6)}\boldsymbol{V} = {} & \boldsymbol{V} + \Phi^x\frac{\partial}{\partial u_x} + \Phi^y\frac{\partial}{\partial u_y} + \Phi^{xx}\frac{\partial}{\partial u_{xx}} \\ & + \Phi^{xy}\frac{\partial}{\partial u_{xy}} + \Phi^{xz}\frac{\partial}{\partial u_{xz}} + \Phi^{xt}\frac{\partial}{\partial u_{xt}} + \Phi^{yy}\frac{\partial}{\partial u_{yy}} \\ & + \Phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \Phi^{xxxx}\frac{\partial}{\partial u_{xxxx}} \\ & + \Phi^{xxxy}\frac{\partial}{\partial u_{xxxy}} + \Phi^{xxxxxx}\frac{\partial}{\partial u_{xxxxxx}}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \Delta = {} & u_{xt} + \left(3u_x^3 + 3u_x u_{xxx} + \frac{1}{5}u_{xxxxx}\right)_x \\ & + 6u_x u_{xx} + u_{xxxx} + u_{xxxy} + 3\left(u_x u_y\right)_x \\ & + u_{xx} + u_{xy} + u_{xz} + u_{yy}. \end{aligned} \tag{11}$$

Then the Lie invariant condition can be obtained as

$$\begin{aligned} & \frac{1}{5}\Phi^{xxxxx} + \Phi^{xxxy} + \Phi^{xxxx} \\ & + \left(3\Phi^{xxx} + 6\Phi^x + 3\Phi^y + 18\Phi^x u_x\right)u_{xx} \\ & + 3\Phi^x u_{xxxx} + 3\Phi^x u_{xy} + 3\Phi^{xx}u_{xxx} \\ & + \left(6\Phi^{xx} + 3\Phi^{xy} + 9\Phi^{xx}u_x + 3\Phi^{xxxx}\right)u_x \\ & + 3\Phi^{xx}u_y + \Phi^{xx} + \Phi^{xy} + \Phi^{xz} + \Phi^{xt} + \Phi^{yy} = 0, \end{aligned} \tag{12}$$

where Φ^x, Φ^y, Φ^{xx}, Φ^{xy}, Φ^{xz}, Φ^{xt}, Φ^{yy}, Φ^{xxx}, Φ^{xxxx}, Φ^{xxxy} and Φ^{xxxxxx} are invariant coefficients. These coefficients can be computed by the standard procedure of Lie group method. Substituting them into the invariant condition (12) and requiring all the different coefficients of u to zero with the help of Eq. (6), one can derive the determined equations for $\{\xi, \eta, \omega, \tau, \Phi\}$. Because it is tedious to write down all the determined equations for $\{\xi, \eta, \omega, \tau, \Phi\}$, we only list the final results below:

$$\begin{aligned} & \xi = \frac{1}{5}xf_2 - yf_5 + f_6, \quad \eta = \frac{3}{5}yf_2 + f_4 + zf_5, \\ & \omega = zf_2 + f_3, \quad \tau = f_1 + zf_2, \\ & \Phi = \frac{y-x}{3}f_5 - \frac{3u+2x}{15}f_2 + \frac{y}{3}\left(f_{6t} + f_{6z}\right) + f_7, \end{aligned} \tag{13}$$

where f_1, f_2, f_3, f_4, f_5 are five arbitrary functions of $t - z$ and f_6, f_7 are two arbitrary functions of z, t. The associated vector fields of the (3+1)-dimensional combined pKP-BKP Eq. (6) take the form

$$\begin{aligned} \boldsymbol{V} = {} & \boldsymbol{V}_1(f_1) + \boldsymbol{V}_2(f_2) + \boldsymbol{V}_3(f_3) + \boldsymbol{V}_4(f_4) \\ & + \boldsymbol{V}_5(f_5) + \boldsymbol{V}_6(f_6) + \boldsymbol{V}_7(f_7), \end{aligned} \tag{14}$$

where

$$\begin{aligned} & \boldsymbol{V}_1 = f_1\frac{\partial}{\partial t}, \quad \boldsymbol{V}_3 = f_3\frac{\partial}{\partial z}, \quad \boldsymbol{V}_4 = f_4\frac{\partial}{\partial y}, \\ & \boldsymbol{V}_6 = \frac{y}{3}\left(f_{6t} + f_{6z}\right)\frac{\partial}{\partial u} + f_6\frac{\partial}{\partial x}, \quad \boldsymbol{V}_7 = f_7\frac{\partial}{\partial u}, \end{aligned}$$

$$\boldsymbol{V}_2 = f_2\left(z\frac{\partial}{\partial t} - \frac{3u+2x}{15}\frac{\partial}{\partial u} + \frac{x}{5}\frac{\partial}{\partial x} + \frac{3y}{5}\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right),$$
$$\boldsymbol{V}_5 = f_5\left(\frac{y-x}{3}\frac{\partial}{\partial u} - y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\right). \tag{15}$$

That is to say, the (3+1)-dimensional pKP-BKP integrable Eq. (6) admits the Lie algebra spanned by the vectors $\boldsymbol{V}_j$ $(j = 1,2,\ldots,7)$.

3. Similarity solutions of Eq. (6)

In this section, we will consider the symmetry reduction and invariant solutions of Eq. (6) by selecting different vectors. In order to obtain similarity solutions of Eq. (6), we should solve the associated characteristic equation to obtain the invariant transformations and then substitute these results into Eq. (6) to determine the corresponding reduction equations. Finally, similarity solutions can be obtained by solving the reduction equations. The characteristic equation of Eq. (6) is in the form

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\eta} = \frac{\mathrm{d}z}{\omega} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}u}{\Phi}, \tag{16}$$

where $\{\xi,\eta,\omega,\tau,\Phi\}$ are given explicitly in Eq. (13). Here we discuss the following cases.

3.1. Vector field $\boldsymbol{V}_2 = f_2\left(z\frac{\partial}{\partial t} - \frac{3u+2x}{15}\frac{\partial}{\partial u} + \frac{x}{5}\frac{\partial}{\partial x} + \frac{3y}{5}\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$

The related characteristic equation of $\boldsymbol{V}_2$ is

$$\frac{\mathrm{d}x}{\frac{x}{5}} = \frac{\mathrm{d}y}{\frac{3y}{5}} = \frac{\mathrm{d}z}{z} = \frac{\mathrm{d}t}{z} = \frac{\mathrm{d}u}{-\frac{3u+2x}{15}}, \tag{17}$$

and we have the similarity form

$$u_1 = \frac{F_1(X,Y,T)}{z^{1/5}} - \frac{x}{3}, \tag{18}$$

where F_1 is a similarity function of the new variables $X = x/z^{1/5}$, $Y = y/z^{3/5}$, $T = t - z$. Substituting Eq. (18) into Eq. (6), we obtain the reduction equation in terms of F_1

$$\begin{aligned}&\left(45F_{1X}^2 + 15F_{1Y} + 15F_{1XXX} - X\right)F_{1XX}\\ &+\left(15F_{1XY} + 15F_{1XXXX} - 2\right)F_{1X} - 3F_{1XY}Y\\ &+5F_{1YY} + 5F_{1XXXY} + F_{1XXXXXX} = 0.\end{aligned} \tag{19}$$

Because the coefficients of Eq. (19) have independent variables X and Y, it is difficult to solve it directly. We will apply the Lie group method to it again and the corresponding infinitesimal generators are given below:

$$\xi_X = g_2, \quad \eta_Y = 0, \quad \tau_T = g_1, \quad \Phi_{F_1} = \frac{g_2}{15}Y + g_3, \tag{20}$$

where g_1, g_2 and g_3 are three arbitrary functions of T. The associated vector fields of Eq. (19) take the form

$$\boldsymbol{W} = \boldsymbol{W}_1(g_1) + \boldsymbol{W}_2(g_2) + \boldsymbol{W}_3(g_3), \tag{21}$$

where

$$\boldsymbol{W}_1 = g_1\frac{\partial}{\partial T}, \quad \boldsymbol{W}_2 = g_2\left(\frac{Y}{15}\frac{\partial}{\partial F_1} + \frac{\partial}{\partial X}\right),$$
$$\boldsymbol{W}_3 = g_3\frac{\partial}{\partial F_1}. \tag{22}$$

3.1.1. Vector field $W_2 = g_2\left(\frac{Y}{15}\frac{\partial}{\partial F_1} + \frac{\partial}{\partial X}\right)$

The related characteristic equation of $\boldsymbol{W}_2$ is

$$\frac{\mathrm{d}X}{1} = \frac{\mathrm{d}Y}{0} = \frac{\mathrm{d}T}{0} = \frac{\mathrm{d}F_1}{\frac{Y}{15}}. \tag{23}$$

Its similarity form is

$$F_{11} = \frac{XY}{15} + M_1(Y_1,T_1), \tag{24}$$

where M_1 is a similarity function of Y_1, T_1 with similarity variables $Y_1 = Y$, $T_1 = T$ and F_{11} is a similarity solution of Eq. (19). Substituting Eq. (24) into Eq. (19), the reduction equation in terms of M_1 can be found

$$5M_{1Y_1Y_1} - \frac{4Y_1}{15} = 0, \tag{25}$$

which has the solution

$$M_1(Y_1,T_1) = \frac{2}{225}Y_1^3 + G_1Y_1 + G_2, \tag{26}$$

with G_1 and G_2 being arbitrary functions of T_1. Then we can obtain the similarity solution of Eq. (6) by substituting Eq. (24) with Eq. (26) into Eq. (18)

$$u_{11} = \frac{xy}{15z} - \frac{x}{3} + \frac{2y^3}{225z^2} + \frac{G_1(t-z)y}{z^{4/5}} + \frac{G_2(t-z)}{z^{1/5}}. \tag{27}$$

3.1.2. Vector field $W_2 + W_3 = \left(g_2\frac{Y}{15} + g_3\right)\frac{\partial}{\partial F_1} + g_2\frac{\partial}{\partial X}$

The related characteristic equation of $\boldsymbol{W}_2 + \boldsymbol{W}_3$ is

$$\frac{\mathrm{d}X}{g_2} = \frac{\mathrm{d}Y}{0} = \frac{\mathrm{d}T}{0} = \frac{\mathrm{d}F_1}{g_2\frac{Y}{15} + g_3}, \tag{28}$$

and the similarity reductions are

$$F_{12} = \frac{XY}{15} + \frac{g_3}{g_2}X + M_2(Y_1,T_1), \quad Y_1 = Y, \quad T_1 = T. \tag{29}$$

Substituting Eq. (29) into Eq. (19), we get the reduction equation in terms of M_2

$$5M_{2Y_1Y_1} - \frac{4}{15}Y_1 - \frac{g_3}{g_2} = 0, \tag{30}$$

which has the solution

$$M_2 = \frac{2}{225}Y_1^3 + \frac{g_3}{10g_2}Y_1^2 + G_1Y_1 + G_2, \tag{31}$$

where G_1 and G_2 are two arbitrary functions of T_1. Hence, we can obtain the similarity solution of Eq. (6) by substituting Eq. (29) with Eq. (31) into Eq. (18)

$$\begin{aligned}u_{12} = &\frac{xy}{15z} - \frac{x}{3} + \frac{2y^3}{225z^2} + \frac{xg_3(t-z)}{z^{2/5}g_2(t-z)}\\ &+\frac{y^2g_3(t-z)}{10z^{7/5}g_2(t-z)} + \frac{yG_1(t-z)}{z^{4/5}} + \frac{G_2(t-z)}{z^{1/5}}.\end{aligned} \tag{32}$$

3.1.3. Vector field $W_1+W_3=g_3\frac{\partial}{\partial F_1}+g_1\frac{\partial}{\partial T}$

The related characteristic equation of $\boldsymbol{W}_1+\boldsymbol{W}_3$ is

$$\frac{\mathrm{d}X}{0}=\frac{\mathrm{d}Y}{0}=\frac{\mathrm{d}T}{g_1}=\frac{\mathrm{d}F_1}{g_3}. \tag{33}$$

It is easy to obatin the similarity function

$$F_{13}=\ln(g_1)+M_3\left(X_1,Y_1\right), \tag{34}$$

where M_3 is a similarity function of $X_1=X$, $Y_1=Y$, under the assumption $g_3=g_1'$. Substituting Eq. (34) into Eq. (19), we get the reduction equation in terms of M_3

$$\begin{aligned}&\left(45M_{3X_1}^2+15M_{3Y_1}+15M_{3X_1X_1X_1}-X_1\right)M_{3X_1X_1}\\&+\left(15M_{3X_1}-3Y_1\right)M_{3X_1Y_1}+5M_{3Y_1Y_1}\\&+\left(15M_{3X_1X_1X_1X_1}-2\right)M_{3X_1}\\&+M_{3X_1X_1X_1X_1X_1X_1}+5M_{3X_1X_1X_1Y_1}=0,\end{aligned} \tag{35}$$

which has a simple solution

$$M_3=X_1+\frac{1}{5}Y_1^2+C_1Y_1+C_2, \tag{36}$$

where C_1 and C_2 are arbitrary constants. Hence we have the similarity solution of Eq. (6) by substituting Eq. (36) with Eq. (34) into Eq. (18)

$$u_{13}=\frac{x}{z^{2/5}}-\frac{x}{3}+\frac{y^2}{5z^{7/5}}+\frac{C_1y}{z^{4/5}}+\frac{C_2}{z^{1/5}}+\frac{\ln\left(g_1(t-z)\right)}{z^{1/5}}. \tag{37}$$

3.2. Vector field $V_4=f_4\frac{\partial}{\partial y}$

With characteristic equation of $\boldsymbol{V}_4$

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{f_4}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{0}, \tag{38}$$

one can obtain similarity form of Eq. (6)

$$u_2=F_2\left(X,Z,T\right), \tag{39}$$

where F_2 is a similarity function of $X=x$, $Z=z$, $T=t$. Substituting Eq. (39) into Eq. (6), we deduce the reduction equation in terms of F_2

$$\begin{aligned}&F_{2XT}+\left(9F_{2X}^2+3F_{2XXX}+6F_{2X}+1\right)F_{2XX}\\&+\left(3F_{2X}+1\right)F_{2XXXX}+\frac{1}{5}F_{2XXXXXX}+F_{2XZ}=0.\end{aligned} \tag{40}$$

Some special solutions of Eq. (40) are given as follows:

$$\begin{aligned}F_{21}&=2\tanh\left(X+Z-\frac{46}{5}T\right)+C_1,\\F_{22}&=XG_1\left(T-Z\right)+G_2\left(Z,T\right),\end{aligned} \tag{41}$$

where G_1, G_2 are two arbitrary functions and C_1 is an arbitrary constant. Then we can get the similarity solutions of Eq. (6)

$$\begin{aligned}u_{21}&=2\tanh\left(x+z-\frac{46}{5}t\right)+C_1,\\u_{22}&=xG_1(t-z)+G_2(z,t).\end{aligned} \tag{42}$$

If fixing

$$\begin{aligned}G_1(t-z)&=t-z,\\G_2(z,t)&=\frac{10z+4t+5}{2t^2+4tz+5z^2+2t+5z+5},\end{aligned} \tag{43}$$

and substituting Eq. (43) into Eq. (42), we have the exact solution of Eq. (6)

$$u_{2a}=(t-z)x+\frac{10z+4t+5}{2t^2+4tz+5z^2+2t+5z+5}. \tag{44}$$

The detailed structure of Eq. (44) is shown in Fig. 1. And when $x=0$, it can be regarded as a lump wave solution of Eq. (6) in (z,t)-plane.

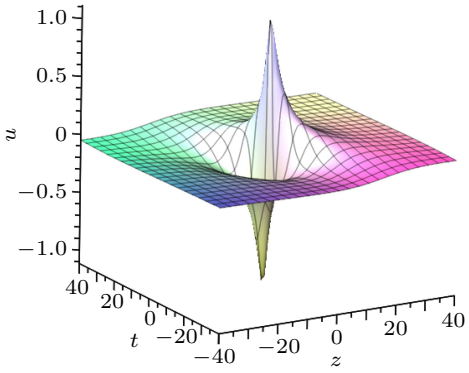


Fig. 1. Lump wave solution (44) for Eq. (6) at $x=0$.

3.3. Vector field $V_5=f_5\left(\frac{y-x}{3}\frac{\partial}{\partial u}-y\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}\right)$

The related characteristic equation of $\boldsymbol{V}_5$ is

$$\frac{\mathrm{d}x}{-y}=\frac{\mathrm{d}y}{z}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{\frac{y-x}{3}}, \tag{45}$$

and the corresponding similarity form of Eq. (6) is

$$u_3=F_3\left(X,Z,T\right)-\frac{xy}{3z}-\frac{y^3}{9z^2}+\frac{y^2}{6z}, \tag{46}$$

where F_3 is a similarity function of $X=x+\frac{y^2}{2z}$, $Z=z$, $T=t$. Substituting Eq. (46) into Eq. (6), we get the reduction equation in terms of F_3

$$\begin{aligned}&\left(45ZF_{3X}^2+30ZF_{3X}+15ZF_{3XXX}-5X+5Z\right)F_{3XX}\\&+\left(15ZF_{3X}+5Z\right)F_{3XXXX}\\&+5Z\left(F_{3XZ}+F_{3XT}\right)+ZF_{3XXXXXX}=0.\end{aligned} \tag{47}$$

It is easy to see that Eq. (47) has the solution

$$F_3=XG_1\left(T-Z\right)+G_2\left(Z,T\right), \tag{48}$$

and the similarity solution of Eq. (6) can be obtained

$$u_{31}=\left(x+\frac{y^2}{2z}\right)G_1(t-z)+G_2(z,t)-\frac{xy}{3z}-\frac{y^3}{9z^2}+\frac{y^2}{6z}. \tag{49}$$

While setting the arbitrary function

$$G_1(t-z)=-\frac{1}{3},\quad G_2=\frac{3\sqrt{3}\,\mathrm{e}^{\sqrt{3}(z+t)-11}+6(z+t)+66}{3\,\mathrm{e}^{\sqrt{3}(z+t)-11}+727+60z+264t+3z^2+(6t+6)z+30t^2}, \tag{50}$$

and substituting Eq. (50) into Eq. (49), we can derive the exact solution of Eq. (6)

$$u_{3a}=\frac{3\sqrt{3}\,\mathrm{e}^{\sqrt{3}(z+t)-11}+6(z+t)+66}{3\,\mathrm{e}^{\sqrt{3}(z+t)-11}+727+60z+264t+3z^2+(6t+6)z+30t^2}-\frac{x}{3}-\frac{xy}{3z}-\frac{y^3}{9z^2}, \tag{51}$$

and the detailed structure of Eq. (51) is given in Fig. 2. Equation (51) can be regarded as a lump-kink wave solution of Eq. (6) when $x=y=0$.

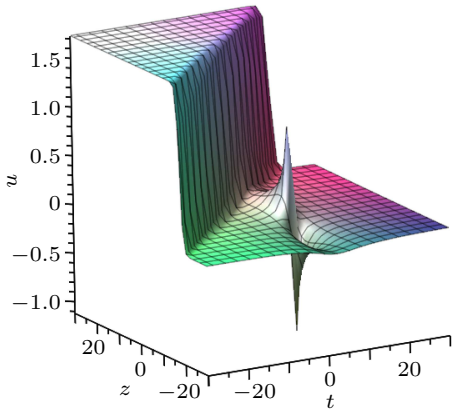


Fig. 2. Lump-kink wave solution (51) for Eq. (6) at $x=0$, $y=0$.

3.4. Vector field $\boldsymbol{V}_6=\frac{y}{3}(f_{6t}+f_{6z})\frac{\partial}{\partial u}+f_6\frac{\partial}{\partial x}$

In a similar way, the related characteristic equation of $\boldsymbol{V}_6$ is

$$\frac{\mathrm{d}x}{f_6}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{\frac{y}{3}(f_{6t}+f_{6z})}. \tag{52}$$

Under the assumption $f_6=\mathrm{e}^{z+t-2}$, the similarity solution and similarity variables can be obtained as

$$u_4=\frac{2xy}{3}+F_4(Y,Z,T),\quad Y=y,\quad Z=z,\quad T=t. \tag{53}$$

Substituting Eq. (53) into Eq. (6), we will derive the reduction equation in terms of F_4

$$F_{4YY}+\frac{4Y}{3}+\frac{2}{3}=0. \tag{54}$$

It is easy to see that Eq. (54) has the solution

$$F_4=G_1(Z,T)Y+G_2(Z,T)-\frac{2Y^3}{9}-\frac{Y^2}{3}, \tag{55}$$

where G_1 and G_2 are two arbitrary functions of Z, T and the corresponding solution of Eq. (6) is given as

$$u_{41}=\frac{2xy}{3}-\frac{2y^3}{9}-\frac{y^2}{3}+G_1(z,t)y+G_2(z,t). \tag{56}$$

Here we list three special exact solutions of Eq. (6):

Case 1:

$$G_1=\tanh(z+t)+\operatorname{sech}^2(t-z),\quad G_2=2\cos(t+2z-2),$$
$$u_{4a}=\frac{2xy}{3}-\frac{2y^3}{9}-\frac{y^2}{3}+y\left(\tanh(z+t)+\operatorname{sech}^2(t-z)\right)+2\cos(t+2z-2). \tag{57}$$

Case 2:

$$G_1=G_2=\frac{30z+8t+5+\mathrm{e}^{z+0.1t+0.1}-\mathrm{e}^{-z-0.1t-0.1}}{4t^2+8tz+15z^2+5t+5z+10+\mathrm{e}^{z+0.1t+0.1}+\mathrm{e}^{-z-0.1t-0.1}},$$
$$u_{4b}=\frac{2xy}{3}-\frac{2y^3}{9}-\frac{y^2}{3}+\frac{30z+8t+5+\mathrm{e}^{z+0.1t+0.1}-\mathrm{e}^{-z-0.1t-0.1}}{4t^2+8tz+15z^2+5t+5z+10+\mathrm{e}^{z+0.1t+0.1}+\mathrm{e}^{-z-0.1t-0.1}}(y+1). \tag{58}$$

Case 3:

$$G_1=\frac{2(t+9z)}{t^2+2zt+9z^2+1},\quad G_2=-\frac{4\sin(t-z+1)}{\mathrm{e}^{1-t}-4\mathrm{e}^{t-1}-4\cos(t-z+1)+30},$$
$$u_{4c}=\frac{2xy}{3}-\frac{2y^3}{9}-\frac{y^2}{3}+\frac{2(t+9z)y}{t^2+2zt+9z^2+1}-\frac{4\sin(t-z+1)}{\mathrm{e}^{1-t}-4\mathrm{e}^{t-1}-4\cos(t-z+1)+30}. \tag{59}$$

The detailed structures of Eqs. (57)–(59) are shown in Fig. 3. It can be regarded as the interaction solution between a solitary wave and periodic waves of Eq. (6) when $x=2$, $y=1$ in Fig. 3(a), lump–two-kink wave solution of Eq. (6) when $x=y=0$ in Fig. 3(b) and lump–breather wave solution of Eq. (6) when $x=0$, $y=1$ in Fig. 3(c).

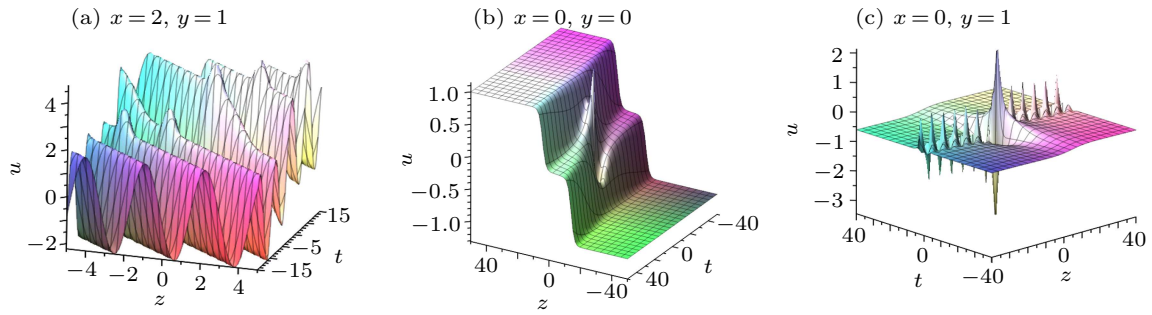


Fig. 3. Similarity solution of Eq. (6): (a) interaction between solitary and periodic wave solution (57) when $x=2$, $y=1$; (b) lump–two-kink solution (58) when $x=y=0$; (c). lump–breather solution (59) when $x=0$, $y=1$.

3.5. Vector field $V_1+V_7=f_1\frac{\partial}{\partial t}+f_7\frac{\partial}{\partial u}$

We solve the characteristic equation of $\boldsymbol{V}_1+\boldsymbol{V}_7$

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{f_1}=\frac{\mathrm{d}u}{f_7}, \tag{60}$$

and obtain the similarity solution and similarity variables of Eq. (6)

$$u_5=\int\frac{f_7}{f_1}\mathrm{d}t+F_5(X,Y,Z),\quad X=x,\quad Y=y,\quad Z=z. \tag{61}$$

Substituting Eq. (61) into Eq. (6) gives the reduction equation in terms of F_5

$$\begin{aligned}&\left(9F_{5X}^2+3F_{5XXX}+6F_{5X}+3F_{5Y}+1\right)F_{5XX}\\&+(3F_{5X}+1)F_{5XXXX}+\frac{1}{5}F_{5XXXXXX}\\&+F_{5XXXY}+(3F_{5X}+1)F_{5XY}+F_{5XZ}+F_{5YY}=0.\end{aligned} \tag{62}$$

It is easy to see that Eq. (62) has the solution

$$\begin{aligned}F_5\ =\ &4C_2\tanh\Big[C_2X-\left(4C_2^3+C_2\right)Y\\&-\left(\frac{16}{5}C_2^5+4C_2^3+C_2\right)Z+C_1\Big]+C_3,\end{aligned} \tag{63}$$

where C_1, C_2, C_3 are arbitrary constants. And we can get the similarity solution of Eq. (6)

$$\begin{aligned}u_{51}\ =\ &\int\frac{f_7}{f_1}\mathrm{d}t+4C_2\tanh\Big[C_2x-\left(4C_2^3+C_2\right)y\\&-\left(\frac{16}{5}C_2^5+4C_2^3+C_2\right)z+C_1\Big]+C_3,\end{aligned} \tag{64}$$

which can be considered as the solitary solution.

In order to find more exact solutions of Eq. (62), we make the traveling wave transformation

$$F_5=\phi(w),\quad w=mX+nY+kZ, \tag{65}$$

where m, n, k are constants. Substituting Eq. (65) into Eq. (62), we obtain the following nonlinear ordinary differential equation:

$$\begin{aligned}&\frac{m^6}{5}\phi^{(6)}+3m^5(\phi^{(4)}\phi'+\phi'''\phi'')+m^3(m+n)\phi^{(4)}\\&+9m^4(\phi')^2\phi''+6m^2(m+n)\phi'\phi''\\&+(km+m^2+mn+n^2)\phi''=0.\end{aligned} \tag{66}$$

We suppose that the solution of Eq. (66) can be written in the form

$$\phi(w)=\sum_{i=-l}^{l}A_iH^i(w), \tag{67}$$

where $H(w)$ should satisfy the Riccati equation

$$H'(w)=1-H^2(w), \tag{68}$$

with the solution

$$H(w)=\tanh(w). \tag{69}$$

The positive integer l will be determined by the homogeneous balance method between the highest order derivative and the highest order nonlinear term appearing in Eq. (66), we can have $l=1$ and so the solution of Eq. (66) is in the form

$$\phi=A_{-1}\tanh^{-1}(w)+A_0+A_1\tanh(w). \tag{70}$$

Substituting Eq. (70) into Eq. (66) and setting the coefficients of $\tanh^i(w)$ to zero, we have three types of exact solutions:

$$\phi_1(w)=A_0+2m\tanh(w), \tag{71}$$

$$\phi_2(w)=A_0+\frac{2m}{\tanh(w)}, \tag{72}$$

with constants m, n, k constraint $5mk+16m^6+20m^4+20m^3n+5m^2+5mn+5n^2=0$, and

$$\phi_3(w)=A_0+\frac{2m}{\tanh(w)}+2m\tanh(w), \tag{73}$$

where A_0 is an arbitrary constant and the constants m,n,k should satisfy $5mk+256m^6+80m^4+80m^3n+5m^2+5mn+5n^2=0$. Then the similarity solutions of Eq. (6) can be acquired as

$$\begin{aligned}u_{52}\ &=\int\frac{f_7}{f_1}\mathrm{d}t+2m\tanh(mx+ny+kz)+A_0,\\u_{53}\ &=\int\frac{f_7}{f_1}\mathrm{d}t+\frac{2m}{\tanh(mx+ny+kz)}+A_0,\\u_{54}\ &=\int\frac{f_7}{f_1}\mathrm{d}t+\frac{2m}{\tanh(mx+ny+kz)}\\&\quad+2m\tanh(mx+ny+kz)+A_0,\end{aligned} \tag{74}$$

with Eqs. (71)–(73), respectively.

3.6. Vector field $\boldsymbol{V}_3+\boldsymbol{V}_7 = f_3\frac{\partial}{\partial z}+f_7\frac{\partial}{\partial u}$

Solving the characteristic equation of $\boldsymbol{V}_3+\boldsymbol{V}_7$

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{f_3}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{f_7}, \tag{75}$$

one can find the similarity solution and similarity variables

$$u_6=\int\frac{f_7}{f_3}\mathrm{d}z+F_6(X,Y,T),\ X=x,\ Y=y,\ T=t, \tag{76}$$

where F_6 is a similarity function of X, Y and T. Substituting Eq. (76) into Eq. (6), the reduction equation in terms of F_6 is obtained as

$$\begin{aligned}&F_{6XT}+\left(9F_{6X}^2+3F_{6XXX}+6F_{6X}+3F_{6Y}+1\right)F_{6XX}\\&+(3F_{6X}+1)F_{6XXXX}+\frac{1}{5}F_{6XXXXXX}\\&+F_{6XXXY}+(3F_{6X}+1)F_{6XY}+F_{6YY}=0.\end{aligned} \tag{77}$$

It is easy to see that Eq. (77) has the solitary wave solution

$$\begin{aligned}F_6 \ =\ &4C_2\tanh\left[C_2X-\left(\frac{16}{5}C_2^5+4C_2^3+C_2\right)T\right.\\&\left.-\left(4C_2^3+C_2\right)Y+C_1\right]+C_5,\end{aligned} \tag{78}$$

and the corresponding similarity solution of Eq. (6) has the form

$$\begin{aligned}u_{61} \ =\ &\int\frac{f_7}{f_3}\mathrm{d}z+4C_2\tanh\left[C_2x-\left(\frac{16}{5}C_2^5+4C_2^3+C_2\right)t\right.\\&\left.-\left(4C_2^3+C_2\right)y+C_1\right]+C_5.\end{aligned} \tag{79}$$

3.7. Vector field $\boldsymbol{V}_4+\boldsymbol{V}_7 = f_4\frac{\partial}{\partial y}+f_7\frac{\partial}{\partial u}$

Solving the characteristic equation of $\boldsymbol{V}_4+\boldsymbol{V}_7$

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{f_4}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{f_7}, \tag{80}$$

one can find the similarity solution and new variables

$$u_7=\frac{f_7}{f_4}y+F_7(X,Z,T),\quad X=x,\ Z=z,\quad T=t. \tag{81}$$

Substituting Eq. (81) into Eq. (6), the (3+1)-dimensional pKP-BKP Eq. (6) can be reduced to

$$\begin{aligned}&F_{7XT}+\left(9F_{7X}^2+3F_{7XXX}+6F_{7X}+\frac{3f_7}{f_4}+1\right)F_{7XX}\\&+(3F_{7X}+1)F_{7XXXX}\\&+\frac{1}{5}F_{7XXXXXX}+F_{7XZ}=0.\end{aligned} \tag{82}$$

It is easy to see that Eq. (82) has a simple solution

$$F_7=G_1(T-Z)X+G_2(Z,T), \tag{83}$$

where G_1 and G_2 are two arbitrary functions of the indicated variables. Then we have the similarity solution of Eq. (6)

$$u_{71}=\frac{f_7}{f_4}y+G_1(t-z)x+G_2(z,t). \tag{84}$$

Here we list three types of exact solutions of Eq. (6) by selecting proper arbitrary functions f_4, f_7, G_1 and G_2.

Case 1:

$$\begin{aligned}&f_4(t-z)=\cot(t-z),\quad f_7(z,t)=-\left(\cot^2(t-z)+1\right),\quad G_1=\operatorname{sech}(t-z),\quad G_2=3\tanh(z+2t),\\&u_{7a}=x\operatorname{sech}(t-z)+3\tanh(2t+z)-\frac{1+\cot(t-z)^2}{\cot(t-z)}y.\end{aligned} \tag{85}$$

Case 2:

$$\begin{aligned}&f_4(t-z)=G_1=1,\quad f_7(z,t)=G_2=-\frac{50\sinh(-50z+2t+1)+20\mathrm{e}^{20z+0.1t+1}-20\mathrm{e}^{-20z-0.1t-1}}{\cosh(-50z+2t+1)+\mathrm{e}^{20z+0.1t+1}+\mathrm{e}^{-20z-0.1t-1}+0.1},\\&u_{7b}=x-\frac{50\sinh(-50z+2t+1)+20\mathrm{e}^{20z+0.1t+1}-20\mathrm{e}^{-20z-0.1t-1}}{\cosh(-50z+2t+1)+\mathrm{e}^{20z+0.1t+1}+\mathrm{e}^{-20z-0.1t-1}+0.1}(y+1).\end{aligned} \tag{86}$$

Case 3:

$$\begin{aligned}&f_4(t-z)=G_1=1,\quad f_7(z,t)=G_2=\frac{2\left[\cos(2(z+t))-\sin(2(z+t))\right]}{\cos(2(z+t))+\sin(2(z+t))+13z^2+5},\\&u_{7c}=x+\frac{2\left[\cos(2(z+t))-\sin(2(z+t))\right]}{\cos(2(z+t))+\sin(2(z+t))+13z^2+5}(y+1).\end{aligned} \tag{87}$$

The detailed structure of Eq. (85) is shown in Fig. 4 and it can be regarded as the interaction solution between a solitary wave and a kink wave of Eq. (6). In addition, the detailed structures of Eqs. (86) and (87) are shown in Fig. 5. It can be regarded as the two-kink wave solution of Eq. (6) when $x=y=0$ in Fig. 5(a) and interaction solution between lump and triangular periodic wave of Eq. (6) when $x=y=0$ in Fig. 5(b).

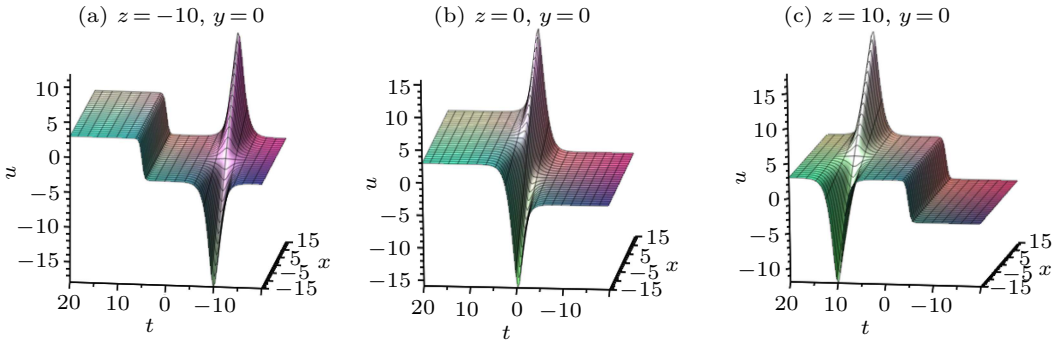


Fig. 4. The interaction solution between a solitary wave and a kink wave (85) for Eq. (6).

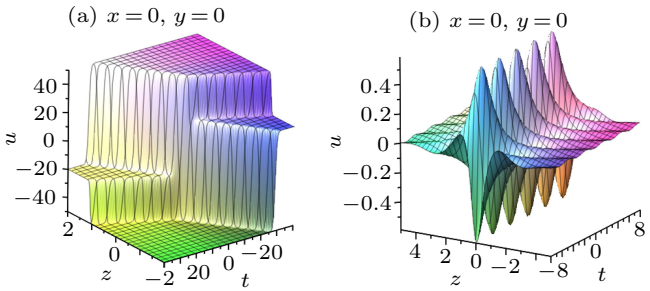


Fig. 5. Similarity solution of Eq. (6): (a) two-kink solution (86) when $x=y=0$; (b) interaction between lump and triangular periodic wave solution (87) when $x=y=0$.

3.8. Vector field $\boldsymbol{V}_6+\boldsymbol{V}_7=\left(\frac{y}{3}\left(f_{6t}+f_{6z}\right)+f_7\right)\frac{\partial}{\partial u}+f_6\frac{\partial}{\partial x}$

The related characteristic equation of $\boldsymbol{V}_6+\boldsymbol{V}_7$ is

$$\frac{\mathrm{d}x}{f_6}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{\frac{y}{3}\left(f_{6t}+f_{6z}\right)+f_7}, \tag{88}$$

we will have the similarity solution and similarity variables by assuming $f_6=f_7=1$,

$$u_8=x+F_8(Y,Z,T),\quad Y=y,\quad Z=z,\quad T=t. \tag{89}$$

Substituting Eq. (89) into Eq. (6), one can derive the reduction equation in terms of F_8

$$F_{8YY}=0. \tag{90}$$

It is easy to see that Eq. (90) has the solution

$$F_8=G_1(Z,T)Y+G_2(Z,T), \tag{91}$$

where G_1 and G_2 are arbitrary functions of Z and T. Then we have the similarity solution of Eq. (6)

$$u_{81}=x+G_1(z,t)y+G_2(z,t). \tag{92}$$

We give two special interesting similarity solutions of Eq. (6) by selecting proper arbitrary functions of G_1 and G_2 in the following cases:

Case 1:

$$G_1(z,t)=G_2(z,t)=-\frac{4\sin(t-z+1)}{\mathrm{e}^{1-t}-4\,\mathrm{e}^{t-1}-4\cos(t-z+1)+30},$$
$$u_{8a}=x-\frac{4\sin(t-z+1)}{\mathrm{e}^{1-t}-4\,\mathrm{e}^{t-1}-4\cos(t-z+1)+30}(y+1). \tag{93}$$

Case 2:

$$G_1(z,t)=G_2(z,t)=-\frac{2\sqrt{3}\sinh\left(\frac{t}{3}-\frac{\sqrt{3}z}{3}\right)}{6\cosh\left(\frac{t}{3}-\frac{\sqrt{3}z}{3}\right)+2\cos\left(\frac{2t}{3}\right)},$$
$$u_{8b}=x-\frac{2\sqrt{3}\sinh\left(\frac{t}{3}-\frac{\sqrt{3}z}{3}\right)}{6\cosh\left(\frac{t}{3}-\frac{\sqrt{3}z}{3}\right)+2\cos\left(\frac{2t}{3}\right)}(y+1). \tag{94}$$

Solution (93) can be regarded as the breather wave of Eq. (6) and solution (94) can be considered as the periodic lump solution of Eq. (6), which are shown in Fig. 6, respectively.

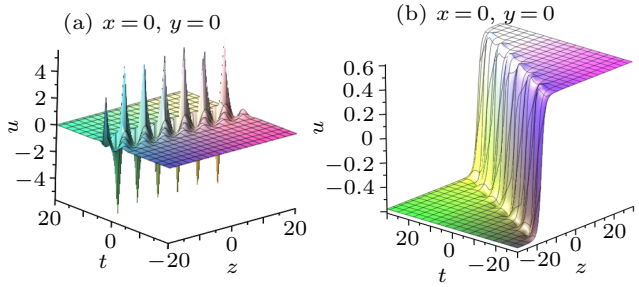


Fig. 6. (a) Breather wave solution (93); (b) periodic lump solution (94).

3.9. Vector field $\boldsymbol{V}_3+\boldsymbol{V}_4+\boldsymbol{V}_6=f_6\frac{\partial}{\partial x}+f_3\frac{\partial}{\partial z}+f_4\frac{\partial}{\partial y}+\frac{y}{3}\left(f_{6t}+f_{6z}\right)\frac{\partial}{\partial u}$

From the related characteristic equation of $\boldsymbol{V}_3+\boldsymbol{V}_4+\boldsymbol{V}_6$

$$\frac{\mathrm{d}x}{f_6}=\frac{\mathrm{d}y}{f_4}=\frac{\mathrm{d}z}{f_3}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{\frac{y}{3}\left(f_{6t}+f_{6z}\right)}, \tag{95}$$

we can obtain the similarity solution and similarity variables of Eq. (6) by assuming $f_6=f_4=f_3'(t-z)$

$$u_9=Q(X,Y,T),\quad X=x+\ln(f_3),$$
$$Y=y+\ln(f_3),\quad T=t. \tag{96}$$

Substituting Eq. (96) into Eq. (6), we arrive at the reduction equation in terms of Q

$$Q_{XT}+\left(9Q_X^2+3Q_{XXX}+6Q_X+3Q_Y+1\right)Q_{XX}$$
$$+\left(3Q_X+1\right)Q_{XXXX}+\frac{1}{5}Q_{XXXXXX}$$
$$+Q_{XXXY}+\left(3Q_X+1\right)Q_{XY}+Q_{YY}=0. \tag{97}$$

It is easy to see that Eq. (97) has the solution

$$Q = -4C_2\tanh\left[\left(\frac{16}{5}C_2^5+4C_2^3+C_2\right)T - C_2X+\left(4C_2^3+C_2\right)Y-C_1\right]+C_3, \quad (98)$$

where C_1, C_2 and C_3 are arbitrary constants. And the similarity solution of Eq. (6) can be obtained as

$$u_{91} = 4C_2\tanh\left((x-y-t)C_2-\frac{16}{5}C_2^5t - 4C_2^3(t+y)-4C_2^3\ln(f_3)+C_1\right)+C_3. \quad (99)$$

Selecting the arbitrary function and constants as

$$f_3=\sin(t-z)^{1/4},\quad C_1=2,\quad C_2=-1,\quad C_3=1, \quad (100)$$

and substituting Eq. (100) into Eq. (99), we have the exact solution of Eq. (6)

$$u_{9a}=-4\tanh\left(\frac{41t}{5}-x+\ln(\sin(t-z))+5y+2\right)+1. \quad (101)$$

4. Summary and discussion

In conclusion, Lie group method has been successfully applied to the (3+1)-dimensional combined pKP-BKP Eq. (6) to obtain abundant similarity solutions with arbitrary functions and constants. Various kinds of similarity solutions of Eq. (6) are obtained by solving different characteristic equations and many interesting similarity solutions are given analytically and graphically. Because there are many arbitrary functions in the solutions of the reduction equations, we can construct more interesting solutions of Eq. (6) as possible, such as the lump–kink solution, the lump–two-kink wave solution, the two-kink wave solution and the periodic lump wave solution. To our knowledge, these similarity solutions of Eq. (6) are not reported in the literature from the point of Lie symmetry method. On the other hand, these new similarity solutions are also plotted directly at different planes to let us know the internal properties of the high-dimensional integrable wave equation. Other integrable properties of the (3+1)-dimensional pKP-BKP equation are worthy of study in future.

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