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The N-periodic wave solutions to the $\mathcal{N}=1$ supersymmetric Sawada–Kotera–Ramani equation

Pengcheng Xin(辛鹏程), Zhonglong Zhao(赵忠龙)†, and Yu Wang(王宇)

School of Mathematics, North University of China, Taiyuan 030051, *China*

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The N-periodic wave solvability problem for the $\mathcal{N}=1$ supersymmetric Sawada–Kotera–Ramani equation is considered by combining the Hirota's bilinear method and the super Riemann theta function. The constraint equations and unknown parameters are redefined, and the numerical calculation process of the N-periodic wave solutions is derived. It has been verified that under certain conditions, the asymptotic relations between N-periodic wave solutions and N-soliton solutions can be established. Some numerical solutions of three-periodic wave are presented. Under the influence of the Grassmann variable, the three-periodic wave solutions will generate an influence band in the middle region, and the amplitude becomes bigger as the distance from the influence band increases.

Keywords: supersymmetry, N-periodic wave solutions, asymptotic relations

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1. Introduction

The theory of supersymmetry[1] has been developed into various fields of physics, such as particle physics,[2–4] nuclear physics,[5,6] and quantum field theory,[7,8] which links the Boson[9] and the Fermion.[10] By adding the Grassmann variables on the basis of standard variables, the supersymmetric mathematical formulas are described.[11] To better understand supersymmetry, the study of super partial derivative and super integral is essential.[12] In the field of mathematical physics, supersymmetry has also attracted wide attention. For some common solving methods, such as Darboux transformation,[13] Bäcklund transformation,[14] Lie symmetry analysis method[15] and Hirota's bilinear method,[16,17] they have also been extended in the supersymmetric field. Liu *et al.* presented the vectorial Darboux transformation for the supersymmetric KdV equation[18] and the supersymmetric KP hierarchies.[19,20] Tian *et al.*[21–23] introduced the supersymmetric reciprocal transformations, which are used to establish a link between two supersymmetric equations. Hamiltonian structures of the supersymmetric equations were introduced.[24–26] Carstea[27] introduced the supersymmetric Hirota operator and obtained the super bilinear forms of the $\mathcal{N}=1$ supersymmetric KdV equation and the $\mathcal{N}=1$ supersymmetric Sawada–Kotera–Ramani equation, but only derived super N-soliton solutions under certain conditions. Meanwhile, Lü *et al.*[28] constructed the supersymmetric Ito's equation and obtained its Bäcklund transformation and super three-soliton solutions. Then, Fan *et al.* used a direct method to obtain one- and two-periodic wave solutions for various supersymmetric equations, such as $\mathcal{N}=2$ supersymmetric KdV equation,[29] supersymmetric Ito's equation,[30] supersymmetric KdV-Sawada–Kotera–Ramani equation,[31] and established the asymptotic relations between the super quasi-periodic waves and super solitons under a small amplitude limit and certain conditions.

The quasi-periodic wave solutions, also known as algebro-geometric solutions, were obtained in the 1970s.[32,33] Due to the complexity of calculations, Nakamura first used a direct method to solve quasi-periodic wave solutions of the single[34] and coupled[35] bilinear equations by combining the Riemann theta functions[36] with the Hirota's bilinear method.[37] Then, Fan *et al.* solved the one- and two-periodic wave solutions of the $(2+1)$ dimensions Bogoyavlenskii's breaking soliton equation[38] and the asymmetrical Nizhnik–Novikov–Veselov equation,[39] and it is shown that the quasi-periodic wave solutions tend to the soliton solutions under a small amplitude limit. However, for the process of solving three- or more periodic waves, the number of equations exceeds the number of unknown parameters, which makes it difficult to find theoretical solutions. Therefore, by defining the constraint equations and unknown parameters, Hu *et al.* transformed the problem of solving three-periodic wave solutions into a least-squares problem, and obtained numerical solutions of the three-periodic waves for some integrable systems by using the Gauss–Newton method,[40,41] such as the KdV-type equations,[42] the Toda-type equations[43] and the coupled KdV–Toda-type equations.[44]

Similar to the case of the general equations,[45–47] it is difficult to solve the three- or more periodic wave solutions of the supersymmetric equations. Thus, the purpose of this paper

†Corresponding author. E-mail: zhaozlhit@163.com, zhaozl@nuc.edu.cn

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is to provide a direct method to obtain three- or more periodic wave solutions of the supersymmetric equations. This paper shall choose $\mathcal{N}=1$ supersymmetric Sawada–Kotera–Ramani equation as an example to illustrate. The equation is derived by Carstea[27] as

$$\psi_t+\mathcal{D}_x^2[10(\mathcal{D}_x\psi)\mathcal{D}_x^4\psi+5(\mathcal{D}_x^5\psi)\psi+15(\mathcal{D}_x\psi)^2\psi]+\mathcal{D}_x^{10}\psi=0, \tag{1}$$

where $\psi=\psi(x,t,\theta)$ is a fermionic superfield, which includes independent variable x, t and Grassmann variable θ. The operator $\mathcal{D}_x=\partial_\theta+\theta\partial_x$ is the superderivative. After substituting

$$\psi=2\mathcal{D}_x^3\ln f(x,t,\theta) \tag{2}$$

into Eq. (1), we obtain the following super Hirota's bilinear form of the $\mathcal{N}=1$ supersymmetric Sawada–Kotera–Ramani equation:

$$(S_xD_t+S_xD_x^5)f\cdot f=0, \tag{3}$$

where the definitions of Hirota bilinear operators D_x, D_t and super-Hirota bilinear operator S_x are

$$S_xD_x^nD_t^mf(x,t,\theta)\cdot g(x,t,\theta)=(\mathcal{D}_\theta-\mathcal{D}_{\theta'})(\partial_x-\partial_{x'})^n\times(\partial_t-\partial_{t'})^mf(x,t,\theta)\cdot g(x',t',\theta')|_{x'=x,t'=t,\theta'=\theta}.$$

When these operators act on the exponential functions, a property can be obtained as follows:

$$Q(D_x,D_t,S_x)\mathrm{e}^{\varpi_1}\cdot\mathrm{e}^{\varpi_2}=Q(a_1'-a_2',\omega_1'-\omega_2',\sigma'(b_1'-b_2')+\theta(a_1'-a_2'))\mathrm{e}^{\varpi_1+\varpi_2}, \tag{4}$$

where $\varpi_m=a_m'x+\omega_m't+\sigma'\theta b_m'+\gamma_m'$ $(m=1,2)$.

Besides, under the constraint $a_ib_j=a_jb_i$ $(i,j=1,2,\ldots,N)$, the super N-soliton solutions[27,48] of Eq. (1) are

$$\psi=2\mathcal{D}_x^3\ln f(x,t,\theta),\quad f=\sum_{\mu=0,1}\exp\left(\sum_{i=1}^N\mu_i\vartheta_i+\sum_{i<j}A_{ij}\mu_i\mu_j\right), \tag{5}$$

where

$$\vartheta_i=a_ix-a_i^5t+\sigma\theta b_i+\gamma_i,\quad \exp(A_{ij})=\left(\frac{a_i-a_j}{a_i+a_j}\right)^2\frac{a_i^2-a_ia_j+a_j^2}{a_i^2+a_ia_j+a_j^2}. \tag{6}$$

The structure of this paper is as follows. In Section 2, we introduce the super Riemann theta function and its periodic characteristics, and provide some preliminary preparations for numerical calculations. In Section 3, a selection for the initial point is provided. Section 4 presents some numerical solutions of three-periodic waves and analyzes their characteristics. Besides, the asymptotic relations between N-periodic wave solutions and N-soliton solutions are theoretically provided. Finally, the conclusion is presented in Section 5.

2. The super Riemann theta function and some preliminary preparations

The theta function is a special function of multiple complex variables that plays an important role in both Abelian varieties and quadratic theory, which is a fundamental issue in elliptic function theory.[49,50] To study quasi-periodic wave solutions, the Riemann theta function is essential. The multi-dimensional super Riemann theta function is defined as

$$\phi(\boldsymbol{\alpha},\boldsymbol{\varepsilon}|\boldsymbol{\tau})=\sum_{\boldsymbol{n}\in Z^N}\exp\{2\pi\mathrm{i}\langle\boldsymbol{\alpha},\boldsymbol{n}+\boldsymbol{\varepsilon}\rangle-\pi\langle\boldsymbol{\tau}(\boldsymbol{n}+\boldsymbol{\varepsilon}),\boldsymbol{n}+\boldsymbol{\varepsilon}\rangle\}, \tag{7}$$

where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_N)^{\mathrm{T}}$, $\alpha_j=k_jx+\omega_jt+l_j\theta\sigma+\eta_j$, $j=1,2,\ldots,N$, θ is the Grassmann variable, $\boldsymbol{n}=(n_1,n_2,\ldots,n_N)^{\mathrm{T}}\in Z^N$ and $\boldsymbol{\varepsilon}=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N)^{\mathrm{T}}\in C^N$. For the inner product of $\boldsymbol{f}=(f_1,f_2,\ldots,f_N)^{\mathrm{T}}$ and $\boldsymbol{g}=(g_1,g_2,\ldots,g_N)^{\mathrm{T}}$, we define $\langle\boldsymbol{f},\boldsymbol{g}\rangle=f_1g_1+f_2g_2+\cdots+f_Ng_N$. Besides, $\boldsymbol{\tau}=(\tau_{pq})$ is a N-dimensional positive definite and real-valued symmetric square matrix. For simplicity, we replace $\phi(\boldsymbol{\alpha},\boldsymbol{\tau})=\phi(\boldsymbol{\alpha},0|\boldsymbol{\tau})$ in certain places.

For N-periodic wave solutions, the periods of the solutions are particularly important. Before studying the periods of the solutions for Eq. (1), we can first understand the periods of the super Riemann theta function. The periods[36] of the super Riemann theta function have the following property:

$$\phi(\boldsymbol{\alpha}+\boldsymbol{e}_j+\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{\tau})=\exp(-2\pi\mathrm{i}\alpha_j+\pi\tau_{jj})\phi(\boldsymbol{\alpha},\boldsymbol{\tau}) \tag{8}$$

due to

$$\begin{aligned}
&\phi(\boldsymbol{\alpha}+\boldsymbol{e}_j+\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{\tau})\\
&=\sum_{\boldsymbol{n}\in Z^N}\exp\{2\pi\mathrm{i}\langle\boldsymbol{\alpha}+\boldsymbol{e}_j+\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{n}\rangle-\pi\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle\}\\
&=\sum_{\boldsymbol{n}\in Z^N}\exp\{2\pi\mathrm{i}\langle\boldsymbol{\alpha}+\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{n}\rangle+2\pi\mathrm{i}n_j-\pi\langle\tau\boldsymbol{n},\boldsymbol{n}\rangle\}\\
&\quad(\text{let }\ \boldsymbol{n}=\boldsymbol{m}-\boldsymbol{e}_j)\\
&=\sum_{\boldsymbol{m}\in Z^N}\exp\{2\pi\mathrm{i}\langle\boldsymbol{\alpha}+\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{m}-\boldsymbol{e}_j\rangle\\
&\quad-\pi\langle\boldsymbol{\tau}(\boldsymbol{m}-\boldsymbol{e}_j),\boldsymbol{m}-\boldsymbol{e}_j\rangle\}\\
&=\sum_{\boldsymbol{m}\in Z^N}\exp\{2\pi\mathrm{i}(\langle\boldsymbol{\alpha},\boldsymbol{m}\rangle-\langle\boldsymbol{\alpha},\boldsymbol{e}_j\rangle+\langle\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{m}\rangle\\
&\quad-\langle\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{e}_j\rangle)-\pi(\langle\boldsymbol{\tau}\boldsymbol{m},\boldsymbol{m}\rangle-\langle\boldsymbol{\tau}\boldsymbol{m},\boldsymbol{e}_j\rangle\\
&\quad-\langle\boldsymbol{\tau}\boldsymbol{e}_j,\boldsymbol{m}\rangle+\langle\boldsymbol{\tau}\boldsymbol{e}_j,\boldsymbol{e}_j\rangle)\}\\
&=\exp(-2\pi\mathrm{i}\alpha_j+\pi\tau_{jj})\phi(\boldsymbol{\alpha},\boldsymbol{\tau}),
\end{aligned}$$

where $\boldsymbol{e}_j$ $(j=1,2,\ldots,N)$ is the jth column of the identity matrix I_N and $\boldsymbol{\tau}_j$ $(j=1,2,\ldots,N)$ is the jth column of $\boldsymbol{\tau}$. In order to get the quasi-periodic wave solutions, we apply a general transformation

$$\psi=\psi_0+2\mathcal{D}_x^3\ln f(x,t,\theta), \tag{9}$$

where $\psi_0 = \psi_0(\theta)$ is a solution of Eq. (1). For Eq. (9), when $f(x,t,\theta) = \phi(\boldsymbol{\alpha},\boldsymbol{\tau})$, we can deduce

$$\psi = \psi_0 + 2\sum_{p,q=1}^{N} k_p(l_q\sigma + \theta k_q)\partial^2_{\alpha_p\alpha_q}\ln\phi(\boldsymbol{\alpha},\boldsymbol{\tau}). \tag{10}$$

For $\partial^2_{\alpha_p\alpha_q}\ln\phi(\alpha,\tau)$, by combining Eq. (8), the following properties can be obtained:

$$\begin{aligned}
&\frac{\partial_{\alpha_p}\phi(\boldsymbol{\alpha}+\boldsymbol{e}_j+\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{\tau})}{\phi(\boldsymbol{\alpha}+\boldsymbol{e}_j+\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{\tau})} = -2\pi\mathrm{i}\delta_{jp} + \frac{\partial_{\alpha_p}\phi(\boldsymbol{\alpha},\boldsymbol{\tau})}{\phi(\boldsymbol{\alpha},\boldsymbol{\tau})},\\
&\partial_{\alpha_p}\ln\phi(\boldsymbol{\alpha}+\boldsymbol{e}_j+\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{\tau}) = -2\pi\mathrm{i}\delta_{jp} + \partial_{\alpha_p}\ln\phi(\boldsymbol{\alpha},\boldsymbol{\tau}),\\
&\partial^2_{\alpha_p\alpha_q}\ln\phi(\boldsymbol{\alpha}+\boldsymbol{e}_j+\mathrm{i}\boldsymbol{\tau}_j,\boldsymbol{\tau}) = \partial^2_{\alpha_p\alpha_q}\ln\phi(\boldsymbol{\alpha},\boldsymbol{\tau}),
\end{aligned} \tag{11}$$

where δ_{jp} $(j = 1,2,\ldots,N)$ represent the Kronecker's delta. Equation (11) shows that $\partial^2_{\alpha_p\alpha_q}\ln\phi(\alpha,\tau)$ has $2N$ basic periods $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_N\}$ and $\{\mathrm{i}\boldsymbol{\tau}_1,\mathrm{i}\boldsymbol{\tau}_2,\ldots,\mathrm{i}\boldsymbol{\tau}_N\}$. It is obvious that ψ also has these $2N$ basic periods.

After substituting Eq. (9) into Eq. (1), we obtain the following super Hirota's bilinear form:[51]

$$\begin{aligned}
F(D_x,D_t,S_x,c)f\cdot f \;=\; &(S_xD_t + S_xD_x^5 + 15\psi_0^2D_x^2\\
&+5\psi_0D_x^4 + c)f\cdot f = 0,
\end{aligned} \tag{12}$$

where $c = c(\theta,t)$ is an integration constant related to x. When D_x, D_t and S_x act on the two theta functions $\phi(\boldsymbol{\alpha},\boldsymbol{\tau})$, we can obtain

$$\begin{aligned}
D_x\phi(\boldsymbol{\alpha},\boldsymbol{\tau})\cdot\phi(\boldsymbol{\alpha},\boldsymbol{\tau}) \;=\; &\left[\sum_{\boldsymbol{\mu}}\partial_x\phi(2\boldsymbol{\alpha},-\boldsymbol{\mu}/2|2\boldsymbol{\tau})|_{\boldsymbol{\alpha}=0}\right]\\
&\times\phi(2\boldsymbol{\alpha},\boldsymbol{\mu}/2|2\boldsymbol{\tau}),
\end{aligned} \tag{13}$$

$$\begin{aligned}
S_x\phi(\boldsymbol{\alpha},\boldsymbol{\tau})\cdot\phi(\boldsymbol{\alpha},\boldsymbol{\tau}) \;=\; &\left[\sum_{\boldsymbol{\mu}}\mathcal{D}_x\phi(2\boldsymbol{\alpha},-\boldsymbol{\mu}/2|2\boldsymbol{\tau})|_{\boldsymbol{\alpha}=0}\right]\\
&\times\phi(2\boldsymbol{\alpha},\boldsymbol{\mu}/2|2\boldsymbol{\tau}),
\end{aligned} \tag{14}$$

where $\boldsymbol{\mu} = (\mu_1,\mu_2,\ldots,\mu_N)^{\mathrm{T}}$ and $\sum_{\boldsymbol{\mu}}$ represents the summation over all possible combinations of $\mu_j = 0,1$ $(j = 1,2,\ldots,N)$.

We provide the following proof for Eq. (13):

$$\begin{aligned}
\Pi \;&\triangleq\; D_x\phi(\boldsymbol{\alpha},\boldsymbol{\tau})\cdot\phi(\boldsymbol{\alpha},\boldsymbol{\tau})\\
&= \sum_{\boldsymbol{p}',\boldsymbol{p}\in Z^N} D_x\exp[2\pi\mathrm{i}\langle\boldsymbol{\alpha},\boldsymbol{p}'\rangle - \pi\langle\boldsymbol{\tau}p',\boldsymbol{p}'\rangle]\\
&\quad\times\exp[2\pi\mathrm{i}\langle\boldsymbol{\alpha},\boldsymbol{p}\rangle - \pi\langle\boldsymbol{\tau}\boldsymbol{p},\boldsymbol{p}\rangle]\\
&= \sum_{\boldsymbol{p}',\boldsymbol{p}\in Z^N} 2\pi\mathrm{i}\langle\boldsymbol{k},\boldsymbol{p}'-\boldsymbol{p}\rangle\exp[2\pi\mathrm{i}\langle\boldsymbol{\alpha},\boldsymbol{p}'+\boldsymbol{p}\rangle\\
&\quad-\pi(\langle\boldsymbol{\tau}\boldsymbol{p}',\boldsymbol{p}'\rangle + \langle\boldsymbol{\tau}\boldsymbol{p},\boldsymbol{p}\rangle)](\text{let }\ \boldsymbol{p} = \boldsymbol{q}'-\boldsymbol{p}')\\
&= \sum_{\boldsymbol{p}',\boldsymbol{q}'\in Z^N} 2\pi\mathrm{i}\langle\boldsymbol{k},2\boldsymbol{p}'-\boldsymbol{q}'\rangle\exp\{2\pi\mathrm{i}\langle\boldsymbol{\alpha},\boldsymbol{q}'\rangle -\\
&\quad\pi(\langle\boldsymbol{\tau}\boldsymbol{p}',\boldsymbol{p}'\rangle + \langle\boldsymbol{\tau}(\boldsymbol{q}'-\boldsymbol{p}'),(\boldsymbol{q}'-\boldsymbol{p}')\rangle)\}\\
&\quad(\text{let }\ \boldsymbol{q}' = 2\boldsymbol{q}+\boldsymbol{\mu})\\
&= \sum_{\boldsymbol{\mu}=0,1}\sum_{\boldsymbol{p}',\boldsymbol{q}\in Z^N} 2\pi\mathrm{i}\langle\boldsymbol{k},2\boldsymbol{p}'-2\boldsymbol{q}-\boldsymbol{\mu}\rangle\\
&\quad\times\exp\{2\pi\mathrm{i}\langle\boldsymbol{\alpha},2\boldsymbol{q}+\boldsymbol{\mu}\rangle - \pi(\langle\boldsymbol{\tau}\boldsymbol{p}',\boldsymbol{p}'\rangle\\
&\quad+\langle\boldsymbol{\tau}(\boldsymbol{p}'-2\boldsymbol{q}-\boldsymbol{\mu}),(\boldsymbol{p}'-2\boldsymbol{q}-\boldsymbol{\mu})\rangle)\},
\end{aligned}$$

if let $\boldsymbol{p}' = \boldsymbol{n}+\boldsymbol{q}$, and with the help of the formulas

$$\begin{aligned}
&\boldsymbol{n}+\boldsymbol{q} = (\boldsymbol{n}-\boldsymbol{\mu}/2)+(\boldsymbol{q}+\boldsymbol{\mu}/2),\\
&\boldsymbol{n}-\boldsymbol{q}-\boldsymbol{\mu} = (\boldsymbol{n}-\boldsymbol{\mu}/2)-(\boldsymbol{q}+\boldsymbol{\mu}/2),
\end{aligned}$$

then

$$\begin{aligned}
\Pi \;=\; &\sum_{\boldsymbol{\mu}=0,1}\Bigg[\sum_{\boldsymbol{n}\in Z^N} 4\pi\mathrm{i}\langle\boldsymbol{k},\boldsymbol{n}-\boldsymbol{\mu}/2\rangle\\
&\times\exp\{-2\pi\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{\mu}/2),(\boldsymbol{n}-\boldsymbol{\mu}/2)\rangle\}\Bigg]\\
&\times\Bigg[\sum_{\boldsymbol{q}\in Z^N}\exp\{2\pi\mathrm{i}\langle 2\boldsymbol{\alpha},\boldsymbol{q}+\boldsymbol{\mu}/2\rangle\\
&-2\pi\langle\boldsymbol{\tau}(\boldsymbol{q}+\boldsymbol{\mu}/2),(\boldsymbol{q}+\boldsymbol{\mu}/2)\rangle\}\Bigg]\\
=\; &\left[\sum_{\boldsymbol{\mu}=\boldsymbol{0},1}\partial_x\phi(2\boldsymbol{\alpha},-\boldsymbol{\mu}/2|2\boldsymbol{\tau})|_{\boldsymbol{\alpha}=\boldsymbol{0}}\right]\phi(2\boldsymbol{\alpha},\boldsymbol{\mu}/2|2\boldsymbol{\tau}).
\end{aligned} \tag{15}$$

We have fully proven that Eq. (14) can be obtained by a similar method.

When the polynomial of operator $F(D_x,D_t,S_x,c)$ acts on two theta functions $\phi(\boldsymbol{\alpha},\boldsymbol{\tau})$, based on Eqs. (13) and (14), we have the following conclusion:

$$\begin{aligned}
&F(D_x,D_t,S_x,c)\phi(\boldsymbol{\alpha},\boldsymbol{\tau})\cdot\phi(\boldsymbol{\alpha},\boldsymbol{\tau})\\
&= \left[\sum_{\boldsymbol{\mu}}B(\boldsymbol{k},\boldsymbol{\omega},\boldsymbol{\sigma},\boldsymbol{\mu})\right]\times\phi(2\boldsymbol{\alpha},\boldsymbol{\mu}/2|2\boldsymbol{\tau}),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
&B(\boldsymbol{k},\boldsymbol{\omega},\boldsymbol{\sigma},\boldsymbol{\mu})\\
&= \sum_{\boldsymbol{n}\in Z^N} F(4\pi\mathrm{i}\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{k}\rangle,4\pi\mathrm{i}\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{\omega}\rangle,\\
&\quad 4\pi\mathrm{i}\langle\boldsymbol{n}-\boldsymbol{\mu}/2,l\boldsymbol{\sigma}+\theta\boldsymbol{k}\rangle,c)\\
&\quad\times\exp[-2\pi\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{\mu}/2),\boldsymbol{n}-\boldsymbol{\mu}/2\rangle].
\end{aligned} \tag{17}$$

From Eqs. (16) and (17), it can be seen that if we make $B(\boldsymbol{k},\boldsymbol{\omega},\boldsymbol{\sigma},\boldsymbol{\mu}) = 0$ for all possible combinations $\mu_j = 0,1$ $(j = 1,2,\ldots,N)$, then $F(S_x,D_x,D_t,c)\phi(\boldsymbol{\alpha},\boldsymbol{\tau})\cdot\phi(\boldsymbol{\alpha},\boldsymbol{\tau}) = 0$. Therefore, we call $B(\boldsymbol{k},\boldsymbol{\omega},\boldsymbol{\sigma},\boldsymbol{\mu}) = 0$ the constraint equations, and there are a total of 2^N equations. For the parameters k_j, ω_j, l_j, σ, ψ_0, c and τ_{ij} $(i\le j,i,j = 1,2,\ldots,N)$, whose total number is $\frac{1}{2}N(N+1)+3N+3$. Among them, k_j, l_j, ψ_0, σ and τ_{jj} are the given parameters, ω_j, c and τ_{ij} $(i<j,i,j = 1,2,\ldots,N)$ are unknown parameters, the number of these unknown parameters is $\frac{1}{2}N(N+1)+1$. When $N = 1,2$, the number of unknown parameters is equal to the number of constraint equations, so the one- and two-periodic wave solutions can be directly calculated. However, if $N\ge 3$, the number of unknown parameters is less than that of the equations, which is an over-determined system. We need to transform the over-determined system into a nonlinear least-squares problem

$$\min\ \boldsymbol{G}(\boldsymbol{x}) = \sum H_i^2(\boldsymbol{x}), \tag{18}$$

where $\boldsymbol{x}=(\omega_j,\tau_{ij},c)^{\mathrm{T}}$ is a vector about all unknown parameters and $\boldsymbol{H}=(H_1,H_2,\ldots,H_{2^N})^{\mathrm{T}}$ is the sequence of all constraint equations. We can use the Gauss–Newton method to solve this problem.

When $n=3$, if σ, k_1, k_2, k_3, l_1, l_2, l_3, τ_{11}, τ_{22}, τ_{33} and $\xi\ll 1$ are given, the detailed calculation process of the Gauss–Newton method is shown as follows.

Step 1 Let $j=0$ and $\boldsymbol{x}^0=(\omega_1^0,\omega_2^0,\omega_3^0,0,0,0,\lambda^0)^{\mathrm{T}}$ are determined. The ω_i $(i=1,2,3)$ can be calculated in Eq. (23).

Step 2 The iterative formula

$$\boldsymbol{x}^{j+1}=\boldsymbol{x}^j-(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J})^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{H}|_{\boldsymbol{x}=\boldsymbol{x}^j} \tag{19}$$

is used to derive new iteration point, where $\boldsymbol{J}$ is the Jacobian matrix of $\boldsymbol{H}$.

Step 3 If $||\boldsymbol{x}^{j+1}-\boldsymbol{x}^j||_2>\xi$ or $||\boldsymbol{H}(\boldsymbol{x}^{j+1})||_2>\xi$, let $j=j+1$, return to Step 2, else go to Step 4.

Step 4 The $\boldsymbol{x}=\boldsymbol{x}^{j+1}$ is adopted as the final result.

However, due to the existence of the Grassmann variable θ, numerical calculation cannot be performed directly. In order to solve the problem, we express c and ψ_0 in the following form:

$$c=c_1+\theta c_2,\quad \psi_0=\psi_1+\theta\psi_2. \tag{20}$$

After substituting Eq. (20) into constraint equation (17), if we extract θ from each constraint equation, two constraint equations that do not include θ are obtained. We redefine 2^{N+1} new constraint equations, and let ψ_1 and ψ_2 be given parameters that can take any constants, c_1 and c_2 be unknown parameters. Thus we have 2^{N+1} equations and $\frac{1}{2}N(N+1)+2$ unknown parameters.

3. The initial point

The Gauss–Newton method is a classic approach for dealing with the least-squares problems, as a second-order optimization algorithm, which has faster convergence speed. However, it is more sensitive to the choice of the initial point. When we use the Gauss–Newton method for iteration, a good initial point can reduce the number of iterations and improve efficiency. For the selection of the initial point, we use the case of $N=1$ to illustrate. The corresponding constraint equations are

$$\begin{aligned}H_1 &= \sum_{n=-\infty}^{+\infty} F(2\pi\mathrm{i}(2n)k,2\pi\mathrm{i}(2n)\omega\\ &\quad\times 2\pi\mathrm{i}(2n)(l\sigma+\theta k),c)\exp\left[-\frac{1}{2}\pi\tau(2n)^2\right],\\ H_2 &= \sum_{n=-\infty}^{+\infty} F(2\pi\mathrm{i}(2n-1)k,2\pi\mathrm{i}(2n-1)\omega\\ &\quad\times 2\pi\mathrm{i}(2n-1)(l\sigma+\theta k),c)\\ &\quad\times\exp\left[-\frac{1}{2}\pi\tau(2n-1)^2\right].\end{aligned} \tag{21}$$

Our goal is to find ω and c making H_1^2 and H_2^2 as small as possible. When n is large, it is obvious that the corresponding exponential functions $\exp[-\frac{1}{2}\pi\tau(2n)^2]$ and $\exp[-\frac{1}{2}\pi\tau(2n-1)^2]$ in Eq. (21) decay to zero very quick. When n is small, if Eq. (21) satisfies the condition

$$F\{2\pi\mathrm{i}k,2\pi\mathrm{i}\omega,2\pi\mathrm{i}(l\sigma+\theta k),c\}=0, \tag{22}$$

then our goal has been achieved. For N-periodic wave solutions, we make the parameters k_j, ω_j and $l_j\sigma+\theta k_j$ $(j=1,2,\ldots,N)$ also satisfy the relationship Eq. (22) to obtain the initial point. For example, when $N=3$, constraint equations should satisfy the conditions

$$F\{2\pi\mathrm{i}k_m,2\pi\mathrm{i}\omega_m,2\pi\mathrm{i}(l_m\sigma+\theta k_m),c\}=0,\quad m=1,2,3. \tag{23}$$

The parameters in Table 1 satisfy Eq. (23).

4. The N-periodic wave solutions and asymptotic relations

On the basis of the above inference, we obtain the numerical solutions of Eq. (1) with the help of Matlab R2018b (for academic purposes) on a computer with a 2.50 GHz CPU and 16.0 GB main memory. The calculation error is around 10^{-14}.

Table 1. Parameters for the three-periodic wave solutions of Eq. (1) when $\sigma=1$, $\psi_1=0$ and $\psi_2=0$.

k_1	k_2	k_3	l_1	l_2	l_3	τ_{11}	τ_{22}	τ_{33}
0.14	0.16	0.18	−0.14	−0.16	−0.18	1.2	1.2	1.2
ω_1	ω_2	ω_3	τ_{12}	τ_{13}	τ_{23}	c_1	c_2	
−0.0993	−0.1646	−0.2786	0.9363	0.7912	0.9554	−0.1592	0.1592	
k_1	k_2	k_3	l_1	l_2	l_3	τ_{11}	τ_{22}	τ_{33}
0.14	0.16	0.18	−0.14	−0.16	−0.18	3.2	3.2	3.2
ω_1	ω_2	ω_3	τ_{12}	τ_{13}	τ_{23}	c_1	c_2	
−0.0838	−0.1634	−0.2945	1.0350	0.8303	1.0752	−0.0000	0.0000	
k_1	k_2	k_3	l_1	l_2	l_3	τ_{11}	τ_{22}	τ_{33}
0.12	0.16	0.2	−0.06	−0.16	0.4	1.5	1.5	1.5
ω_1	ω_2	ω_3	τ_{12}	τ_{13}	τ_{23}	c_1	c_2	
−0.1259	−0.3014	−0.4511	0.5900	0.4227	0.5986	0.1013	0.0679	

In Table 1, since the three-soliton solution needs to satisfy condition $a_p b_q = a_p b_q$ $(p,q=1,2,3)$, we also let the first set of data satisfy $k_p l_q = k_q l_p$. The second set of data is based on the first set of data, where τ_{pp} become very large. From the results, it can be seen that c approaches 0. By combining with the theoretical analysis, the asymptotic relations between quasi-periodic waves and solitons are verified. We select the third group that does not meet $k_p l_q = k_q l_p$ to compare with the first set of data. At the same time, three-dimensional plots and density plots corresponding to each set of data are plotted. By employing the analytical method[52,53] related to the characteristic lines, we can obtain some characteristics of the three-periodic waves.

(I) It has six fundamental periods (e_1,e_2,e_3) and $(\mathrm{i}\tau_1,\mathrm{i}\tau_2,\mathrm{i}\tau_3)$ in the phase variables $(\alpha_1,\alpha_2,\alpha_3)$.

(II) Under the influence of the Grassmann variable θ, there will be an influencing band in the three-periodic wave, as shown in Fig. 1(a). As the distance from the influencing band increases, the amplitude of the three-periodic wave will become bigger. When θ is very small, the super three-periodic wave declines into a general three-periodic wave, as shown in Fig. 1(b).

(III) We add characteristic lines to the density plots of the three-periodic waves to analyze the dynamic characteristics. In Figs. 1(c) and 2(c), the three characteristic lines are $\alpha_1 = 0.14x - 0.14\theta = 0$, $\alpha_2 = 0.16x - 0.16\theta = 0$ and $\alpha_3 = 0.18x - 0.18\theta = 0$, respectively. Due to condition $k_p l_q = k_q l_p$ $(p,q=1,2,3)$, the slopes of the three characteristic lines are equal, the three characteristic lines overlap together, so the three-periodic wave presents a pattern similar to the one-periodic wave. As τ_{jj} $(j=1,2,3)$ increase, it can be seen from Figs. 2(a) and 2(b) that the amplitude of the wave decreases, which proves that τ_{jj} affect the amplitude.

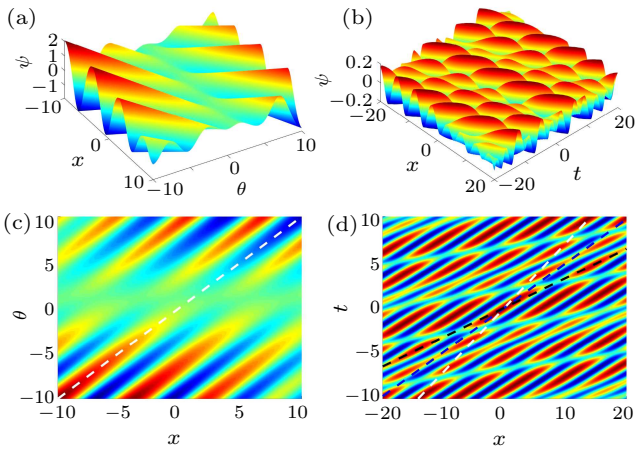


Fig. 1. Three-periodic wave related to the first set of data in Table 1. (a) Three-dimensional plot of three-periodic wave when $t=0$. (b) Three-dimensional plot of three-periodic wave when $\theta=0$. (c) Vertical view of plot (a). (d) Vertical view of plot (b).

(IV) The peaks and valleys in Fig. 3(a) are not propagating in the same direction. In Fig. 3(c), the characteristic lines represented by the red line, white line and black line are $\alpha_1 = 0.12x - 0.06\theta = 0$, $\alpha_2 = 0.16x - 0.16\theta = 0$ and $\alpha_3 = 0.2x - 0.4\theta = 0$, the three-periodic wave is formed through the overlap and collision of one-periodic wave corresponding to each characteristic line. The general three-periodic waves on the plane of (x,t) shown in Figs. 1(d), 2(d) and 3(d) have already been analyzed in Refs. [54,55].

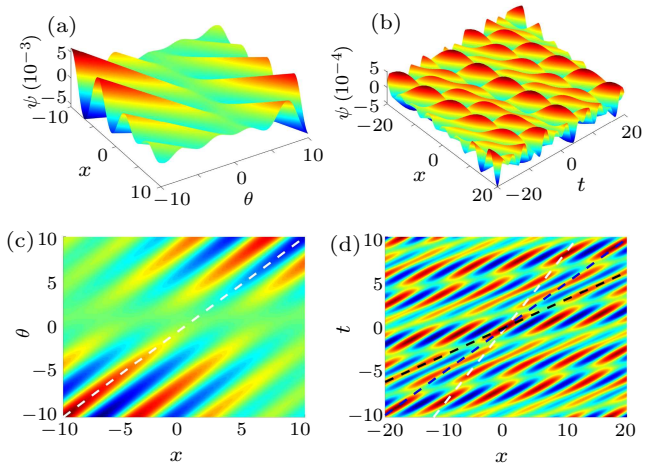


Fig. 2. Three-periodic wave related to the second set of data in Table 1. (a) Three-dimensional plot of three-periodic wave when $t=0$. (b) Three-dimensional plot of three-periodic wave when $\theta=0$. (c) Vertical view of plot (a). (d) Vertical view of plot (b).

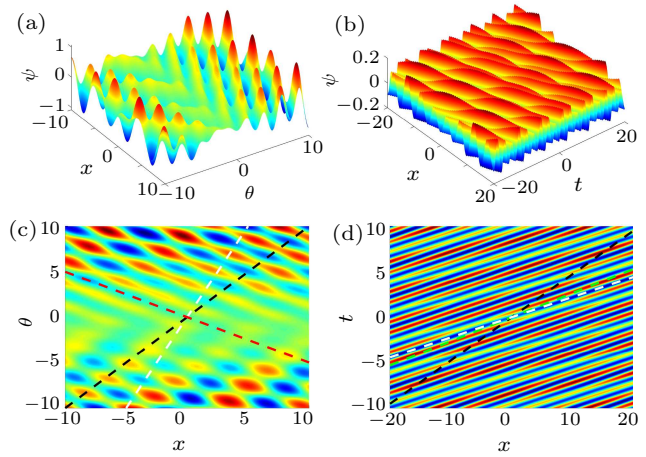


Fig. 3. Three-periodic wave related to the third set of data in Table 1. (a) Three-dimensional plot of three-periodic wave when $t=0$. (b) Three-dimensional plot of three-periodic wave when $\theta=0$. (c) Vertical view of plot (a). (d) Vertical view of plot (b).

The asymptotic relations of N-periodic wave solutions and N-soliton solutions are established as follows. If we let

$$k_p = \frac{a_p}{2\pi\mathrm{i}}, \quad l_p = \frac{b_p}{2\pi\mathrm{i}}, \quad \eta_p = \frac{\gamma_p + \pi\tau_{pp}}{2\pi\mathrm{i}},$$
$$\tau_{pq} = -\frac{A_{pq}}{2\pi}, \quad (p<q,\ p,\ q=1,2,\ldots,N), \tag{24}$$

under condition $k_p l_q = k_q l_p$, we have the following conclusion:

$$\psi_0 \to 0, \quad c \to 0, \quad \alpha_p \to \frac{\vartheta_p + \pi\tau_{pp}}{2\pi\mathrm{i}},$$
$$\phi(\boldsymbol{\alpha},\boldsymbol{\tau}) \to f, \quad \text{as} \quad \lambda_{pp} \to 0. \tag{25}$$

The corresponding proof is provided as follows.

Let

$$\lambda_{pq}=\mathrm{e}^{-\pi\tau_{pq}/2},\quad (p<q,p,q=1,2,\ldots,N) \tag{26}$$

and

$$\begin{aligned}&\phi_j(\boldsymbol{\alpha},\boldsymbol{\lambda})=\phi(2\boldsymbol{\alpha},-\boldsymbol{r}_j/2|2\boldsymbol{\tau})\\ =&\sum_{\boldsymbol{n}\in Z^N}\exp(4\pi\mathrm{i}\langle\boldsymbol{\alpha},\boldsymbol{n}-\boldsymbol{r}_j/2\rangle)\\ &\times\prod_{p,q=1}^{N}\lambda_{pq}^{(2n_p-r_{j,p})(2n_q-r_{j,q})},\quad j=1,2,\ldots,2^N,\end{aligned} \tag{27}$$

where $\boldsymbol{r}_j=(r_{j,1},r_{j,2},\ldots,r_{j,N})^{\mathrm{T}}$ represent all possible combinations of $r_{j,m}=0,1$ $(m=1,2,\ldots,N)$. We expand ω_i,c and ψ_0 as follows:

$$\begin{aligned}\omega_i&=\omega_{i,0}+\omega_{i,1}\lambda_{11}+\omega_{i,2}\lambda_{22}+\omega_{i,3}\lambda_{33}+\cdots\\&=\omega_{i,0}+o(\lambda_{11},\lambda_{22},\ldots,\lambda_{NN}),\\ c&=c_{0,0}+c_{0,1}\lambda_{11}+c_{0,2}\lambda_{22}+c_{0,3}\lambda_{33}+\cdots\\&=c_{0,0}+o(\lambda_{11},\lambda_{22},\ldots,\lambda_{NN}),\\ \psi_0&=\psi_{0,0}+\psi_{0,1}\lambda_{11}+\psi_{0,2}\lambda_{22}+\psi_{0,3}\lambda_{33}+\cdots\\&=\psi_{0,0}+o(\lambda_{11},\lambda_{22},\ldots,\lambda_{NN}),\\ &i=1,2,\ldots,N.\end{aligned} \tag{28}$$

We also expand $\phi_j(\boldsymbol{\alpha},\boldsymbol{\lambda})$ into a polynomial of λ_{pq}

$$\begin{aligned}\phi_1(\boldsymbol{\alpha},\boldsymbol{\lambda})&=\sum_{\boldsymbol{n}\in Z^N}\exp(4\pi\mathrm{i}\langle\boldsymbol{\alpha},\boldsymbol{n}-\boldsymbol{r}_1/2\rangle)\\&\quad\times\prod_{p,q=1}^{N}\lambda_{pq}^{(2n_p-r_{1,p})(2n_q-r_{1,q})}\\&=1+\exp(4\pi\mathrm{i}\alpha_1)\lambda_{11}^4+\exp(4\pi\mathrm{i}\alpha_2)\lambda_{22}^4+\cdots,\\ \phi_2(\boldsymbol{\alpha},\boldsymbol{\lambda})&=\sum_{\boldsymbol{n}\in Z^N}\exp(4\pi\mathrm{i}\langle\boldsymbol{\alpha},\boldsymbol{n}-\boldsymbol{r}_2/2\rangle)\\&\quad\times\prod_{p,q=1}^{N}\lambda_{pq}^{(2n_p-r_{2,p})(2n_q-r_{2,q})}\\&=\exp(-2\pi\mathrm{i}\alpha_1)\lambda_{11}+\exp(2\pi\mathrm{i}\alpha_1)\lambda_{11}\\&\quad+\exp(-2\pi\mathrm{i}\alpha_1+4\pi\mathrm{i}\alpha_2)\lambda_{11}\lambda_{12}^{-4}\lambda_{22}^4+\cdots,\\ \phi_3(\boldsymbol{\alpha},\boldsymbol{\lambda})&=\sum_{\boldsymbol{n}\in Z^N}\exp(4\pi\mathrm{i}\langle\boldsymbol{\alpha},\boldsymbol{n}-\boldsymbol{r}_3/2\rangle)\\&\quad\times\prod_{p,q=1}^{N}\lambda_{pq}^{(2n_p-r_{3,p})(2n_q-r_{3,q})}=\exp(-2\pi\mathrm{i}\alpha_2)\lambda_{22}\\&\quad+\exp(-2\pi\mathrm{i}\alpha_2+4\pi\mathrm{i}\alpha_1)\lambda_{11}^4\lambda_{12}^{-4}\lambda_{22}\\&\quad+\exp(2\pi\mathrm{i}\alpha_2)\lambda_{22}+\cdots,\\&\quad\vdots\end{aligned} \tag{29}$$

where $r_1=(0,0,\ldots,0)^{\mathrm{T}}$, $r_2=(1,0,\ldots,0)^{\mathrm{T}}$, $r_3=(0,1,\ldots,0)^{\mathrm{T}}$, $\cdots,r_{2^N}=(1,1,\ldots,1)^{\mathrm{T}}$. After substituting Eqs. (28) and (29) into Eq. (17) and letting $\lambda_{pp}\to 0$, then

$$B(\omega_{j,0},c_{0,0},\psi_{0,0})=0 \tag{30}$$

is obtained. By solving Eq. (30), we can get

$$c_{0,0}=0,\quad \psi_{0,0}=0,\quad \omega_{i,0}=-16\pi^4k_i^5, \tag{31}$$

which imply that

$$\begin{aligned}c&=o(\lambda_{11},\lambda_{22},\ldots,\lambda_{NN})\to 0,\\ \psi_0&=o(\lambda_{11},\lambda_{22},\ldots,\lambda_{NN})\to 0,\\ \omega_i&=-16\pi^4k_i^5+o(\lambda_{11},\lambda_{22},\ldots,\lambda_{NN})\\&\to-16\pi^4k_i^5,\ \text{ as }\ \lambda_{pp}\to 0.\end{aligned} \tag{32}$$

Hence we obtain

$$\alpha_p\to\frac{\vartheta_p+\pi\tau_{pp}}{2\pi\mathrm{i}},\ \text{ as }\ \lambda_{pp}\to 0. \tag{33}$$

If we expand $\phi(\boldsymbol{\alpha},\boldsymbol{\tau})$ as follows:

$$\begin{aligned}\phi(\boldsymbol{\alpha},\boldsymbol{\tau})&=1+\sum_{p=1}^{N}(\mathrm{e}^{2\pi\mathrm{i}\alpha_p}+\mathrm{e}^{-2\pi\mathrm{i}\alpha_p})\mathrm{e}^{-\pi\tau_{pp}}\\&\quad+\sum_{p,q=1,p<q}^{N}(\mathrm{e}^{2\pi\mathrm{i}(\alpha_p+\alpha_q)}+\mathrm{e}^{-2\pi\mathrm{i}(\alpha_p+\alpha_q)})\\&\quad\times\mathrm{e}^{-\pi(\tau_{pp}+2\tau_{pq}+\tau_{qq})}+\cdots,\end{aligned} \tag{34}$$

and substitute Eqs. (24), (32) and (33) into Eq. (34), the conclusion

$$\phi(\boldsymbol{\alpha},\boldsymbol{\tau})\to f \tag{35}$$

is obtained.

5. Conclusions

In summary, we have provided a direct method to calculate the N-periodic wave solutions for the $\mathcal{N}=1$ supersymmetric Sawada–Kotera–Ramani equation, which is based on the bilinear method and the super Riemann theta functions. During this process, due to the presence of Grassmann variable θ, we redefine new constraint equations and unknown variables and successfully transform the problem into a least-squares problem. For the least-squares problem, in order to seek the global optimal solution by using the Gauss–Newton method, we provide a selection of the initial point. This method can also be used to solve other supersymmetric equations. On the plane of (x,θ), it can be seen that there is also an influencing band in the three-periodic waves, and as the distance from the influencing band increases, the amplitude also becomes greater. We also find that the three-periodic waves in Figs. 1(a) and 2(a) decay to the one-periodic waves, but Fig. 3(a) obviously shows the overlap and collision of several waves. After adding the characteristic lines, it is clear that this phenomenon is officially caused by the slopes of the characteristic lines. The slopes of the three characteristic lines in Figs. 1(a) and 2(a) are equal, while the three characteristic lines in Fig. 3(a) intersect. We have theoretically

proven the asymptotic relations between N-periodic wave solutions and N-soliton solutions. Under conditions $a_i b_j = a_j b_i$ $(i, j = 1, 2, 3)$, the three-soliton solutions of Eq. (1) can be obtained, we make sure that the three-periodic waves also satisfy the conditions $k_i l_j = k_j l_i$ to verify the asymptotic relations. We list some examples of numerical solutions for the three-periodic waves in Table 1. By comparing the first two sets of data, we find c infinitely close to 0 under the diagonal elements of τ being large, which precisely corresponds to our theoretical proof. We hope that the theory presented in this paper will provide benefits for studying the complex wave structures of the supersymmetric integrable systems.

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