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Hawking radiation and Page curve of regular black holes

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We discuss the information paradox and its implications for regular black holes. Our primary focus is on Page curve using the island treatment and analyzing relevant parameters like Page time and scrambling time. Calculations without considering the island show that the entanglement entropy increases linearly and continues to infinity. When we consider the generalized entropy, we find that the island extends just beyond the horizon, leading to a constant entanglement entropy. Specifically, we find that in the early stages, the island never forms, regardless of the charge and mass configuration of the black hole.

Keywords: black holes, AdS/CFT correspondence, Page curve

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1. Introduction

Since the mature development of black hole thermodynamics,[1–4] the black hole information paradox has emerged as a significant issue in quantum gravity. The black hole information paradox, which involves both quantum mechanics and general relativity, is a key for exploring quantum gravity and deepening our understanding of it. The origin of the information paradox can be traced back to the proposal of Hawking radiation by Stephen Hawking in 1975.[2] On one hand, Hawking radiation represents a type of thermal radiation that lacks information, leading to the continuous increase of entanglement entropy in a black hole. On the other hand, the unitarity of quantum mechanics asserts that the initial and final stages of the black hole evaporation process must remain in a pure state. This contrast indicates that the entanglement entropy increases in the initial stages, then begins to decrease at a certain time, and ultimately tends to zero as the evaporation completes. The process described above, where the entropy first increases and then decreases, starting from zero and returning to zero, is commonly known as the Page curve, introduced by Page in Ref. [5].

The AdS/CFT duality, first proposed by Juan Maldacena in Ref. [6] is a profound idea and provides a powerful tool for understanding black holes and the information paradox. The insights from the holographic principle have generated a wealth of significant work, including several early works have thoroughly examined the information paradox,[7–9] and provides a solid foundation for further studies in this area. Recently, one of the most well-known work related to it is the discovery of "island".[10] The idea can be traced back to the study of quantum corrections via Ryu–Takayanagi (RT) formula[11] and further in-depth studies.[12,13] The general entropy proposed as the sum of the area term and the matter term. When considering higher-order quantum corrections, quantum extremal surfaces (QES) are used to represent the entropy.[14–16] The entanglement entropy is the minimal extreme of general entropy as follows:

$$S_{\mathrm{Rad}} = \min\left\{\mathrm{ext}\left[\frac{\mathrm{Area}(\partial I)}{4G_{\mathrm{N}}} + S_{\mathrm{matter}}(R\cup I)\right]\right\}. \tag{1}$$

Here I stands for the island, the boundary ∂I of which is the quantum extremal surface. We have two entities need to be addressed using the same approach: the matter term has a UV cutoff because of the divergences[17,18] and the Newton constant should also be renormalized for the same reason.[19] Although the formula is proposed for calculating the entanglement entropy in AdS spacetime, its applicability is widely broadened with further research.[20–22] Firstly it is used to calculate the entropy under the two-dimensional (2D) JT gravity.[10,23] The holographic CFT (HCFT) model represents a significant advancement in this field, allowing for the direct calculation of entropy dynamics without the OPE approximation as discussed in Refs. [15,23,24]. Apart from holographic method above, the island rules can also be verified and derived not only in CFT but also in the gravitational inertial using the replica wormhole method. Additionally, low-dimensional SYK models play an essential role in these studies by offering a microscopic explanation for gravitational dynamics.[21,25–27]

The original calculation of the Page curve depends on the factorization of the Hilbert space. Some studies have shown that the assumption of a factorizable Hilbert space in the calculation of entanglement entropy no longer holds.[28,29] Coupling a 'bath' system has successfully addressed this is-

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sue and recently research has started exploring how this approach might work in the more challenging case of asymptotically flat spacetime.[30] In Ref. [24], the island is shown to be located outsside the horizon in eternal black hole under the 2D JT gravity, and its Page curve can be derived. The island concept has since been extended beyond 2D JT gravity, making significant contributions to a variety of spacetime settings. The asymptotically flat spacetime case was discussed in Refs. [31–33] under different conditions. A family of other metrics for eternal black holes including Schwarzschild black holes,[34] Reissner–Nordström (RN) black holes,[35] BTZ black holes,[36] has also been explored in Refs. [37–45] These developments suggested that the island configuration may be applicable to a wide range of black hole types, which can help us better understand black hole entropy and the information paradox. This implies that the island configuration has the potential to be used in a wide range of black hole types.

Regular black holes (also written as RBH) are a specific category of black holes. Unlike typical black holes, RBHs have coordinate singularities at the horizon but lack the central singularity.[46,47] Bardeen introduced the first RBH model.[48] Subsequently, many other RBH metrics have been discovered, such as the Hayward solution[49] and the Frolov solution.[50] Due to the de Sitter core inside the black hole, singularities do not emerge. The entropy of a regular black hole has been calculated and proved to obey the area law.[51] Recently, another class of metrics has been found where singularities are replaced, not by de Sitter spacetime, but by Minkowski spacetime.[52–56] As a simple model, the Bardeen black hole shares a similar inner causal structure with the RN black hole, but the inner singularity is replaced by a regular center.[49] Additionally, its metric resembles the Schwarzschild black hole at large distances. The black holes mentioned above have all been studied, as reported in Refs. [34,35]. This asymptotic behavior guides our study, and allows us to compare with RN or Schwarzschild black holes when necessary. Unlike the common black holes, regular black holes, with a unique structure, offering new insights to understand the information paradox. This enables us to explore deeper connections and enrich our understanding of black hole entropy.

Regular black holes (RBHs) are interesting for their absence of a central singularity, a feature that distinguishes them from classical black hole models like the Schwarzschild and RN black holes. Hayward, for instance, argued that if no singularity exists, the information paradox might not arise.[49,55] By studying RBHs, we are able to investigate the role singularities play in the black hole information paradox, offering new perspectives on how information may be preserved or lost in the presence or absence of a singularity. We will provide further comments on this situation and explain why it is meaningful to apply the island configuration to regular black holes. Additionally, we will provide comparative analysis and commentary on Hayward's claim based on our results.

In this paper, we analyze the entanglement entropy of Hawking radiation for an eternal regular black holes, focusing on the influence of island contributions on the entropy calculation. The structure of our work will be as follows: In Section 2, we will present a brief review of the regular black hole to ensure the setup prepared. In Section 3, we will derive the entanglement entropy under the island configuration on various occasions. Section 4 will address the Page time and the scrambling time, followed by a brief discussion about the Page curve. Finally, in Section 5, we will engage in a discussion on the Page curve within the black hole we are studying, and give an oppinon about the requirements of information paradox for regular black holes.

2. Review of regular black hole

In this section, we will give a brief review of the regular black hole with specific attention to the Bardeen black hole. Then we will calculate the Kruskal coordinate that will be used in subsequent calculations. We will also briefly introduce the Hayward statement about regular black holes and information paradox.

Regular black hole (also known as non-singular black hole) is a collection of black hole that does not exist singularity. A simple example is the Bardeen metric introduced by Bardeen.[48] It is similar to the RN black hole, except that the singularity inside is replaced by the regular center at $r = 0$.

A spherically symmetric, static black hole with metric is as follows:

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\,\mathrm{d}\Omega^2, \tag{2}$$

where the symbols are defined as

$$\mathrm{d}\Omega^2 = r\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2 \sin^2\theta. \tag{3}$$

The shape function of Bardeen metric satisfies

$$f(r) = 1 - 2m\frac{r^2}{(r^2+q^2)^{3/2}}, \tag{4}$$

where m represents the mass of the black hole and q represents the charge. Another example is the Hayward metric, represented as

$$f(r) = 1 - 2m\frac{r^2}{r^3+\alpha_0 m}. \tag{5}$$

Both metrics share a similar form of $f(r) = 1 - 2mr^2$ term, except for the denominator.

Let us have a quick review of the asymptotic behavior. For small values of r, the Bardeen metric behaves like de Sitter metric

$$f(r) \sim 1 - 2\frac{m}{q^3}r^2. \tag{6}$$

Similarly, the Hayward metric obeys

$$f(r) \sim 1 - 2\frac{m}{\alpha_0^2} r^2. \tag{7}$$

When r tends to infinity or q tends to 0, both of these metrics behaves like Schwarzschild metric as[47]

$$f(r) \sim 1 - \frac{2m}{r}. \tag{8}$$

To analyze the zero points of $f(r)$, we introduce the critical mass m_* and its corresponding critical radius r_*. When $m = m_*$, it represents an extremal black hole and the function $f(r) = 0$ has one (or two equal) zero points at radius r_*. As shown in Fig. 1, we can see the conditions of regular black holes with varying mass. When $m < m_*$, there is no black hole since the function $f(r) = 0$ has no zero points. When $m > m_*$ the black hole has two positive zero points: the outer horizon at r_+ and the inner horizon at r_-.

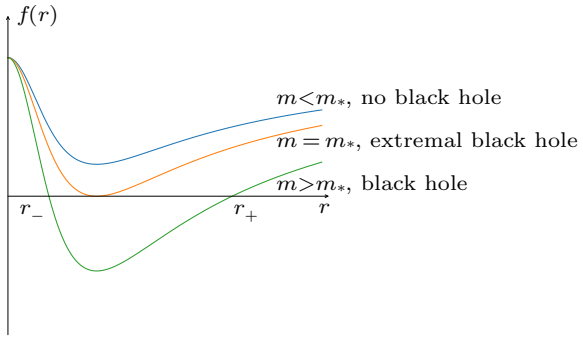


Fig. 1. The diagram of metric with shape function $f(r)$. We can see when the mass $m > m_*$, the shape function has two zero points, corresponding to a formed black hole. When mass equals the critical mass $m = m_*$, the black hole becomes an extremal black hole, with two horizons shrink to one point. When $m > m_*$, there is no formed black holes.

Minimizing the shape function, we find that the critical mass satisfies $m = \frac{3\sqrt{3}}{4}q$ at r_*. and the critical radius is solved as $r_0 = \sqrt{2}q$.

We can have a quick summary from Fig. 2, we image a line parallel to the r-axis representing the mass of the black hole. When the black hole forms, the red line will increase, when two horizons extend. When the black hole lose mass through Hawking radiation, the red line decrease while the two horizon shrink. Once the mass is less than the critical mass corresponding to the lowest point in the figure, the black hole can no longer exist in the spacetime.

By setting $f(r) = 0$, the mass–radius relation could be evaluated as follows:

$$m = \frac{1}{2}\frac{(r^2+q^2)^{3/2}}{r^2}. \tag{9}$$

For $m \ll m_*$, we have $r_- \sim \sqrt{q^3/m}$ and $r_+ \sim 2m$, which could also be seen in Fig. 2.

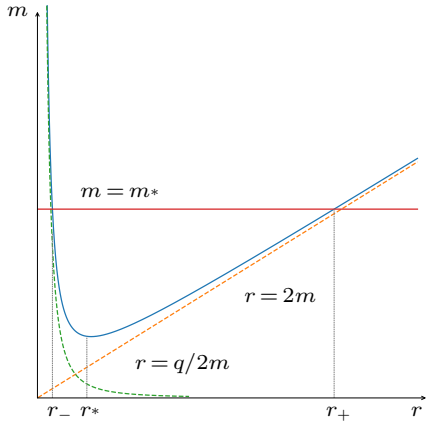


Fig. 2. The relationship between mass and radius shown above. The blue color describes the metric. It behaves like tow asymptotic curves showing the asymptotic behavior in $r \to 0$, $m = q^3/2r^2$, and $r \to \infty$ with $r = 2m$.

The similarity of the two shape functions in Eqs. (4) and (5) pursuit us to expand the $\left(r^2/(r^2+q^2)\right)^{3/2}$ term in the Bardeen metric to the third order of r. Expanding the function gives us the function of $f(r)$ as a polynomial in r as

$$f(r) = 1 - \frac{2m}{r} + \frac{3mq^2}{r^3}. \tag{10}$$

Following the approach employed in Ref. [56] in the study of RN black hole, we multiply both the right-hand side (RHS) and the left-hand side (LHS) by r^3 and get a polynomial in r

$$r^3 f(r) = r^3 - 2mr^2 + 3mq^2. \tag{11}$$

We seek solutions to the equation $f(r) = 0$ to locate the horizon. Hence, we can rewrite the equation as

$$r^3 f(r) = (r - r_1)(r - r_2)(r - r_3). \tag{12}$$

Here, r_1, r_2, and r_3 are the three roots of the equation, which make the RHS of Eq. (12) equals zero.

Now let us shift our focus to Kruskal coordinates. The tortoise coordinates are defined as follows:

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{1}{f(r)}. \tag{13}$$

For further clarification, substituting $f(r)$ into it yields

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{r^3}{(r-r_1)(r-r_2)(r-r_3)}. \tag{14}$$

Integrating both sides gives the tortoise coordinates as

$$\begin{aligned} r^* = r &+ \frac{r_1^3}{(r_1-r_2)(r_1-r_3)}\log(r-r_1) \\ &+ \frac{r_2^3}{(r_2-r_1)(r_2-r_3)}\log(r-r_2) \\ &+ \frac{r_3^3}{(r_3-r_1)(r_3-r_2)}\log(r-r_3). \end{aligned} \tag{15}$$

A simplified expression could be

$$r^* = r + \sum_i A_i \log(r - r_i), \tag{16}$$

where the coefficient is defined by

$$A_i = \frac{r_i^3}{(r_i - r_j)(r_i - r_k)}. \tag{17}$$

As a third order equation, the fundamental theorem of algebra guarantees 3 real roots if the discriminant discriminant Δ is positive. The charge denoted by q is assumed not be too large ensuring that the $f(r)$ has two zero points where $r > 0$ denoted as r_1 and r_2. For convenience, the positive solutions will be written as r_+ and r_-. The coefficient A_i can be obtained through the following Vieta's formulas:

$$\begin{aligned} &r_1 + r_2 + r_3 = 2m, \\ &r_1 r_2 + r_2 r_3 + r_1 r_3 = 0, \\ &r_1 r_2 r_3 = 3mq^2. \end{aligned} \tag{18}$$

Following Eq. (18), we can always use m and q to denote the coefficient A_i.

Another important physical quantity of the black hole is Hawking temperature. The Hawking temperature is determined by the surface gravity at the outer horizon

$$T_{\rm H} = \frac{1}{\beta} = \frac{\kappa_+}{2\pi} = \frac{f'(r_+)}{4\pi}. \tag{19}$$

Here κ_+ is the surface gravity at the outer horizon r_+. If no further explainations, all κ refer to the surface gravity at the outer horizon.

Now we can apply the surface gravity on the Kruskal coordinates. The Kruskal coordinates are given by

$$\begin{aligned} v &= t + r^*, \\ u &= t - r^*. \end{aligned} \tag{20}$$

By using $\mathrm{d}r^* = \mathrm{d}r/f(r)$, the metric can be written as

$$\begin{aligned} \mathrm{d}s^2 &= f(r)(\mathrm{d}t^2 - \mathrm{d}r^{*2}) + r^2 \mathrm{d}\Omega^2 \\ &= -f(r)\mathrm{d}u\mathrm{d}v + r^2 \mathrm{d}\Omega^2. \end{aligned} \tag{21}$$

With the representation of the surface gravity, the Kruskal coordinate in Bardeen black hole can be represented as

$$\begin{aligned} U &\equiv -\mathrm{e}^{-\kappa_+ u}, \\ V &\equiv \mathrm{e}^{\kappa_+ v}. \end{aligned} \tag{22}$$

Similarly the metric can be expressed by the new coordinate with

$$\mathrm{d}s^2 = -f(r)\frac{\mathrm{d}U\,\mathrm{d}V}{\kappa^2 UV} + r^2 \mathrm{d}\Omega^2, \tag{23}$$

we can obtain the metric in the new coordinate using $UV = -\mathrm{e}^{2\kappa r^*}$

$$\mathrm{d}s^2 = f(r) \cdot \frac{\mathrm{e}^{-2\kappa_+ r^*}}{\kappa_+^2} \mathrm{d}U\,\mathrm{d}V + r^2 \Omega^2. \tag{24}$$

Finally, we obtain the following relationship:

$$\mathrm{d}s^2 = W^2(r)\mathrm{d}U\,\mathrm{d}V + r^2\Omega^2, \tag{25}$$

with the conformal factor

$$W^2(r) = -f(r)\frac{\mathrm{e}^{-2\kappa_+ r^*}}{\kappa_+^2}. \tag{26}$$

The conformal factor has similarity to RN black hole. From the coordinates we can obtain the entanglement entropy throughout the Kruskal coordinate in the next section.

3. Entanglement entropy of regular black hole

In this section, we will first complete groundwork for calculating the entanglement entropy, mainly focusing on the coordinates. Then, we will calculate the entanglement entropy in two situations: The entropy without island contribution in Subsection 3.1 and the entropy when the island closed to the horizon location at Subsection 3.2.

Notice that the generalized entanglement entropy including the island is given by

$$S = \min\left\{\mathrm{ext}\left[\frac{\mathrm{Area}(\partial I)}{4G_{\rm N}} + S_{\rm matter}(R \cup I)\right]\right\}. \tag{27}$$

the matter term of the entanglement entropy follows

$$S_{\rm matter} = \frac{c}{6}\log d^2(b_+, b_-), \tag{28}$$

where b_+ and b_- represent the cutoff as shown in Fig. 3.

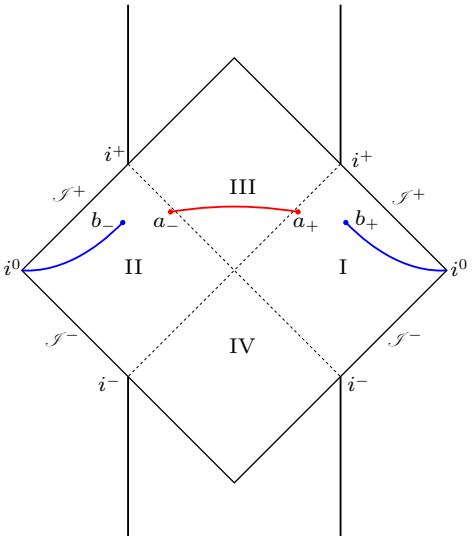


Fig. 3. The Penrose diagram depicts a portion of an eternal Bardeen black hole. It is RN-like, but the bolded line is the regular center corresponding the singularity of non-regular black hole. The boundary of island stands outside the horizon marked with symbol a, while the symbol b represents the cutoff where the radiation is all going beyond. The dashed line denotes the horizon of the black hole.

The position of the island and the cutoff distance in different regions are noted by

$$a_+ = (t_a, a), \quad a_- = \left(-t_a + \frac{\mathrm{i}\beta}{2}, a\right),$$

$$b_+ = (t_b, b), \quad b_- = \left(-t_b + \frac{\mathrm{i}\beta}{2}, b\right). \tag{29}$$

Given any two positions with x and y, the distance between them in the specific metric expressed by Kruscal coordinates is calculated by

$$\mathrm{d}^2(x,y) = \frac{(U(y)-U(x))(V(x)-V(y))}{\mathcal{W}(x)\mathcal{W}(y)}, \tag{30}$$

where $\mathcal{W}$ is defined as the reciprocal $1/W$.

Notice that in different wedges, the Kruscal coordinates have inverse values shown by

$$\begin{aligned} U_{\mathrm{R}} &= -\mathrm{e}^{-\kappa(t-r^*)}, \quad V_{\mathrm{R}} = +\mathrm{e}^{+\kappa(t+r^*)}, \\ U_{\mathrm{L}} &= +\mathrm{e}^{-\kappa(t-r^*)}, \quad V_{\mathrm{L}} = -\mathrm{e}^{+\kappa(t+r^*)}. \end{aligned} \tag{31}$$

Substituting the coordinate expressions we can conclude the following.

(i) If x and y have the same sign in the subscript, for example $\mathrm{d}^2(a_+, b_+)$, we obtain

$$\mathrm{d}^2(x,y) = \frac{\mathrm{e}^{2\kappa x^*} + \mathrm{e}^{2\kappa y^*} - \mathrm{e}^{2\kappa(x+y)^*} \cdot 2\cosh\left[\kappa(t_x - t_y)\right]}{\mathcal{W}(x)\mathcal{W}(y)}. \tag{32}$$

(ii) If x and y have opposite signs in the subscript position, we obtain

$$\mathrm{d}^2(x,y) = \frac{\mathrm{e}^{2\kappa x^*} + \mathrm{e}^{2\kappa y^*} + \mathrm{e}^{2\kappa(x^*+y^*)} \cdot 2\cosh\left[\kappa(t_x + t_y)\right]}{\mathcal{W}(x)\mathcal{W}(y)}. \tag{33}$$

Notice that

$$\frac{\mathrm{e}^{2\kappa x^*} + \mathrm{e}^{2\kappa y^*}}{2\,\mathrm{e}^{\kappa(x^*+y^*)}} = \cosh\kappa(x^* - y^*). \tag{34}$$

The common factors $2\,\mathrm{e}^{\kappa(x^*+y^*)}$ can be extracted from the above equation, so we can derive the expression about distance between x and y as follows:

(iii) If x and y have the same sign, we have

$$\mathrm{d}^2(x,y) = \frac{2\,\mathrm{e}^{\kappa(x^*+y^*)}\left(\cosh\left[\kappa(x^* - y^*)\right] - \cosh\left[\kappa(t_x - t_y)\right]\right)}{\mathcal{W}(x)\mathcal{W}(y)}. \tag{35}$$

(iv) If x and y have the opposite signs, we obtain

$$\mathrm{d}^2(x,y) = \frac{2\,\mathrm{e}^{\kappa(x^*+y^*)}\left(\cosh\left[\kappa(x^* - y^*)\right] + \cosh\left[\kappa(t_x + t_y)\right]\right)}{\mathcal{W}(x)\mathcal{W}(y)}. \tag{36}$$

3.1. Entanglement entropy without island

In this section we will discuss the entanglement entropy in the absence of the island. Recalling entanglement entropy represented as

$$S_{\mathrm{Rad}} = \min\left\{\mathrm{ext}\left[\frac{\mathrm{Area}(\partial I)}{4G_{\mathrm{N}}} + S_{\mathrm{matter}}(R \cup I)\right]\right\}. \tag{37}$$

According to expression (27), it is clear that the entropy has two components: the area term and the contribution of matter. It is evident that without island, the entanglement entropy is solely determined by the matter term.

Applying the method from CFT, the entanglement entropy without the contribution of island can be expressed as

$$S_{\mathrm{matter}} = \frac{c}{6}\log \mathrm{d}^2(b_+, b_-), \tag{38}$$

which solely depends on the cutoff $b_\pm$.

Substituting the previously mentioned coordinates into Eq. (35), we obtain

$$S_{\mathrm{Rad}} = \frac{c}{6}\log\frac{2}{\mathcal{W}^2(b)} \cdot 2\,\mathrm{e}^{\kappa 2b^*} \cdot (1 + \cosh 2\kappa t_b). \tag{39}$$

Applying a more compact term with

$$\cosh^2\kappa t_b = \frac{1}{2}[\cosh(2\kappa t_b) + 1], \tag{40}$$

we obtain a simplified expression about entanglement entropy as

$$S_{\mathrm{Rad}} = \frac{c}{6}\log\frac{4f(b)}{\kappa^2}\cosh^2\kappa t_b. \tag{41}$$

The squared term can be moved outside the logarithm and the behavior of the entropy only depends on the non-constant term $\log\cosh(\kappa t)$. In the early time the entropy grows slowly but will rapidly explode when t_b is sufficient large. In the later time, notice that the term $\cosh\kappa t_b \sim \mathrm{e}^{\kappa t_b}$. It shows that the entropy will grow linearly to infinity as

$$S \sim \frac{c}{3}\kappa t_b. \tag{42}$$

So taking an overview, the entropy increases linearly with time. This solution does not exhibit Page time or structure of Page curve. That aligns with the primal understanding of the black hole information paradox. In the next subsection, we will demonstrate that the Page curve will appear correctly when the island is considered.

3.2. Entanglement entropy with island

In this section we will calculate the entanglement entropy in the presence of an island. We will divide our discussion into two parts: early time and later time.

$$S_{\mathrm{R}} = \min\left\{\mathrm{ext}\left[\frac{\mathrm{Area}(\partial I)}{4G_{\mathrm{N}}} + S_{\mathrm{matter}}(R \cup I)\right]\right\}. \tag{43}$$

The entanglement entropy of the matter term is determined by

$$S_{\mathrm{matter}} = \frac{c}{3}\log\frac{d(a_+,a_-)d(b_+,b_-)d(a_+,b_+)d(a_-,b_-)}{d(a_+,b_-)d(a_-,b_+)}. \tag{44}$$

First of all, in order to obtain the entanglement entropy, substituting the distance function (33) in the above expresion

$$\begin{aligned} S_{\mathrm{gen}} = {} & \frac{2\pi a^2}{G_{\mathrm{N}}} + \frac{c}{3}\log[4W(a)W(b)\,\mathrm{e}^{\kappa+(a^*+b^*)} \\ & \times \cosh(\kappa_+ t_a)\cosh(\kappa_+ t_b)] \end{aligned}$$

$$+\frac{c}{3}\log\left[\frac{\cosh(\kappa_+(a^*-b^*))-\cosh(\kappa_+(t_a-t_b))}{\cosh(\kappa_+(a^*-b^*))+\cosh(\kappa_+(t_a+t_b))}\right]. \tag{45}$$

The first term represents the area contribution of the island, while the remaining part represents the contribution from the matter entropy. In order to study the behavior of entropy, we expect the entanglement entropy only depends on a single variable particularly with t. For the location of the cutoff is known, entropy without island relies solely on the unknown parameter t_b. That is precisely what we expected. Apart from the configuration without the island, when the island is considered, the entropy expression will introduce 4 variables denoted as a, t_a, b, and t_b, demonstrating the impact of the island on entropy. To obtain where the island locates, we must extremize the generalized entropy represented by known parameters like b. Furthermore, for extremizing the entropy, we have fortunately discovered that the time coordinate of both a and b must take the same value. Through this process, we have successfully derived the expression for the entropy only depending on b and t, which is similar to the previous section. Thanks to the somewhat arbitrary choice of b, we can always select an easily calculable value. This enables us to study the entropy's behavior varying overtime. In the upcoming subsections, we will evaluate the entropy and the location of the island in different settings.

3.2.1. Early time

In the early time, the magnitude of radiation is small. Therefore, we will make the following assumptions:[34]

The early time assumption allows $t_a, t_b \ll r_+$.

The cutoff is far away enough, assumption allows $b \gg r_+$.

Considering t is large enough, the third term of the entropy can be neglected. So we obtain

$$\begin{aligned} S_{\text{gen}} &= \frac{2\pi a^2}{G_{\text{N}}}+\frac{c}{3}\log[4W(a)W(b)\,\mathrm{e}^{\kappa_+(a^*+b^*)} \\ &\quad\times\cosh(\kappa_+ t_a)\cosh(\kappa_+ t_b)]. \end{aligned} \tag{46}$$

Substituting the expression of W in Eq. (26), and noticing that in order to get the location of a, our primary concern is the expression of a in the S_{gen}. The whole expression can be decomposed into the sum of the part depending on a and the part independent of a.

$$\begin{aligned} S_a &= \frac{2\pi a^2}{G_{\text{N}}}+\frac{c}{3}[\kappa a^*+\log W(a)] \\ &= \frac{2\pi a^2}{G_{\text{N}}}+\frac{c}{6}\log[f(a)]. \end{aligned} \tag{47}$$

Extremizing the entropy leads to

$$\frac{\partial S}{\partial a}=0. \tag{48}$$

Consequently, we derive the unresolved equation involving a as follows:

$$a\frac{f(a)}{f'(a)}+\frac{cG_{\text{N}}}{24\pi}=0. \tag{49}$$

We define $p=cG_{\text{N}}/(24\pi)$ to simplify future expressions due to its frequent appearance in calculations. Substituting the original expression of $f(a)$, we get a quadratic equation in terms of a^2

$$a^4+(q^2-p)a^2+2pq^2=0. \tag{50}$$

The solution to this quadratic equation yields

$$a^2=\frac{1}{2}\left[p-q^2+\sqrt{p^2-10pq^2+q^4}\right]. \tag{51}$$

We use the parameter k defined by $k=q^2/p$ to characterize the relative magnitudes q and p. Extracting p outside the square root, let

$$a^2=\frac{1}{2}\cdot\left[p-q^2+p\left(\sqrt{k^2-10k+1}\right)\right]. \tag{52}$$

Our object is to obtain an equation involving only p or q (possibly k), so we rewrite the square root term $\sqrt{k^2-10k+1}$ as a simple parameter λ, the final solution for a is as follows:

$$a=\sqrt{\frac{\lambda-k}{2}p}. \tag{53}$$

Hence, it is crucial to immediately discuss the behavior of λ for different values of k. λ is a function of k with two zero points at $k_1=5-2\sqrt{6}$ and $k_2=5+2\sqrt{6}$. It takes $[0,1]$ and decreases monotonically when $k<k_1$, it has no significance when $k_1<k<k_2$, and keeps positive when $k>k_2$. Remember that we have definition: $kp=q^2$. When $k=0$ which gives $\lambda=1$, the equation leads to

$$a=\sqrt{\frac{cG_{\text{N}}}{24\pi}}, \tag{54}$$

when $k=k_1$, $\lambda=0$, we obtain

$$a=\sqrt{\frac{cG_{\text{N}}}{48\pi}}. \tag{55}$$

For λ takes $[0,1]$, we can conclude that a will have a magnitute of $\sqrt{cG_{\text{N}}/(48\pi)}$. Furthermore, in geometrized unit system, it is demonstrated that

$$\sqrt{G_{\text{N}}}=l_p. \tag{56}$$

The solution of the island is observed at several Planck lengths. It is important to note that it does not carry sufficient physical significance, indicating the absence of any non-zero quantum extremal surface or the extremization of the entanglement entropy.[35] So when $k<k_1$, the island locates at the same magnitude of Planck length, without any physical meaning.

When k is even more larger, equation (50) has no solutions. This is self-evident and requires no further explanations. Therefore, $k_1 < k < k_2$ causes the equation to have no solutions.

Considering the case when $q \gg p$ which implies $k \gg k_2$, the square root term λ will behave as $\lambda \sim k-5$, for sufficiently large k. Now the coordinate can be solved as

$$a \sim \sqrt{-\frac{cG_{\rm N}}{12\pi}}. \tag{57}$$

However, this solution does not have physical meaning. If k is sufficiently big to guarantee the existence of solutions to the equation, but the solution is physical meaningless. Therefore, we consider that the island does not emerge.

In summary, we only consider the cases where $q < (\sqrt{3}-\sqrt{2})\frac{cG_{\rm N}}{24\pi}$ allows the equation to have solutions of a. Any other configuration will directly result in the non-existence of the island.

Hayward stated that the information paradox is not applicable in a non-singular black hole. A regular black hole evaporate completely when two horizons approach each other and reunite, leaving no information rest. If a black hole has no singularity, the inner horizon is supposed to be a near-Planck value.[55] Note that we already have the solution for a, the location of island, we realized that q and inner horizon r_- are both very small. We will establish the relationship between them. For a small enough r, with the approximation of

$$f(r) \sim 1 - 2\frac{mr^2}{q^3}. \tag{58}$$

We can compare r_- with a by

$$\frac{r_-}{a} = \sqrt{\frac{kq}{m}}. \tag{59}$$

Note that we always have $m > q$ to ensure the black hole could form, so r_- will always be less than a. Even in the extremal black holes, the critical radius $r_*^2 = 2ka^2$ will also give $r_* < a$ for $k < k_1 < 1/2$. This means that, if we take a proper (*i.e.* small enough) value of q, we can let the island locate at a magitute of Planck length. The corresponding inner horizon will remain less than where the island locates, and also must be a near the Planck value. From the above discussion we can see that we obtained a similar conclusion using the island configuration.

All discussion above suggests equivalently that the island has not yet emerged at early time, consistent with our expectations. In this case, the entropy grows linearly with time, as in the absence of an island obeying Eq. (41).

3.2.2. Late time

In this section, we will recalculate the entropy of radiation under the conditions of later time. The equation has already been obtained by

$$\begin{aligned} S_{\rm gen} =& \frac{2\pi a^2}{G_{\rm N}} + \frac{c}{3}\log[4W(a)W(b)\,{\rm e}^{\kappa_+(a^*+b^*)} \\ &\times \cosh(\kappa_+ t_a)\cosh(\kappa_+ t_b)] \\ &+ \frac{c}{3}\log\left[\frac{\cosh(\kappa_+(a^*-b^*)) - \cosh(\kappa_+(t_a-t_b))}{\cosh(\kappa_+(a^*-b^*)) + \cosh(\kappa_+(t_a+t_b))}\right]. \end{aligned} \tag{60}$$

For convenience, we have decomposed the expression into four parts and introduced the following symbols:

$$S_{\rm gen}^{(1)} = \frac{2\pi a^2}{G_{\rm N}}, \tag{61}$$

$$S_{\rm gen}^{(2)} = \frac{c}{3}\log[W(a)W(b)], \tag{62}$$

$$S_{\rm gen}^{(3)} = \frac{c}{3}\log[2^2\,{\rm e}^{\kappa_+((r^*a+r^*b))}\cosh(\kappa_+ t_a)\cosh(\kappa_+ t_b)], \tag{63}$$

$$S_{\rm gen}^{(4)} = \frac{c}{3}\log\left[\frac{\cosh(\kappa_+ r^*(a) - r^*(b) - \cosh(\kappa_+ t_a - t_b))}{\cosh(\kappa_+ r^*(a) - r^*(b) + \cosh(\kappa_+ t_a + t_b))}\right]. \tag{64}$$

At later time, we apply an approximation that allows t to be sufficiently large to dominate the position coordinates throughout the entire term.[34] Therefore, the following assumptions are utilized:

$$t \gg b \gg a, \quad t \gg b \gg r_+, \tag{65}$$

which immediately means the following approximations

$$\cosh\kappa_+ t \simeq \frac{1}{2}{\rm e}^{\kappa_+ t}, \tag{66}$$

$$\cosh\kappa_+(b^*-a^*) \simeq \frac{1}{2}{\rm e}^{\kappa_+(b^*-a^*))}, \tag{67}$$

$$\cosh\kappa_+(t_a+t_b) >> \cosh\kappa_+(b^*-a^*). \tag{68}$$

It is evident that $S_{\rm gen}$ will always take the minimum value only when $t_a = t_b$, so we can replace every t_a and t_b by a single varible t and notice the term of Eq. (67). By using these approximations, they will have the following forms solved as

$$S_{\rm gen}^{(3)} = \frac{c}{3}\log\left[{\rm e}^{\kappa_+((a^*+b^*))}\cdot{\rm e}^{2\kappa_+ t}\right], \tag{69}$$

$$S_{\rm gen}^{(4)} = \frac{c}{3}\log\left[\frac{\frac{1}{2}{\rm e}^{\kappa_+(r^*(b)-r^*(a))}-1}{\frac{1}{2}{\rm e}^{(2\kappa_+ t)}}\right]. \tag{70}$$

By utilizing the approximation $\log[1+(-2{\rm e}^{-\kappa(b^*-a^*)})] \simeq -2{\rm e}^{-\kappa(b^*-a^*)}$ for sufficiently small ${\rm e}^{-\kappa(b^*-a^*)}$, the sum can be represented in a simplified form as

$$S_{\rm gen}^{(3)} + S_{\rm gen}^{(4)} = \frac{2c}{3}(\kappa b^* - {\rm e}^{-\kappa_+(b^*-a^*)}). \tag{71}$$

In conclusion, the sum of all the above expressions will yield the complete expression of entropy

$$\begin{aligned} S_{\rm gen} =& \frac{2\pi a^2}{G_{\rm N}} + \frac{c}{6}\log[f(a)f(b)] - \frac{2c}{3}\log[\kappa_+] \\ &+ \frac{c}{3}\kappa_+(b^*-a^*) - \frac{2c}{3}{\rm e}^{-\kappa_+(b^*-a^*)}. \end{aligned} \tag{72}$$

In order to find the extremum of entropy for the island, we need to take the partial derivative with respect to a. The final expression for the partial derivative of entropy $\partial S/\partial a$ is shown as

$$\frac{\partial S}{\partial a}=\frac{4\pi a}{G_{\rm N}}+\frac{c}{6}\frac{f'(a)}{f(a)}-\frac{\kappa_+ c}{3f(a)}-\frac{2c}{3}\frac{\kappa}{f(a)}\,{\rm e}^{\kappa_+ a^*}. \tag{73}$$

The analysis of spherically symmetric black holes under near-horizon conditions allows us to expect the island locating very close to the horizon. Consequently, under near-horizon conditions, the island's position can be expressed as

$$a=r_+ +\varepsilon, \tag{74}$$

where ε is a parameter which is very small. Utilizing near-horizon approximations, we make the following expansions and relationships, as noted in previous studies.[30,42]

(i) The shape function $f(r)$ can be expanded near the horizon: $f(r)=f(r_+)+f'(r_+)\cdot(r-r_+)$.

(ii) The relationship between surface gravity and the Hawking temperature at the horizon is given by $T_{\rm H}=\frac{\kappa_+}{2\pi}=\frac{f'(r_+)}{4\pi}$.

(iii) The Kruskal coordinate approximation near the horizon is r^*: $r^*\simeq\frac{1}{2\kappa}\log\left[\frac{r}{r_+}-1\right]$.

Using these approximations, we derive the following expression:

$$\begin{aligned}\frac{\partial S}{\partial a}&=\frac{4\pi a}{G_{\rm N}}-\frac{2c}{3}\frac{\kappa}{f(a)}\,{\rm e}^{\kappa a^*}\,{\rm e}^{-\kappa b^*}\\&=\frac{12\pi a}{G_{\rm N}}-\frac{c}{\varepsilon}\sqrt{\frac{\varepsilon}{r_+}}\,{\rm e}^{-\kappa b^*}=0.\end{aligned} \tag{75}$$

Simplifying further, we obtain

$$12\pi(r_+ +\varepsilon)\sqrt{r_+\varepsilon}=cG_{\rm N}\,{\rm e}^{-\kappa b^*}. \tag{76}$$

Squaring the LHS and neglecting the higher order terms of ε, we find

$$r_+^3\varepsilon+\mathcal{O}(\varepsilon^2)=\frac{cG_{\rm N}}{12\pi}\,{\rm e}^{-2\kappa b^*}\Rightarrow\varepsilon=\left(\frac{cG_{\rm N}}{12\pi}\right)^2\frac{{\rm e}^{-2\kappa b^*}}{r_+^3}. \tag{77}$$

Thus, the location of the island is

$$a=r_+ +\left(\frac{cG_{\rm N}}{12\pi}\right)^2\frac{{\rm e}^{-2\kappa b^*}}{r_+^3}. \tag{78}$$

It shows that the island locates near the outer horizon at a close distance. The distance ε is determined by the cutoff we selected. Replacing the location a, the expression for the generalized entropy in Eq. (72) is given below:

$$\begin{aligned}S_{\rm gen}&=\frac{2\pi a^2}{G_{\rm N}}+\frac{c}{3}\log[W(a)W(b)]\\&\quad+\frac{2c}{3}\kappa b^*+\frac{2c}{3}\,{\rm e}^{-\kappa(b^*-a^*)}.\end{aligned} \tag{79}$$

Note that the conformal factor in Kruskal coordinates can be simplified as

$$W(a)=2\kappa\varepsilon\cdot\frac{r_+/\varepsilon}{\kappa}=\frac{2r_+}{\kappa}. \tag{80}$$

Substituting this into the entropy expression, we obtain

$$\begin{aligned}S_{\rm gen}&=\frac{2\pi a^2}{G_{\rm N}}+\frac{c}{6}\log\left[\frac{2r_+}{\kappa^3}f(b)\,{\rm e}^{2\kappa b^*}\right]-\frac{2c}{3}\,{\rm e}^{-\kappa_+(b^*-a^*)}\\&\simeq 2S_{\rm BH}+\mathcal{O}(G_{\rm N}^2).\end{aligned} \tag{81}$$

The higher orders of $cG_{\rm N}$ is ignored in the discussion. The expression without time term means we obtain a time-independent solution for the Bardeen black hole. The entropy concerns a constant, at later time, indicating that the entropy remains static.

In summary, for the eternal black hole discussed above, the entropy will continue to increase regardless of the presence of the island. However, if the influence of the island is considered throughout the entire duration, the entropy will exhibit linear growth in the early stages, and eventually stabilize at a specific value, equal twice the Bekenstein–Hawking entropy of the black hole. This phenomenon arises from different contributions of the island at different stages. In the early stage, the contribution of the island becomes negligible due to its absence. Conversely, in the later stages, the emergence of the island at the vicinity of the horizon results in the generalized entropy being precisely twice the Bekenstein–Hawking entropy. It is difficult to utilize the island formula to directly explain how could information paradox be avoid in Hayward's claim in Refs. [49,55]. We do not find sufficient connections at later time. Therefore, it remains doubtful whether the regular center can essentially resolve the information paradox.

4. Page time and scrambling time

In this paper, we have calculated entanglement entropy of the Hawking radiation of the asymptotically flat eternal black hole. In the following we will focus on two important values namely the Page time and the scrambling time.

The Page time indicates the moment when the entanglement entropy has a phase transition. This transition occurs when the entanglement entropy reaches to a maximum value and begins to decline (for an evaporating black hole) or does not increase any more (for a eternal black hole). The scrambling time quantifies the time in which the information taken into the black hole could be recovered from the Hawking radiation. First we will handle the Page time and then analyze the scrambling time.

To obtain the Page time, simply let $S_{\text{with Island}}=S_{\text{without Island}}$ from Eqs. (81) and (42)

$$t=\frac{6\pi r_+^2}{\kappa cG_{\rm N}}. \tag{82}$$

Recalling the relationship between surface gravity and the temperature we have

$$T = \frac{\kappa}{2\pi}. \tag{83}$$

It is easy to obtain

$$t = \frac{3S_{\rm BH}}{c\pi T}, \tag{84}$$

where the $S_{\rm BH}$ represents the Bekenstein entropy of the black hole.

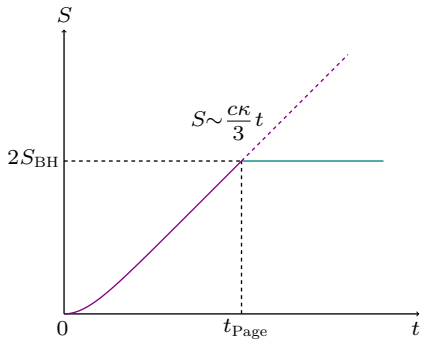


Fig. 4. The Page curve shown in the eternal Bardeen black hole. The solid purple line shows the entropy before the Page time and the dashed one shows the ongoing trend if the island is never constructed. The solid green line shows the entropy of the eternal black hole keeps a constant value after the Page time.

Simultaneously, we can obtain the scrambling time by using the island configuration. "Scrambling time" describes the duration required for an object to be completely "scrambled" throughout the entire black hole and then emitted through the Hawking radiation. Essentially, it signifies when this information reaches the "island". So the scrambling time can be derived from the distance cutoff to the island expressed by $\Delta t = b - a$. Recalling that the location of the island is given by

$$a = r_+ + \left(\frac{cG_{\rm N}}{12\pi}\right)^2 \frac{{\rm e}^{-2\kappa b^*}}{r_h^3}, \tag{85}$$

and the time between two events is determined by

$$\Delta t = b^* - a^* = \int_a^b \frac{1}{f(r)}\,{\rm d}r. \tag{86}$$

By using the approximation about $r^* \simeq \frac{1}{2\kappa}\log[\frac{r}{r_+} - 1]$ when r is near the horizon, the time is given by

$$\Delta t = b^* - a^* \simeq \frac{1}{2\kappa}\log\frac{r_b - r_+}{r_a - r_+}. \tag{87}$$

We can substitute $r_b = 2r_+$ into the equation for r_b and r_+, they have the same magnitude. This will result in a trivial value for ${\rm e}^{-2\kappa b^*} = 1$. Therefore we obtain the scrambling time from

$$t_{\rm scr} \simeq \frac{1}{2\kappa}\log\frac{144\pi^2 r_+^4}{c^2 G_{\rm N}^2} = \frac{1}{2\pi T}\log\left[\frac{r_+^2}{cG_{\rm N}}\right], \tag{88}$$

or in simplified form as

$$t_{\rm scr} \simeq \frac{1}{2\pi T}\log[S_{\rm BH}]. \tag{89}$$

Based on the above discussion, we find that the scrambling time has a logarithmic relationship to the Bekenstein–Hawking entropy. The relationship between the scrambling time is consistent with the results in Refs. [30,34,35].

5. Discussion and conclusion

The original black hole information paradox arises from an evaporating black hole, primarily concerning that a black hole loses all information through thermal Hawking radiation, leaving nothing behind, thus violating the unitarity of quantum mechanics. However, for an eternal black hole, the stability of the black hole's mass and charge leads to infinite radiation resulting in the entropy approaching infinity. Our discussion on the island configuration ensures that the black hole boundary satisfies the Page proposal.

In this paper, we investigated a regular black hole that lacks a singularity and intended a de Sitter core inside. Similar to the RN black hole, it has a regular center replacing the inner horizon but does not possess a singularity. Our findings regarding this are as follows. In the early stage, the island has not yet appeared regardless of the amount of charge it contained. We analyzed three situations: when k is small, when k leads to no solution, and when k is sufficiently large. In the first occasion, the island locates in the scale compatible with the Planck length, which does not have sufficient physical meaning. In the second occasion, the island cannot be solved from the extremelized equation. In the third occasion the solution should be neglected. Neither of them will produce an island in the early stage. As time progresses, the radiation increases. The island term becomes dominant in the entanglement entropy. The calculation confirms that the island is located within the stretch of the horizon, allowing us to obtain a fixed boundary to limit the entropy from approaching infinity.

In order to investigate the entropy behavior in the regular behavior using the island configuration, we presented an argument supporting the island proposal in the field of information paradox. Our findings suggest that while the absence of a singularity significantly impacts the internal structure of the black hole, particularly on small scales, the behavior of the entanglement entropy remains similar to that of Schwarzschild and RN black holes[34,35] for their similarities in near horizon region and asymptotic flat spacetime. The discussion of information paradox could be a clarification of ER = EPR.[57] Furthermore, the answer to how to explain the emergence of the island and to how could the information be reconstructed from the black hole still requires further discussion and research. Perhaps we are advancing along a route with enchanting scenery that could lead us to the destination of information paradox.

We also discuss necessity of the island method in resolving the information paradox for non-singular black holes. There is a viewpoint that regular black holes do not really hold the information problem. Our research offers a solution, inspired by the island idea, for regular black holes. While we have found some similarities, there is currently insufficient evidence to definitively conclude that the information paradox can be avoided solely due to the absence of a singularity. Further and more in-depth study is required to understand this issue.

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