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Multiparameter generalized universal characters and multiparameter generalized B-type universal characters

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We construct the quantum fields presentation of the generalized universal character and the generalized B-type universal character, and by acting the quantum fields presentations to the constant 1, the generating functions are derived. Furthermore, we introduce two integrable systems known as the generalized UC (GUC) hierarchy and the generalized B-type UC (GBUC) hierarchy satisfied by the generalized universal character and the generalized B-type universal character, respectively. Based on infinite sequences of complex numbers, we further establish the multiparameter generalized universal character and the multiparameter generalized B-type universal character, which have been proved to be solutions of the GUC hierarchy and the GBUC hierarchy, respectively.

Keywords: symmetric functions, quantum fields, generating functions, multiparameter generalized universal characters

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1. Introduction

Symmetric functions[1] attract much attention in mathematics and physics due to their applications in combinatorics, representation theory, and integrable systems.[2,3] Schur functions and Schur Q-functions are two significant symmetric functions that serve as the irreducible characters of the symmetric groups. The Kyoto School established a fundamental connection between the Schur functions, the Schur Q-functions, and the Kadomtsev–Petviashvili (KP) hierarchy, as well as the Kadomtsev–Petviashvili sub-hierarchy of B-type (BKP hierarchy).[4–6] Kac *et al.*[7,8] demonstrated the polynomial tau-functions of the KP and BKP hierarchies can be expressed by Schur functions and Schur Q-functions.

Koike proposed the universal character, which is an extension of the Schur function.[9] Tsuda[10] established a generalized KP hierarchy called the UC hierarchy characterized by the universal character. Moreover, the relations between the UC hierarchy and Painlevé equations have been extensively investigated. Ogawa[11] constructed an integrable UC hierarchy of B-type (BUC hierarchy) characterized by the generalized Q-function expressed as a Pfaffian. By using the vertex operators realizations of the symplectic universal character and orthogonal universal character, the authors[12,13] constructed the symplectic UC (SUC) hierarchy and orthogonal UC (OUC) hierarchy. Huang *et al.*[14] presented a novel symmetric functions called symplectic Grothendieck universal character which are proved as the polynomial tau-functions of the symplectic UC hierarchy. Furthermore, the π-type Boson–Fermion correspondence and fermions expectation values have been emplyed to study the π-type KP and UC hierarchies.[15–19] Recently, Li *et al.*[20] developed the vertex operator realizations and generating functions of generalized universal characters and generalized B-type of universal characters, leading to the analysis of the corresponding integrable systems called generalized UC (GUC) hierarchy and generalized B-type of UC (GBUC) hierarchy. Quantum fields presentation plays a crucial role in studying the representation theory of infinite-dimensional algebras and integrable systems. By utilizing the quantum fields of operators, the twisted quantum fields presentation of Hall–Littlewood polynomials has been derived.[21] It is showed the generalized Hall–Littlewood functions can be reduced to the universal characters with $t = 0$ and generalized Q-functions with $t = -1$.[22] Additionally, by developing the quantum fields presentation of symplectic and orthogonal universal characters, the corresponding integrable systems have been investigated.[23,24]

Multiparameter symmetric functions are a wider class of symmetric functions that provide a rich set of approaches in the representation theory and integrable systems.[25,26] Multiparameter Schur functions $s_{\mu;a}$ of the form s_μ with lower terms lead to the classical Schur functions when $a_i = 0$.[27] Here $a = (a_i)_{i\in\mathbb{Z}}$ is an arbitrary doubly infinite sequence of parameters. By using the interpolation analogs for Schur Q-functions, Ivanov[28] introduced the multiparameter Schur Q-functions. Korotkikh[29] presented dual multiparameter Schur Q-functions with a multiparameter Cauchy identity. Later, Rozhkovskaya[30] presented the multiparameter Schur Q-functions, including specializations of factorial Schur Q-

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functions and classical Schur Q-functions, which showed the solutions of the BKP hierarchy. Furthermore, the multiparameter B-type universal characters have been constructed, proving the tau-functions of BUC hierarchy.[31] Our goal is to construct the multiparameter generalized universal characters and B-type ones by using quantum fields presentation, satisfying the GUC and GBUC hierarchies, respectively.

2. Preliminaries

We review some facts about symmetric functions, including universal characters, B-type universal characters, the UC hierarchy and the BUC hierarchy.[1,10,11,31]

2.1. Symmetric functions

Let $\Lambda(\boldsymbol{x})$ be a ring of symmetric functions in variables $\boldsymbol{x}=(x_1,x_2,\ldots)$. The definition of r-th elementary symmetric function e_r, complete symmetric function h_r and power sum p_r are

$$\begin{aligned} e_r(\boldsymbol{x}) &= \sum_{i_1<i_2<\cdots<i_r<\infty} x_{i_1}x_{i_2}\ldots x_{i_r}, \quad \text{for} \quad r\geq 1,\\ h_r(\boldsymbol{x}) &= \sum_{i_1\leq i_2\leq\cdots\leq i_r\leq\infty} x_{i_1}x_{i_2}\ldots x_{i_r}, \quad \text{for} \quad r\geq 1,\\ p_r(\boldsymbol{x}) &= \sum_i x_i^r, \end{aligned}$$

with the convention $h_r(\boldsymbol{x})=e_r(\boldsymbol{x})=p_r(\boldsymbol{x})=0$ for $r<0$ and $h_0=e_0=p_0=1$.

The generating functions of $e_r(\boldsymbol{x})$, $h_r(\boldsymbol{x})$, and $p_r(\boldsymbol{x})$ are defined as

$$L(u)=\sum_{k\geqslant 0}(-1)^k e_k(\boldsymbol{x})u^k, \quad M(u)=\sum_{k\geqslant 0}h_k(\boldsymbol{x})u^k,$$

$$P(u)=\sum_{k\geqslant 1}p_k(\boldsymbol{x})u^k.$$

The following relations hold

$$L(u) = \prod_{i\geqslant 1}(1-x_iu)=\exp\left(-\sum_{n\geqslant 1}\frac{p_n}{n}u^n\right), \tag{1}$$

$$M(u) = \prod_{i\geqslant 1}\frac{1}{1-x_iu}=\exp\left(\sum_{n\geqslant 1}\frac{p_n}{n}u^n\right). \tag{2}$$

The adjoint of a symmetric function f can be defined as

$$\langle f^{\perp}g,\boldsymbol{\omega}\rangle=\langle g,f\boldsymbol{\omega}\rangle, \quad g,f,\boldsymbol{\omega}\in\Lambda(\boldsymbol{x}).$$

The adjoint operators of $L(u)$ and $M(u)$ are

$$L^{\perp}(u)=\sum_{k\geqslant 0}(-1)^k e_k^{\perp}u^k, \quad M^{\perp}(u)=\sum_{k\geqslant 0}h_k^{\perp}u^k.$$

Then, we get

$$\begin{aligned} L^{\perp}(u) &= \exp\left(-\sum_{k\geqslant 1}u^k\frac{\partial}{\partial p_k}\right),\\ M^{\perp}(u) &= \exp\left(\sum_{k\geqslant 1}u^k\frac{\partial}{\partial p_k}\right). \end{aligned} \tag{3}$$

2.2. Universal characters and the UC hierarchy

A partition $\lambda=(\lambda_1,\lambda_2,\ldots)$ is a sequence of non-negative in decreasing order $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant 0$ and $\lambda_i=0$ for $i\gg 0$. The number of non-zero λ_i is the length of λ denoted by $l(\lambda)$. The sum $|\lambda|=\lambda_1+\lambda_2+\cdots+\lambda_{l(\lambda)}$ is called the weight of λ.

For two partitions $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_l)$ and $\mu=(\mu_1,\mu_2,\ldots,\mu_m)$, the universal characters $S_{[\lambda,\mu]}(\boldsymbol{x},y)$ are defined as

$$S_{[\lambda,\mu]}(\boldsymbol{x},y)=\det\begin{pmatrix} h_{\mu_{m-i+1}+i-j}(\boldsymbol{y}), & 1\leqslant i\leqslant m\\ h_{\lambda_{i-m}-i+j}(\boldsymbol{x}), & m+1\leqslant i\leqslant l+m\end{pmatrix}_{1\leqslant i,j\leqslant l+m}. \tag{4}$$

Define the operators $R_{\lambda_i\lambda_j}$ and $D_{\lambda_i\mu_a}$

$$\begin{aligned} &R_{\lambda_i\lambda_j}\cdot(\lambda_1,\ldots,\lambda_i,\ldots,\lambda_j,\ldots,\lambda_l)\\ =\ &(\lambda_1,\ldots,\lambda_i+1,\ldots,\lambda_j-1,\ldots,\lambda_l),\\ &D_{\lambda_i\mu_a}h_{[\lambda,\mu]}(\boldsymbol{x},y)=h_{[\lambda',\mu']}(\boldsymbol{x},y), \end{aligned}$$

where $[\lambda',\mu']=[(\lambda_1,\ldots,\lambda_i-1,\ldots,\lambda_l),(\mu_1,\ldots,\mu_a-1,\ldots,\mu_m)]$, $h_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})=h_\lambda(\boldsymbol{x})h_\mu(\boldsymbol{y})$ and $h_\lambda=h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_l}$.

In terms of two operators $R_{\lambda_i\lambda_j}$ and $D_{\lambda_i\mu_a}$, the universal characters $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ can also be written as

$$\begin{aligned} S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = &\prod_{1\leqslant i<j\leqslant l}(1-R_{\lambda_i\lambda_j})\prod_{1\leqslant a<b\leqslant m}(1-R_{\mu_a\mu_b})\\ &\times\prod_{\substack{1\leqslant i\leqslant l\\ 1\leqslant a\leqslant m}}(1-D_{\lambda_i\mu_a})h_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}). \end{aligned} \tag{5}$$

Let $\Lambda(\boldsymbol{x},\boldsymbol{y})=\mathbb{C}[p_1(\boldsymbol{x}),p_2(\boldsymbol{x}),\ldots,p_1(\boldsymbol{y}),p_2(\boldsymbol{y}),\ldots]$ be the symmetric polynomial ring. We consider the bononic Fock space $\mathcal{B}=\mathbb{C}[z_1^{\pm},z_2^{\pm}]\otimes\Lambda(\boldsymbol{x},\boldsymbol{y})$, which can be decomposed by

$$\mathcal{B}=\bigoplus_{m,n\in\mathbb{Z}}\mathcal{B}^{(m,n)},$$

where $\mathcal{B}^{(m,n)}=z_1^mz_2^n\Lambda(\boldsymbol{x},\boldsymbol{y})$.

Define the operators $R^{\pm}(u),R'^{\pm}(u):\mathcal{B}\to\mathcal{B}$ as

$$\begin{aligned} &R(u)(z_1^mz_2^nf)=z_1^{m+1}z_2^nu^{m+1}f,\\ &R^{-1}(u)(z_1^mz_2^nf)=z_1^{m-1}z_2u^nu^{-m}f,\\ &R'(u)(z_1^mz_2^nf)=z_1^mz_2^{n+1}u^{n+1}f,\\ &R'^{-1}(u)(z_1^mz_2^nf)=z_1^mz_2^{n-1}u^{-n}f. \end{aligned} \tag{6}$$

Introduce the quantum fields of the universal characters

$\Psi^{\pm}(u)$ and $\widehat{\Psi}^{\pm}(u)$

$$\begin{aligned}
\Psi^{+}(u) &= u^{-1}R(u^{-1})M(u^{-1})L'^{\perp}(u^{-1})L^{\perp}(u)\\
&= \sum_{i\in\mathbb{Z}+1/2}\Psi_i^{+}u^{-i-1/2},\\
\Psi^{-}(u) &= R^{-1}(u^{-1})L(u^{-1})M'^{\perp}(u^{-1})M^{\perp}(u)\\
&= \sum_{i\in\mathbb{Z}+1/2}\Psi_i^{-}u^{-i-1/2},\\
\widehat{\Psi}^{+}(u) &= u^{-1}R'(u^{-1})M'(u^{-1})L^{\perp}(u^{-1})L'^{\perp}(u)\\
&= \sum_{i\in\mathbb{Z}+1/2}\widehat{\Psi}_i^{+}u^{-i-1/2},\\
\widehat{\Psi}^{-}(u) &= R'^{-1}(u^{-1})L'(u^{-1})M^{\perp}(u^{-1})M'^{\perp}(u)\\
&= \sum_{i\in\mathbb{Z}+1/2}\widehat{\Psi}_i^{-}u^{-i-1/2}. \qquad (7)
\end{aligned}$$

Definition 1 For an unknown function $\tau=\tau(\boldsymbol{x},\boldsymbol{y})$, the bilinear equation

$$\Omega_{\mathrm{UC}}(\tau\otimes\tau)=\widehat{\Omega}_{\mathrm{UC}}(\tau\otimes\tau)=0, \qquad (8)$$

is called the UC hierarchy, where

$$\Omega_{\mathrm{UC}}=\sum_{k\in\mathbb{Z}+\frac{1}{2}}\Psi_k^{-}\otimes\Psi_{-k}^{+},\quad \widehat{\Omega}_{\mathrm{UC}}=\sum_{k\in\mathbb{Z}+\frac{1}{2}}\widehat{\Psi}_k^{-}\otimes\widehat{\Psi}_{-k}^{+}. \qquad (9)$$

Proposition 1 All the universal characters $S_{[\lambda,\mu]}(\boldsymbol{x},y)$ are solutions of the UC hierarchy.

2.3. B-type universal characters and the BUC hierarchy

Let

$$H(u)=M(u^{-1})=\exp\left(\sum_{n\geqslant 1}\frac{p_n}{nu^n}\right),$$

$$\begin{aligned}
H^{\perp}(u)&=M^{\perp}(u)=\exp\left(\sum_{n\geqslant 1}u^n\frac{\partial}{\partial p_n}\right),\\
E(u)&=L(-u^{-1})=\exp\left(\sum_{n\geqslant 1}\frac{(-1)^{n-1}p_n}{nu^n}\right),\\
E^{\perp}(u)&=L^{\perp}(-u)=\exp\left(\sum_{n\geqslant 1}(-1)^{n-1}u^n\frac{\partial}{\partial p_n}\right). \qquad (10)
\end{aligned}$$

The generating function of symmetric functions $\{Q_k=Q_k(x_1,x_2,\ldots)|k=0,1,2,\ldots\}$ is given by

$$Q(u)=\sum_{k\geqslant 0}\frac{Q_k}{u^k}=\prod_i\frac{u+x_i}{u-x_i}. \qquad (11)$$

Then, we obtain

$$\begin{aligned}
&Q(u)=E(u)H(u)=R^2(u),\\
&R(u)=\exp\left(\sum_{n\in\mathbb{N}_{\mathrm{odd}}}\frac{p_n}{nu^n}\right),\\
&Q^{\perp}(u)=E^{\perp}(u)H^{\perp}(u)=(R^{\perp}(u))^2,\\
&R^{\perp}(u)=\exp\left(\sum_{n\in\mathbb{N}_{\mathrm{odd}}}u^n\frac{\partial}{\partial p_n}\right),\\
&Q(u)Q(-u)=1, \qquad (12)
\end{aligned}$$

where $\mathbb{N}_{\mathrm{odd}}=\{1,3,5,\ldots\}$.

Definition 2 Let $(\boldsymbol{x},\boldsymbol{y})=(x_1,\ldots,x_M,y_1,\ldots,y_N)$ be independent variables and $\lambda=(\lambda_1,\ldots,\lambda_l)$, $\mu=(\mu_1,\ldots,\mu_s)$ be strict partitions. The symmetric functions called B-type universal characters are defined as

$$\begin{aligned}
Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) &= \frac{2^{r+s}}{(M-r)!(N-s)!}\sum_{\sigma\in S_N}\prod_{i=1}^{s}\frac{(\sigma(i)y_{\sigma(i)}-2\partial_{x_{\sigma(i)}})^{\mu_i}}{(\sigma(i))^{\mu_i}}\\
&\times\prod_{i<j}\frac{\sigma(i)\sigma(j)y_{\sigma(i)}+\sigma(i)\sigma(j)y_{\sigma(j)}-2\sigma(j)\partial_{x_{\sigma(i)}}-2\sigma(i)\partial_{x_{\sigma(j)}}}{\sigma(i)\sigma(j)y_{\sigma(i)}-\sigma(i)\sigma(j)y_{\sigma(j)}-2\sigma(j)\partial_{x_{\sigma(i)}}+2\sigma(i)\partial_{x_{\sigma(j)}}}\cdot\left(\sum_{\sigma\in S_M}\prod_{i=1}^{r}x_{\sigma(i)}^{\lambda_i}\prod_{i<j}\frac{x_{\sigma(i)}+x_{\sigma(j)}}{x_{\sigma(i)}-x_{\sigma(j)}}\right), \qquad (13)
\end{aligned}$$

where $\partial_x=\partial/\partial x$.

Let $\mathcal{B}[\boldsymbol{x},\boldsymbol{y}]=\mathcal{B}[x_1,x_2,x_3,,\ldots,y_1,y_2,y_3,\ldots]$ be the polynomial ring of infinitely many variables. Let us consider $\mathcal{B}_{\mathrm{odd}}=\mathcal{B}_{\mathrm{odd}}[x_1,x_3,x_5,,\ldots,y_1,y_3,y_5,\ldots]$, which is subalgebra of $\mathcal{B}[\boldsymbol{x},\boldsymbol{y}]$. It is known that the B-type universal characters forms a linear basis of $\mathcal{B}_{\mathrm{odd}}$.

Introduce two operators $\{\phi_k\}_{k\in\mathbb{Z}}$ and $\{\widehat{\phi}_k\}_{k\in\mathbb{Z}}$ acting on the coefficients of generating functions: $Q(u_1,\ldots,u_l,v_1,\ldots,v_s)$ as follows:

$$\begin{aligned}
&\phi_m: Q_{[\lambda,\mu]}\mapsto Q_{[(m,\lambda),\mu]},\\
&\widehat{\phi}_n: Q_{[\lambda,\mu]}\mapsto Q_{[\lambda,(n,\mu)]},\\
&\phi_m\widehat{\phi}_n(\text{ or }\widehat{\phi}_n\phi_m): Q_{[\lambda,\mu]}\mapsto Q_{[(m,\lambda),(n,\mu)]}. \qquad (14)
\end{aligned}$$

Definition 3 For an unknown function $\tau=\tau(\boldsymbol{x},\boldsymbol{y})$, the bilinear identities

$$\Omega_{\mathrm{BUC}}(\tau\otimes\tau)=\widehat{\Omega}_{\mathrm{BUC}}(\tau\otimes\tau)=\tau\otimes\tau \qquad (15)$$

are called the BUC hierarchy, where $\Omega_{\mathrm{BUC}}=\sum_n\phi_n\otimes(-1)^n\phi_{-n}$ and $\widehat{\Omega}_{\mathrm{BUC}}=\sum_n\widehat{\phi}_n\otimes(-1)^n\widehat{\phi}_{-n}$.

Proposition 2 The B-type universal characters are the solutions of the BUC hierarchy.

3. Generalized universal characters and GUC hierarchy

In this section, we construct the quantum fields presentation of the generalized universal character and define the generalized UC (GUC) hierarchy satisfied by the generalized uni-

versal characters. In terms of infinite sequences of complex numbers, we propose the multiparaeter generalized universal characters and prove these symmetric functions are tau functions of the GUC hierarchy.

3.1. Quantum fields presentation of the generalized universal character and the GUC hierarchy

For n partitions $\lambda_{[1]}=(\lambda_{1,1},\lambda_{1,2},\ldots,\lambda_{1,l_1}),\lambda_{[2]}=(\lambda_{2,1},\lambda_{2,2},\ldots,\lambda_{2,l_2}),\ldots,\lambda_{[n]}=(\lambda_{n,1},\lambda_{n,2},\ldots,\lambda_{n,l_n})$, define the operators $R_{\lambda_{m,i}\lambda_{m,j}}$ and $D_{\lambda_{a,i}\lambda_{b,j}}(1\leqslant a,b,m\leqslant n;a\neq b)$

$$\begin{aligned}&R_{\lambda_{m,i}\lambda_{m,j}}\cdot h_{[\lambda_{[1]},\ldots,\lambda_{[m]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\\ =\;& h_{[\lambda_{[1]},\ldots,R_{ij}\cdot\lambda_{[m]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n),\\ &D_{\lambda_{a,i}\lambda_{b,j}}\cdot h_{[\lambda_{[1]},\ldots,\lambda_{[a]},\ldots,\lambda_{[b]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\\ =\;& h_{[\lambda_{[1]},\ldots,\widetilde{\lambda}_{[a]},\ldots,\widetilde{\lambda}_{[b]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n),\end{aligned}$$

where

$$\begin{aligned}&R_{ij}\cdot(\lambda_{m,1},\ldots,\lambda_{m,i},\ldots,\lambda_{m,j},\ldots,\lambda_{m,l_m})\\ =\;&(\lambda_{m,1},\ldots,\lambda_{m,i}+1,\ldots,\lambda_{m,j}-1,\ldots,\lambda_{m,l_m}),\\ &\widetilde{\lambda}_{[a]}=(\lambda_{a,1},\ldots,\lambda_{a,i}-1,\ldots,\lambda_{a,l_a}),\\ &\widetilde{\lambda}_{[b]}=(\lambda_{b,1},\ldots,\lambda_{b,j}-1,\ldots,\lambda_{b,l_b}),\\ &h_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)=h_{\lambda_{[1]}}(\boldsymbol{x}_1)h_{\lambda_{[2]}}(\boldsymbol{x}_2)\cdots h_{\lambda_{[n]}}(\boldsymbol{x}_n).\end{aligned}$$

Definition 4 For the variables $\boldsymbol{x}_i=(x_{i,1},x_{i,2},\ldots)$ $(i=1,\ldots,n)$, in $\mathbb{C}[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n]$ and n partitions $[\lambda_{[1]},\ldots,\lambda_{[n]}]$, the function

$$\begin{aligned}&S_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\\ =\;&\prod_{m=1}^{n}\prod_{1\leqslant i<j\leqslant l_m}\left(1-R_{\lambda_{m,i}\lambda_{m,j}}\right)\prod_{\substack{1\leqslant a,b\leqslant n\\ a\neq b}}\prod_{\substack{1\leqslant i\leqslant l_a\\ 1\leqslant j\leqslant l_b}}\left(1-D_{\lambda_{a,i}\lambda_{b,j}}\right)\\ &\times h_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\end{aligned}\tag{16}$$

are called generalized universal characters. Introduce

$$\begin{aligned}&L_i(u)=\sum_{k\geqslant 0}(-1)^k e_k(\boldsymbol{x}_i)u^k=\exp\left(-\sum_{n\geqslant 1}\frac{p_n(\boldsymbol{x}_i)}{n}u^n\right),\\ &M_i(u)=\sum_{k\geqslant 0}h_k(\boldsymbol{x}_i)u^k=\exp\left(\sum_{n\geqslant 1}\frac{p_n(\boldsymbol{x}_i)}{n}u^n\right),\\ &L_i^{\perp}(u)=\sum_{k\geqslant 0}(-1)^k e_k^{\perp}(\boldsymbol{x}_i)u^k=\exp\left(-\sum_{n\geqslant 1}u^n\frac{\partial}{\partial p_n(\boldsymbol{x}_i)}\right),\\ &M_i^{\perp}(u)=\sum_{k\geqslant 0}h_k^{\perp}(\boldsymbol{x}_i)u^k\exp\left(\sum_{n\geqslant 1}u^n\frac{\partial}{\partial p_n(\boldsymbol{x}_i)}\right),\end{aligned}\tag{17}$$

satisfying

$$\begin{aligned}&(1-uv)L_i^{\perp}(u)L_i(v)=L_i(v)L_i^{\perp}(u),\\ &(1-uv)M_i^{\perp}(u)M_i(v)=M_i(v)M_i^{\perp}(u),\\ &M_i^{\perp}(u)L_i(v)=(1-uv)L_i(v)M_i^{\perp}(u),\\ &L_i^{\perp}(u)M_i(v)=(1-uv)M_i(v)L_i^{\perp}(u).\end{aligned}\tag{18}$$

If $i\neq j$, we have

$$\begin{aligned}[M_i(u),M_j^{\perp}(v)]&=[L_i(u),M_j^{\perp}(v)]=[M_i(u),L_j^{\perp}(v)]\\ &=[L_i(u),L_j^{\perp}(v)]=0,\end{aligned}\tag{19}$$

where $[A,B]=AB-BA$.

We consider the bosonic Fock space

$$\overline{\mathcal{B}}=\mathbb{C}[z_1^{\pm},\ldots,z_n^{\pm}]\otimes\Lambda(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)=\bigoplus_{m_i\in\mathbb{Z}}\overline{\mathcal{B}}^{(m_1,\ldots,m_n)},$$

where $\Lambda(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)=\mathbb{C}[p_1(\boldsymbol{x}_1),\ p_2(\boldsymbol{x}_1),\ \ldots,\ p_1(\boldsymbol{x}_2),\ p_2(\boldsymbol{x}_2),\ \ldots,\ p_1(\boldsymbol{x}_n),\ p_2(\boldsymbol{x}_n),\ldots]$ is a symmetric polynomial ring and $\overline{\mathcal{B}}^{(m_1,\ldots,m_n)}=z_1^{m_1}\ldots z_n^{m_n}\Lambda(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$.

Let $R_i^{\pm}:\overline{\mathcal{B}}^{(m_1,\ldots,m_i,\ldots,m_n)}\to\overline{\mathcal{B}}^{(m_1,\ldots,m_i\pm1,\ldots,m_n)}$ act on $z_1^{m_1}\ldots z_n^{m_n}f,f\in\Lambda(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$ by the rule

$$\begin{aligned}&R_i(u):z_1^{m_1}\cdots z_n^{m_n}f\mapsto z_1^{m_1}\cdots z_i^{m_i+1}\cdots z_n^{m_n}u^{m_i+1}f,\\ &R_i^{-1}(u):z_1^{m_1}\cdots z_n^{m_n}f\mapsto z_1^{m_1}\cdots z_i^{m_i-1}\cdots z_n^{m_n}u^{-m_i}f,\end{aligned}$$

where $i\in\{1,2,\ldots,n\}$.

Define the formal distributions $\{\varphi_i^{\pm}(u)\}_{i=1,\ldots,n}$ as

$$\begin{aligned}\varphi_i^{+}(u)&=u^{-1}R_i(u^{-1})M_i(u^{-1})\prod_{\substack{1\leqslant j\leqslant n\\ j\neq i}}L_j^{\perp}(u^{-1})L_i^{\perp}(u)\\ &=\sum_{j\in\mathbb{Z}+\frac{1}{2}}\varphi_{i,j}^{+}u^{-j-1/2},\\ \varphi_i^{-}(u)&=R_i^{-1}(u^{-1})L_i(u^{-1})\prod_{\substack{1\leqslant j\leqslant n\\ j\neq i}}M_j^{\perp}(u^{-1})M_i^{\perp}(u)\\ &=\sum_{j\in\mathbb{Z}+\frac{1}{2}}\varphi_{i,j}^{-}u^{-j-1/2}.\end{aligned}\tag{20}$$

Proposition 3 The following relations of charged free fermions hold

$$\begin{aligned}&[\varphi_{i,j}^{\pm},\varphi_{i,k}^{\pm}]_+=0,\quad[\varphi_{i,j}^{+},\varphi_{i,k}^{-}]_+=\delta_{j+k,0},\\ &[\varphi_{i,j}^{\pm},\varphi_{l,k}^{\pm}]=[\varphi_{i,j}^{\pm},\varphi_{l,k}^{\mp}]=0,\quad(i\neq l),\end{aligned}\tag{21}$$

where $j,k\in\mathbb{Z}+1/2$.

Proof For any $m_i\in\mathbb{Z}$ $(i=1,\ldots,n)$ and $f\in\Lambda(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$, we have

$$\begin{aligned}&(\varphi_i^{+}(u_1)\varphi_i^{+}(u_2)+\varphi_i^{+}(u_2)\varphi_i^{+}(u_1))f\\ =\;&\left(\left(1-\frac{u_1}{u_2}\right)u_1^{-m_i-3}u_2^{-m_i-2}\right.\\ &\left.+\left(1-\frac{u_2}{u_1}\right)u_1^{-m_i-2}u_2^{-m_i-3}\right)z_1^{m_1}\cdots z_i^{m_i+2}\cdots z_n^{m_n}\\ &\times M_i(u_1^{-1})M_i(u_2^{-1})\prod_{\substack{1\leqslant j\leqslant n\\ j\neq i}}L_j^{\perp}(u_1^{-1})L_j^{\perp}(u_2^{-1})L_i^{\perp}(u_1)L_i^{\perp}(u_2)f\\ =\;&((u_2-u_1)+(u_1-u_2))z_1^{m_1}\cdots z_i^{m_i+2}\cdots\\ &z_n^{m_n}u_1^{-m_i-3}u_2^{-m_i-3}M_i(u_1^{-1})M_i(u_2^{-1})\\ &\times\prod_{\substack{1\leqslant j\leqslant n\\ j\neq i}}L_j^{\perp}(u_1^{-1})L_j^{\perp}(u_2^{-1})L_i^{\perp}(u_1)L_i^{\perp}(u_2)f\\ =\;&0,\end{aligned}\tag{22}$$

$$
\begin{aligned}
&(\varphi_i^+(u_1)\varphi_i^-(u_2)+\varphi_i^-(u_2)\varphi_i^+(u_1))f\\
=&\left(\left(1-\frac{u_1}{u_2}\right)^{-1}u_1^{-m_i-1}u_2^{m_i}\right.\\
&\left.+\left(1-\frac{u_2}{u_1}\right)^{-1}u_1^{-m_i-2}u_2^{m_i+1}\right)z_1^{m_1}\cdots z_n^{m_n}\\
&\times M_i(u_1^{-1})L_i(u_2^{-1})\prod_{\substack{1\leqslant j\leqslant n\\ j\neq i}}L_j^{\perp}(u_1^{-1})M_j^{\perp}(u_2^{-1})L_i^{\perp}(u_1)M_i^{\perp}(u_2)f\\
=&\left(\frac{1}{u_2-u_1}+\frac{1}{u_1-u_2}\right)z_1^{m_1}\cdots z_n^{m_n}u_1^{-m_i-1}\\
&\times u_2^{m_i+1}M_i(u_1^{-1})L_i(u_2^{-1})\\
&\times\prod_{\substack{1\leqslant j\leqslant n\\ j\neq i}}L_j^{\perp}(u_1^{-1})M_j^{\perp}(u_2^{-1})L_i^{\perp}(u_1)M_i^{\perp}(u_2)f\\
=&\,\delta(u_1,u_2)\left(\frac{u_2}{u_1}\right)^{m_i+1}z_1^{m_1}\cdots z_n^{m_n}M_i(u_1^{-1})L_i(u_2^{-1})\\
&\times\prod_{\substack{1\leqslant j\leqslant n\\ j\neq i}}L_j^{\perp}(u_1^{-1})M_j^{\perp}(u_2^{-1})L_i^{\perp}(u_1)M_i^{\perp}(u_2)f\\
=&\,\delta(u_1,u_2)z_1^{m_1}\cdots z_n^{m_n}f,
\end{aligned}
$$

where $\delta(u,v)=\sum_{r\in\mathbb{Z}}u^rv^{-(r+1)}$ is a formal delta distribution with property $\delta(u,v)a(u)=\delta(u,v)a(v)$. Hence, we obtain

$$
\begin{aligned}
&\varphi_i^+(u_1)\varphi_i^+(u_2)+\varphi_i^+(u_1)\varphi_i^+(u_2)=0,\\
&\varphi_i^+(u_1)\varphi_i^-(u_2)+\varphi_i^-(u_1)\varphi_i^+(u_2)=\delta(u_1,u_2).
\end{aligned}\tag{23}
$$

If $i\neq l$, we get

$$
\begin{aligned}
\varphi_i^+(u_1)\varphi_l^+(u_2)&=u_1^{-1}R_i(u_1^{-1})M_i(u_1^{-1})\prod_{\substack{1\leqslant j\leqslant n\\ j\neq i}}L_j^{\perp}(u_1^{-1})L_i^{\perp}(u_1)u_2^{-1}R_l(u_2^{-1})L_l(u_2^{-1})\prod_{\substack{1\leqslant k\leqslant n\\ k\neq l}}M_k^{\perp}(u_2^{-1})M_l^{\perp}(u_2)\\
&=\varphi_l^+(u_2)\varphi_i^+(u_1).
\end{aligned}\tag{24}
$$

From Eq. (20), taking the coefficient of $u_1^{-j-1/2}u_2^{-k-1/2}$ in Eqs. (23) and (24) yields the proposition.

Let $\boldsymbol{u}_i=(u_{i,1},u_{i,2},\ldots,u_{i,l_1})$ $(i=1,\ldots,n)$, we have the following statement.

Theorem 1 Let a series of quantum fields act on the constant function 1, we obtain

$$
\prod_{i=1}^{n}\prod_{j=1}^{l_i}\varphi_i^+(u_{i,j})\cdot(1)=\prod_{i=1}^{n}\prod_{j=1}^{l_i}z_i^{l_i}u_{i,j}^{j-l_i-2}G(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n),\tag{25}
$$

where

$$
G(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)=\prod_{m=1}^{n}\prod_{1\leqslant i<j\leqslant l_m}\left(1-\frac{u_{m,i}}{u_{m,j}}\right)\prod_{\substack{1\leqslant a,b\leqslant n\\ a\neq b}}\prod_{\substack{1\leqslant i\leqslant l_a\\ 1\leqslant j\leqslant l_b}}\left(1-\frac{1}{u_{a,i}u_{b,j}}\right)\prod_{i=1}^{n}\prod_{j=1}^{l_i}M_i(u_{i,j}^{-1})\tag{26}
$$

is a generating function of the generalized universal characters $S_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$.

Proof

$$
\begin{aligned}
G(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)&=\prod_{m=1}^{n}\prod_{1\leqslant i<j\leqslant l_m}\left(1-\frac{u_{m,i}}{u_{m,j}}\right)\prod_{\substack{1\leqslant a,b\leqslant n\\ a\neq b}}\prod_{\substack{1\leqslant i\leqslant l_a\\ 1\leqslant j\leqslant l_b}}\left(1-\frac{1}{u_{a,i}u_{b,j}}\right)\prod_{i=1}^{n}\prod_{j=1}^{l_i}\left(\sum_{\lambda_{i,j}\geqslant 0}h_{\lambda_{i,j}}(\boldsymbol{x}_i)u_{i,j}^{-\lambda_{i,j}}\right)\\
&=\prod_{m=1}^{n}\prod_{1\leqslant i<j\leqslant l_m}\left(1-R_{\lambda_{m,i},\lambda_{m,j}}\right)\prod_{\substack{1\leqslant a,b\leqslant n\\ a\neq b}}\prod_{\substack{1\leqslant i\leqslant l_a\\ 1\leqslant j\leqslant l_b}}\left(1-D_{\lambda_{a,i},\lambda_{b,j}}\right)h_{\lambda_{[1]}}(\boldsymbol{x}_1),\ldots,h_{\lambda_{[n]}}(\boldsymbol{x}_n)\prod_{i=1}^{n}\prod_{j=1}^{l_i}u_{i,j}^{-\lambda_{i,j}}\\
&=S_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\prod_{i=1}^{n}\prod_{j=1}^{l_i}u_{i,j}^{-\lambda_{i,j}}.
\end{aligned}
$$

Corollary 1 By Eqs. (20) and (25), we have

$$
\prod_{i=1}^{n}\prod_{j=1}^{l_i}\varphi^+_{\lambda_{i,j}-1/2}\cdot 1=\prod_{i=1}^{n}\prod_{j=1}^{l_i}z_i^{l_i}u_{i,j}^{j-l_i-2}S_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n).\tag{27}
$$

Definition 5 For any unknown function $\tau=\tau(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$, the system of bilinear equations

$$
\Omega_1(\tau\otimes\tau)=\Omega_2(\tau\otimes\tau)=\cdots=\Omega_n(\tau\otimes\tau)=0\tag{28}
$$

is called the generalized UC hierarchy, where

$$
\Omega_i=\sum_{k\in\mathbb{Z}+\frac{1}{2}}\varphi^-_{i,k}\otimes\varphi^+_{i,-k},\quad i=1,\ldots,n.\tag{29}
$$

Let $X_i = \sum_{j>M} C_j \varphi^+_{i,j}, C_j \in \mathbb{C}, M \in \mathbb{Z}$. It is easy to get $X_i^2 = 0$.

Proposition 4 The following relations hold.

$$[\Omega_i, X_i \otimes X_i] = 0, \quad i = 1, \dots, n. \tag{30}$$

Proof Form Eq. (21), we have

$$\begin{aligned}\Omega_i(X_i \otimes X_i) &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \varphi^-_{i,k} X_i \otimes \varphi^+_{i,-k} X_i \\ &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \left(\sum_{j>0} C_j \delta_{k+j,0} - X_i \varphi^-_{i,k}\right) \otimes \left(-X_i \varphi^+_{i,-k}\right) \\ &= (X_i \otimes X_i)\Omega_i.\end{aligned}$$

Corollary 2 $\tau_i = X_i \tau$ is a solution of the GUC hierarchy if $\tau = \tau(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ satisfies Eq. (28).

Proof If $\tau = \tau(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ satisfies Eq. (28), we get

$$\begin{aligned}\Omega_i(\tau_i \otimes \tau_i) &= \Omega_i(X_i\tau \otimes X_i\tau) = \Omega_i(X_i \otimes X_i)(\tau \otimes \tau) \\ &= (X_i \otimes X_i)\Omega_i(\tau \otimes \tau) = 0.\end{aligned}$$

Proposition 5 All the generalized universal characters are solutions of the GUC hierarchy.

Proof Obviously, $\tau = 1$ is a solution. From Eq. (27), we obtain

$$\begin{aligned}S_{[\lambda_{[1]},\dots,\lambda_{[n]}]}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) &= \prod_{i=1}^{n}\prod_{j=1}^{l_i} z_i^{-l_i} u_{i,j}^{2-j+l_i} \varphi^+_{\lambda_{i,j}-1/2} \cdot 1 \\ &= \prod_{i=1}^{n}\prod_{j=1}^{l_i} z_i^{-l_i} u_{i,j}^{2-j+l_i} X_{\lambda_{i,j}} \cdot 1,\end{aligned}$$

where $X_{\lambda_{i,j}} = \varphi^+_{\lambda_{i,j}-1/2}$.

3.2. Multiparameter generalized universal characters

For a fixed infinite sequence $a_i = (a_{i,0}, a_{i,1}, a_{i,2}, \dots)$ $(a_{i,j} \in \mathbb{C})$, we consider the following power of a variable $\boldsymbol{x}$:

$$(x|a_i)^k = (x - a_{i,0})(x - a_{i,1}) \cdots (x - a_{i,k-1}).$$

Using the shift operation $\tau : a_{i,k} \mapsto a_{i,k+1}$, we obtain

$$(x|\tau a_i)^k = (x - a_{i,1})(x - a_{i,2}) \cdots (x - a_{i,k}).$$

Definition 6 Let $\boldsymbol{\alpha}_{[i]} = (\alpha_{i,1}, \dots, \alpha_{i,l_i})$ $(i = 1, \dots, n,\ \alpha_{i,j} \in \mathbb{Z}_{\geqslant 0})$ be n partitions. The multiparameter generalized universal character in variables $\boldsymbol{x}_1 = (x_{1,1}, \dots, x_{1,N_1}), \dots, \boldsymbol{x}_n = (x_{n,1}, \dots, x_{n,N_n})$ with $l_i < N_i$ is defined by

$$\begin{aligned}S^{(a_1,\dots,a_n)}_{[\boldsymbol{\alpha}_{[1]},\dots,\boldsymbol{\alpha}_{[n]}]}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) &= \prod_{i=1}^{n-1} \sum_{\sigma_i \in S_{N_i}} \prod_{j=1}^{l_i} \frac{((\sigma_i(j) x_{i,\sigma_i(j)} - \partial_{x_{1,\sigma_i(j)}} - \cdots - \widehat{\partial}_{x_{i,\sigma_i(j)}} - \cdots - \partial_{x_{n,\sigma_i(j)}})|a_i)^{\alpha_{i,j}}}{(\sigma_i(j))^{\alpha_{i,j}}} \\ &\times \prod_{j<k} \frac{(\boldsymbol{x}_i - \widetilde{\partial}_{\boldsymbol{x}_1} - \cdots - \widehat{\widehat{\partial}}_{\boldsymbol{x}_i} - \cdots - \widetilde{\partial}_{\boldsymbol{x}_n})_{\sigma_i(j)}}{(\boldsymbol{x}_i - \widetilde{\partial}_{\boldsymbol{x}_1} - \cdots - \widehat{\widehat{\partial}}_{\boldsymbol{x}_i} - \cdots - \widetilde{\partial}_{\boldsymbol{x}_n})_{\sigma_i(j)} - (\boldsymbol{x}_i - \widetilde{\partial}_{\boldsymbol{x}_1} - \cdots - \widehat{\widehat{\partial}}_{\boldsymbol{x}_i} - \cdots - \widetilde{\partial}_{\boldsymbol{x}_n})_{\sigma_i(k)}} \\ &\times \sum_{\sigma_n \in S_{N_n}} \prod_{j=1}^{l_n} (x_{n,\sigma_n(j)}|a_n)^{\alpha_{n,j}} \prod_{j<k} \frac{x_{n,\sigma_n(j)}}{x_{n,\sigma_n(j)} - x_{n,\sigma_n(k)}},\end{aligned} \tag{31}$$

where $\widetilde{\partial}_{\boldsymbol{x}_i}$ stands for $\left(\frac{\partial}{\partial x_{i,1}}, \frac{1}{2}\frac{\partial}{\partial x_{i,2}}, \frac{1}{3}\frac{\partial}{\partial x_{i,3}}, \dots\right)$ and $\widehat{\partial}_{\boldsymbol{x}_i}$ represents the absence of this item.

Remark 1 The multiparameter generalized universal characters can be reduced to generalized universal characters with $a_i = (0, 0, \dots)$ and $\boldsymbol{\alpha}_{[i]}$ are strict partitions.

Lemma 1 For any sequence $a_i = (0, a_{i,1}, a_{i,2}, \dots)$,

$$\begin{aligned}&(u - a_{i,1}) \cdots (u - a_{i,n}) \\ &= \sum_{k=0}^{\infty} (-1)^{n-k} e_{n-k}(a_{i,1}, \dots, a_{i,n}) u^k, \\ &\frac{1}{(u - a_{i,1}) \cdots (u - a_{i,n})} \\ &= \sum_{k=0}^{\infty} h_{k-n}(a_{i,1}, \dots, a_{i,n}) u^{-k}, \\ u^n &= \sum_{k=0}^{\infty} h_{n-k}(a_{i,1}, \dots, a_{i,k+1})(u - a_{i,1}) \cdots (u - a_{i,k}), \\ \frac{1}{u^n} &= \sum_{k=0}^{\infty} (-1)^{n-k} e_{k-n}(a_{i,1}, \dots, a_{i,k-1}) \\ &\times \frac{1}{(u - a_{i,1}) \cdots (u - a_{i,k})},\end{aligned} \tag{32}$$

where $n \in \mathbb{N}$.

Lemma 2 For any sequence $a_i = (0, a_{i,1}, a_{i,2}, \dots)$,

$$\sum_{m=0}^{\infty} \frac{(x|a_i)^m}{(u|\tau a_i)^m} = \sum_{m=0}^{\infty} \frac{x^m}{u^m}. \tag{33}$$

Considering the positive values of $\lambda_{i,j}$ in Eq. (26), we have

$$G^+(\boldsymbol{u}_1, \dots, \boldsymbol{u}_l) = \sum_{\lambda_{i,j} \in \mathbb{Z}_{>0}} \frac{S_{[\lambda_{[1]},\dots,\lambda_{[n]}]}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)}{u_{1,1}^{\lambda_{1,1}} \cdots u_{1,l_1}^{\lambda_{1,l_1}} \cdots u_{n,1}^{\lambda_{n,1}} \cdots u_{n,l_n}^{\lambda_{n,l_n}}}. \tag{34}$$

Theorem 2 For sequences $a_i = (0, a_{i,1}, a_{i,2}, \dots)$ $(i = 1, \dots, n)$,

$$G^+(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_l)=\sum_{\lambda_{i,j}\in\mathbb{Z}_{>0}}\frac{S^{(a_1,\ldots,a_n)}_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)}{(u_{1,1}|\tau a_1)^{\lambda_{1,1}}\cdots(u_{1,l_1}|\tau a_1)^{\lambda_{1,l_1}}\cdots(u_{n,1}|\tau a_n)^{\lambda_{n,1}}\cdots(u_{n,l_n}|\tau a_n)^{\lambda_{n,l_n}}}. \tag{35}$$

Proof From Eqs. (32) and (33), we have

$$\begin{aligned}
&\sum_{\lambda_{i,j}\in\mathbb{Z}_{>0}}\frac{S^{(a_1,\ldots,a_n)}_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)}{(u_{1,1}|\tau a_1)^{\lambda_{1,1}}\cdots(u_{1,l_1}|\tau a_1)^{\lambda_{1,l_1}}\cdots(u_{n,1}|\tau a_n)^{\lambda_{n,1}}\cdots(u_{n,l_n}|\tau a_n)^{\lambda_{n,l_n}}}\\
=&\sum_{\lambda_{i,j}\in\mathbb{Z}_{>0}}\prod_{i=1}^{n-1}\sum_{\sigma_i\in S_{N_i}}\prod_{j=1}^{l_i}\frac{\left(\frac{\sigma_i(j)x_{i,\sigma_i(j)}-\partial_{x_{1,\sigma_i(j)}}-\cdots-\widehat{\partial}_{x_{i,\sigma_i(j)}}-\cdots-\partial_{x_{n,\sigma_i(j)}}}{\sigma_i(j)}|a_i\right)^{\lambda_{i,j}}}{(u_{i,j}|\tau a_i)^{\lambda_{i,j}}}\\
&\times\prod_{j<k}\frac{(\boldsymbol{x}_i-\widetilde{\partial}_{\boldsymbol{x}_1}-\cdots-\widehat{\widetilde{\partial}}_{\boldsymbol{x}_i}-\cdots-\widetilde{\partial}_{\boldsymbol{x}_n})_{\sigma_i(j)}}{(\boldsymbol{x}_i-\widetilde{\partial}_{\boldsymbol{x}_1}-\cdots-\widehat{\widetilde{\partial}}_{\boldsymbol{x}_i}-\cdots-\widetilde{\partial}_{\boldsymbol{x}_n})_{\sigma_i(j)}-(\boldsymbol{x}_i-\widetilde{\partial}_{\boldsymbol{x}_1}-\cdots-\widehat{\widetilde{\partial}}_{\boldsymbol{x}_i}-\cdots-\widetilde{\partial}_{\boldsymbol{x}_n})_{\sigma_i(k)}}\\
&\times\sum_{\sigma_n\in S_{N_n}}\prod_{j=1}^{l_n}\frac{(x_{n,\sigma_n(j)}|a_n)^{\lambda_{n,j}}}{(u_{n,j}|\tau a_n)^{\lambda_{n,j}}}\prod_{j<k}\frac{x_{n,\sigma_n(j)}}{x_{n,\sigma_n(j)}-x_{n,\sigma_n(k)}}\\
=&\sum_{\lambda_{i,j}\in\mathbb{Z}_{>0}}\prod_{i=1}^{n-1}\sum_{\sigma_i\in S_{N_i}}\prod_{j=1}^{l_i}\frac{\left(\sigma_i(j)x_{i,\sigma_i(j)}-\partial_{x_{1,\sigma_i(j)}}-\cdots-\widehat{\partial}_{x_{i,\sigma_i(j)}}-\cdots-\partial_{x_{n,\sigma_i(j)}}\right)^{\lambda_{i,j}}}{(u_{i,j}\sigma_i(j))^{\lambda_{i,j}}}\\
&\times\prod_{j<k}\frac{(\boldsymbol{x}_i-\widetilde{\partial}_{\boldsymbol{x}_1}-\cdots-\widehat{\widetilde{\partial}}_{\boldsymbol{x}_i}-\cdots-\widetilde{\partial}_{\boldsymbol{x}_n})_{\sigma_i(j)}}{(\boldsymbol{x}_i-\widetilde{\partial}_{\boldsymbol{x}_1}-\cdots-\widehat{\widetilde{\partial}}_{\boldsymbol{x}_i}-\cdots-\widetilde{\partial}_{\boldsymbol{x}_n})_{\sigma_i(j)}-(\boldsymbol{x}_i-\widetilde{\partial}_{\boldsymbol{x}_1}-\cdots-\widehat{\widetilde{\partial}}_{\boldsymbol{x}_i}-\cdots-\widetilde{\partial}_{\boldsymbol{x}_n})_{\sigma_i(k)}}\cdot\sum_{\sigma_n\in S_{N_n}}\prod_{j=1}^{l_n}\frac{x_{n,\sigma_n(j)}^{\lambda_{n,j}}}{u_{n,j}^{\lambda_{n,j}}}\prod_{j<k}\frac{x_{n,\sigma_n(j)}}{x_{n,\sigma_n(j)}-x_{n,\sigma_n(k)}}.
\end{aligned}$$

Corollary 3 For n partitions $\alpha_{[i]}=(\alpha_{i,1},\ldots,\alpha_{i,l_i})$ and sequences of complex numbers $a_i=(0,a_{i,1},a_{i,2},\ldots)$,

$$\begin{aligned}
S^{(a_1,\ldots,a_n)}_{[\boldsymbol{\alpha}_{[1]},\ldots,\boldsymbol{\alpha}_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)=&\sum_{\lambda_{i,j}\in\mathbb{Z}_{>0}}(-1)^{\sum_i^{l_i}\sum_j(\lambda_{i,j}-\alpha_{i,j})}e_{\alpha_{1,1}-\lambda_{1,1}}(a_{1,1},\ldots,a_{1,\alpha_{1,1}-1})\\
&\times\cdots e_{\alpha_{1,l_1}-\lambda_{1,l_1}}(a_{1,1},\ldots,a_{1,\alpha_{1,l_1}-1})\cdots e_{\alpha_{n,1}-\lambda_{n,1}}(a_{n,1},\ldots,a_{n,\alpha_{n,1}-1})\\
&\times\cdots e_{\alpha_{n,l_n}-\lambda_{n,l_n}}(a_{n,1},\ldots,a_{n,\alpha_{n,l_n}-1})\cdot S_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n).
\end{aligned}\tag{36}$$

Proof By Eq. (35), we obtain

$$\sum_{\lambda_{i,j}\in\mathbb{Z}_{>0}}\frac{S_{[\lambda_{[1]},\ldots,\lambda_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)}{u_{1,1}^{\lambda_{1,1}}\cdots u_{1,l_1}^{\lambda_{1,l_1}}\cdots u_{n,1}^{\lambda_{n,1}}\cdots u_{n,l_n}^{\lambda_{n,l_n}}}=\sum_{\alpha_{i,j}\in\mathbb{Z}_{>0}}\frac{S^{(a_1,\ldots,a_n)}_{[\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)}{(u_{1,1}|\tau a_1)^{\alpha_{1,1}}\cdots(u_{1,l_1}|\tau a_1)^{\alpha_{1,l_1}}\cdots(u_{n,1}|\tau a_n)^{\alpha_{n,1}}\cdots(u_{n,l_n}|\tau a_n)^{\alpha_{n,l_n}}}. \tag{37}$$

With

$$\frac{1}{u_{1,1}^{\lambda_{1,1}}}=\sum_{\alpha_{1,1}=0}^{\infty}(-1)^{\lambda_{1,1}-\alpha_{1,1}}e_{\alpha_{1,1}-\lambda_{1,1}}(a_{1,1},\ldots,a_{1,\alpha_{1,1}-1})\frac{1}{(u_{1,1}|\tau a_1)^{\alpha_{1,1}}},$$

the corollary can be proved by taking the coefficients of

$$(u_{1,1}|\tau a_1)^{-\alpha_{1,1}}\cdots(u_{1,l_1}|\tau a_1)^{-\alpha_{1,l_1}}\cdots(u_{n,1}|\tau a_1)^{-\alpha_{n,1}}\cdots(u_{n,l_n}|\tau a_1)^{-\alpha_{n,l_n}}$$

in Eq. (37).

Theorem 3 For n partitions $\alpha_{[i]}=(\alpha_{i,1},\ldots,\alpha_{i,l_i})$ $(\alpha_{i,j}\in\mathbb{Z}_{>0})$ and sequences of complex numbers $a_i=(0,a_{i,1},a_{i,2},\ldots)$, the multiparameter generalized universal characters $S^{(a_1,\ldots,a_n)}_{[\boldsymbol{\alpha}_{[1]},\ldots,\boldsymbol{\alpha}_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$ are solutions of the GUC hierarchy.

Proof By means of Eqs. (31) and (36), we have

$$S^{(a_1,\ldots,a_n)}_{[\boldsymbol{\alpha}_{[1]},\ldots,\boldsymbol{\alpha}_{[n]}]}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)=\prod_{i=1}^{n}\prod_{j=1}^{l_i}z_i^{l_i}u_{i,j}^{l_i+2-j}\overline{X}_{\lambda_{i,j}}\cdot 1,$$

where

$$\overline{X}_{\lambda_{i,j}}=\sum_{\alpha_{i,j}\in\mathbb{Z}_{>0}}(-1)^{\lambda_{i,j}-\alpha_{i,j}}e_{\alpha_{i,j}-\lambda_{i,j}}(a_{i,1},\ldots,a_{i,\alpha_{i,j}-1})\varphi^{+}_{\lambda_{i,j}+1/2}.$$

According to Corollary 2, we find that multiparameter generalized universal characters satisfy the GUC hierarchy.

4. Generalized B-type universal characters and GBUC hierarchy

In this section, based on quantum fields representation, the generalized B-type UC (GBUC) hierarchy is proposed. Meanwhile we construct the generalized B-type universal characters to its multiparameter form with infinite sequences of complex numbers. It can be proved that both of them are solutions of the GBUC hierarchy.

4.1. Quantum fields presentation of generalized B-type universal character and GBUC hierarchy

Let

$$\begin{aligned}
H_i(u)&=M_i(u^{-1})=\exp\left(\sum_{n\geqslant1}\frac{p_n(\boldsymbol{x}_i)}{nu^n}\right),\\
H_i^{\perp}(u)&=M_i^{\perp}(u)=\exp\left(\sum_{n\geqslant1}u^n\frac{\partial}{\partial p_n(\boldsymbol{x}_i)}\right),\\
E_i(u)&=L_i(-u^{-1})=\exp\left(\sum_{n\geqslant1}\frac{(-1)^{n-1}p_n(\boldsymbol{x}_i)}{nu^n}\right),\\
E_i^{\perp}(u)&=L_i^{\perp}(-u)=\exp\left(\sum_{n\geqslant1}(-1)^{n-1}u^n\frac{\partial}{\partial p_n(\boldsymbol{x}_i)}\right).
\end{aligned}\tag{38}$$

Obviously, we have

$$\begin{aligned}
Q_i(u)&=E_i(u)H_i(u)=R_i^2(u),\\
R_i(u)&=\exp\left(\sum_{n\in\mathbb{N}_{\rm odd}}\frac{p_n(\boldsymbol{x}_i)}{nu^n}\right),\\
Q_i^{\perp}(u)&=E_i^{\perp}(u)H_i^{\perp}(u)=(R_i^{\perp}(u))^2,\\
R_i^{\perp}(u)&=\exp\left(\sum_{n\in\mathbb{N}_{\rm odd}}u^n\frac{\partial}{\partial p_n(\boldsymbol{x}_i)}\right),
\end{aligned}\tag{39}$$

where $\mathbb{N}_{\rm odd}=\{1,3,5,\ldots\}$. Besides, we get $Q_i(u)Q_i(-u)=1$. Furthermore, we also obtain

$$\begin{aligned}
\left(1-\frac{u}{v}\right)H_i^{\perp}(u)H_i(v)&=H_i(v)H_i^{\perp}(u),\\
\left(1-\frac{u}{v}\right)E_i^{\perp}(u)E_i(v)&=E_i(v)E_i^{\perp}(u),\\
H_i^{\perp}(u)E_i(v)&=\left(1+\frac{u}{v}\right)E_i(v)H_i^{\perp}(u),\\
E_i^{\perp}(u)H_i(v)&=\left(1+\frac{u}{v}\right)H_i(v)E_i^{\perp}(u),\\
H_i^{\perp}(u)Q_i(v)&=\frac{v+u}{v-u}Q_i(v)H_i^{\perp}(u),\\
E_i^{\perp}(u)Q_i(v)&=\frac{v+u}{v-u}Q_i(v)E_i^{\perp}(u),\\
R_i^{\perp}(u)Q_i(v)&=\frac{v+u}{v-u}Q_i(v)R_i^{\perp}(u),
\end{aligned}\tag{40}$$

these operators can exchange freely if $i=j$.

By use of the Corollary in Ref. [20], we present the following definition.

Definition 7 For n strict partitions $\mu_{[i]}=(\mu_{i,1},\ldots,\mu_{i,r_i})$ $(i=1,\ldots,n)$, the generalized B-type universal characters in independent variables $\boldsymbol{y}_1=(y_{1,1},y_{1,3},\ldots,y_{1,M_1}),\ldots,\boldsymbol{y}_n=(y_{n,1},y_{n,3},\ldots,y_{n,M_n})$ $(M_i\in\mathbb{N}_{\rm odd})$ is defined by

$$\begin{aligned}
&Q_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)\\
&=\prod_{i=1}^{n-1}\frac{2^{r_i}}{(M_i-r_i)!}\sum_{\sigma_i\in S_{M_i}}\prod_{j=1}^{r_i}\frac{(\sigma_i(j)y_{i,\sigma_i(j)}-2\partial_{y_{1,\sigma_i(j)}}-\cdots-2\widehat{\partial}_{y_{i,\sigma_i(j)}}-\cdots-2\partial_{y_{n,\sigma_i(j)}})^{\mu_{i,j}}}{(\sigma_i(j))^{\mu_{i,j}}}\\
&\times\prod_{j<k}\frac{(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(j)}+(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(k)}}{(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(j)}-(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(k)}}\\
&\times\frac{2^{r_n}}{(M_n-r_n)!}\sum_{\sigma_n\in S_{M_n}}\prod_{j=1}^{r_n}y_{n,\sigma_n(j)}^{\mu_{n,j}}\prod_{j<k}\frac{y_{n,\sigma_n(j)}+y_{n,\sigma_n(k)}}{y_{n,\sigma_n(j)}-y_{n,\sigma_n(k)}},
\end{aligned}\tag{41}$$

where $\widetilde{\partial}_{\boldsymbol{y}_i}$ stands for $\left(\dfrac{\partial}{\partial y_{i,1}},\dfrac{1}{3}\dfrac{\partial}{\partial y_{i,3}},\dfrac{1}{5}\dfrac{\partial}{\partial y_{i,5}},\cdots\right)$.

Set $\boldsymbol{v}_i=(v_{i,1},\ldots,v_{i,r_i})$ $(i=1,\ldots,n)$. The generating function of $Q_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)$ for $M_i>r_i$ is

$$Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)=\prod_{i=1}^{n}\prod_{1\leqslant j<k\leqslant r_i}\frac{v_{i,k}-v_{i,j}}{v_{i,k}+v_{i,j}}\prod_{\substack{1<i\leqslant n\\1\leqslant m<n}}\prod_{\substack{1\leqslant j\leqslant r_i\\1\leqslant k\leqslant r_m}}\frac{v_{i,j}v_{m,k}-1}{v_{i,j}v_{m,k}+1}\prod_{i=1}^{n}\prod_{j=1}^{r_i}Q_i(v_{i,j}),\tag{42}$$

where

$$\frac{v_{i,k}-v_{i,j}}{v_{i,k}+v_{i,j}}=1+2\sum_{r\geqslant 1}(-1)^r v_{i,k}^r v_{i,j}^{-r}.$$

Taking the coefficient of $v_{1,1}^{-\mu_{1,1}}\cdots v_{1,r_1}^{-\mu_{1,r_1}}\cdots v_{n,1}^{-\mu_{n,1}}\cdots v_{n,r_n}^{-\mu_{n,r_n}}$ in Eq. (42), we get generalized B-type universal characters. That is

$$Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)=\sum_{\mu_{i,j}\in\mathbb{Z}}\frac{Q_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)}{v_{1,1}^{\mu_{1,1}}\cdots v_{1,r_1}^{\mu_{1,r_1}}\cdots v_{n,1}^{\mu_{n,1}}\cdots v_{n,r_n}^{\mu_{n,r_n}}}. \quad (43)$$

Let $\overline{\mathcal{B}}_{\rm odd}=\mathcal{B}_{\rm odd}[x_{1,1},x_{1,3},x_{1,5},\ldots,\ x_{2,1},x_{2,3},x_{2,5},\ldots,x_{n,1},x_{n,3},x_{n,5},\ldots]$ be a subalgebra of $\mathcal{B}[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n]$, which is the polynomial algebra of infinitely many variables.

Define the quantum fields $\psi_i(v)=\sum_{m\in\mathbb{Z}}\psi_{i,m}v^{-m}$ $(i=1,\ldots,n)$ acting on $Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)$ by rule

$$\begin{aligned}&\psi_i(v)Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_i,\ldots,\boldsymbol{v}_n)=Q(\boldsymbol{v}_1,\ldots,\overline{\boldsymbol{v}}_i,\ldots,\boldsymbol{v}_n),\\&\overline{\boldsymbol{v}}_i=(v,v_{i,1},v_{i,2},\ldots,v_{i,r_i}).\end{aligned} \quad (44)$$

By Eq. (43), we also have

$$\begin{aligned}&\psi_{i,m}Q_{[\mu_{[1]},\ldots,\mu_{[i]},\ldots,\mu_{[n]}]}=Q_{[\mu_{[1]},\ldots,\overline{\mu}_{[i]},\ldots,\mu_{[n]}]},\\&\overline{\mu}_{[i]}=(m,\mu_{i,1},\ldots,\mu_{i,r_i}).\end{aligned} \quad (45)$$

Proposition 6 The quantum fields $\psi_i(v)$ $(i=1,\ldots,n)$ satisfy

$$[\psi_i(u),\psi_i(v)]_+=2v\delta(v,-u),\ \ [\psi_i(u),\psi_j(v)]=0, \quad (46)$$

which are equivalent to

$$[\psi_{i,m},\psi_{i,n}]_+=2(-1)^m\delta_{m+n,0},\ \ [\psi_{i,m},\psi_{j,n}]=0, \quad (47)$$

where $[A,B]_+=AB+BA$.

Proof It is easy to derive $[\psi_i(u),\psi_j(v)]=0$. From Eq. (44), we get

$$\begin{aligned}&(\psi_i(u)\psi_i(v)+\psi_i(v)\psi_i(u))Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)\\&=\left(\frac{v-u}{v+u}+\frac{u-v}{u+v}\right)C(u,v,\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)\\&=2\left(1+\sum_{r\geqslant 1}(-1)^r u^r v^{-r}+\sum_{r\geqslant 1}(-1)^r v^r u^{-r}\right)C(u,v,\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)\\&=2v\delta(v,-u)C(u,v,\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)\\&\quad\times 2v\delta(v,-u)Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n),\end{aligned} \quad (48)$$

where

$$\begin{aligned}&C(u,v,\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)\\&=Q_i(u)Q_i(v)\prod_{j=1}^{r_i}\frac{v_{i,j}-u}{v_{i,j}+u}\frac{v_{i,j}-v}{v_{i,j}+v}Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n).\end{aligned}$$

Expanding $[\psi_i(u),\psi_i(v)]_+=2v\delta(v,-u)$ and $[\psi_i(u),\psi_j(v)]=0$ into

$$\begin{aligned}&\sum_{m\in\mathbb{Z}}\psi_{i,m}u^{-m}\sum_{n\in\mathbb{Z}}\psi_{i,n}u^{-n}+\sum_{n\in\mathbb{Z}}\psi_{i,n}u^{-n}\sum_{m\in\mathbb{Z}}\psi_{i,m}u^{-m}\\&=\sum_{r\in\mathbb{Z}}\left(\frac{-u}{v}\right)^r,\\&\sum_{m\in\mathbb{Z}}\psi_{i,m}u^{-m}\sum_{n\in\mathbb{Z}}\psi_{j,n}u^{-n}+\sum_{n\in\mathbb{Z}}\psi_{j,n}u^{-n}\sum_{m\in\mathbb{Z}}\psi_{i,m}u^{-m}\\&=0,\end{aligned} \quad (49)$$

by taking coefficient of $u^{-m}v^{-n}$ in Eq. (49), we obtain Eq. (47).

For n strict partitions $\mu_{[1]},\ldots,\mu_{[n]}$, we get

$$Q_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)=\prod_{i=1}^{n}\prod_{j=1}^{r_i}\psi_{i,j}\cdot 1, \quad (50)$$

which can be rewritten as

$$Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)=\prod_{i=1}^{n}\prod_{j=1}^{r_i}\psi_i(v_{i,j})\cdot(1). \quad (51)$$

Proposition 7 The quantum fields $\psi_i(v)$ $(i=1,\ldots,n)$ can be expressed by

$$\psi_i(v)=Q_i(v)\prod_{\substack{j=1\\j\neq i}}^{n}R_j^{\perp}\left(-\frac{1}{v}\right)R_i^{\perp}(-v).$$

Proof

$$\begin{aligned}&Q_i(v)\prod_{\substack{j=1\\j\neq i}}^{n}R_j^{\perp}\left(-\frac{1}{v}\right)R_i^{\perp}(-v)Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)\\&=Q_i(v)\prod_{i=1}^{n}\prod_{1\leqslant j<k\leqslant r_i}\frac{v_{i,k}-v_{i,j}}{v_{i,k}+v_{i,j}}\prod_{\substack{1<i\leqslant n\\1\leqslant m<n}}\prod_{\substack{1\leqslant j\leqslant r_i\\1\leqslant k\leqslant r_m}}\\&\quad\times\frac{v_{i,j}v_{m,k}-1}{v_{i,j}v_{m,k}+1}\left(\prod_{\substack{j=1\\j\neq i}}^{n}R_j^{\perp}\left(-\frac{1}{v}\right)\prod_{t=1}^{r_j}Q_j(v_{j,t})\right)\\&\quad\times\left(R_i^{\perp}(-v)\prod_{t=1}^{r_i}Q_i(v_{i,t})\right)\\&=Q_i(v)\prod_{i=1}^{n}\prod_{1\leqslant j<k\leqslant r_i}\frac{v_{i,k}-v_{i,j}}{v_{i,k}+v_{i,j}}\prod_{\substack{1<i\leqslant n\\1\leqslant m<n}}\prod_{\substack{1\leqslant j\leqslant r_i\\1\leqslant k\leqslant r_m}}\\&\quad\times\frac{v_{i,j}v_{m,k}-1}{v_{i,j}v_{m,k}+1}\prod_{\substack{j=1\\j\neq i}}^{n}\prod_{t=1}^{r_j}\frac{vv_{j,t}-1}{vv_{j,t}+1}\prod_{t=1}^{r_i}\frac{v_{i,t}-v}{v_{i,t}+v}\prod_{i=1}^{n}\prod_{j=1}^{r_i}Q_i(v_{i,j})\\&=Q(\boldsymbol{v}_1,\ldots,\overline{\boldsymbol{v}}_i,\ldots,\boldsymbol{v}_n),\end{aligned}$$

where $\overline{\boldsymbol{v}}_i=(v,v_{i,1},\ldots,v_{i,r_i})$.

Definition 8 For an unknown function $\overline{\tau}=\tau(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)$, the GBUC hierarchy is given by

$$\overline{\Omega}_1(\overline{\tau}\otimes\overline{\tau})=\overline{\Omega}_2(\overline{\tau}\otimes\overline{\tau})=\cdots=\overline{\Omega}_n(\overline{\tau}\otimes\overline{\tau})=\overline{\tau}\otimes\overline{\tau}, \quad (52)$$

where

$$\overline{\Omega}_i=\sum_{n\in\mathbb{Z}}\psi_{i,n}\otimes(-1)^n\psi_{i,-n}.$$

Set $Y_i=\sum_{j>M}D_j\psi_{i,j}, D_j\in\mathbb{C}, M\in\mathbb{Z}$. Obviously, $Y_i^2=0$.

Proposition 8 The commutation of $\overline{\Omega}_i$ and Y_i is

$$[\overline{\Omega}_i,Y_i\otimes Y_i]=0. \quad (53)$$

Proof Using the definition of $\overline{\Omega}_i$, we have

$$\begin{aligned}\overline{\Omega}_i(Y_i\otimes Y_i) &= \sum_{n\in\mathbb{Z}}\psi_{i,n}Y_i\otimes(-1)^n\psi_{i,-n}Y_i\\ &= \sum_{n\in\mathbb{Z}}\left(-Y_i\psi_{i,n}+2\sum_{j>0}D_j\delta_{j+n,0}\right)\\ &\quad\otimes(-1)^n\left(-Y_i\psi_{i,-n}+2\sum_{k>0}D_k\delta_{n-k,0}\right)\\ &= (Y_i\otimes Y_i)\overline{\Omega}_i-\sum_{j>0}2(-1)^jD_j\otimes(-1)^jY_i\psi_{i,j}\\ &\quad-\sum_{k>0}(-1)^kY_i\psi_{i,k}\otimes 2(-1)^kD_k\\ &\quad+4\sum_{n\in\mathbb{Z}}\sum_{j,k>0}(-1)^jD_j\delta_{j+n,0}\otimes(-1)^kD_k\delta_{n-k,0}\\ &= (Y_i\otimes Y_i)\overline{\Omega}_i.\end{aligned}$$

Corollary 4 If $\overline{\tau}=\tau(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)\in\overline{\mathcal{B}}_{\rm odd}$ is a tau-function of the GBUC hierarchy, $\overline{\tau}_i=Y_i\overline{\tau}$ also satisfies the GBUC hierarchy.

Proof Taking $\overline{\tau}_i$ into Eq. (52), we obtain

$$\begin{aligned}\overline{\Omega}_i(\overline{\tau}_i\otimes\overline{\tau}_i) &= \overline{\Omega}_i(Y_i\otimes Y_i)(\overline{\tau}\otimes\overline{\tau})\\ &= (Y_i\otimes Y_i)\overline{\Omega}_i(\overline{\tau}\otimes\overline{\tau})=\overline{\tau}_i\otimes\overline{\tau}_i.\end{aligned}$$

Since $\overline{\tau}=1$ satisfies the GBUC hierarchy, we imply that all B-type generalized universal characters are solutions of the GBUC hierarchy with Eq. (50).

4.2. Multiparameter generalized B-type universal characters

Definition 9 Set $b_i=(0,b_{i,1},b_{i,2},\ldots)(b_{i,j}\in\mathbb{C},i=1,\ldots,n)$ be n sequences of complex numbers. For n partitions $\beta_{[1]}=(\beta_{1,1},\ldots,\beta_{1,r_1}),\ldots,\beta_{[n]}=(\beta_{n,1},\ldots,\beta_{n,r_n})$, the definition of multiparameter generalized B-type universal characters in variables $\boldsymbol{y}_i=(y_{i,1},\ldots,y_{i,M_i})$ $(i=1,\ldots,n)$ with $r_i<M_i$ is

$$\begin{aligned}Q^{(b_1,\ldots,b_n)}_{[\beta_{[1]},\ldots,\beta_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n) &= \prod_{i=1}^{n-1}\frac{2^{r_i}}{(M_i-r_i)!}\sum_{\sigma_i\in S_{M_i}}\prod_{j=1}^{r_i}\frac{\left((\sigma_i(j)y_{i,\sigma_i(j)}-2\partial_{y_{1,\sigma_i(j)}}-\cdots-2\widehat{\partial}_{y_{i,\sigma_i(j)}}-\cdots-2\partial_{y_{n,\sigma_i(j)}})|b_i\right)^{\beta_{i,j}}}{(\sigma_i(j))^{\mu_{i,j}}}\\ &\quad\times\prod_{j<k}\frac{(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(j)}+(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(k)}}{(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(j)}-(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(k)}}\\ &\quad\times\frac{2^{r_n}}{(M_n-r_n)!}\sum_{\sigma_n\in S_{M_n}}\prod_{j=1}^{r_n}(y_{n,\sigma_n(j)}|b_n)^{\beta_{n,j}}\prod_{j<k}\frac{y_{n,\sigma_n(j)}+y_{n,\sigma_n(k)}}{y_{n,\sigma_n(j)}-y_{n,\sigma_n(k)}}.\end{aligned}\tag{54}$$

Taking the part of $\mu_{i,j}>0$ in $Q(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)$, we obtain

$$Q^+(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)=\sum_{\mu_{i,j}\in\mathbb{Z}_{>0}}\frac{Q_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)}{v_{1,1}^{\mu_{1,1}}\cdots v_{1,r_1}^{\mu_{1,r_1}}\cdots v_{n,1}^{\mu_{n,1}}\cdots v_{n,r_n}^{\mu_{n,r_n}}}.\tag{55}$$

Theorem 4

$$Q^+(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n)=\sum_{\mu_{i,j}\in\mathbb{Z}_{>0}}\frac{Q^{(b_1,\ldots,b_n)}_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)}{(v_{1,1}|\tau b_1)^{\mu_{1,1}}\cdots(v_{1,r_1}|\tau b_1)^{\mu_{1,r_1}}\cdots(v_{n,1}|\tau b_n)^{\mu_{n,1}}\cdots(v_{n,r_n}|\tau b_n)^{\mu_{n,r_n}}}.\tag{56}$$

Proof In terms of Eqs. (33) and (54), we have

$$\begin{aligned}&\sum_{\mu_{i,j}\in\mathbb{Z}_{>0}}\frac{Q^{(b_1,\ldots,b_n)}_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)}{(v_{1,1}|\tau b_1)^{\mu_{1,1}}\cdots(v_{1,r_1}|\tau b_1)^{\mu_{1,r_1}}\cdots(v_{n,1}|\tau b_n)^{\mu_{n,1}}\cdots(v_{n,r_n}|\tau b_n)^{\mu_{n,r_n}}}\\ &=\sum_{\mu_{i,j}\in\mathbb{Z}_{>0}}\prod_{i=1}^{n-1}\frac{2^{r_i}}{(M_i-r_i)!}\sum_{\sigma_i\in S_{M_i}}\prod_{j=1}^{r_i}\frac{\left(\frac{\sigma_i(j)y_{i,\sigma_i(j)}-2\partial_{y_{1,\sigma_i(j)}}-\cdots-2\widehat{\partial}_{y_{i,\sigma_i(j)}}-\cdots-2\partial_{y_{n,\sigma_i(j)}}}{\sigma_i(j)}|b_i\right)^{\mu_{i,j}}}{(v_{i,j}|\tau b_i)^{\mu_{i,j}}}\\ &\quad\times\prod_{j<k}\frac{(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(j)}+(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(k)}}{(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(j)}-(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(k)}}\\ &\quad\times\frac{2^{r_n}}{(M_n-r_n)!}\sum_{\sigma_n\in S_{M_n}}\prod_{j=1}^{r_n}\frac{(y_{n,\sigma_n(j)}|b_n)^{\mu_{n,j}}}{(v_{n,j}|\tau b_n)^{\mu_{n,j}}}\prod_{j<k}\frac{y_{n,\sigma_n(j)}+y_{n,\sigma_n(k)}}{y_{n,\sigma_n(j)}-y_{n,\sigma_n(k)}}\\ &=\sum_{\mu_{i,j}\in\mathbb{Z}_{>0}}\prod_{i=1}^{n-1}\frac{2^{r_i}}{(M_i-r_i)!}\sum_{\sigma_i\in S_{M_i}}\prod_{j=1}^{r_i}\frac{\left(\sigma_i(j)y_{i,\sigma_i(j)}-2\partial_{y_{1,\sigma_i(j)}}-\cdots-2\widehat{\partial}_{y_{i,\sigma_i(j)}}-\cdots-2\partial_{y_{n,\sigma_i(j)}}\right)^{\mu_{i,j}}}{(v_{i,j}\sigma_i(j))^{\mu_{i,j}}}\end{aligned}$$

$$\times\prod_{j<k}\frac{(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(j)}+(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(k)}}{(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(j)}-(\boldsymbol{y}_i-2\widetilde{\partial}_{\boldsymbol{y}_1}-\cdots-2\widehat{\widetilde{\partial}}_{\boldsymbol{y}_i}-\cdots-2\widetilde{\partial}_{\boldsymbol{y}_n})_{\sigma_i(k)}}$$
$$\times\frac{2^{r_n}}{(M_n-r_n)!}\sum_{\sigma_n\in S_{M_n}}\prod_{j=1}^{r_n}\frac{y_{n,\sigma_n(j)}^{\mu_{n,j}}}{v_{n,j}^{\mu_{n,j}}}\prod_{j<k}\frac{y_{n,\sigma_n(j)}+y_{n,\sigma_n(k)}}{y_{n,\sigma_n(j)}-y_{n,\sigma_n(k)}}. \tag{57}$$

Corollary 5 For n partitions $\beta_{[i]}=(\beta_{i,1},\ldots,\beta_{i,r_i})$ $(i=1,\ldots,n)$ and sequences of complex numbers $b_i=(0,b_{i,1},b_{i,2},\ldots)$,

$$\begin{aligned}Q^{(b_1,\ldots,b_n)}_{[\beta_{[1]},\ldots,\beta_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)=&\sum_{\mu_{i,j}\in\mathbb{Z}_{>0}}(-1)^{\sum_i\sum_j^{r_i}(\mu_{i,j}-\beta_{i,j})}e_{\beta_{1,1}-\mu_{1,1}}(b_{1,1},\ldots,b_{1,\beta_{1,1}-1})\\&\times\cdots e_{\beta_{1,r_1}-\mu_{1,r_1}}(b_{1,1},\ldots,b_{1,\beta_{1,r_1}-1})\cdots e_{\beta_{n,1}-\mu_{n,1}}(b_{n,1},\ldots,b_{n,\beta_{n,1}-1})\\&\times\cdots e_{\beta_{n,r_n}-\mu_{n,r_n}}(b_{n,1},\ldots,b_{n,\beta_{n,l_n}-1})\cdot Q_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n).\end{aligned} \tag{58}$$

Proof Using Eq. (56), we get

$$\sum_{\mu_{i,j}\in\mathbb{Z}_{>0}}\frac{Q_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)}{v_{1,1}^{\mu_{1,1}}\cdots v_{1,r_1}^{\mu_{1,r_1}}\cdots v_{n,1}^{\mu_{n,1}}\cdots v_{n,r_n}^{\mu_{n,r_n}}}=\sum_{\mu_{i,j}\in\mathbb{Z}_{>0}}\frac{Q^{(b_1,\ldots,b_n)}_{[\mu_{[1]},\ldots,\mu_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)}{(v_{1,1}|\tau b_1)^{\mu_{1,1}}\cdots(v_{1,r_1}|\tau b_1)^{\mu_{1,r_1}}\cdots(v_{n,1}|\tau b_n)^{\mu_{n,1}}\cdots(v_{n,r_n}|\tau b_n)^{\mu_{n,r_n}}}. \tag{59}$$

Taking the coefficients of $(v_{1,1}|\tau b_1)^{-\beta_{1,1}}\cdots(v_{1,r_1}|\tau b_1)^{-\beta_{1,r_1}}\cdots(v_{n,1}|\tau b_1)^{-\beta_{n,1}}\cdots(v_{n,r_n}|\tau b_1)^{-\beta_{n,r_n}}$ in Eq. (59) with

$$\begin{aligned}\frac{1}{v_{1,1}^{\mu_{1,1}}}=&\sum_{\beta_{1,1}=0}^{\infty}(-1)^{\mu_{1,1}-\beta_{1,1}}e_{\beta_{1,1}-\mu_{1,1}}(b_{1,1},\ldots,b_{1,\beta_{1,1}-1})\\&\times\frac{1}{(u_{1,1}|\tau b_1)^{\beta_{1,1}}},\end{aligned}$$

we obtain Eq. (58).

Theorem 5 For n partitions $\beta_{[i]}=(\beta_{i,1},\ldots,\beta_{i,r_i})$ $(i=1,\ldots,n)$ and sequences of complex numbers $b_i=(0,b_{i,1},b_{i,2},\ldots)$, multiparameter generalized B-type universal characters $Q^{(b_1,\ldots,b_n)}_{[\beta_{[1]},\ldots,\beta_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)$ are tau functions of the GBUC hierarchy.

Proof Using Eqs. (58) and (46), we can rewrite multiparameter generalized B-type universal characters as

$$Q^{(b_1,\ldots,b_n)}_{[\beta_{[1]},\ldots,\beta_{[n]}]}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)=\prod_{i=1}^{n}\prod_{j=1}^{r_i}Y_{\mu_{i,j}}\cdot 1,$$

where

$$Y_{\mu_{i,j}}=\sum_{\beta_{i,j}>0}(-1)^{\mu_{i,j}-\beta_{i,j}}e_{\beta_{i,j}-\mu_{i,j}}(b_{i,1},\ldots,b_{i,\beta_{i,j}-1})\psi_{i,j}. \tag{60}$$

According to Corollary 4, it is proved that multiparameter generalized B-type universal characters are solutions of the GBUC hierarchy.

5. Summary and discussion

In this paper, we construct the quantum fields presentation of the generalized universal character and B-type of generalized universal character and propose the GUC and GBUC hierarchies. Moreover, two types of multiparameter symmetric functions are established which satisfy the GUC and GBUC hierarchies, respectively. It should be pointed out that dual analogs of symmetric functions have been investigated.[29] We will concentrate on the dual multiparameter generalized universal character and B-type of generalized universal character and the corresponding integrable systems in the near future.

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