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Self-similarity of multilayer networks

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Research on the self-similarity of multilayer networks is scarce, when compared to the extensive research conducted on the dynamics of these networks. In this paper, we use entropy to determine the edge weights in each sub-network, and apply the degree–degree distance to unify the weight values of connecting edges between different sub-networks, and unify the edges with different meanings in the multilayer network numerically. At this time, the multilayer network is compressed into a single-layer network, also known as the aggregated network. Furthermore, the self-similarity of the multilayer network is represented by analyzing the self-similarity of the aggregate network. The study of self-similarity was conducted on two classical fractal networks and a real-world multilayer network. The results show that multilayer networks exhibit more pronounced self-similarity, and the intensity of self-similarity in multilayer networks can vary with the connection mode of sub-networks.

Keywords: multilayer networks, self-similarity, degree–degree distance, entropy

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1. Introduction

From power network to the world wide network, from ecological network to the social interactions within of animal groups, there exists a large number of complex systems in nature and daily life. These complex systems can be modeled as complex networks for analysis.[1–5] The research of complex networks has emerged as a frontier in science and engineering, and significant achievements have been made in investigating their self-similarity, small-world, and scale-free properties.[6] The self-similarity of complex networks, in comparison to its other two properties, poses challenges in comprehension due to its less intuitive manifestation in real-world networks. Consequently, the theoretical framework surrounding this characteristic requires further refinement and development. As multilayer complex networks offer a more nuanced reflection of real-world systems, they have gradually emerged as a research hotspot in the field of complex networks in recent years.[7–10] However, the lack of research into the self-similarity of multilayer complex networks undermines the completeness of fractal theory within the realm of complex networks.

The self-similarity is a ubiquitous topological characteristic of hierarchical complex networks, which necessitates the employment of fractal properties of complex networks for its description. The official recognition of fractality in this context emerged with the publication of a paper titled "Self-similarity of Complex Networks" in Nature magazine by Song *et al.* in 2005.[11] Through renormalization techniques, this work unveiled that numerous real-world networks exhibit self-similarity and possess finite fractal dimensions, a phenomenon collectively referred to as the "fractal properties" of complex networks.[12] Since then, research on the fractal properties of single-layer complex networks has garnered widespread attention.

The studies on the fractal properties of complex networks are primarily categorized into two areas: one focuses on measuring fractal properties and calculating fractal dimensions, while the other explores the origins of fractal properties as well as the growth and evolution mechanisms of networks.[13,14] The design of the fractal algorithm for the complex network reveals various fractal dimensions of the complex network and whether the complex network exhibits self-similarity, such as the box-covering algorithm of complex networks (BCAN), the box-covering algorithm of weighted complex networks (BCANw), the information dimension of complex networks, and the volume dimension of complex networks.[10,15–17] For the origin of fractal properties, Yook *et al.* conducted a study from the perspective of the degree correlation of network and pointed out that the occurrence of fractal structure in the network is related to the degree negative correlation or disassortativity of the network;[18] Song *et al.* studied the connection correlation based on the network and pointed out that the symbol of the fractal properties of the network is the strong mutual exclusion or disassortativity between the hub nodes under any length scale;[19] Goh *et al.* studied from the potential of tree structure and pointed out that the fractal scale was derived from a special type of growing tree, which was called "skeleton".[20] These two types of research depend on each other and promote each other, which is the link that connects

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the fractal theory of the single layer networks with other subjects. The research on fractal properties and self-similarity of the single-layer complex networks enables the application of network theory in practical networks to become increasingly widespread.

However, for multilayer networks, there are few reports about the self-similarity of that. Demeester *et al.* published an article on multilayer network information transmission in 1999, in which they described how to solve the problem of mutual coordination between different message propagation layers.[21] The paper of Buldyrev *et al.* attracted the attention of researchers in the field of complex networks to multilayer networks.[22] At present, the research of the multilayer networks has evolved from initially extending the concept and method of single layer networks to focusing on the structure, topological properties, and dynamic behavior of the multilayer networks.[23,24] Therefore, this paper focuses on the fractal property of the multilayer networks. In multilayer networks, nodes and edges in each sub-networks represent different meanings. How to compress multilayer networks with multiple functions into aggregate networks has become a key point. Given that Shannon entropy serves as a straightforward yet powerful tool in depicting the complexity and statistical properties of the overall structure of complex networks.[25] This paper adopts Shannon entropy as a metric to evaluate individual sub-networks and subsequently assigns varying weight values to these sub-networks based on their entropy values. Specifically, the more intricate a single-layer network is, the greater weight it receives. Besides, the degree–degree distance, which solely considers the similarity between two nodes without factoring in their topological positions, offers a superior representation of the scale-free nature of networks compared to traditional distance measures.[26] Leveraging the degree–degree distance as a linking bridge between layers, this study compresses the multilayer network with different weights into an aggregate network for analysis. Subsequently, a comparative analysis is conducted between the fractal properties of the aggregate network and those exhibited within the sub-networks of the multilayer network. The overarching goal is twofold: firstly, to affirm the presence of fractal properties within multilayer networks; secondly, to delve into the potential correlations between the fractal properties of multilayer networks and those observed in single-layer networks.

The rest of this paper is organized as follows. In Section 2, the preliminaries briefly introduce some works about the multilayer networks, the aggregate networks, the Shannon entropy, the box-covering algorithm of complex networks fractal, and the degree–degree distance. In Section 3, we consider degree–degree distance as the effective distance of compressed multilayer networks, and then study the self-similarity of the multilayer networks. In Section 4, examples of the Zachary–Karate club network and the classical fractal networks are given, and the self-similarity of these networks are studied. In Section 5, the conclusions are given.

2. Preliminaries

In this section, some preliminaries are briefly introduced, including the multilayer networks, the aggregate networks, the Shannon entropy, the box-covering algorithm of complex networks, and the degree–degree distance.

2.1. Multilayer networks

The concept of the multilayer networks originates from the study of the supernetworks in complex networks theory. The multilayer networks are characterized by the presence of a sub-network in each layer. The node set of each sub-network is the same or has a certain intersection, and there are no connecting edges between sub-networks, or there may be connecting edges between the same nodes across layers. At the same time, the connected edges in each sub-network of the multilayer networks have different properties.

The multilayer networks consists of N nodes and M sub-networks. Nodes and edges in each sub-network represent different meanings. Let $l = 1,2,\ldots,M$ represent each sub-network, then the structure of the multilayer networks can be expressed as $MG = [G^1, G^2, \ldots, G^l]$. Here, G^l represents a sub-network on l-th layer. The adjacency matrix of G^l is expressed as follows:

$$a_{ij}^l = \begin{cases} 1, & \text{there is an edge between nodes } i \text{ and } j \text{ in } l\text{-layer}, \\ 0, & \text{there is no edge between nodes } i \text{ and } j \text{ in } l\text{-layer}. \end{cases} \tag{1}$$

For describing the multilayer networks, an example of multilayer networks is given and shown in Fig. 1. From Fig. 1, there are three sub-networks ($M = 3$). Each sub-network has different topology, and the nodes and edges in the sub-network have different meanings.

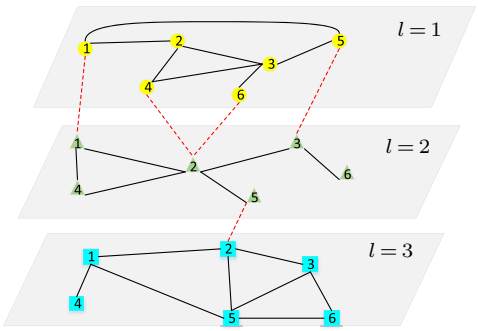


Fig. 1. The multilayer network topology structure. $l = 1,2,3$ represents three sub-networks of the multilayer network, respectively. The nodes and edges in each sub-network represent a different meaning. And these three sub-networks are interconnected and interdependent, forming a three-layer network.

2.2. The aggregate networks

The aggregate multilayer networks is a simplified form of the multilayer networks. This modeling method reduces the research difficulty of the multilayer networks.[27] In other words, the multilayer networks are compressed into a single-layer network (the single-layer network is called the aggregate network). The adjacency matrix of the aggregate networks is defined as follows:

$$a_{ij}^{M} = \begin{cases} 1, & \exists l,\ a_{ij}^{l} = 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

That is, for a_{ij}^{M}, as long as there is a connection between nodes i and j at any layer in M-layer network, there is a connection between nodes i and j. Therefore, the degree of node i on the aggregate networks is defined as follows:

$$k_i^{M} = \sum_j a_{ij}^{M}. \tag{3}$$

2.3. The Shannon entropy

Entropy is a measure of disorder in systems. If the network is randomly connected and the importance of each node is approximately equal, the network is considered to be disorder. In contrast, if the network is scale-free and the importance of the nodes varies, the network can be considered ordered. The application of entropy concepts can measure the order state of complex networks more succinctly.[28,29]

Shannon entropy is proposed by Shannon in 1948 to measure the uncertainty of information.[28] Shannon entropy not only lays the foundation of information theory, but also builds a bridge between the uncertainty of information and the uncertainty of information source. Corresponding to a signal source X, there are n kinds of signals U emitted by it, as follows:

$$U = [u_1, u_2, \ldots, u_i, \ldots, u_n]. \tag{4}$$

The corresponding occurrence probability of each signal is as follows:

$$P = [p_1, p_2, \ldots, p_i, \ldots, p_n]. \tag{5}$$

According to the definition of Shannon entropy, the uncertainty of signal source X is the statistical average of the uncertainties of a single signal. The obtained value is the information entropy of signal source X, which is defined as follows:

$$E_s = -\sum_{i=1}^{n} p_i \log p_i. \tag{6}$$

The Shannon entropy is applied to the complex network, and the calculation formula of p_i is as follows:[28]

$$p_i = \frac{k_i}{\sum_i^n k_i}, \tag{7}$$

where p_i represents the degree distribution of the complex networks, and k_i represents the value of degree of the i-th node in the network.

2.4. The box-covering algorithm of complex networks

In Euclidean geometry, dimension is an important property that quantities geometric objects and refers to the number of independent space–time coordinates.[30–32] In self-similarity theory, dimension is an important physical quantity used to numerically measure self-similarity. The BCANw is a common method for calculating the self-similarity fractal dimension of the complex networks.[33–35]

In the BCANw, the distance between two nodes i and j is defined as the shortest distance between the two nodes.[36–38] For a given network $G = (N, E, W)$, W is the weight value of edge (when $W = 1$, the network is an unweighted network), $A = (a_{ij})_{n\times n}$ represents the adjacency matrix of G, where $a_{ij} = 1$ means that nodes i and j have connected edges otherwise there are no connected edges.[33] The shortest distance between nodes i and j is defined as d_{ij} and shown as follows:

$$d_{ij} = \min\{a_{ie_1} + a_{e_1e_2} + \cdots + a_{e_{k-1}e_k} + a_{e_kj}\}, \tag{8}$$

where $i, e_1, e_2, \ldots, e_{k-1}, e_k, j$ are the labels of nodes. And then, for the given box size l_b, the process of greedy coloring algorithm is shown in Fig. 2.[15] The main steps of the greedy coloring algorithm are given as follows:[39]

Step 1 For a given network G, the nodes labels in G are randomly from 1 to N.

Step 2 Given the box size l_{b}, the double network G_1 of G is constructed. The construction rule is as follows: if the distance between two nodes in G is greater than or equal to box l_{b}, then the two nodes are connected in G_1.

Step 3 When G_1 is colored, the two connected nodes have different colors then we get network G_2. Therefore, the number of colors of nodes in network G_2 is the number of boxes $N_{\mathrm{b}}(l_{\mathrm{b}})$ needed to cover network G. G_3 colors the original network diagram.

Changing the size of the box l_{b}, the minimum number of boxes $N_{\mathrm{b}}(l_{\mathrm{b}})$ is also changed. If the network has fractal property, then l_{b} and $N_{\mathrm{b}}(l_{\mathrm{b}})$ should meet the relationship as follows:

$$N_{\mathrm{b}}(l_{\mathrm{b}}) \sim l_{\mathrm{b}}^{-d_{\mathrm{b}}}, \tag{9}$$

where d_{b} is the dimension of complex networks. The d_{b} is shown as follows:[39]

$$d_{\mathrm{b}} = -\lim_{l_{\mathrm{b}}\to 0} \frac{\ln N_{\mathrm{b}}(l_{\mathrm{b}})}{\ln l_{\mathrm{b}}}. \tag{10}$$

From Fig. 2, there are at least two boxes to cover the network G when $l_{\mathrm{b}} = 3$. The BCANw has been widely used to calculate the fractal dimension of the complex networks.

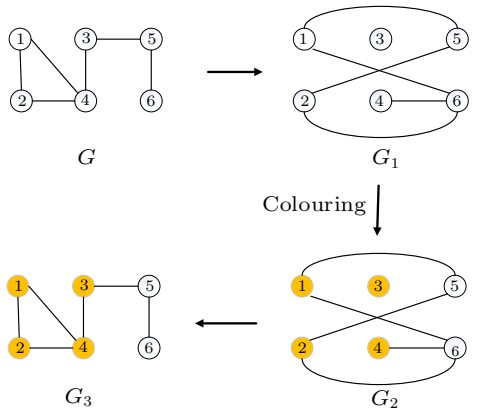


Fig. 2. The process of greedy coloring algorithm.[39] Here G is the original network graph; when $l_b = 3$, G_1 is a double network formed by connecting nodes whose distance between nodes is no less than 3 in the original graph; G_2 is to color the network, and the connected nodes are colored with different colors; G_3 restores G_2 to the original graph.

2.5. Degree–degree distance

For a given network $G = (N,E)$, the degree k_i is only determined by the network topology, so it is very difficult to study the link statistics of complex networks. Therefore, Zhou *et al*. proposed a property to regain the statistical importance of links, namely degree–degree distance.[26] The degree–degree distance $\eta(i,j)$ is defined as $\log \eta(i,j)$, specifically calculated as follows:

$$\log \eta(i,j) = \log \eta(j,i) = |\log k_i - \log k_j|, \quad (i,j) \in E. \tag{11}$$

The degree–degree distance is a nature scale, in the sense that $\eta(i,j)$ is also determined solely by network topology. Meanwhile, in some cases degree–degree distance is also dimensionless, and for the sake of understanding and calculating, the definition of degree–degree distance can be redefined as follows:[26]

$$\eta(i,j) = \max\{k_i,k_j\}/\min\{k_i,k_j\}. \tag{12}$$

Equation (12) is the ratio between the degrees of the nodes at both ends after ordering.

3. The self-similarity of the multilayer networks

In this section, the self-similarity of the multilayer network is studied. For the multilayer network $MG = [G^1,G^2,\ldots,G^l]$ $(l = 1,2,\ldots,M)$, the number of nodes and edges in each sub-network of the multilayer network are N^l and V^l, respectively. Where MG represents a multilayer network with M sub-networks, G^l represents l-th sub-network in the multilayer network. The number of nodes of the multilayer networks, which is denoted as N:

$$N = N^1 + N^2 + \cdots + N^l, \quad (l = 1,2,\ldots,M). \tag{13}$$

For the number of edges of the multilayer networks, it is denoted as V:

$$V = V^1 + V^2 + \cdots + V^l + \sum_{l=1,a,b\in l \text{ and } a\neq b}^{M} V^{a,b}, \quad (l = 1,2,\ldots,M), \tag{14}$$

where $V^{a,b}$ represents the number of connecting edges between sub-network.

The $E_s(G^l)$ represents the value of the Shannon entropy of l-layer network. Ranking each sub-network according to the $E_s(G^l)$, and assign the value of weight to each sub-network. The value of weight is defined as follows: the greater the complexity of the sub-network, the greater the value of weight is. In other words, the larger Shannon entropy of the sub-network is, the larger the value of weight is. Therefore, for convenience of calculations, the value of weight in this paper is defined as follows:

$$W_{G^l} = \begin{cases} E_s(\max) = M, \\ E_s(\max-1) = M-1, \\ \quad \vdots \\ E_s(\min) = 1, \end{cases} \tag{15}$$

where W_{G^l} represents the value of weight of the sub-network G^l.

The degree–degree distance as the effective distance of the connection between sub-networks, the multilayer network is compressed into a aggregate network. The degree–degree distance of the connecting edges of nodes between sub-networks is expressed by $\eta^{a,b}(i,j)$, and is defined as follows:

$$\eta^{a,b}(i,j) = \max\{k_i^a,k_j^b\}/\min\{k_i^a,k_j^b\}, \quad (a,b \in l \text{ and } a \neq b), \tag{16}$$

where k_i^a represents the value of degree of node i in a-layer network, and k_j^b represents the value of degree of node j in b-layer network.

After aggregation, the number of nodes and edges in the aggregate network remain the same, while the value of edge weights changes. The original multilayer networks are compressed into an aggregate network. Then, the BCANw method is applied to analyze the self-similarity of the aggregate network. Based on the analysis results of the aggregate networks, the self-similarity of the original multilayer networks is detrmined.

For illustrating the process of self-similarity of multilayer network, an example which shown in Fig. 1 is given as follows:

Step 1 Shannon entropy is used to calculate the complexity of the three-layer network G^1, G^2, and G^3, and the results are $E_s(G^1) = 1.7105$, $E_s(G^2) = 1.6762$ and $E_s(G^3) = 1.7214$.

Step 2 According to the Shannon entropy of the three sub-networks, each sub-network is assigned a different weight, and the weighted multilayer network is obtained as shown in Fig. 3.

Step 3 The degree–degree distance between connections across sub-networks is computed, with the results depicted in

Fig. 4. By treating the computed degree–degree distance as an effective distance between interconnected sub-networks, the three-layer network is compressed into an aggregate network.

Step 4 The BCANw is used to fit the aggregate network, and the results are shown in Fig. 5(a).

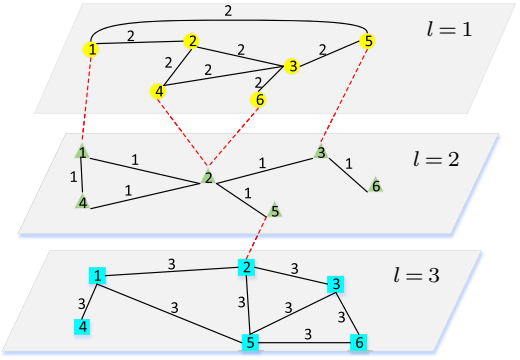


Fig. 3. The weighted multilayer network. According to the order of Shannon entropy, three sub-networks are given by different values of weights. The values of weights are assigned to each sub-network as the edge weight of the network.

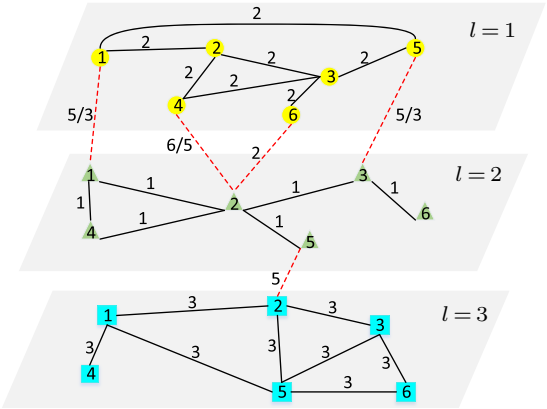


Fig. 4. The unified network structure model. First, the nodes of G^1-layer, G^2-layer, and G^3-layer networks are re-numbered in order. Then, calculate the degree–degree distance of the edge between sub-networks as $\eta(1,7) = 5/3$, $\eta(4,8) = 6/5$, $\eta(6,8) = 2$, $\eta(5,9) = 5/3$, $\eta(11,14) = 5$. Therefore, the edge weight between node 1 and node 7 in the aggregate network is $5/3$. Similarly, other edge weights can be obtained. The multilayer network is compressed into an aggregate network.

Figures 5(b)–5(d) illustrate the fitting results obtained using the BCANw algorithm for three respective sub-networks within a multilayer network. From Figs. 5(b)–5(d), it is evident that due to the shorter path lengths within each sub-network, there are limited box sizes that can be defined during the computation process. Consequently, only a few data points are present in the fitting plots, insufficient to provide a reasonable explanation for the presence or absence of self-similarity in the networks. In contrast, figure 5(a) demonstrates that by condensing the multilayer network into an aggregate network, the aggregate network exhibits significantly longer shortest path values. This allows for a more diverse range of box sizes to be utilized during the fitting process, resulting in a much richer set of data points. Notably, most of these data points cluster around the fitting line, indicating a pronounced fitting effect compared to the individual sub-networks' results. This clearly suggests that the aggregate network possesses self-similarity, which is not as evident in the analysis of individual sub-networks.

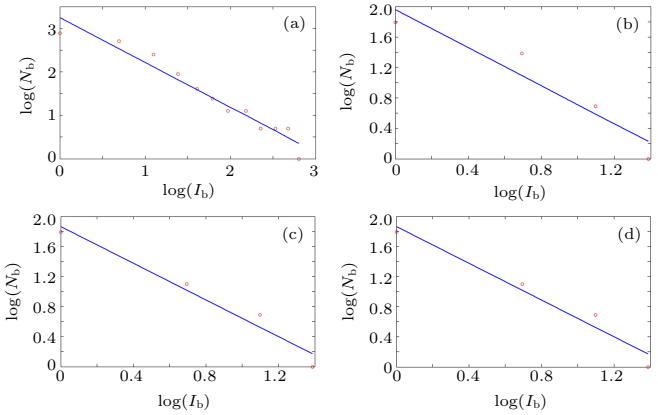


Fig. 5. The log–log plots of box number versus box size for the G^1-layer network, G^2-layer network, G^3-layer network, and aggregate network. (a) The fitting results for the aggregate network, (b) the fitting results for the G^1-layer network, (c) the fitting results for the G^2-layer network network, (d) the fitting results for the G^3-layer network.

4. Experimental analysis

4.1. The classical fractal networks are taken as example

In this section, two classical fractal networks connected as a multilayer network are discussed. The two classical fractal networks in this paper are generated by imitating the construction method of classical weighted fractal networks. The classical weighted fractal networks are built by iterated function systems, whose Hausdorff dimension is completely determined by two parameters.[40] One of these parameters is the number of copies of the graph, denoted by s ($s > 1$); and the other is the scaling factor, denoted by f ($0 < f < 1$). And the fractal dimension of the classical weighted fractal networks is called similarity dimension, and is defined as follows:

$$d_{\mathrm{b}} = -\frac{\log(s)}{\log(f)}. \tag{17}$$

To further study the self-similarity of the multilayer networks, we construct two classical fractal networks W_1 and W_2, using the iterative method. However, the scaling factors of the two constructed classical fractal networks are 1, that is, $f = 1$. The rule for forming the W_1 network structure is simple. At the beginning, there is only 1 isolated node in the network, and then 3 isolated nodes are replicates and connected to the initial node in the network in the first time step. In the second time step, 3^2 nodes are replicated, and every 3 nodes from a group that is connected to the nodes generated in the previous time step. This process continues, and at the step t, 3^t nodes will be replicated, with every 3 nodes forming a group connected to the nodes generated in the previous time step. The value of edge-weight between each pair of nodes is 1, that is,

W_1 is an unweighted network. The W_1 network is generated by iterating this process four times, as shown in Fig. 6(a). The W_2 network is generated by iterating this process four times and replicating 4^t isolated nodes at step t, as depicted in Fig. 6(b). The W_2 network is also an unweighted network.

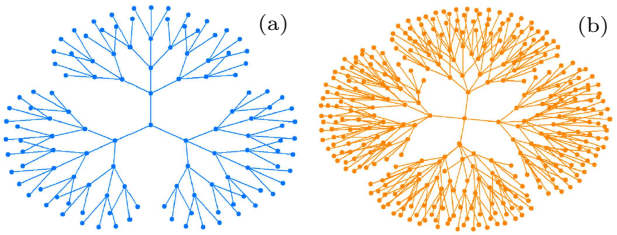


Fig. 6. The classical fractal networks: (a) a fractal network with three nodes in each branch and an edge weight of 1 between each pair of nodes, (b) a fractal network with four nodes per branch and an edge weight of 1 between each pair of nodes.

The topological properties and the fractal dimensions of the classical fractal networks are shown in the second and third columns of Table 1, respectively. From the second column of Table 1, the W_1 network has 121 nodes and 120 edges. In the W_1 network, there are 39 nodes with the largest value of degree, whose degree is 4; there are 81 nodes with the smallest value of degree, whose degree are 1. We use BCANw algorithm to calculate the fractal dimension of the W_1 network, and the result is 2.0903. From the third column of Table 1, the W_2 has 341 nodes and 340 edges. In the W_2 network, there are 84 nodes with the largest value of degree, whose degree is 5; there are 256 nodes with the smallest value of degree, whose degree are 1. The fractal dimension of the W_2 network is 2.5632. Besides, the fitting results of the two classical networks obtained using BCANw are shown in Figs. 7(a) and 7(b). It is shown from Figs. 7(a) and 7(b), the fitting effect of these two classical networks is good and they have obvious fractal property.

Table 1. The topological properties of the deterministic networks.

Networks	W_1	W_2	Aggregate (max–max)	Aggregate (min–min)	Aggregate (max–min)	Aggregate (min–max)
Nodes	121	341	462	462	462	462
Edges	120	340	3736	21092	10444	7264
Diameter	8	8	7.5918	15.0964	15.3415	7.0709
Max degree	4	5	–	–	–	–
Min degree	1	1	–	–	–	–
Average degree	1.9835	1.9941	28.9323	181.4154	277.5478	5.0061
Fractal dimension	2.0903	2.5632	2.0737	2.6675	2.3120	2.6664

If the two classical fractal networks are sub-networks of the multilayer network, does the multilayer network composed of the two classical fractal networks exhibit self-similarity? Next, in this section, we study the problem of whether the multilayer network composed of two classical fractal networks exhibits self-similarity. To ensure the accuracy of the experiment to the greatest extent possible, we use three methods to connect these two classical fractal networks. The specific connection modes of the classical fractal networks are as follows.

4.1.1. Connecting the nodes with the max–max or min–min

In this section, we connect the nodes with the maximum degree of the two networks and the nodes with the minimum degree of the two networks to form two multilayer networks with different connection modes, respectively. From the fourth column and fifth column of Table 1, the number of nodes in the multilayer networks formed by these two connection methods is 462, while the number of edges are 3736 and 21092, respectively (see Eqs. (13) and (14) for the calculation methods used to determine the number of nodes and edges in the multilayer network). And the maximum degree and minimum degree of nodes in the aggregate network are not given in Table 1, as the connection relationship between sub-networks does not need to be considered in the aggregate network. The fractal dimensions of the two multilayer networks are 2.0737 and 2.6675, respectively. Based on the numerical results, the fractal dimensions of both the aggregate network (max–max) and the aggregate network (min–min) do not fall strictly between the fractal dimensions of the two original networks. Specifically, the fractal dimension of the aggregate network (max–max) closely resembles that of the W_1 network, while the fractal dimension of the aggregate network (min–min) is similar to the fractal dimension of the W_2 network. The fitting results of the two aggregate networks obtained using BCANw are shown in Figs. 7(c) and 7(d). From Figs. 7(c) and 7(d), the fitting result in Fig. 7(d) is better than that in Fig. 7(c), as most of the data points are distributed near the fitting line. In addition, by comparison, the diameter of the aggregate network (min–min) is larger than that of the two fractal networks, while the diameter of the aggregate network (max–max) is smaller. Due to the large diameter of the aggregate network (min–min), more data points are obtained during the process of network fitting, resulting in a more obvious final fitting result.

4.1.2. Connecting the nodes with the max–min or min–max

In this section, we connect the nodes with the maximum degree in one sub-network to the nodes with the minimum degree in another sub-network. From the sixth column and seventh column of Table 1, the number of nodes of the multilayer networks connected by these two methods are 462, and the number of edges are 10444 and 7264, respectively. The fractal dimensions of the two aggregate networks are 2.3120 and 2.6664, respectively. The numerical results show that the fractal dimension of the aggregate network (max–min) is between the fractal dimensions of the two deterministic networks, while the fractal dimension of the aggregate network (min–max) is not between the fractal dimension of the two deterministic networks and it is close to the fractal dimension of the W_2 network. The fitting results of the two aggregate networks obtained by using BCANw are shown in Figs. 7(e) and 7(f). From Fig. 7(e), due to the large diameter of the aggregate network (max–min), more data points are obtained, and the final fitting result is also obvious. Meanwhile, the minimum number of boxes required to cover the network remains the same across different box sizes.

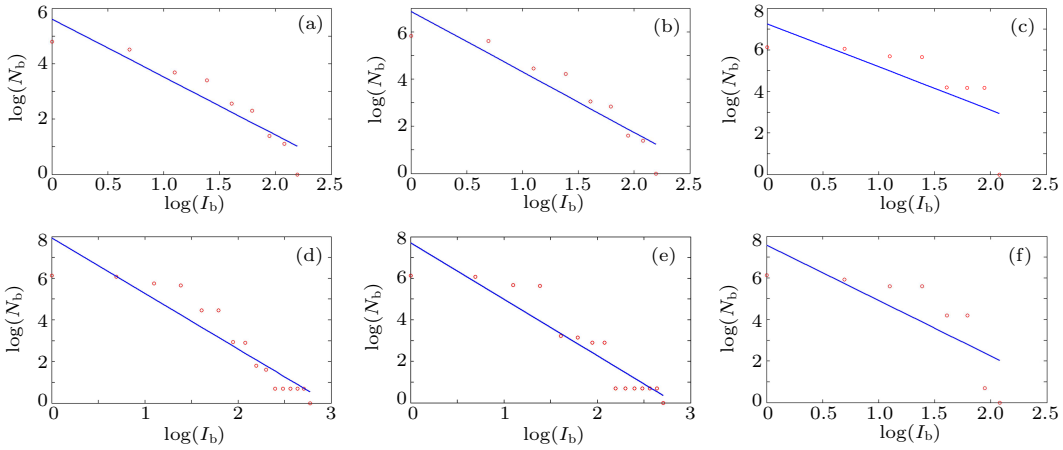


Fig. 7. The log–log plots of box number versus box size for the fractal networks: (a) fitting results for W_1 network, (b) fitting results for W_2 network, (c) fitting results for aggregate network (max–max), (d) fitting results for aggregate network (min–min), (e) fitting results for aggregate network (max–min), (f) fitting results for aggregate network (min–max).

4.1.3. Connecting the nodes with random probability

In this section, we connect the nodes of two classical fractal networks with a random probability. For the random connection of the two networks, we conducted experiments with connection probabilities ranging from $p = 0.9 \sim 0.1$, from $p = 0.09 \sim 0.01$, and from $p = 0.009 \sim 0.001$ for each node in the two classical fractal networks. It is found that when the connection probability is greater than 0.1, the connectivity of the aggregate network is good, while results in the aggregate network's fractal property becoming insignificant when the diameter of the aggregate network falls between 5 and 9. However, when the connection probability is less than $p = 0.1$, the diameter of the aggregate networks are larger. With the decrease of the connection probability p, the diameter of the aggregate networks increase, leading to more obvious fractal properties of the aggregate networks. To ensure the accuracy of the experiment, we conducted numerous experiments with connection probabilities ranging from $p = 0.9 \sim 0.001$. The results show that, as the connection probability decreases, the fitting effect of the aggregate networks improves. In this paper, topology properties and fractal dimension of the aggregate networks with connection probability p were randomly selected and recorded, as shown in Table 2. From Table 2, with the continuous decrease of the connection probability p, all the statistical information of the aggregate networks changed. For example, when p is 0.7 and 0.04, the number of edges of the network is 29468 and 2068, respectively. Besides, from the last line of Table 2, as the connection probability p decreases, the fractal dimension of the aggregate networks also decreases. However, due to the randomness of the connections, the values of fractal dimension may fluctuate to some extent, but this does not affect the overall downward trend. From Table 2, We also find that when $p = 0.7$, the value of the fractal dimension of the aggregate network is 2.8769, which is close to the fractal dimension of the W_2 network; when $p = 0.001$, the value of the the fractal dimension of the aggregate network is 2.1536, which is between the fractal dimension of the two fractal networks and close to the fractal dimension of the W_1 network. Meanwhile, from Figs. 8(a)–8(i), the random aggregate networks have more and more data points, and the fitting effect improves. The data points are distributed close to the fitting line.

To sum up, when the specified nodes of multiple networks are connected, the resulting multilayer network does not necessarily exhibit self-similarity. However, when multiple networks are randomly connected, the multilayer network is aggregated according to the method proposed in this paper. If the diameter of the aggregate network is large, most of the fit-

ting results obtained by using BCANw are well. This indicates that when multiple networks are randomly connected with an appropriate probability, the multilayer networks are likely to possess self-similarity. The method proposed in this paper is reasonable for verifying whether the multilayer network exhibits self-similarity.

Table 2. The topological properties of the random aggregate networks.

Networks	Aggregate ($p = 0.7$)	Aggregate ($p = 0.4$)	Aggregate ($p = 0.1$)	Aggregate ($p = 0.07$)	Aggregate ($p = 0.04$)	Aggregate ($p = 0.01$)	Aggregate ($p = 0.007$)	Aggregate ($p = 0.004$)	Aggregate ($p = 0.001$)
Nodes	462	462	462	462	462	462	462	462	462
Edges	29468	17026	4567	3318	2068	875	750	634	497
Diameter	6.2072	6.8155	9.0763	9.1405	11.3455	15.9	16	16	17.5
Max degree	–	–	–	–	–	–	–	–	–
Min degree	–	–	–	–	–	–	–	–	–
Average degree	345.1936	194.6485	45.3261	31.1893	17.6617	6.8917	5.9577	4.9007	3.8067
Fractal dimension	2.8769	2.6720	2.6951	2.6467	2.5558	2.2264	2.1554	2.0934	2.1536

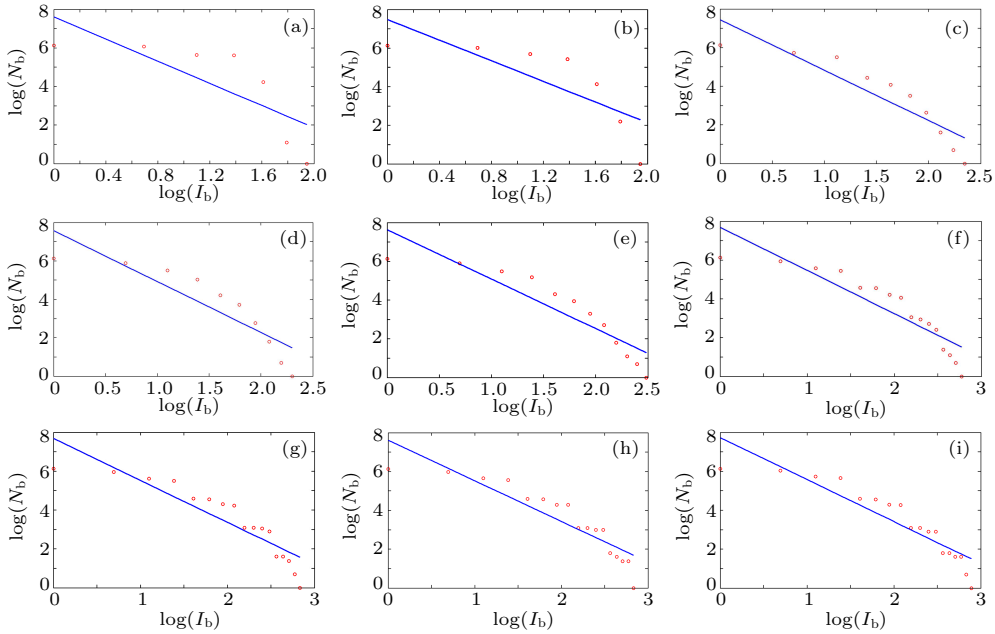


Fig. 8. The log–log plots of box number versus box size for the aggregate network: (a) fitting results for the aggregate network ($p = 0.7$), (b) fitting results for the aggregate network ($p = 0.4$), (c) fitting results for the aggregate network ($p = 0.1$), (d) fitting results for the aggregate network ($p = 0.07$), (e) fitting results for the aggregate network ($p = 0.04$), (f) fitting results for the aggregate network ($p = 0.01$), (g) fitting results for the aggregate network ($p = 0.007$), (h) fitting results for the aggregate network ($p = 0.004$), (i) fitting results for the aggregate network ($p = 0.001$).

4.2. The self-similarity of a real multilayer network

In this section, the self-similarity of the Zachary–Karate club network is discussed, which is a 2-layer network. The Zachary–Karate club network is a classical data set in the social analysis.[41] The Zachary–Karate club network constructed by observing a karate club of an American university. The Zachary–Karate club network consists of 34 nodes and 78 edges, in which each node represents the members of the club and each edge represents the communication between members. It is found that node 1 is the coach of the club and node 34 is the president of the club. Due to a conflict between the coach and the president, the club was divided into two communities. However, the two community members are still communicated to each other, so the Zachary–Karate club network can be constructed as a 2-layer network.[42] The Zachary–Karate club network is shown in Fig. 9. The topological characteristics and the fractal dimension of the Zachary–Karate club network are shown in Table 3. Besides, the fitting results of these two single layer networks and the aggregate network are shown in Fig. 10.

From Fig. 9, the Zachary–Karate club network can be divided into two single layer networks. The yellow nodes form one layer of the network, and the green nodes form another layer of the network. From Table 3, the number of nodes and edges in the yellow-layer network and the green-layer network are different. Specifically, the number of nodes is 18 and 16, and the number of edges is 35 and 33, respectively. The number of nodes and edges in the aggregate network are 34 and 78. Besides, from Table 3, the diameter of the aggregate network is 7.4545. Compared to the aggregate network, both single-layer networks have relatively small diameters of 3.

This also results in few data points in Figs. 10(a)

and 10(b), when the BCANw algorithm is used to fit the two single-layer networks. Therefore, we cannot determine whether the two single-layer networks exhibit self-similarity, and the values of fractal dimensions in Table 3 are not recorded. On the contrary, from Fig. 10(c), there are numerous data points in the fitting graph after compressing of the two single-layer networks using our method, and most of them are distributed close to the fitting line. This shows that the network has self-similarity, and the value of fractal dimension of the network is 1.7255. This indicates that it is reasonable to consider degree–degree distance as the effective distance to connect two single-layer networks and obtain an aggregate network, thus studying the fractal properties of the aggregate network.

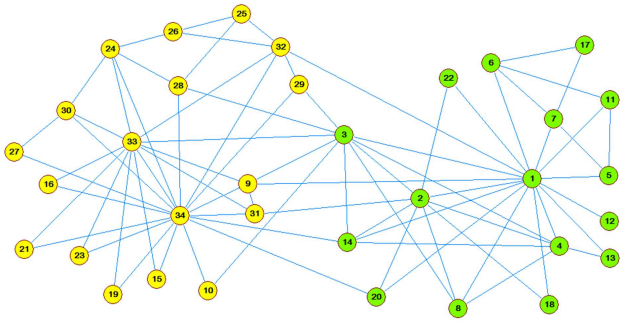


Fig. 9. The Zachary–Karate club network. The Zachary–Karate club network consists of 34 nodes and 78 edges, in which each node represents the members of the club and the edge represents the communication between members. The Zachary–Karate club network is constructed as a 2-layer network. One layer of the network is the yellow node, and the node 34 as the core node; the other layer of the network is a green node, and the node 1 as the core node.

Table 3. The dataset topological properties of the Zachary–Karate club network.

Networks	Yellow-layer network	Green-layer network	Aggregate network
Nodes	18	16	34
Edges	35	33	78
Diameter	3	3	7.4545
Average degree	3.8889	4.1250	7.9482
Clustering coefficient	0.6515	0.7197	0.8561
Fractal dimension	–	–	1.7255

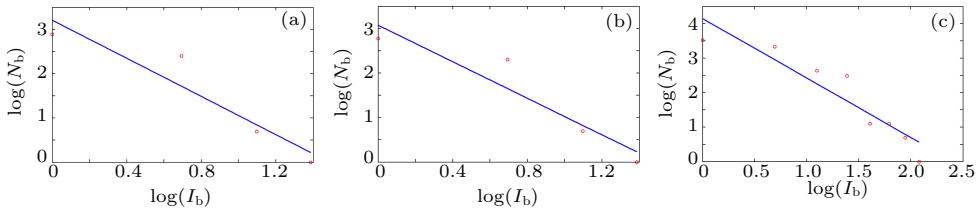


Fig. 10. The log–log plots of box number versus box size for the Zachary–Karate club network: (a) fitting results for the yellow-layer network, (b) fitting results for the green-layer network, (c) fitting results for the aggregate network.

5. Conclusions

The self-similarity is one of the most important aspects in research of the complex networks, and many research achievements have been made. However, most current research methods are focused on single layer networks, with less attention paid to multilayer networks. This ignores the existence of multiple network interactions in real complex systems. Therefore, we discuss the self-similarity of the multilayer complex networks. Based on the multilayer networks structure, the complexity of each sub-network is quantified using Shannon entropy. Assign different values of weights to each sub-network according to the quantization results, while keeping the proprietary topology of each sub-network as much as possible. Then, the degree–degree distance is used to connect the sub-networks in order to build an aggregate network. The self-similarity of the aggregate network is studied to determine whether the original multilayer network exhibits self-similarity. To illustrate the effectiveness of the proposed method, two classical fractal networks and the real social network of the Zachary–Karate club network are studied by the proposed method. The results show that this method can reveal the self-similarity of the obtained multilayer networks. However, when the connecting ways of sub-networks change, the fitting graphs obtained using BCANw algorithm differ, and the values of the fractal dimensions of the multilayer networks also vary. In addition, we find that, under the appropriate connection probability, the smaller the connection probability between the sub-networks (the weaker the connectivity of the constructed multilayer networks), the smaller the value of fractal dimension of the multilayer networks. The above shows that whether the constructed multilayer network exhibits self-similarity is probably related to the connecting ways of sub-networks. Furthermore, the value of the fractal dimensions of the multilayer networks also correlates with the connecting ways of the sub-networks.

The method proposed in this paper provides a novel perspective for the study of self-similarity in multilayer networks and also opens up new avenues for our future research. In the

coming future, we will continue to search for datasets that conform to the multilayer network model presented herein, and conduct research on the self-similarity of multilayer networks with a large number of data points. Furthermore, we will focus on further optimizing the aggregation method for multilayer networks to ensure that the unique topological features of each layer are preserved to the greatest extent possible during the aggregation process.

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