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Hengchun Hu(胡恒春) and Jiali Kang(康佳丽)

Citation: Chin. Phys. B, 2024, 33 (**11**): 110206. DOI: 10.1088/1674-1056/ad7fd1
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Abundant invariant solutions of extended (3+1)-dimensional KP–Boussinesq equation

Hengchun Hu(胡恒春)[†] and Jiali Kang(康佳丽)

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, *China*

(Received 18 August 2024; revised manuscript received 24 September 2024; accepted manuscript online 26 September 2024)

Lie group analysis method is applied to the extended $(3+1)$-dimensional Kadomtsev–Petviashvili–Boussinesq equation and the corresponding similarity reduction equations are obtained with various infinitesimal generators. By selecting suitable arbitrary functions in the similarity reduction solutions, we obtain abundant invariant solutions, including the trigonometric solution, the kink-lump interaction solution, the interaction solution between lump wave and triangular periodic wave, the two-kink solution, the lump solution, the interaction between a lump and two-kink and the periodic lump solution in different planes. These exact solutions are also given graphically to show the detailed structures of this high dimensional integrable system.

Keywords: extended $(3+1)$-dimensional KP–Boussinesq equation, Lie group method, similarity reduction, invariant solution

PACS: 02.30.Ik, 05.45.Yv, 02.30.Jr **DOI:** 10.1088/1674-1056/ad7fd1

1. Introduction

The classical completely integrable KP equation

$$(u_t + 6uu_x + u_{xxx})_x - \lambda u_{yy} = 0 \tag{1}$$

is a well-known nonlinear partial differential equation, which describes the evolution of nonlinear small amplitude long waves with the constant $\lambda = \pm 1$.[1] In fact, one of the extension forms of the famous Kadomtsev–Petviashvili (KP) equation is the generalized (3+1)-dimensional KP equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{xt} + u_{yt} - u_{zz} = 0, \tag{2}$$

which has the multi-soliton solutions and the multiple wave solutions with specific phase shifts with Hirota bilinear method and multi exp-function algorithm in Refs. [2,3] respectively. In recent years, the research on the KP-like equations or the KP combined other nonlinear equation has attracted much attention in soliton theory.[4–8] For example, the integrable (3+1)-dimensional KP–Boussinesq equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{xt} + u_{yt} + u_{tt} - u_{zz} = 0 \tag{3}$$

has been studied extensively. The soliton solutions, the higher-order breathing waves and rogue waves and the similarity reduction solutions of Eq. (3) are obtained explicitly in Refs. [9–12] with the bilinear method and Lie symmetry analysis, etc.

In this paper, we aim to study abundant invariant solutions to the special extended (3+1)-dimensional KP–Boussinesq equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{xt} + u_{yt} + u_{tt} - u_{xz} = 0, \tag{4}$$

which can describe the long wavelength water wave in hydrodynamics. Equation (4) has been proved to be not integrable in the Painlevé sense and its periodic solution, n-soliton solution and variable separation solution have been derived by Hirota bilinear method.[13] To the best of our knowledge, there has been no study on the symmetry reduction and similarity solutions of Eq. (4) in the literature. We will use the classical Lie group method to obtain various similarity solutions of Eq. (4) with arbitrary functions, including the trigonometric solution, the interaction solution between a kink and a soliton, the periodic solution, the interaction solution between lump waves and triangular periodic waves, the two-kink solution, the lump solution, the lump-kink solution, the interaction between a lump and two-soliton solution, the periodic lump solution and the breather solution.

The paper is organized as follows. We conduct the Lie symmetry analysis of Eq. (4) and give the corresponding infinitesimal generators with different arbitrary functions in Section 2. In Section 3, various vector fields are used to find many kinds of reduction equations and abundant similarity solutions of Eq. (4) are obtained by solving these reduction equations. Some similarity solutions are also exhibited directly by selecting proper arbitrary functions. Summary and conclusion are given in the last section.

2. Lie group analysis for Eq. (4)

In this section, the classical Lie group method is applied to the extended (3+1)-dimensional KP–Boussinesq equation (4) and the infinitesimal generators are calculated according to the standard Lie group procedure. Firstly, let us consider

[†]Corresponding author. E-mail: hhengchun@163.com

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a one-parameter Lie group transformation for Eq. (4)

$$\begin{aligned} &x \to x+\varepsilon\xi_1(x,y,z,t,u),\\ &y \to y+\varepsilon\xi_2(x,y,z,t,u),\\ &z \to z+\varepsilon\xi_3(x,y,z,t,u),\\ &t \to t+\varepsilon\xi_4(x,y,z,t,u),\\ &u \to u+\varepsilon\eta(x,y,z,t,u), \end{aligned} \tag{5}$$

where ε is a small parameter and the infinitesimal generators $\{\xi_1,\xi_2,\xi_3,\xi_4,\eta\}$ are undetermined functions depending on x, y, z, t, u. From the standard steps of Lie group method, once we find the infinitesimals of the group transformation, then the corresponding vector field

$$\boldsymbol{V}=\xi_1\frac{\partial}{\partial x}+\xi_2\frac{\partial}{\partial y}+\xi_3\frac{\partial}{\partial z}+\xi_4\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u}, \tag{6}$$

is a Lie point symmetry of Eq. (4) if and only if

$$Pr^{(4)}\boldsymbol{V}(\Delta)\mid_{\Delta=0}=0, \tag{7}$$

with

$$\Delta\equiv u_{xxxy}+3(u_xu_y)_x+u_{xt}+u_{yt}+u_{tt}-u_{xz}=0.$$

Here $Pr^{(4)}\boldsymbol{V}$ is the fourth prolongation of Eq. (6) as follows:

$$\begin{aligned} Pr^{(4)}\boldsymbol{V} =\ & \boldsymbol{V}+\eta^x\frac{\partial}{\partial u_x}+\eta^y\frac{\partial}{\partial u_y}+\eta^{xx}\frac{\partial}{\partial u_{xx}}\\ &+\eta^{xy}\frac{\partial}{\partial u_{xy}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{yt}\frac{\partial}{\partial u_{yt}}\\ &+\eta^{tt}\frac{\partial}{\partial u_{tt}}+\eta^{xz}\frac{\partial}{\partial u_{xz}}+\eta^{xxxy}\frac{\partial}{\partial u_{xxxy}} \end{aligned}$$

and the functions $\{\eta^{xt},\eta^{yt},\eta^{tt},\eta^{xxxy},\eta^{xx},\eta^{y},\eta^{x},\eta^{xy},\eta^{xz}\}$ are invariant coefficients, which can be computed by the prolongation formulae.

Then the invariant condition of Eq. (4) is given in the following:

$$\begin{aligned} &\eta^{xxxy}+\eta^{xt}+\eta^{yt}+\eta^{tt}+3\eta^{xx}u_y+3\eta^{y}u_{xx}\\ &+3\eta^{x}u_{xy}+3\eta^{xy}u_x-\eta^{xz}=0. \end{aligned} \tag{8}$$

Substituting the explicit expressions of invariant coefficients into Eq. (8), we can arrive at the determined equations and the infinitesimal generators $\{\xi_1,\xi_2,\xi_3,\xi_4,\eta\}$ can be solved directly with Eq. (4)

$$\begin{aligned} \xi_1 &= \frac{t}{2}f_{1z}+f_3+\frac{(t+x)}{5}c_1,\ \ \xi_2=\frac{3}{5}c_1y+f_2,\\ \xi_3 &= c_1z+c_2,\ \ \xi_4=\frac{3}{5}c_1t+f_1,\\ \eta &= \frac{t(t-y)}{6}f_{1zz}-\frac{t^2}{6}f_{2zz}+\frac{(x+y)}{6}f_{1z}-\frac{x}{3}f_{2z}\\ &\quad -\frac{y}{3}f_{3z}+f_4t+f_5+\frac{(x+y-3u)}{15}c_1, \end{aligned} \tag{9}$$

where f_1, f_2, f_3, f_4, f_5 are five arbitrary functions of z and c_1, c_2 are arbitrary constants. Taking $f_1(z)=c_3z$, $f_2(z)=c_4z$, $f_3(z)=c_5z$, $f_4(z)=c_6z$, $f_5(z)=c_7z$ for simplicity, the infinitesimal generators Eq. (9) can be rewritten as

$$\begin{aligned} \xi_1 &= \frac{t}{2}c_3+c_5z+\frac{(t+x)}{5}c_1,\ \ \xi_2=\frac{3}{5}c_1y+c_4z,\\ \xi_3 &= c_1z+c_2,\ \ \xi_4=\frac{3}{5}c_1t+c_3z,\\ \eta &= \frac{(x+y)}{6}c_3-\frac{1}{3}c_4x-\frac{1}{3}c_5y\\ &\quad +c_6zt+c_7z+\frac{(x+y-3u)}{15}c_1, \end{aligned} \tag{10}$$

and the corresponding vector field is given as

$$\boldsymbol{V}=c_1\boldsymbol{V}_1+c_2\boldsymbol{V}_2+c_3\boldsymbol{V}_3+c_4\boldsymbol{V}_4+c_5\boldsymbol{V}_5+c_6\boldsymbol{V}_6+c_7\boldsymbol{V}_7,$$

where

$$\begin{aligned} \boldsymbol{V}_1 &= \frac{(t+x)}{5}\frac{\partial}{\partial x}+\frac{3y}{5}\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\frac{3t}{5}\frac{\partial}{\partial t}+\frac{(x+y-3u)}{15}\frac{\partial}{\partial u},\\ \boldsymbol{V}_2 &= \frac{\partial}{\partial z},\ \ \boldsymbol{V}_4=z\frac{\partial}{\partial y}-\frac{x}{3}\frac{\partial}{\partial u},\ \ \boldsymbol{V}_5=z\frac{\partial}{\partial x}-\frac{y}{3}\frac{\partial}{\partial u},\\ \boldsymbol{V}_3 &= \frac{t}{2}\frac{\partial}{\partial x}+z\frac{\partial}{\partial t}+\frac{(x+y)}{6}\frac{\partial}{\partial u},\\ \boldsymbol{V}_6 &= zt\frac{\partial}{\partial u},\ \ \boldsymbol{V}_7=z\frac{\partial}{\partial u}. \end{aligned} \tag{11}$$

That is to say, equation (4) admits the Lie algebra spanned by the vectors $\boldsymbol{V}_j$ $(j=1,2,\ldots,7)$.

3. Similarity solutions of Eq. (4)

In this section, we will study the symmetry reduction and invariant solutions of Eq. (4) by choosing different vector fields based on Eq. (11). From the corresponding characteristic equations, different vector fields are used to obtain the reduction equations and similarity solutions of Eq. (4). The results will be given in some cases as given below.

3.1. Vector field $\boldsymbol{V}_3=\frac{t}{2}\frac{\partial}{\partial x}+z\frac{\partial}{\partial t}+\frac{x+y}{6}\frac{\partial}{\partial u}$

From the characteristic equation

$$\frac{\mathrm{d}x}{\frac{t}{2}}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{z}=\frac{\mathrm{d}u}{(x+y)/6}, \tag{12}$$

we can have the similarity solution

$$u=F_1(X,Y,Z)+\frac{6xzt+6yzt-t^3}{36z^2}, \tag{13}$$

where $X=(4xz-t^2)/4z$, $Y=y$, $Z=z$ are similarity variables by solving Eq. (12). Substituting Eq. (13) into Eq. (4), we have the reduction equation in terms of F_1

$$\begin{aligned} &6F_{1XXXY}Z+18F_{1Y}F_{1XX}Z\\ &+18F_{1X}F_{1XY}Z-6F_{1XZ}Z-3F_{1X}+2=0. \end{aligned} \tag{14}$$

Though it is not easy to solve the reduction equation (14), we still can take a particular solution

$$F_1(X,Y,Z)=L_1(Z)+\frac{\sqrt{3Z}}{3Z}L_2\left(\frac{X}{3}+Z\right)+Y+\frac{2X}{3}+\frac{4Z}{3}, \tag{15}$$

and the exact explicit solution of Eq. (4) is

$$u_1 = L_1(z)+\frac{\sqrt{3z}}{3z}L_2\left(z+\frac{4xz-t^2}{12z}\right)+y + \frac{4z}{3}+\frac{4xz-t^2}{6z}+\frac{6xzt+6yzt-t^3}{36z^2}, \tag{16}$$

where L_1, L_2 are two arbitrary functions of the indicated variables. In addition, to obtain more solutions of Eq. (14), we apply Lie group method on it and a new set of infinitesimals is solved out directly

$$\begin{aligned}
\xi_X &= \frac{X}{2}a_1-\frac{X}{2}a_3+\frac{X\sqrt{Z}}{2}a_2+g_1,\\
\xi_Z &= Za_1+Z^{3/2}a_2,\\
\xi_Y &= (2Z+Y)a_3-2Za_1+\left(\frac{Y\sqrt{Z}}{2}-Z^{3/2}\right)a_2+2\sqrt{Z}a_4+a_5,\\
\eta_U &= g_2-\frac{Y}{3}g_{1Z}+\frac{4X-3F_1}{6}a_1\\
&\quad+\frac{6XZ\sqrt{Z}-XY\sqrt{Z}-6F_1Z\sqrt{Z}}{12Z}a_2\\
&\quad+\frac{3F_1-4X}{6}a_3-\frac{X\sqrt{Z}}{3Z}a_4,
\end{aligned} \tag{17}$$

where g_1, g_2 are arbitrary functions of Z and a_1, a_2, a_3, a_4, a_5 are arbitrary constants. If taking $g_1(Z)=g_2(Z)=0$ for simplicity, equation (17) can be rewritten as

$$\begin{aligned}
\xi_X &= \frac{X}{2}a_1-\frac{X}{2}a_3+\frac{X\sqrt{Z}}{2}a_2,\\
\xi_Z &= Za_1+Z^{3/2}a_2,\\
\xi_Y &= (2Z+Y)a_3-2Za_1+\left(\frac{Y\sqrt{Z}}{2}-Z^{3/2}\right)a_2+2\sqrt{Z}a_4+a_5,\\
\eta_U &= \frac{4X-3F_1}{6}a_1+\frac{6XZ\sqrt{Z}-XY\sqrt{Z}-6F_1Z\sqrt{Z}}{12Z}a_2\\
&\quad+\frac{3F_1-4X}{6}a_3-\frac{X\sqrt{Z}}{3Z}a_4.
\end{aligned} \tag{18}$$

The vector field of the infinitesimal generators (18) can be expressed by

$$\boldsymbol{W}=a_1\boldsymbol{W}_1+a_2\boldsymbol{W}_2+a_3\boldsymbol{W}_3+a_4\boldsymbol{W}_4+a_5\boldsymbol{W}_5,$$

where

$$\begin{aligned}
\boldsymbol{W}_1 &= \frac{X}{2}\frac{\partial}{\partial X}-2Z\frac{\partial}{\partial Y}+Z\frac{\partial}{\partial Z}+\frac{(4X-3F_1)}{6}\frac{\partial}{\partial F_1},\\
\boldsymbol{W}_5 &= \frac{\partial}{\partial Y},\\
\boldsymbol{W}_3 &= (2Z+Y)\frac{\partial}{\partial Y}-\frac{X}{2}\frac{\partial}{\partial X}+\frac{(3F_1-4X)}{6}\frac{\partial}{\partial F_1},\\
\boldsymbol{W}_4 &= 2\sqrt{Z}\frac{\partial}{\partial Y}-\frac{X\sqrt{Z}}{3Z}\frac{\partial}{\partial F_1},\\
\boldsymbol{W}_2 &= \frac{X\sqrt{Z}}{2}\frac{\partial}{\partial X}+\left(\frac{Y\sqrt{Z}}{2}-Z^{3/2}\right)\frac{\partial}{\partial Y}+Z^{3/2}\frac{\partial}{\partial Z}\\
&\quad+\frac{6XZ\sqrt{Z}-XY\sqrt{Z}-6ZF_1\sqrt{Z}}{12Z}\frac{\partial}{\partial F_1}.
\end{aligned} \tag{19}$$

3.1.1. Vector field $\boldsymbol{W}_1=\frac{X}{2}\frac{\partial}{\partial X}-2Z\frac{\partial}{\partial Y}+Z\frac{\partial}{\partial Z}+\frac{4X-3F_1}{6}\frac{\partial}{\partial F_1}$

The related characteristic equation is

$$\frac{\mathrm{d}X}{\frac{X}{2}}=\frac{\mathrm{d}Y}{-2Z}=\frac{\mathrm{d}Z}{Z}=\frac{\mathrm{d}F_1}{\frac{4X-3F_1}{6}}, \tag{20}$$

then we can get the similarity form

$$F_{11}=\frac{M_1(X_1,Y_1)}{\sqrt{Z}}+\frac{2X}{3}, \tag{21}$$

where M_1 is a similarity function of variables $X_1=X/\sqrt{Z}$, $Y_1=Y+2Z$ and F_{11} is an invariant solution of Eq. (14). Substituting Eq. (21) into Eq. (14), we get the reduction equation in terms of M_1

$$\begin{aligned}
&2M_{1X_1X_1X_1Y_1}+M_{1X_1X_1}X_1+6M_{1X_1X_1}M_{1Y_1}\\
&+6M_{1X_1Y_1}M_{1X_1}+M_{1X_1}=0.
\end{aligned} \tag{22}$$

It is easy to see that $H_1(Y_1)$ is a solution of Eq. (22) and we can have the exact solution of Eq. (14)

$$F_1(X,Y,Z)=\frac{H_1(2Z+Y)}{\sqrt{Z}}+\frac{2X}{3}. \tag{23}$$

Substituting Eq. (23) into Eq. (13), we obtain the similarity solution of Eq. (4)

$$u_2=\frac{H_1(2z+y)}{\sqrt{z}}+\frac{24z^2x-6zt^2-t^3+6txz+6tyz}{36z^2}. \tag{24}$$

3.1.2. Vector field $\boldsymbol{W}_2=\frac{X\sqrt{Z}}{2}\frac{\partial}{\partial X}+(\frac{Y\sqrt{Z}}{2}-Z^{3/2})\frac{\partial}{\partial Y}+Z^{3/2}\frac{\partial}{\partial Z}+\frac{6XZ\sqrt{Z}-XY\sqrt{Z}-6ZF_1\sqrt{Z}}{12Z}\frac{\partial}{\partial F_1}$

The related characteristic equation of $\boldsymbol{W}_2$ is

$$\begin{aligned}
&\frac{\mathrm{d}X}{X\sqrt{Z}/2}=\frac{\mathrm{d}Y}{Y\sqrt{Z}/2-Z^{3/2}}\\
&=\frac{\mathrm{d}Z}{Z^{3/2}}=\frac{\mathrm{d}F_1}{(6XZ\sqrt{Z}-ZY\sqrt{Z}-6ZF_1\sqrt{Z})/12Z},
\end{aligned} \tag{25}$$

then we can have the similarity form

$$F_{12}=\frac{M_2(X_1,Y_1)}{\sqrt{Z}}+\frac{X}{3}-\frac{XY}{6Z}, \tag{26}$$

where M_2 is a similarity function of variables $X_1=X/\sqrt{Z}$, $Y_1=(2Z+Y)/\sqrt{Z}$ and F_{12} is an invariant solution of Eq. (14). Substituting Eq. (26) into Eq. (14), we arrive at the corresponding reduction equation in terms of M_2

$$M_{2X_1X_1X_1Y_1}+3M_{2X_1X_1}M_{2Y_1}+3M_{2X_1Y_1}M_{2X_1}=0. \tag{27}$$

It is easy to see that $H_2(Y_1)$ is a solution of Eq. (27) and we can have the solution of Eq. (14)

$$F_1(X,Y,Z)=\frac{H_2(\frac{2Z+Y}{\sqrt{Z}})}{\sqrt{Z}}+\frac{X}{3}-\frac{XY}{6Z}. \tag{28}$$

Substituting Eq. (28) into Eq. (13), we obtain the similarity solution of Eq. (4)

$$u_3 = \frac{H_2\left(\frac{2z+y}{\sqrt{z}}\right)}{\sqrt{z}} + \frac{3t^2y - 2t^3 - 6zt^2 + 12txz + 12tyz - 12xyz + 24z^2x}{72z^2}. \tag{29}$$

3.1.3. Vector field $W_5 = \frac{\partial}{\partial Y}$

The related characteristic equation is

$$\frac{\mathrm{d}X}{0} = \frac{\mathrm{d}Y}{1} = \frac{\mathrm{d}Z}{0} = \frac{\mathrm{d}F_1}{0}, \tag{30}$$

then we can get the similarity form

$$F_{13} = M_3(X_1, Z_1), \tag{31}$$

where M_3 is a similarity function of variables $X_1 = X, Z_1 = Z$ and F_{13} is an invariant solution of Eq. (14). Substituting Eq. (31) into Eq. (14), we obtain the reduction equation in terms of M_3

$$2 - 6M_{3X_1Z_1}Z_1 - 3M_{3X_1} = 0. \tag{32}$$

The solution of Eq. (32) can be written as

$$M_3(X_1, Z_1) = M_4(Z_1) + \frac{M_5(X_1)}{\sqrt{Z_1}} + \frac{2X_1}{3}, \tag{33}$$

where M_4 and M_5 are arbitrary functions of indicated variables. Substituting Eq. (33) into Eq. (13), we obtain the invariant solution of Eq. (4)

$$\begin{aligned} u_4 \;=\; & M_4(z) + \frac{M_5\left(\frac{4xz - t^2}{4z}\right)}{\sqrt{z}} \\ & + \frac{24z^2x - 6zt^2 - t^3 + 6txz + 6tyz}{36z^2}. \end{aligned} \tag{34}$$

3.2. Vector field $V_4 = z\frac{\partial}{\partial y} - \frac{x}{3}\frac{\partial}{\partial u}$

In the similar way, the related characteristic equation of $\boldsymbol{V}_4$ is

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{z} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}u}{-x/3}, \tag{35}$$

then we can obtain similarity solution

$$u = F_2(X, Z, T) - \frac{xy}{3z}, \tag{36}$$

where $X = x, Z = z, T = t$ are similarity variables. Substituting Eq. (36) into Eq. (4), we get the reduction equation in terms of F_2

$$F_{2XT}Z - F_{2XZ}Z + F_{2TT}Z - F_{2XX}X - F_{2X} = 0. \tag{37}$$

Though it is not easy to solve the reduction equation (37), we still can take a particular solution to Eq. (37)

$$F_2 = T + Q_1(Z) + \int \frac{Q_2(Z/X)}{X}\,\mathrm{d}X. \tag{38}$$

Hence, the similarity solution of Eq. (4) is

$$u_5 = Q_1(z) + \int \frac{Q_2(z/x)}{x}\,\mathrm{d}x + t - \frac{xy}{3z}, \tag{39}$$

where $Q_1(z)$ and $Q_2(z/x)$ are arbitrary functions.

3.3. Vector field $V_5 = z\frac{\partial}{\partial x} - \frac{y}{3}\frac{\partial}{\partial u}$

The related characteristic equation of $\boldsymbol{V}_5$ is

$$\frac{\mathrm{d}x}{z} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}u}{-y/3}, \tag{40}$$

then we can obtain similarity solution

$$u = F_3(Y, Z, T) - \frac{xy}{3z}, \tag{41}$$

where $Y = y, Z = z, T = t$ are similarity variables. Substituting Eq. (41) into Eq. (4), we get the reduction equation in terms of F_3

$$F_{3YT} + F_{3TT} = 0, \tag{42}$$

and equation (42) has the solution

$$F_3(Y, Z, T) = Q_3(Z, Y) + Q_4(Z, T - Y), \tag{43}$$

hence, the similarity solution of Eq. (4) is

$$u_6 = Q_3(z, y) + Q_4(z, t - y) - \frac{xy}{3z}, \tag{44}$$

where Q_3, Q_4 are two arbitrary functions of the indicated variables.

While setting two arbitrary functions

$$\begin{aligned} Q_3(z, y) &= \arctan(2z - 4y), \\ Q_4(z, t - y) &= \frac{3}{2}\sin(y - t), \end{aligned} \tag{45}$$

and substituting Eq. (45) into Eq. (44), we will obtain the trigonometric solution of the extended $(3+1)$-dimensional KP–Boussinesq equation (4) and the detailed structures on different planes are shown in Fig. 1.

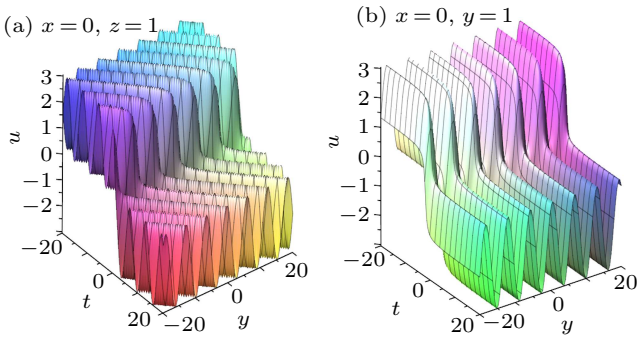


Fig. 1. Trigonometric solution (44) with Eq. (45).

3.4. Vector field $V_2+V_4=z\frac{\partial}{\partial y}+\frac{\partial}{\partial z}-\frac{x}{3}\frac{\partial}{\partial u}$

Solving the related characteristic equation

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{z}=\frac{\mathrm{d}z}{1}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{-x/3}, \tag{46}$$

we can obtain similarity solution

$$u=F_4(X,Y,T)-\frac{xz}{3}, \tag{47}$$

where $X=x, Y=y-z^2/2, T=t$ are similarity variables. Substituting Eq. (47) into Eq. (4), we have the reduction equation in terms of F_4

$$\begin{aligned}&3F_{4XXXY}+9F_{4XX}F_{4Y}+9F_{4XY}F_{4X}\\&+3F_{4XT}+3F_{4YT}+3F_{4TT}+1=0.\end{aligned} \tag{48}$$

Though it is not easy to solve the reduction equation (48), we can still take a particular solution to Eq. (48)

$$F_4(X,Y,T)=Q_5(Y)+Q_6(T-Y)+\frac{Y(Y-2T)}{6}, \tag{49}$$

and the similarity solution of Eq. (4) is

$$\begin{aligned}u_7 \;=\; & Q_5(y-\frac{z^2}{2})+Q_6\left(t+\frac{z^2}{2}-y\right)\\&-\frac{xz}{3}+\frac{(z^2/2-y)(z^2/2-y+2t)}{6},\end{aligned} \tag{50}$$

where Q_5, Q_6 are two arbitrary functions of the indicated variables.

3.5. Vector field $V_2+V_5=z\frac{\partial}{\partial x}+\frac{\partial}{\partial z}-\frac{y}{3}\frac{\partial}{\partial u}$

Solving the related characteristic equation

$$\frac{\mathrm{d}x}{z}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{1}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{-y/3}, \tag{51}$$

we can obtain similarity solution

$$u=F_5(X,Y,T)-\frac{yz}{3}, \tag{52}$$

where $X=x-z^2/2, Y=y, T=t$ are similarity variables. Substituting Eq. (52) into Eq. (4), we gain the reduction equation in terms of F_5

$$\begin{aligned}&F_{5XXXY}+3F_{5X}F_{5XY}+3F_{5XX}F_{5Y}\\&+F_{5XT}+F_{5YT}+F_{5TT}=0.\end{aligned} \tag{53}$$

Some special solutions of Eq. (53) and the corresponding solutions of Eq. (4) are given as follows:

$$\begin{aligned}&F_{51}(X,Y,T)=Q_7(X)+Q_8(T-X),\\&u_8=Q_7\left(x-\frac{z^2}{2}\right)+Q_8\left(t+\frac{z^2}{2}-x\right)-\frac{yz}{3},\\&F_{52}(X,Y,T)=X+Q_9(Y)+Q_{10}(T-Y),\\&u_9=Q_9(y)+Q_{10}(t-y)-\frac{yz}{3}-\frac{z^2}{2}+x,\end{aligned} \tag{54}$$

where Q_7, Q_8, Q_9, Q_{10} are four arbitrary functions of the indicated variables.

While setting four arbitrary functions

$$Q_7\left(x-\frac{z^2}{2}\right)=-\tanh\left(x-\frac{z^2}{2}\right),$$

$$Q_8\left(t+\frac{z^2}{2}-x\right)=\frac{1}{\cosh^2(t+\frac{z^2}{2}-x)}, \tag{56}$$

$$Q_9(y)=\mathrm{e}^{\cos(y)},\quad Q_{10}(t-y)=\arctan(\mathrm{e}^{2t-2y}), \tag{57}$$

we can obtain the corresponding exact solutions of Eq. (4) explicitly

$$u_{81}=\frac{1}{\cosh^2(t+\frac{z^2}{2}-x)}-\tanh\left(x-\frac{z^2}{2}\right)-\frac{yz}{3}, \tag{58}$$

$$u_{91}=\mathrm{e}^{\cos(y)}+\arctan(\mathrm{e}^{2t-2y})-\frac{yz}{3}-\frac{z^2}{2}+x. \tag{59}$$

It is known that equation (58) is the interaction solution between a kink and a soliton and equation (59) is the periodic-like wave solution of Eq. (4) and the corresponding structures are shown in Fig. 2 in different planes.

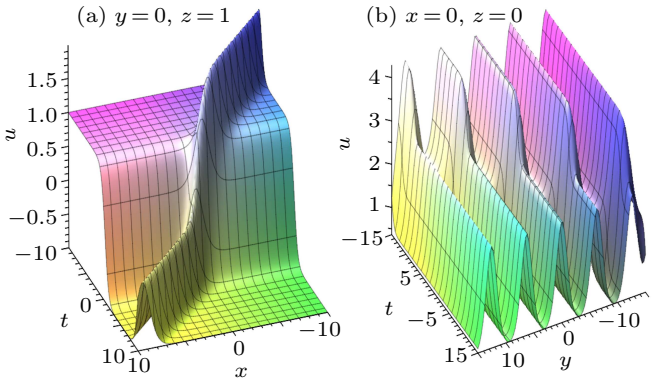


Fig. 2. (a) The interaction solution between a kink and a soliton (58) of Eq. (4), (b) the periodic solution (59) of Eq. (4).

3.6. Vector field $V_3+V_4=\frac{t}{2}\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}+z\frac{\partial}{\partial t}+\frac{y-x}{6}\frac{\partial}{\partial u}$

Solving the related characteristic equation

$$\frac{\mathrm{d}x}{\frac{t}{2}}=\frac{\mathrm{d}y}{z}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{z}=\frac{\mathrm{d}u}{\frac{y-x}{6}}, \tag{60}$$

we can obtain similarity solution

$$u=F_6(X,Y,Z)+\frac{(6y-3t-6x)zt+t^3}{36z^2}, \tag{61}$$

where $X=(4xz-t^2)/4z, Y=y-t, Z=z$ are similarity variables. Substituting Eq. (61) into Eq. (4), we have the reduction equation in terms of F_6

$$\begin{aligned}&6F_{6XXXY}Z+18F_{6Y}F_{6XX}Z+18F_{6XY}F_{6X}Z\\&-6F_{6XZ}Z-6F_{6XY}Z-3F_{6X}-1=0.\end{aligned} \tag{62}$$

A particular solution to Eq. (62) has the form

$$F_6(X,Y,Z) \;=\; Q_{11}(Z)+\frac{\sqrt{3Z}Q_{12}(X/3+Z)}{3Z}+Y$$

$$-\frac{X}{3}-\frac{2Z}{3}, \tag{63}$$

and the similarity solution of Eq. (4) is

$$\begin{aligned} u_{10} &= Q_{11}(z)+\frac{\sqrt{3z}}{3z}Q_{12}\left(\frac{4xz+12z^2-t^2}{12z}\right) \\ &\quad +y-t-\frac{x}{3}-\frac{2z}{3}+\frac{t^3}{36z^2}-\frac{tx}{6z}+\frac{ty}{6z}, \end{aligned} \tag{64}$$

where $Q_{11}(z)$ and $Q_{12}(\frac{4xz+12z^2-t^2}{12z})$ are arbitrary functions with indicated variables respectively.

3.7. Vector field $V_5+V_6=z\frac{\partial}{\partial x}+\frac{3zt-y}{3}\frac{\partial}{\partial u}$

Solving the related characteristic equation

$$\frac{\mathrm{d}x}{z}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{zt-y/3}, \tag{65}$$

we can obtain similarity solution

$$u=F_7(Y,Z,T)+xt-\frac{xy}{3z}, \tag{66}$$

where $Y=y$, $Z=z$, $T=t$ are similarity variables. Substituting Eq. (66) into Eq. (4), we obtain the reduction equation in terms of F_7

$$F_{7YT}Z+F_{7TT}Z-T+Z=0. \tag{67}$$

It is not difficult to find the solution for Eq. (67) has the form

$$\begin{aligned} F_7(Y,Z,T) &= Q_{13}(Z,Y)+Q_{14}(Z,T-Y) \\ &\quad +\frac{T^3}{6Z}+\frac{Y^2}{2}-TY, \end{aligned} \tag{68}$$

where Q_{13}, Q_{14} are two arbitrary functions of the indicated variables. Substituting Eq. (68) into Eq. (66), one can obtain the similarity solution of Eq. (4)

$$u_{11}=Q_{13}(z,y)+Q_{14}(z,t-y)+xt-\frac{xy}{3z}+\frac{t^3}{6z}+\frac{y^2}{2}-ty. \tag{69}$$

Because of the arbitrariness of the two functions Q_{13} and Q_{14}, we can have different selection of the arbitrary functions to find new exact solutions of Eq. (4). If fixing

$$\begin{aligned} Q_{13} &= \frac{2(10\sin(10z/7-5y)-10\cos(10z/7-5y))}{8z^2+9+2\cos(10z/7-5y)+2\sin(10z/7-5y)}, \\ Q_{14} &= -\frac{(t-y)^2}{2}, \end{aligned} \tag{70}$$

$$\begin{aligned} Q_{13} &= \frac{436\,\mathrm{e}^{6-2z}-436\,\mathrm{e}^{18-8y-6z}-1744\,\mathrm{e}^{24-8y-8z}+112}{109\,\mathrm{e}^{6-2z}+109\,\mathrm{e}^{18-8y-6z}+218\,\mathrm{e}^{24-8y-8z}+14}, \\ Q_{14} &= -\frac{(t-y)^2}{2}, \end{aligned} \tag{71}$$

we will have the exact solutions of Eq. (4)

$$\begin{aligned} u_{12} &= \frac{2(10\sin(10z/7-5y)-10\cos(10z/7-5y))}{8z^2+9+2\cos(10z/7-5y)+2\sin(10z/7-5y)} \\ &\quad +\frac{t^3-2xy-3t^2z+6xzt}{6z}, \end{aligned} \tag{72}$$

$$\begin{aligned} u_{13} &= \frac{436\,\mathrm{e}^{6-2z}-436\,\mathrm{e}^{18-8y-6z}-1744\,\mathrm{e}^{24-8y-8z}+112}{109\,\mathrm{e}^{6-2z}+109\,\mathrm{e}^{18-8y-6z}+218\,\mathrm{e}^{24-8y-8z}+14} \\ &\quad +\frac{t^3-2xy-3t^2z+6xzt}{6z}, \end{aligned} \tag{73}$$

respectively. The solution (72) can be regarded as an interaction solution between lump wave and triangular periodic wave solution of Eq. (4) when $x=0$, $t=0$. The solution (73) of Eq. (4) is two-kink solution when $x=0$, $t=0$. The corresponding structures are shown in Fig. 3.

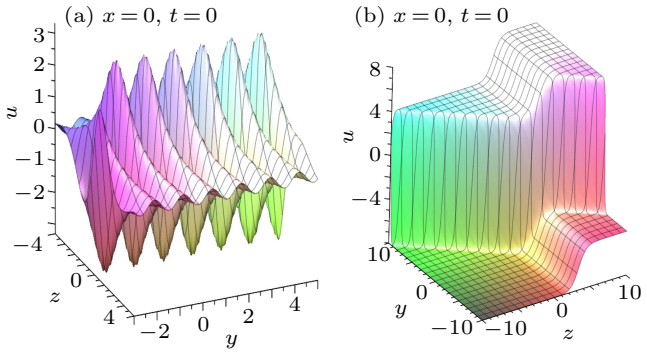


Fig. 3. (a) Interaction solution between lump wave and triangular periodic wave solution (72) when $x=0$, $t=0$, (b) two-kink solution (73) when $x=0$, $t=0$.

3.8. Vector field $V_5+V_7=z\frac{\partial}{\partial x}+\frac{3z-y}{3}\frac{\partial}{\partial u}$

Solving the related characteristic equation

$$\frac{\mathrm{d}x}{z}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{z-y/3}, \tag{74}$$

we can obtain similarity solution

$$u=F_8(Y,Z,T)+x-\frac{xy}{3z}, \tag{75}$$

where $Y=y, Z=z, T=t$ are similarity variables. Substituting Eq. (75) into Eq. (4), we have the reduction equation in terms of F_8

$$F_{8YT}Z+F_{8TT}Z-1=0, \tag{76}$$

which has the solution

$$F_8=Q_{15}(Z,Y)+Q_{16}(Z,T-Y)+\frac{Y(2T-Y)}{2Z}. \tag{77}$$

Then the corresponding similarity solution of Eq. (4) is

$$u_{14}=Q_{15}(z,y)+Q_{16}(z,t-y)+x-\frac{xy}{3z}+\frac{y(2t-y)}{2z}, \tag{78}$$

where Q_{15} and Q_{16} are arbitrary functions of indicated variables. Then we can have abundant exact solutions by fixing three sets of arbitrary functions Q_{15} and Q_{16}.

Case 1

$$\begin{aligned} Q_{15} &= \frac{24548z-74176}{6137z^2+1088y^2-37088z+3904y+60980}, \\ Q_{16} &= \frac{(t-y)^2}{2z}, \\ u_{15} &= \frac{24548z-74176}{6137z^2+1088y^2-37088z+3904y+60980} \\ &\quad +\frac{6xz-2xy+3t^2}{6z}. \end{aligned} \tag{79}$$

Case 2

$$Q_{15}=\frac{2458\,\mathrm{e}^{2z-4y/7}+46118z+37647-5647y}{614\,\mathrm{e}^{2z-4y/7}+11529z^2+18823z-2823yz+71085y^2-2304y+9412},\quad Q_{16}=\frac{(t-y)^2}{2z},$$
$$u_{16}=\frac{2458\,\mathrm{e}^{2z-4y/7}+46118z+37647-5647y}{614\,\mathrm{e}^{2z-4y/7}+11529z^2+18823z-2823yz+71085y^2-2304y+9412}+\frac{6xz-2xy+3t^2}{6z}. \tag{80}$$

Case 3

$$Q_{15}=\frac{18\,\mathrm{e}^{2\sqrt{3}y/3+2z}-96+(48\sqrt{3}y+144z)\,\mathrm{e}^{\sqrt{3}y/3+z}}{9\,\mathrm{e}^{2\sqrt{3}y/3+2z}+48+(24\sqrt{3}zy+48+36z^2+120y^2)\,\mathrm{e}^{\sqrt{3}y/3+z}},\quad Q_{16}=\frac{(t-y)^2}{2z},$$
$$u_{17}=\frac{18\,\mathrm{e}^{2\sqrt{3}y/3+2z}-96+(48\sqrt{3}y+144z)\,\mathrm{e}^{\sqrt{3}y/3+z}}{9\,\mathrm{e}^{2\sqrt{3}y/3+2z}+48+(24\sqrt{3}zy+48+36z^2+120y^2)\,\mathrm{e}^{\sqrt{3}y/3+z}}+\frac{6xz-2xy+3t^2}{6z}. \tag{81}$$

It can be seen that the lump solution, lump-kink solution, and interaction solution between two kink and lump are given in Eqs.(79)–(81) when $x=0$ and $t=0$, which are shown directly in Fig. 4.

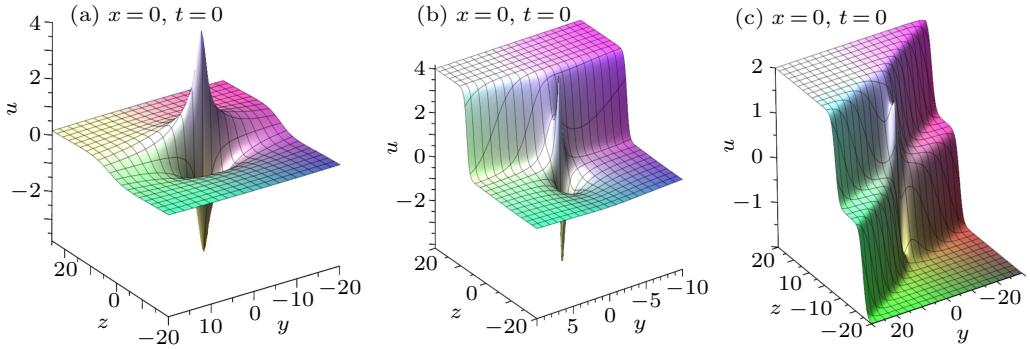


Fig. 4. (a) Lump solution (79), (b) lump-kink solution (80), (c) interaction solution between two-kink and lump (81).

3.9. Vector field $V_5+V_6+V_7=z\frac{\partial}{\partial x}+\frac{3z+3zt-y}{3}\frac{\partial}{\partial u}$

Solving the related characteristic equation

$$\frac{\mathrm{d}x}{z}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{0}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}u}{z+zt-\frac{y}{3}}, \tag{82}$$

we can obtain similarity solution

$$u=F_9(Y,Z,T)+x+xt-\frac{xy}{3z}, \tag{83}$$

where $Y=y,Z=z,T=t$ are similarity variables. Substituting Eq. (83) into Eq. (4), we get the reduction equation in terms of F_9

$$F_{9YT}Z+F_{9TT}Z-T+Z-1=0, \tag{84}$$

which has the solution

$$F_9=Q_{17}(Z,Y)+Q_{18}(Z,T-Y)+\frac{Y(3T^2-3TY-6ZT+Y^2+3YZ+6T-3Y)}{6Z}, \tag{85}$$

and the similarity solution of Eq. (4) is

$$u_{18}=Q_{17}(z,y)+Q_{18}(z,t-y)+x+xt+\frac{y(3t^2-3ty-6zt-2x+y^2+3yz+6t-3y)}{6z}, \tag{86}$$

where $Q_{17}(z,y)$ and $Q_{18}(z,t-y)$ are arbitrary functions. Here we list two types of exact solutions of Eq. (4) by selecting proper arbitrary functions Q_{17} and Q_{18}

Case 1

$$Q_{17}=\frac{\sinh(2z/3+y/3)}{4\cosh(2z/3+y/3)+3\cos(2y/3)},$$
$$Q_{18}=\frac{3(y-t)^2-(y-t)^3-3z(y-t)^2}{6z},$$
$$u_{19}=\frac{\sinh(2z/3+y/3)}{4\cosh(2z/3+y/3)+3\cos(2y/3)}+\frac{t^3+(3-3z)t^2+6txz-2xy+6xz}{6z}. \tag{87}$$

Case 2

$$Q_{17}=\frac{2\mathrm{i}(\mathrm{e}^{4-\mathrm{i}+2\mathrm{i}z-8y}-\mathrm{e}^{4+\mathrm{i}}+2\mathrm{e}^{2\mathrm{i}z+8-8y})}{\mathrm{e}^{4-\mathrm{i}+2\mathrm{i}z-8y}+\mathrm{e}^{4+\mathrm{i}}+2\mathrm{e}^{2\mathrm{i}z+8-8y}},$$
$$Q_{18}=\frac{3(y-t)^2-(y-t)^3-3z(y-t)^2}{6z},$$
$$u_{20}=\frac{2\mathrm{i}(\mathrm{e}^{4-\mathrm{i}+2\mathrm{i}z-8y}-\mathrm{e}^{4+\mathrm{i}}+2\mathrm{e}^{2\mathrm{i}z+8-8y})}{\mathrm{e}^{4-\mathrm{i}+2\mathrm{i}z-8y}+\mathrm{e}^{4+\mathrm{i}}+2\mathrm{e}^{2\mathrm{i}z+8-8y}}+\frac{t^3+(3-3z)t^2+6txz-2xy+6xz}{6z}, \tag{88}$$

where i is an imaginary unit. The solution (87) can be regarded as a periodic lump solution of Eq. (4) when $x=0,\ t=0$. The solution (88) of Eq. (4) is breather solution when $x=0,t=0$. The corresponding structures are shown in Fig. 5.

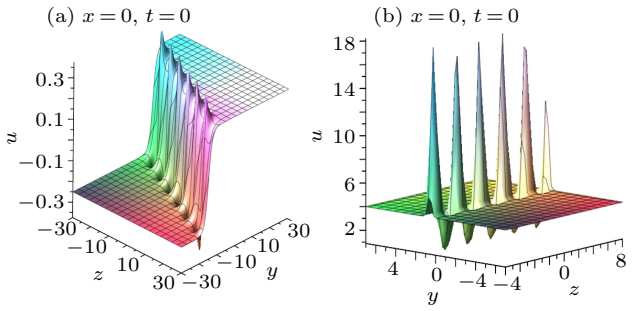


Fig. 5. (a) Periodic lump solution (87), (b) the module of breather solution (88).

4. Summary and discussion

In conclusion, Lie group method has been applied to the extended $(3+1)$-dimensional KP–Boussinesq equation (4) to find abundant similarity solutions with arbitrary functions. Various kinds of similarity solutions of Eq. (4) are obtained by solving different characteristic equations and many interesting similarity solutions are studied explicitly, which include the trigonometric solution, the kink-lump interaction solution, the interaction solution between lump wave and triangular periodic wave, the lump solution, and the breather solution, etc. Some detailed structures of these exact solutions are shown graphically in different planes. From these exact solutions of Eq. (4), one can learn that the Lie group method is very effective to solve high dimensional nonlinear models. Other integrable systems may also be studied by means of Lie group method in future.

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