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Asymptotic analysis on bright solitons and breather solutions of a generalized higher-order nonlinear Schrödinger equation in an optical fiber or a planar waveguide

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We study a generalized higher-order nonlinear Schrödinger equation in an optical fiber or a planar waveguide. We obtain the Lax pair and N-fold Darboux transformation (DT) with N being a positive integer. Based on Lax pair obtained by us, we derive the infinitely-many conservation laws. We give the bright one-, two-, and N-soliton solutions, and the first-, second-, and Nth-order breather solutions based on the N-fold DT. We conclude that the velocities of the bright solitons are influenced by the distributed gain function, $g(z)$, and variable coefficients in equation, $h_1(z)$, $p_1(z)$, $r_1(z)$, and $s_1(z)$ via the asymptotic analysis, where z represents the propagation variable or spatial coordinate. We also graphically observe that: the velocities of the first- and second-order breathers will be affected by $h_1(z)$, $p_1(z)$, $r_1(z)$, and $s_1(z)$, and the background wave depends on $g(z)$.

Keywords: optical fiber, planar waveguide, Darboux transformation, bright soliton, breather

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1. Introduction

Optical fibers have been studied to support many guided waveforms called modes. Especially, single-mode fibers have been important for communication applications where extremely high signal bandwidth is required.[1,2] A planar waveguide has comprised the thin layers of optically transparent materials, featuring a core with a higher refractive index compared to the surrounding substrate and cladding layers in planar waveguide devices.[3] Researchers have been interested in the propagation of an optical beam and superradiance in atom within planar waveguide.[4] Propagation of the multiple optical waves in the nonlinear media has been described by the nonlinear Schrödinger (NLS)-type equations.[5] Many of researchers have studied the solitons occurred with balancing the self-phase modulation and group velocity dispersion.[6–9]

Taking into account that the medium's inhomogeneity impacts wave propagation and gain or loss affects wave intensity, the author in Ref. [10] has presented a generalized higher-order NLS equation for optical fibers or planar waveguides, i.e.,

$$\begin{aligned}&\mathrm{i}q_z+s_1(z)q_{tt}+s_2(z)q\,|q|^2-\mathrm{i}[h_1(z)q_{ttt}+h_2(z)\,|q|^2\,q_t]\\&+p_1(z)q_{tttt}+p_2(z)q\,|q|^4\\&+p_3(z)\,|q|^2\,q_{tt}+p_4(z)q\,|q_t|^2+p_5(z)q_t^2q^*\\&+p_6(z)q^2q_{tt}^*-\mathrm{i}[r_1(z)q_{ttttt}+r_2(z)\,|q|^4\,q_t\\&+r_3(z)\,|q|^2\,q_{ttt}+r_4(z)qq_tq_{tt}^*+r_5(z)qq_t^*q_{tt}\\&+r_6(z)q^*q_tq_{tt}+r_7(z)q_t^2q_t^*]-\mathrm{i}\frac{g(z)}{2}q=0,\end{aligned}\tag{1}$$

which describes the attosecond pulse propagation in an optical fiber with variables z for propagation and t for time, and propagation of a continuous wave beam in a planar waveguide with the subscripts z being spatial variable and t being the time variable, where the modulus of the $q(z,t)$ signifies the continuous wave beam or envelope of the attosecond pulse, $g(z)$ is the distributed gain function, $s_1(z)$, $s_2(z)$, $h_1(z)$, $h_2(z)$, $p_\varsigma(z)$'s $(\varsigma=1,2,3,\ldots,6)$, $r_\varsigma(z)$'s, and $r_7(z)$ are the real functions of z, and "$*$" represents the complex conjugate.

For Eq. (1), the author in Ref. [10] reported the solitons, breathers, and bound waves via the Hirota bilinear method. When $p_\varsigma(z)=r_\varsigma(z)=r_7(z)=g(z)=0$, $2s_1(z)=s_2(z)=1$, and $h_2(z)=6h_1(z)=\alpha$, equation (1) has been simplified to the Hirota equation, which models the propagation of pulse in an optical fiber and the propagation of wave in an ocean, where α is a real coefficient.[11,12]

However, Lax pair, conservation laws, N-fold Darboux transformation (DT), bright soliton, and breather solutions different from that in Ref. [10] for Eq. (1) have not been reported. In Section 2, we will give the Lax pair for Eq. (1). We will obtain the conservation laws for Eq. (1) in Section 3. In Section 4, the N-fold DT for Eq. (1) based on the Lax pair obtained by us. In Section 5, bright one and two solitons solutions different from that in Ref. [10] and asymptotic analysis on the bright two solitons for Eq. (1) will be studied. In Section 6, breather solutions different from that in Ref. [10] for

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Eq. (1) will be derived. Impacts of variable coefficients on the bright solitons, breathers for Eq. (1) will be discussed in Section 7. In Section 8, we will give conclusions.

2. Lax pair for Eq. (1)

We obtain the Lax pair for Eq. (1) as

$$\Phi_t = U\Phi, \quad \Phi_z = V\Phi, \tag{2}$$

where $\Phi = (\phi_1, \phi_2)^{\mathrm{T}}$ represents the 2×1 vector eigenfunction, the superscript T denotes the transpose for a vector, ϕ_1 and ϕ_2 are all the complex functions of λ, z, and t, and U and V are all the 4×4 matrices expressed as

$$U = -\mathrm{i}\lambda U_1 + U_0, \quad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \tag{3}$$

where

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_0 = \begin{pmatrix} 0 & Lq \\ Hq^* & 0 \end{pmatrix},$$

$$\begin{aligned}
V_{11} =\ & -16\mathrm{i}r_1(z)\lambda^5 + 8\mathrm{i}\lambda^4 p_1(z) + 4\mathrm{i}\lambda^3[h_1(z) - 2HLr_1(z)|q|^2] + 2\lambda^2\{-\mathrm{i}s_1(z) + 2HL[\mathrm{i}p_1(z)|q|^2 + r_1(z)q^*q_t - r_1(z)qq_t^*]\} \\
& + 2\mathrm{i}HL\lambda[h_1(z)|q|^2 - 3HLr_1(z)|q|^4 + \mathrm{i}p_1(z)q_tq^* - r_1(z)|q_t|^2 + r_1(z)q^*q_{tt} - \mathrm{i}p_1(z)qq_t^* + r_1(z)qq_{tt}^*] \\
& + HL[-\mathrm{i}s_1(z)|q|^2 + h_1(z)(-q^*q_t + qq_t^*) + \mathrm{i}p_1(z)(3HL|q|^4 + |q_t|^2 - q^*q_{tt} - qq_{tt}^*) \\
& + r_1(z)(6HL|q|^2q^*q_t - 6HL|q|^2qq_t^* + q_t^*q_{tt} - q_tq_{tt}^* - q^*q_{ttt} + qq_{ttt}^*)] + \frac{1}{4}g(x), \\
V_{12} =\ & 16L\lambda^4 r_1(z)q + L\lambda^3[-8p_1(z)q + 8\mathrm{i}r_1(z)q_t] + L\lambda^2[-4h_1(z)q + 8r_1(z)HL|q|^2q - 4\mathrm{i}p_1(z)q_t - 4r_1(z)q_{tt}] \\
& + L\lambda[-4HLp_1(z)|q|^2q + 2s_1(z)q - 2\mathrm{i}h_1(z)q_t + 12\mathrm{i}HLr_1(z)|q|^2q_t + 2p_1(z)q_{tt} - 2\mathrm{i}r_1(z)q_{ttt}] \\
& + L\{\mathrm{i}s_1(z)q_t + h_1(z)(-2HL|q|^2q + q_{tt}) - \mathrm{i}p_1(z)(6HL|q|^2q_t - q_{ttt}) \\
& + r_1(z)[6H^2L^2|q|^4q - 6HLq^*q_t^2 - 4HLq(|q_t|^2 + 2q^*q_{tt}) - 2HLq^2q_{tt}^* + q_{tttt}]\}, \\
V_{21} =\ & 16H\lambda^4 r_1(z)q^* + H\lambda^3[-8p_1(z)q^* - 8\mathrm{i}r_1(z)q_t^*] + H\lambda^2[-4h_1(z)q^* + 8r_1(z)HL|q|^2q^* + 4\mathrm{i}p_1(z)q_t^* - 4r_1(z)q_{tt}^*] \\
& + H\lambda[-4HLp_1(z)|q|^2q^* + 2s_1(z)q^* + 2\mathrm{i}h_1(z)q_t^* - 12\mathrm{i}HLr_1(z)|q|^2q_t^* + 2p_1(z)q_{tt}^* + 2\mathrm{i}r_1(z)q_{ttt}^*] \\
& + H\{-\mathrm{i}s_1(z)q_t^* + h_1(z)(-2HL|q|^2q^* + q_{tt}^*) + \mathrm{i}p_1(z)(6HL|q|^2q_t^* - q_{ttt}^*) + r_1(z)[6H^2L^2|q|^4q^* - 4HLq^*|q_t|^2 \\
& - 2HLq(3q_t^{*2} + 4q^*q_{tt}^*) - 2HLq^{*2}q_{tt} + q_{tttt}^*]\}, \\
V_{22} =\ & -V_{11},
\end{aligned}$$

with H and L being the real constants.

Based on the compatibility condition $U_z - V_t + UV - VU = 0$, we can derive Eq. (1) under the constraints

$$\begin{aligned}
& r_2(z) = 30H^2L^2r_1(z), \\
& r_3(z) = r_4(z) = r_5(z) = r_7(z) = -10HLr_1(z), \\
& r_6(z) = -20HLr_1(z), \quad s_2(z) = -2HLs_1(z), \\
& h_2(z) = -6HLh_1(z), \\
& p_2(z) = 6H^2L^2p_1(z), \quad p_3(z) = -8HLp_1(z), \\
& p_4(z) = -4HLp_1(z), \\
& p_5(z) = -6HLp_1(z), \quad p_6(z) = -2HLp_1(z).
\end{aligned} \tag{4}$$

3. Conservation laws for Eq. (1)

We can give the infinitely-many conservation laws through the functions $\Gamma(x,t) = \phi_2/\phi_1$ for Eq. (1) and derive the Ricatti-type equation as

$$\Gamma_t = Hq^* + 2\mathrm{i}\lambda\Gamma - Lq\Gamma^2. \tag{5}$$

Motivated by Ref. [13], we expand

$$q\Gamma = \sum_{n=1}^{\infty} \frac{F_n}{\lambda^n}, \tag{6}$$

where F_n's $(n = 1,2,3,\ldots)$ denote the complex functions about z and t. Substituting Eq. (6) into Eq. (5), and setting the coefficients corresponding to equal powers of λ to zero, we construct the recurrence relations

$$\begin{aligned}
& F_1 = \frac{\mathrm{i}}{2}H|q|^2, \\
& F_2 = \frac{\mathrm{i}}{2}\left(\frac{F_1q_t}{q} - F_{1,t}\right), \\
& F_3 = \frac{\mathrm{i}}{2}\left(\frac{F_2q_t}{q} - LF_1^2 - F_{2,t}\right), \\
& \cdots, \\
& F_{\rho+1} = \frac{\mathrm{i}}{2}\left(\frac{F_\rho q_t}{q} - L\sum_{w=1}^{\rho-1} F_wF_{\rho-w} - F_{\rho,t}\right), \\
& (\rho = 3,4,\ldots).
\end{aligned} \tag{7}$$

According to the compatibility condition $(\ln\phi_1)_{zt} = (\ln\phi_1)_{tz}$, we can draw the conservation form for Eq. (1) as

$$(-\mathrm{i}\lambda + Lq\Gamma)_z = (V_{11} + V_{12}\Gamma)_t. \tag{8}$$

By substituting Eqs. (6) and (7) into Eq. (8) and consolidating the coefficients corresponding to identical powers of λ, we derive an infinite sequence of the conservation laws for Eq. (1) as

$$\frac{\partial \boldsymbol{R}_\kappa}{\partial z} = \frac{\partial \boldsymbol{J}_\kappa}{\partial t}, \quad (\kappa = 1,2,3,\ldots), \tag{9}$$

where $\boldsymbol{R}_\kappa$'s denote the conserved densities and $\boldsymbol{J}_\kappa$'s denote the conserved fluxes, and

$$
\begin{aligned}
\boldsymbol{R}_1 &= LF_1 = \frac{\mathrm{i}}{2}HL|q|^2,\\
\boldsymbol{J}_1 &= \frac{1}{2}HL\big[s_1(z)(-q^*q_t+qq_t^*)+h_1(z)(-3\mathrm{i}HL|q|^4-\mathrm{i}\,|q_t|^2+\mathrm{i}q^*q_{tt}+\mathrm{i}qq_{tt}^*)\\
&\quad +p_1(z)(7HL|q|^2q^*q_t-2HL|q|^2qq_t^*+q_t^*q_{tt}-q_tq_{tt}^*-q^*q_{ttt})\\
&\quad +r_1(z)(8\mathrm{i}H^2L^2|q|^6-6\mathrm{i}HLq^{*2}q_t^2+2\mathrm{i}HL|q|^2|q_t|^2-9\mathrm{i}HL|q|^2q^*q_{tt}-4\mathrm{i}HL|q|^2qq_{tt}^*+\mathrm{i}\,|q_{tt}|^2-\mathrm{i}q_t^*q_{ttt}+\mathrm{i}q^*q_{tttt})\big],\\
\boldsymbol{R}_2 &= LF_2 = \frac{\mathrm{i}}{4}HLqq_t^*,\\
\boldsymbol{J}_2 &= \frac{1}{4}HL\big[\mathrm{i}s_1(z)(HL|q|^4+|q_t|^2-qq_{tt}^*)+h_1(z)(HL|q|^2q^*q_t-2HL|q|^2qq_t^*\\
&\quad +q_t^*q_{tt}-q_tq_{tt}^*)-\mathrm{i}p_1(z)(2H^2L^2|q|^6+6HL|q|^2|q_t|^2-HL|q|^2q^*q_{tt}-2HL|q|^2qq_{tt}^*+|q_{tt}|^2-q_t^*q_{ttt})\\
&\quad +r_1(z)(6H^2L^2|q|^4qq_t^*-6H^2L^2|q|^4q^*q_t-2HLq^2q_t^*q_{tt}^*-q_{tt}^*q_{ttt}-4HLq|q_t|^2q_t^*+6HL|q|^2q_tq_{tt}^*-8HL|q|^2q_t^*q_{tt}\\
&\quad +HL|q|^2q^*q_{ttt}-6HL|q_t|^2q^*q_t+q_t^*q_{tttt})\big],\\
\boldsymbol{R}_3 &= LF_3 = \frac{\mathrm{i}}{8}HL(HL|q|^4-qq_{tt}^*),\\
\boldsymbol{J}_3 &= \frac{1}{8}HL(HL|q|^2q^*-q_{tt}^*)\{-s_1(z)q_t+\mathrm{i}h_1(z)(-2HL|q|^2q+q_{tt})+\mathrm{i}p_1(z)(-6\mathrm{i}HL|q|^2q_t+\mathrm{i}q_{ttt})\\
&\quad +\mathrm{i}r_1(z)\big[6H^2L^2|q|^4q-6HLq^*q_t^2-4HLq(|q_t|^2+2q^*q_{tt})-2HL|q|^4+q_{tttt}\big]\},\\
&\quad \cdots.
\end{aligned}
\tag{10}
$$

4. The N-fold DT for Eq. (1)

When $\Phi_1=(\psi_1[0],\phi_1[0])^{\mathrm{T}}$ is a vector solution for Lax pair (2) with $q[0]$ being the seed solutions for Eq. (1) at $\lambda=\lambda_1$, λ_1 is a given parameter, and I is a 2×2 identity matrix, we will construct one-fold DT for Eq. (1) based on Lax pair (2) as

$$
\Phi[1]=D[1]\Phi,\quad D[1]=\lambda I-M[0]\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^*\end{pmatrix}M[0]^{-1},
\tag{11}
$$

$$
\begin{aligned}
M[0] &= \begin{pmatrix}\psi_1[0] & \phi_1[0]^*\\ \phi_1[0] & -\psi_1[0]^*\end{pmatrix},\\
q[1] &= q[0]+\frac{1}{L}2\mathrm{i}\frac{(\lambda_1^*-\lambda_1)\phi_1[0]^*\psi_1[0]}{|\psi_1[0]|^2+|\phi_1[0]|^2},
\end{aligned}
\tag{12}
$$

where $H=-L$, $M[0]$ is a 2×2 nonsingular matrix, the superscript "-1" denotes inverse of a matrix. Conditions,

$$
\begin{aligned}
&\Phi[1]_t=U[1]\Phi[1],\quad U[1]=D[1]_tD[1]^{-1}+D[1]UD[1]^{-1},\\
&\Phi[1]_z=V[1]\Phi[1],\quad V[1]=D[1]_zD[1]^{-1}+D[1]VD[1]^{-1},
\end{aligned}
\tag{13}
$$

should be satisfied, which means that Lax pair (2) keeps invariant under transformation (11), $U[1]$ and $V[1]$ are all the 2×2 nonsingular matrices, and possess the same form of U or V, while $q[0]$ in U or V is mapped into $q[1]$ in $U[1]$ or $V[1]$.

Similarly, we assume that $\Phi_j=(\psi_j[0],\phi_j[0])^{\mathrm{T}}$ $(j=1,2,\ldots,N)$ are the solutions of Lax pair (2) at $\lambda=\lambda_j$ with $q[0]$ being the seed solutions, where $\psi_j[0]$'s, $\phi_j[0]$'s are the scalar functions about z and t. $\psi_j[j-1]$ and $\phi_j[j-1]$ represent the scalar functions about z and t for the j-fold DT, j represents the j-th iteration. Iterating the one-fold DT N times, we derive the N-fold DT for Eq. (1) as

$$
\begin{aligned}
&\Phi[N]=D[N]D[N-1]\cdots D[1]\Phi,\\
&D[j]=\lambda I-M[j-1]\Lambda_jM[j-1]^{-1},
\end{aligned}
$$

$$
q[N]=q[0]+\sum_{j=1}^{N}\frac{1}{L}2\mathrm{i}\frac{(\lambda_j^*-\lambda_j)\phi_j[j-1]^*\psi_j[j-1]}{|\psi_j[j-1]|^2+|\phi_j[j-1]|^2},
\tag{14}
$$

where

$$
\begin{aligned}
\Phi_j[j-1] &= (\psi_j[j-1],\phi_j[j-1])^{\mathrm{T}}\\
&= D[j-1]D[j-2]\ldots D[1]|_{\lambda=\lambda_j}\Phi_j,\\
M[j-1] &= \begin{pmatrix}\psi_j[j-1] & \phi_j[j-1]^*\\ \phi_j[j-1] & -\psi_j[j-1]^*\end{pmatrix},\\
\Lambda_j &= \begin{pmatrix}\lambda_j & 0\\ 0 & \lambda_j^*\end{pmatrix}.
\end{aligned}
$$

5. Asymptotic analysis on the bright two solitons for Eq. (1) under constraints (4)

To construct the bright soliton solutions, we take the seed solutions as $q[0]=0$ of Eq. (1) and derive the solutions of Lax pair (2) at $\lambda=\lambda_j$ as

$$
\Phi_j=\begin{pmatrix}l_{j1}\,\mathrm{e}^{-\mathrm{i}\lambda_jt+f_j}\\ l_{j2}\,\mathrm{e}^{\mathrm{i}\lambda_jt-f_j}\end{pmatrix},
\tag{15}
$$

where l_{j1}'s and l_{j2}'s are all the real constants and

$$
\begin{aligned}
f_j &= \frac{1}{4}\int\{g(z)-8\mathrm{i}\lambda_j^2[-2h_1(z)\lambda_j-4p_1(z)\lambda_j^2\\
&\quad +8r_1(z)\lambda_j^3+s_1(z)]\}\,\mathrm{d}z.
\end{aligned}
$$

Substituting solutions (15) with $N=1$ into DT (12), we obtain the bright one-soliton solutions of Eq. (1)

$$
q[1]=2\frac{\lambda_{1I}\,\mathrm{sech}\,[\theta_1-\ln\frac{l_{12}}{l_{11}}]}{L\mathrm{e}^{\mathrm{i}\theta_2}},
\tag{16}
$$

where

$$\theta_1 = 2t\lambda_{1\mathrm{I}} + \int \left\{ \frac{g(z)}{2} + 8\lambda_{1\mathrm{I}}[(\lambda_{1\mathrm{I}}^2 - 3\lambda_{1\mathrm{R}}^2)h_1(z) + 8(\lambda_{1\mathrm{I}}^2 - \lambda_{1\mathrm{R}}^2)\lambda_{1\mathrm{R}}p_1(z) + 4(\lambda_{1\mathrm{I}}^4 - 10\lambda_{1\mathrm{I}}^2\lambda_{1\mathrm{R}}^2 + 5\lambda_{1\mathrm{R}}^4)r_1(z) + \lambda_{1\mathrm{R}}s_1(z)] \right\} \mathrm{d}z,$$

$$\theta_2 = 2t\lambda_{1\mathrm{R}} + 4\int [h_1(z)(6\lambda_{1\mathrm{I}}^2\lambda_{1\mathrm{R}} - 2\lambda_{1\mathrm{R}}^3) - 4p_1(z)(\lambda_{1\mathrm{I}}^4 - 6\lambda_{1\mathrm{I}}^2\lambda_{1\mathrm{R}}^2 + \lambda_{1\mathrm{R}}^4) + 8\lambda_{1\mathrm{R}}r_1(z)(5\lambda_{1\mathrm{I}}^4 - 10\lambda_{1\mathrm{I}}^2\lambda_{1\mathrm{R}}^2 + \lambda_{1\mathrm{R}}^4) + (\lambda_{1\mathrm{R}}^2 - \lambda_{1\mathrm{I}}^2)s_1(z)]\mathrm{d}z,$$

$\lambda_{1\mathrm{R}}$ and $\lambda_{1\mathrm{I}}$ represent the real and imaginary part of λ_1, respectively.

Substituting solutions (15) with $N = 2$ into DT (14), we obtain the bright two-soliton solutions of Eq. (1) as

$$q[2] = \frac{\alpha_{11}\mathrm{e}^{\omega_1} + \alpha_{12}\mathrm{e}^{\omega_{22}} + \alpha_{21}\mathrm{e}^{\omega_3} + \alpha_{22}\mathrm{e}^{\omega_4} + \alpha_3\mathrm{e}^{\omega_5} + \alpha_4\mathrm{e}^{\omega_{21}} + \alpha_5\mathrm{e}^{\omega_6}}{LB}, \tag{17}$$

where

$$\begin{aligned}
B &= \beta_{11}\mathrm{e}^{\nu_{12}} + \beta_{22}\mathrm{e}^{\nu_{22}} + \beta_{21}\mathrm{e}^{\nu_3} + \beta_{31}\mathrm{e}^{\nu_4} + \beta_{32}\mathrm{e}^{\nu_5} \\
&\quad + \beta_{11}\mathrm{e}^{\nu_{11}} + \beta_{12}\mathrm{e}^{\nu_{21}} + \beta_{12}\mathrm{e}^{\nu_{62}} + \beta_{41}\mathrm{e}^{\nu_7} + \beta_{42}\mathrm{e}^{\nu_{61}}, \\
\omega_1 &= 2[\mathrm{i}t(\lambda_1^* + \lambda_2^*) + 2f_1 + f_1^* + f_2 + f_2^*], \\
\omega_{2\iota} &= 2[\mathrm{i}t(\lambda_1 + \lambda_\iota^*) + f_1 + f_2 + f_\iota^*], \quad \iota = 1,2, \\
\omega_3 &= 2[\mathrm{i}t(\lambda_1^* + \lambda_2) + 2f_1 + f_1^*], \\
\omega_4 &= 2[\mathrm{i}t(\lambda_1 + \lambda_2) + f_1], \\
\omega_5 &= 4(\mathrm{i}t\lambda_1^* + f_1 + f_1^*) + 2f_2, \quad \omega_6 = 2(2\mathrm{i}t\lambda_1 + f_2), \\
\nu_{1\iota} &= 2[\mathrm{i}t(\lambda_1^* + \lambda_\iota + \lambda_\iota^*) + f_1 + f_1^* + f_\iota^* + f_\ell], \\
&\quad \ell = 1,2, \ell \neq \iota, \\
\nu_{2\iota} &= 2[\mathrm{i}t(2\lambda_1 + \lambda_\iota^*) + f_2 + f_\iota^*], \\
\nu_3 &= 2\mathrm{i}t(2\lambda_1^* + \lambda_2) + 4f_1 + 4f_1^*, \\
\nu_4 &= 2[\mathrm{i}t(2\lambda_1^* + \lambda_2^*) + 2f_1 + 2f_1^* + f_2 + f_2^*], \\
\nu_5 &= 2\mathrm{i}t(2\lambda_1 + \lambda_2), \\
\nu_{6\iota} &= 2[\mathrm{i}t(\lambda_1 + \lambda_2 + \lambda_\iota^*) + f_1 + f_\iota^*], \\
\nu_7 &= 2[\mathrm{i}t(\lambda_1 + \lambda_2^* + \lambda_1^*) + f_1 + f_1^* + f_2 + f_2^*], \\
\alpha_{1\iota} &= -2\mathrm{i}l_{1\iota}^3 l_{1\ell} l_{21}^2(\lambda_1 - \lambda_1^*)(\lambda_1 - \lambda_2)(\lambda_1^* - \lambda_2), \\
\alpha_{2\iota} &= -2\mathrm{i}l_{1\iota}^3 l_{1\ell} l_{22}^2(\lambda_1 - \lambda_1^*)(\lambda_1 - \lambda_2^*)(\lambda_1^* - \lambda_2^*), \\
\alpha_3 &= -2\mathrm{i}l_{11}^4 l_{21} l_{22}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2^*)(\lambda_2 - \lambda_2^*), \\
\alpha_5 &= -2\mathrm{i}l_{12}^4 l_{21} l_{22}(\lambda_1^* - \lambda_2)(\lambda_1^* - \lambda_2^*)(\lambda_2 - \lambda_2^*), \\
\alpha_4 &= -2\mathrm{i}l_{11}^2 l_{12}^2 l_{21} l_{22}(\lambda_2 - \lambda_2^*) \\
&\quad \times [\lambda_1^2 + \lambda_1^{*2} + 2\lambda_2\lambda_2^* - (\lambda_1 + \lambda_1^*)(\lambda_2 + \lambda_2^*)], \\
\beta_{1\iota} &= l_{1\iota}^3 l_{1\ell} l_{21} l_{22}(\lambda_1 - \lambda_1^*)(\lambda_2 - \lambda_2^*), \\
\beta_{2\iota} &= l_{1\iota}^4 l_{2\ell}^2(\lambda_1^* - \lambda_2)(\lambda_1 - \lambda_2^*), \\
\beta_{3\iota} &= l_{1\iota}^4 l_{2\iota}^2(\lambda_1 - \lambda_2)(\lambda_1^* - \lambda_2^*), \\
\beta_{4\iota} &= l_{11}^2 l_{12}^2 l_{2\iota}^2[\lambda_1(2\lambda_1^* - \lambda_2 - \lambda_2^*) + 2\lambda_2\lambda_2^* - \lambda_1^*(\lambda_2 + \lambda_2^*)].
\end{aligned}$$

Based on solutions (17), we study the interactions between the bright two solitons from their amplitudes, velocities, and phase shifts via the asymptotic analysis and give the asymptotics in the following expressions.

Before collision ($t \to -\infty$),

$$\begin{aligned}
q[2]^{1-} &\to \frac{A_{11}}{2\sqrt{A_{21}A_{32}}}\mathrm{e}^{\mathrm{i}\xi_{11}}\,\mathrm{sech}\left[\xi_{21} - \ln\sqrt{\frac{A_{32}}{A_{21}}}\right], \\
&\quad (f_1 + f_1^* \sim 0, \quad f_2 + f_2^* \to -\infty), \\
q[2]^{2-} &\to \frac{T_{11}}{2\sqrt{T_{22}T_{31}}}\mathrm{e}^{\mathrm{i}\xi_{12}}\,\mathrm{sech}\left[\xi_{22} - \ln\sqrt{\frac{T_{22}}{T_{31}}}\right], \\
&\quad (f_1 + f_1^* \to +\infty, \quad f_2 + f_2^* \sim 0),
\end{aligned} \tag{18}$$

After collision ($t \to +\infty$),

$$\begin{aligned}
q[2]^{1+} &\to \frac{A_{12}}{2\sqrt{A_{31}A_{22}}}\mathrm{e}^{\mathrm{i}\xi_{11}}\,\mathrm{sech}\left[\xi_{21} - \ln\sqrt{\frac{A_{22}}{A_{31}}}\right], \\
&\quad (f_1 + f_1^* \sim 0, \quad f_2 + f_2^* \to +\infty), \\
q[2]^{2+} &\to \frac{T_{12}}{2\sqrt{T_{32}T_{21}}}\mathrm{e}^{\mathrm{i}\xi_{12}}\,\mathrm{sech}\left[\xi_{22} - \ln\sqrt{\frac{T_{32}}{T_{21}}}\right], \\
&\quad (f_1 + f_1^* \to -\infty, \quad f_2 + f_2^* \sim 0),
\end{aligned} \tag{19}$$

where $q[2]^{1-}$, $q[2]^{1+}$, $q[2]^{2-}$, and $q[2]^{2+}$ indicate the important physical quantities of the 1st or 2nd bright two solitons, $q[2]$, before and after collisions and

$$\begin{aligned}
\xi_{1\iota} &= -2\lambda_{\iota\mathrm{R}}t - 4\int [h_1(z)(6\lambda_{\iota\mathrm{I}}^2\lambda_{\iota\mathrm{R}} - 2\lambda_{\iota\mathrm{R}}^3) \\
&\quad - 4p_1(z)(\lambda_{\iota\mathrm{I}}^4 - 6\lambda_{\iota\mathrm{I}}^2\lambda_{\iota\mathrm{R}}^2 + \lambda_{\iota\mathrm{R}}^4) \\
&\quad + 8\lambda_{\iota\mathrm{R}}r_1(z)(5\lambda_{\iota\mathrm{I}}^4 - 10\lambda_{\iota\mathrm{I}}^2\lambda_{\iota\mathrm{R}}^2 + \lambda_{\iota\mathrm{R}}^4) \\
&\quad + s_1(z)(\lambda_{\iota\mathrm{R}}^2 - \lambda_{\iota\mathrm{I}}^2)]\mathrm{d}z, \\
\xi_{2\iota} &= 2\lambda_{\iota\mathrm{I}}t + \int \frac{1}{2}\{g(z) + 16\lambda_{\iota\mathrm{I}}[h_1(z)(\lambda_{\iota\mathrm{I}}^2 - 3\lambda_{\iota\mathrm{R}}^2) \\
&\quad + 8p_1(z)\lambda_{\iota\mathrm{R}}(\lambda_{\iota\mathrm{I}}^2 - \lambda_{\iota\mathrm{R}}^2) \\
&\quad + 4r_1(z)(\lambda_{\iota\mathrm{I}}^4 - 10\lambda_{\iota\mathrm{I}}^2\lambda_{\iota\mathrm{R}}^2 + 5\lambda_{\iota\mathrm{R}}^4) + s_1(z)\lambda_{\iota\mathrm{R}}]\}\mathrm{d}z, \\
A_{11} &= -2\mathrm{i}l_{11}l_{12}(\lambda_1 - \lambda_1^*)(\lambda_1 - \lambda_2^*)(\lambda_1^* - \lambda_2^*), \\
A_{12} &= -2\mathrm{i}l_{11}l_{12}(\lambda_1 - \lambda_1^*)(\lambda_1 - \lambda_2)(\lambda_1^* - \lambda_2), \\
A_{2\iota} &= Ll_{1\iota}^2(\lambda_1^* - \lambda_2)(\lambda_1 - \lambda_2^*), \\
A_{3\iota} &= Ll_{1\iota}^2(\lambda_1 - \lambda_2)(\lambda_1^* - \lambda_2^*), \\
T_{11} &= 2\mathrm{i}l_{21}l_{22}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2^*)(\lambda_2 - \lambda_2^*), \\
T_{12} &= 2\mathrm{i}l_{21}l_{22}(\lambda_1^* - \lambda_2)(\lambda_1^* - \lambda_2^*)(\lambda_2 - \lambda_2^*), \\
T_{2\iota} &= Ll_{2\iota}^2(\lambda_1^* - \lambda_2)(\lambda_2^* - \lambda_1), \\
T_{3\iota} &= Ll_{2\iota}^2(\lambda_1 - \lambda_2)(-\lambda_1^* + \lambda_2^*); \\
V_\iota &= -\frac{g(z)}{4\lambda_{\iota\mathrm{I}}} - 4[h_1(z)(\lambda_{\iota\mathrm{I}}^2 - 3\lambda_{\iota\mathrm{R}}^2) + 8p_1(z)\lambda_{\iota\mathrm{R}}(\lambda_{\iota\mathrm{I}}^2 - \lambda_{\iota\mathrm{R}}^2) \\
&\quad + 4r_1(z)(\lambda_{\iota\mathrm{I}}^4 - 10\lambda_{\iota\mathrm{I}}^2\lambda_{\iota R}^2 + 5\lambda_{\iota\mathrm{R}}^4) + s_1(z)\lambda_{\iota\mathrm{R}}].
\end{aligned}$$

Consequently, by referring to the aforementioned asymptotic expressions (18) and (19), the key physical attributes of

the two bright solitons prior to and post-collision can be succinctly summarized in the following Table 1. For the bright two solitons, similar to the analysis of the one soliton, we can derive that: the velocities of the two solitons depend on λ_ι, $r_1(z)$, $g_1(z)$, $h_1(z)$, $p_1(z)$, and $s_1(z)$; widths of the solitons are $1/(2\lambda_{\rm I})$.

Table 1. Physical quantities of solitons $q[2]$ before and after collisions.

Soliton	Amplitude	Velocity	Width	Initial phase
$q[2]^{1-}$	$A_{11}/(2\sqrt{A_{21}A_{32}})$	V_1	$1/(2\lambda_{11})$	$-\ln\sqrt{A_{32}/A_{21}}$
$q[2]^{2-}$	$T_{11}/(2\sqrt{T_{22}T_{31}})$	V_2	$1/(2\lambda_{21})$	$-\ln\sqrt{T_{22}/T_{31}}$
$q[2]^{1+}$	$A_{12}/(2\sqrt{A_{31}A_{22}})$	V_1	$1/(2\lambda_{11})$	$-\ln\sqrt{A_{22}/A_{31}}$
$q[2]^{2+}$	$T_{12}/(2\sqrt{T_{32}T_{21}})$	V_2	$1/(2\lambda_{21})$	$-\ln\sqrt{T_{32}/T_{21}}$

Iterating the above procedures, substituting solution (15) into DT (14), we obtain the bright N-soliton solutions of Eq. (1) as

$$q[N]=\sum_{j=1}^{N}\frac{1}{L}2\mathrm{i}\frac{(\lambda_j^*-\lambda_j)\phi_j[j-1]^*\psi_j[j-1]}{\left|\psi_j[j-1]\right|^2+\left|\phi_j[j-1]\right|^2}, \tag{20}$$

where

$$(\psi_j[j-1],\phi_j[j-1])^{\rm T}=D[j-1]D[j-2]\cdots D[1]|_{\lambda=\lambda_j}\Phi_j,$$

Φ_j's have been given in expression (15).

6. Breather solutions for Eq. (1) under constraints (4)

We investigate the breather solutions for Eq. (1) and select the seed solutions

$$q[0]=k_1\,\mathrm{e}^{\mathrm{i}[m_1(z)]}, \tag{21}$$

where

$$m_1(z)=\int\left\{-\frac{\mathrm{i}}{2}g(z)+2Hk_1^2L\left[3Hk_1^2Lp_1(z)-s_1(z)\right]\right\}\mathrm{d}z, \tag{22}$$

and k_1 is a real constant.

Via the transformation

$$\Phi=G\Psi,\quad G=\begin{pmatrix}1 & 0\\ 0 & \mathrm{e}^{-\mathrm{i}m_1(z)}\end{pmatrix}, \tag{23}$$

we derive

$$\Psi_t=[G^{-1}UG+(G^{-1})_tG]\Psi=U_0'\Psi,$$
$$\Psi_z=[G^{-1}VG+(G^{-1})_zG]\Psi=V_0'\Psi,$$

where

$$U_0'=\begin{pmatrix}-\mathrm{i}\lambda & k_1L\\ Hk_1 & \mathrm{i}\lambda\end{pmatrix},$$
$$V_0'=y_1(z)U_0'+y_2(z)I, \tag{24}$$
$$\begin{aligned}y_1(z)&=2[-(HLk_1^2+2\lambda^2)[h_1(z)+2\lambda p_1(z)]\\&\quad+3(H^2k_1^4L^2+4Hk_1^2L\lambda^2+8\lambda^4)r_1(z)+\lambda s_1(z)],\\ y_2(z)&=\frac{1}{4}g(z)+\mathrm{i}Hk_1^2L[3Hk_1^2Lp_1(z)-s_1(z)].\end{aligned} \tag{25}$$

Solutions for Lax pair (2) at $\lambda=\lambda_j$ are derived as

$$\Phi_j=GP_jT_jZ_j, \tag{26}$$

where

$$P_j=\begin{pmatrix}1 & 1\\ \frac{Hk_1}{-\mathrm{i}\lambda_j+\rho_{j1}} & \frac{Hk_1}{-\mathrm{i}\lambda_j+\rho_{j2}}\end{pmatrix},\quad \Delta_j=\begin{pmatrix}\delta_{j1}\\ \delta_{j2}\end{pmatrix},$$
$$T_j=\begin{pmatrix}\mathrm{e}^{\eta(\rho_{j1})} & 0\\ 0 & \mathrm{e}^{\eta(\rho_{j2})}\end{pmatrix},$$
$$\eta(\rho_{j\iota})=\rho_{j\iota}t+\int y_1(z)\rho_{j\iota}+y_2(z)\mathrm{d}z,$$
$$\rho_{j1}=-\sqrt{HLk_1^2-\lambda_j^2},\quad \rho_{j2}=\sqrt{HLk_1^2-\lambda_j^2}.$$

When $N=1$, setting $\delta_{11}=\delta_{12}=1$, substituting solutions (26) into N-fold DT (14), the first-order breather solutions for Eq. (1) are given as

$$q[1]=\mathrm{e}^{\mathrm{i}m_1(z)}k_1\left(1-\frac{2\mathrm{i}}{L}\frac{\Xi_1\Xi_2^*H(\lambda_1-\lambda_1^*)}{|\Xi_1|^2+H^2k_1^2|\Xi_2|^2}\right), \tag{27}$$

where

$$\begin{aligned}\Xi_1&=\exp\left[\rho_{11}t+\int\rho_{11}y_1(z)+y_2(z)\mathrm{d}z\right]\\&\quad+\exp\left[\rho_{12}t+\int\rho_{12}y_1(z)+y_2(z)\mathrm{d}z\right],\\ \Xi_2&=\frac{1}{\rho_{11}-\mathrm{i}\lambda_1}\exp\left[\rho_{11}t+\int\rho_{11}y_1(z)+y_2(z)\mathrm{d}z\right]\\&\quad+\frac{1}{\rho_{12}-\mathrm{i}\lambda_1}\exp\left[\rho_{12}t+\int\rho_{12}y_1(z)+y_2(z)\mathrm{d}z\right].\end{aligned}$$

When $N=2$, the second-order breather solutions for Eq. (1) via substituting solution (26) into N-fold DT (14) are derived as

$$\begin{aligned}q[2]&=k_1\,\mathrm{e}^{\mathrm{i}m_1(z)}+\frac{1}{L}2\mathrm{i}\frac{(\lambda_1^*-\lambda_1)\phi_1[0]^*\psi_1[0]}{|\psi_1[0]|^2+|\phi_1[0]|^2}\\&\quad+\frac{1}{L}2\mathrm{i}\frac{(\lambda_2^*-\lambda_2)\phi_2[1]^*\psi_2[1]}{|\psi_2[1]|^2+|\phi_2[1]|^2},\end{aligned} \tag{28}$$

where $(\psi_2[1],\phi_2[1])^{\rm T}=D[1]|_{\lambda=\lambda_2}\Phi_2$, Φ_2 has been given in expression (26).

Iterating the above procedures, substituting solution (26) into DT (14), we obtain the Nth-order breather solutions for

Eq. (1)

$$q[N] = k_1 \mathrm{e}^{\mathrm{i}m_1(z)} + \sum_{j=1}^{N} \frac{1}{L} 2\mathrm{i} \frac{(\lambda_j^* - \lambda_j)\phi_j[j-1]^* \psi_j[j-1]}{\left|\psi_j[j-1]\right|^2 + \left|\phi_j[j-1]\right|^2}, \quad (29)$$

where $(\psi_j[j-1], \phi_j[j-1])^{\mathrm{T}} = D[j-1]D[j-2]\cdots D[1]|_{\lambda=\lambda_j}\Phi_j$, Φ_j's have been given in expression (26).

7. Discussion on the bright solions and breathers for Eq. (1)

From solution (16), we obtain that the velocity of the soliton depends on λ_1, $r_1(z)$, $g_1(z)$, $h_1(z)$, $p_1(z)$, and $s_1(z)$; the amplitude of the soliton is $2\lambda_{1\mathrm{I}}/L$; the initial phase is $-\ln(l_{12}/l_{11})$, and the width of the soliton is $1/(2\lambda_{1\mathrm{I}})$. For convenience, we only show the figures when the variable coefficients $h_1(z)/s_1(z)/g(z)$ as quadratic/sine/cosin functions in Fig. 1. It is obvious that $h_1(z)$, $s_1(z)$, and $g(z)$ can affect the velocities of the bright solitons, which is consistent with the conclusions obtained from solution (16). $p_1(z)$ and $r_1(z)$ share the same conclusion, and we will not show them one by one.

Figure 2 shows the periodic interaction between the two bright solitons when the variable coefficients $s_1(z)$ and $r_1(z)$ are cosin functions. Figure 2(c) depicts the bound state solitons with the same velocity of the two bright solitons via Table 1.

Via solution (27), the first-order breathers are shown in Fig. 3. We can obtain that $s_1(z)$, $r_1(z)$, $h_1(z)$, and $p_1(z)$ affect the velocity of the breather, as shown in Fig. 3. As shown in Fig. 4, we study the interaction trend of the second-order breathers.

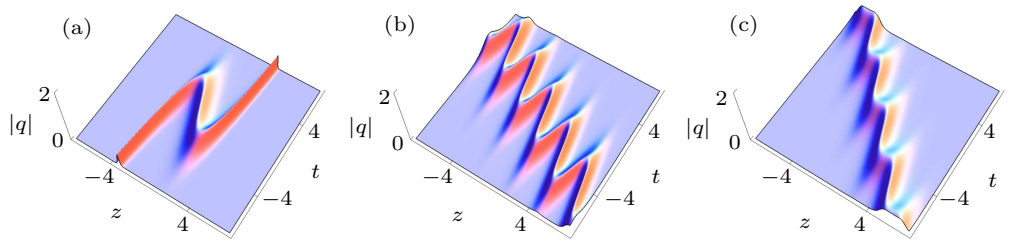


Fig. 1. One solitons via solution (16) with $H = -L = -2$, $l_{11} = -l_{12} = 1$, $\lambda_1 = 1 - \mathrm{i}/2$, $p_1(z) = r_1(z) = 1/8$: (a) $g(z) = s_1(z) = 1/8$, $h_1(z) = x^2/8$; (b) $g(z) = h_1(z) = 1/8$, $s_1(z) = \sin 2x$; and (c) $s_1(z) = h_1(z) = 1/8$, $g(z) = 3\cos 2x$.

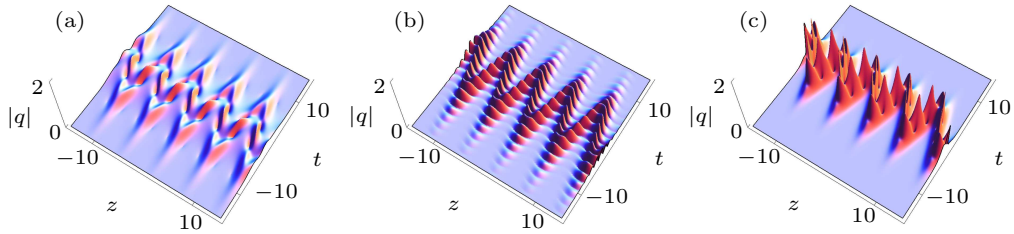


Fig. 2. Interactions between the two solitons via solution (17) with $H = -L = -2$, $l_{11} = l_{12} = l_{21} = l_{22} = 1$, $g(z) = h_1(z) = p_1(z) = 0$, $s_1(z) = r_1(z) = (1/15)\cos z$: (a) $\lambda_1 = 1 - \mathrm{i}/4$, $\lambda_2 = 1/2 - \mathrm{i}/4$; (b) $\lambda_1 = 1 - \mathrm{i}/4$, $\lambda_2 = -1 + \mathrm{i}/4$; and (c) $\lambda_1 = 1 + \mathrm{i}/2$, $\lambda_2 = -(1/2)5^{1/4}\sqrt{3}\mathrm{i}$.

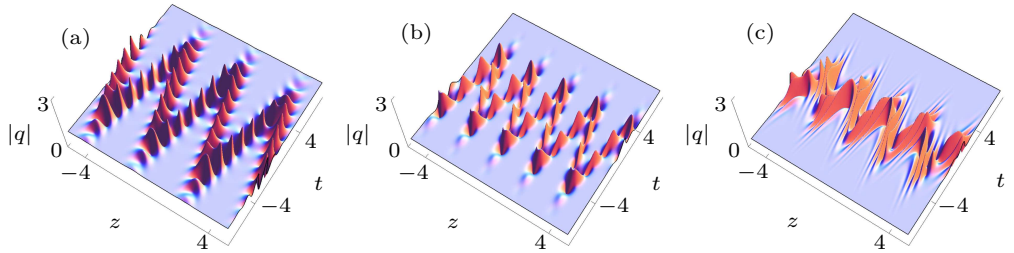


Fig. 3. The first-order breathers via solution (27) with $H = -L = -2$, $\delta_{11} = \delta_{12} = 1$, $\lambda_1 = -1 - \mathrm{i}$, $g(z) = 0$, $s_1(z) = (1/8)\cos 3z$: (a) $k_1 = 1$, $p_1(z) = h_1(z) = 0$, $r_1(z) = (1/30)\cos 2z$; (b) $k_1 = 1/2$, $p_1(z) = 0$, $h_1(z) = (1/20)\cos 3z$, $r_1(z) = (1/20)\cos 3z$; and (c) $k_1 = 1/2$, $h_1(z) = 0$, $p_1(z) = (1/3)\cos 5z/2$, $r_1(z) = (1/20)\cos 3z$.

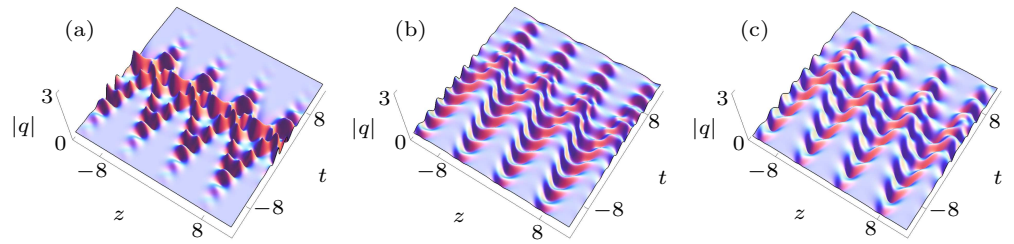


Fig. 4. The second-order breathers via solution (28) with $H = -L = -2$, $\delta_{11} = \delta_{12} = \delta_{21} = \delta_{22} = 1$, $k_1 = 1/2$, $p_1(z) = 0$, $g(z) = 0$, $s_1(z) = 1/8\cos z$, $r_1(z) = (1/15)\cos z$: (a) $h_1(z) = 0$, $\lambda_1 = 1 - \mathrm{i}/2$, $\lambda_2 = 1/4 + \mathrm{i}$; (b) $h_1(z) = 0$, $\lambda_1 = 1 - \mathrm{i}/4$, $\lambda_2 = -1 + \mathrm{i}/4$; and (c) $h_1(z) = (1/8)\cos 2z$, $\lambda_1 = 1 - \mathrm{i}/4$, $\lambda_2 = -1 + \mathrm{i}/4$.

8. Conclusions

Asymptotic analysis of the solitons has been beneficial for governing the propagation of dispersion in optical fibers. The main advantages of the asymptotic formalism are its simplicity and physical transparency. Asymptotic expression obtained through asymptotic analysis has been more conducive to studying the influences of the variable coefficients in equation on solitons propagation in an optical fiber or a planar waveguide. In this paper, we have studied a generalized higher-order NLS equation in an optical fiber or a planar waveguide, i.e., Eq. (1), and obtained the Lax pair (2) under constraint (4), based on which we derived the infinitely-many conservation laws (9). We have given the N-fold DT (14) and constructed the bright one-soliton solutions, bright two-soliton solutions, the first-order breather solutions, and the second-order breather solutions, i.e., solutions (16), (17), (27), and (28). The asymptotic analyses (18) and (19) for the bright two solitons have also been studied.

Figure 1 has depicted the propagation of the bright one soliton. Interactions between the two bright solitons have been depicted in Fig. 2. Figure 2(c) has shown the bound state solitons with the same velocity of two bright solitons via Table 1. Figure 3 has depicted the first-order breathers from solution (27). We have found that $p_1(z)$, $r_1(z)$, $s_1(z)$, and $h_1(z)$ affect the velocity of the breather, as shown in Fig. 3. Via solution (28), figure 4 has shown the motion trajectory of the two breathers.

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