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Freezing imaginarity of quantum states based on ℓ_1-norm

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We discuss freezing of quantum imaginarity based on ℓ_1-norm. Several properties about a quantity of imaginarity based on ℓ_1-norm are revealed. For a qubit (2-dimensional) system, we characterize the structure of real quantum operations that allow for freezing the quantity of imaginarity of any state. Furthermore, we characterize the structure of local real operations which can freeze the quantity of imaginarity of a class of N-qubit quantum states.

Keywords: imaginarity freezing, ℓ_1-norm, real operation

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1. Introduction

In quantum information theory, complex numbers play a crucial role.[1] Quantum states are typically represented by wave functions, which are complex-valued functions. The real and imaginary parts of the wave function respectively provide the amplitude and phase information of the state. This information is crucial in quantum computing and quantum communication because it determines the probability distribution of measurement outcomes and the evolution of the state.

Quantum coherence, as a vital physical resource for the superposition of quantum states,[1–5] is highly significant because it directly impacts the effectiveness and performance of applications such as quantum computing, quantum communication, and quantum sensing. Hence the quantification of quantum coherence has been extensively discussed.[6–9] To maximize coherence of available resources, there are many references about coherence freezing (preserving) by using ℓ_1-norm.[10–19] Furthermore, as special quantum coherence, quantum imaginarity is also considered as a resource,[20–22] and its quantification has garnered widespread attention.[23–26]

In mathematics, the ℓ_1-norm of a matrix is relatively simple and efficient because it involves only the sum of absolute values of entries. And the ℓ_1-norm of imaginary part of a matrix can be useful in applications such as numerical stability analysis, where it is important to measure the magnitude of imaginary components in matrix computations. It may also be relevant in signal processing or control theory to analyze the stability and performance of systems with complex-valued matrices. In Ref. [23], Chen *et al.* introduce the quantification about imaginarity by ℓ_1-norm. We will discuss freezing imaginarity of quantum states based on ℓ_1-norm.

The main structure of our paper is as follows. In Section 2, we present some properties of the quantity of imaginarity by ℓ_1-norm. In Section 3, we characterize real quantum operations that can freeze imaginarity of qubit states by ℓ_1-norm. Furthermore, we provide structure of local real quantum operations that can freeze imaginarity of a class of N-qubit states by ℓ_1-norm. The concise conclusion is presented in Section 4. The proofs of main theorems are listed in Appendix A.

2. Quantification of quantum imaginarity based on ℓ_1-norm

Let H be a d-dimensional Hilbert space. We use $\mathcal{B}(H)$ and $\mathcal{D}(H)$ to denote sets of all bounded linear operators and all trace one positive semidefinite operators (quantum states) on H, respectively. A pure state is a unit vector in H. For $\rho \in \mathcal{D}(H)$ and a fixed orthonormal basis $\{|m\rangle\}_{m=1}^{d}$ for H, we say that ρ is incoherent if $\langle m|\rho|n\rangle = 0$ for any $m \neq n$. An important coherence quantification is based on the ℓ_1-norm[3]

$$C_{\ell_1}(\rho) = \sum_{m\neq n} |\rho_{mn}|,$$

for any quantum state ρ with $\rho_{mn} = \langle m|\rho|n\rangle$. We say that ρ is real if $\langle m|\rho|n\rangle \in \mathbb{R}$ for any m,n. Otherwise, we say that ρ is nonreal. We use $\mathcal{R}(H)$ to denote the set of all real quantum states on H. A quantum operation Φ is a trace-preserving completely positive mapping on $\mathcal{B}(H)$, which has a Kraus operator representation

$$\Phi(X) = \sum_{\alpha=1}^{r} K_\alpha X K_\alpha^\dagger,$$

where $\sum_{\alpha=1}^{r} K_\alpha^\dagger K_\alpha = I_{\rm d}$ and $I_{\rm d}$ means the identity operator on H. $X^\dagger$ and $X^{\rm T}$ denote the conjugate transpose and transpose of the matrix X, respectively. A quantum operation is said to be real if it maps any real state to a real state. Furthermore, in Ref. [25], real operations are just ones that have a family of Kraus operators K_α $(\alpha = 1,2,\ldots,r)$ with $\langle m|K_\alpha|n\rangle \in \mathbb{R}$ for any m,n,α.

For any $\rho = [\rho_{mn}] \in \mathcal{D}(H)$ with $\rho_{mn} = \langle m|\rho|n\rangle$, it is natural to discuss the quantification of quantum imaginarity based

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on the ℓ_1-norm, which is defined in Ref. [23] by

$$\mathcal{M}_{\ell_1}(\rho)=\sum_{m\neq n}|\mathrm{Im}(\rho_{mn})|,$$

where $\mathrm{Im}(z)$ denotes the imaginary part of a complex number z. Clearly, $\mathcal{M}_{\ell_1}(\rho)\geq 0$ and further $\mathcal{M}_{\ell_1}(\rho)=0$ if and only if $\rho\in\mathcal{R}(H)$.[23] We call the quantity $\mathcal{M}_{\ell_1}(\rho)$ ℓ_1-norm imaginarity of the quantum state ρ. In the following, we will discuss some properties of $\mathcal{M}_{\ell_1}$.

A measure of quantum imaginarity based on trace norm is as follows:[25]

$$\mathcal{M}_1(\rho)=\frac{1}{2}\|\rho-\rho^{\mathrm{T}}\|_1,$$

which satisfies (i) $\mathcal{M}_1(\rho)\geq 0$; $\mathcal{M}_1(\rho)=0$ if and only if ρ is real, (ii) $\mathcal{M}_1(\Phi(\rho))\leq\mathcal{M}_1(\rho)$ for any real operation Φ and any state ρ.

For any qubit (2-dimensional) state

$$\rho=\begin{pmatrix} a & b+\mathrm{i}c \\ b-\mathrm{i}c & 1-a\end{pmatrix},$$

it is easy to compute that $\mathcal{M}_{\ell_1}(\rho)=\mathcal{M}_1(\rho)=2|c|\leq d-1=1$ and further the maximum is attained if and only if $a=1/2$, $b=0,|c|=1/2$. In the higher dimensional case, whether the maximum $\mathcal{M}_1(\rho)$ is $d-1$? The following theorem gives a negative answer, whose proof is listed in Appendix A.

Theorem 1 (i) $\mathcal{M}_{\ell_1}$ is convex on $\mathcal{D}(H)$.

(ii) For $d=3$ and $\rho\in\mathcal{D}(H)$, it holds that

$$0\leq\mathcal{M}_{\ell_1}(\rho)\leq\sqrt{3}=\mathcal{M}_{\ell_1}(|\psi(\beta)\rangle\langle\psi(\beta)|),$$

where

$$|\psi(\beta)\rangle=\frac{1}{\sqrt{3}}\mathrm{e}^{\mathrm{i}\beta}\left(\mathrm{e}^{\mathrm{i}\pi/3}|1\rangle+\mathrm{e}^{\mathrm{i}2\pi/3}|2\rangle+\mathrm{e}^{\mathrm{i}\pi}|3\rangle\right),\ \beta\in\mathbb{R}.$$

Let H_s $(s=1,2,\ldots,N)$ be finite dimensional Hilbert spaces and $H_1\otimes H_2\otimes\ldots\otimes H_N$ be the composite Hilbert space of $H_s(s=1,2,\ldots,N)$. In the following, we discuss several properties about $\mathcal{M}_{\ell_1}$ on the multipartite quantum system.

Theorem 2 (i) For any state $\tilde{\rho}\in\mathcal{D}(H_1\otimes H_2\otimes\cdots\otimes H_N)$, it holds that

$$\begin{aligned}&\mathcal{M}_{\ell_1}(\mathrm{tr}_{H_1}(\tilde{\rho}))+\mathcal{M}_{\ell_1}(\mathrm{tr}_{H_2}(\tilde{\rho}))+\cdots\\&+\mathcal{M}_{\ell_1}(\mathrm{tr}_{H_N}(\tilde{\rho}))\leq\mathcal{M}_{\ell_1}(\tilde{\rho}).\end{aligned}$$

(ii) For any state $\rho_1\in\mathcal{D}(H_1)$ and $\rho_s\in\mathcal{R}(H_s)(s=2,3,\ldots,N)$, it holds that

$$\mathcal{M}_{\ell_1}(\rho_1\otimes\rho_2\otimes\rho_3\otimes\cdots\otimes\rho_N)=\mathcal{M}_{\ell_1}(\rho_1)\prod_{s=2}^{N}(1+C_{\ell_1}(\rho_s)).$$

The proof of Theorem 2 will be given in Appendix A.

3. Freezing of quantum imaginarity based on $\mathcal{M}_{\ell_1}$

In general, real quantum operations alter imaginarity of quantum states. Hence, we investigate which real quantum operations preserve imaginarity of quantum states, known as the freezing phenomenon in quantum mechanics. Based on the quantity $\mathcal{M}_{\ell_1}$ of imaginarity, we provide the following definition.

Definition 1 For a nonreal state ρ and a real operation Φ, if $\mathcal{M}_{\ell_1}(\Phi(\rho))=\mathcal{M}_{\ell_1}(\rho)$, then we say that the ℓ_1-norm imaginarity of ρ is frozen by Φ.

In the following, we consider a real quantum operation $\Phi(X)=\sum_{\alpha=1}^{r}K_\alpha XK_\alpha^\dagger$. Under the fixed basis $\{|s\rangle\}_{s=1}^{d}$, denote $K_\alpha=\sum_{s,j=1}^{d}q_{sj}^{(\alpha)}|s\rangle\langle j|$, where $q_{sj}^{(\alpha)}\in\mathbb{R}$.

Theorem 3 When $d=2$, for a real quantum operation Φ, the following statements are equivalent.

(i) For any state ρ, $\mathcal{M}_{\ell_1}(\Phi(\rho))=\mathcal{M}_{\ell_1}(\rho)$.

(ii) For a nonreal state ρ, $\mathcal{M}_{\ell_1}(\Phi(\rho))=\mathcal{M}_{\ell_1}(\rho)$.

(iii) The Kraus operators of Φ satisfy

$$\left|\sum_\alpha\left(q_{11}^{(\alpha)}q_{22}^{(\alpha)}-q_{12}^{(\alpha)}q_{21}^{(\alpha)}\right)\right|=1.$$

(iv) The Kraus operators of Φ are of the form

$$\begin{aligned}&K_\alpha=q_{11}^{(\alpha)}I_2+\mathrm{i}q_{12}^{(\alpha)}\sigma_2(\forall\alpha)\ \text{ or}\\&K_\alpha=q_{11}^{(\alpha)}\sigma_3+q_{12}^{(\alpha)}\sigma_1(\forall\alpha),\end{aligned}\tag{1}$$

where $\sum_\alpha\left(|q_{11}^{(\alpha)}|^2+|q_{12}^{(\alpha)}|^2\right)=1$ and $\sigma_1,\sigma_2,\sigma_3$ are Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\sigma_2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},$$
$$\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

We present the proof of Theorem 3 in Appendix A.

Remark 1 From Theorem 3 (4), we can get the Kraus operators of Φ satisfy

$$\left|\sum_\alpha\left(q_{11}^{(\alpha)}q_{21}^{(\alpha)}+q_{12}^{(\alpha)}q_{22}^{(\alpha)}\right)\right|=0.\tag{2}$$

But this equation does not imply that

$$\left|\sum_\alpha\left(q_{11}^{(\alpha)}q_{22}^{(\alpha)}-q_{12}^{(\alpha)}q_{21}^{(\alpha)}\right)\right|=1.$$

For example, let

$$K_1=\begin{pmatrix}-1/2&1/2\\1/2&-1/2\end{pmatrix},\quad K_2=\begin{pmatrix}1/2&1/2\\1/2&1/2\end{pmatrix}.$$

Then

$$\left|\sum_\alpha\left(q_{11}^{(\alpha)}q_{21}^{(\alpha)}+q_{12}^{(\alpha)}q_{22}^{(\alpha)}\right)\right|=0,$$

but

$$\left|\sum_\alpha\left(-q_{11}^{(\alpha)}q_{22}^{(\alpha)}+q_{12}^{(\alpha)}q_{21}^{(\alpha)}\right)\right|=0<1.$$

This shows that equation (2) is a necessary but not sufficient condition for operations to freeze the ℓ_1-norm imaginarity in the qubit system.

By Theorem 3, we obtain the structure of real unitary operations that freeze ℓ_1-norm imaginarity of any state in a qubit system.

Corollary 1 When $d=2$, if Φ is a real unitary operation with $\Phi(X)=UXU^\dagger$ for any X, then Φ can freeze ℓ_1-norm imaginarity of any qubit state if and only if the unitary operator U is of the form $\cos\theta I_2+\mathrm{i}\sin\theta\sigma_2$ or $\cos\theta\sigma_3+\sin\theta\sigma_1(\theta\in[0,2\pi))$.

In the following, we will discuss the structure of local real quantum operations under the product basis $\{|1\rangle,|2\rangle\}^{\otimes N}$ of

$$\Phi=\Phi_1\otimes\Phi_2\otimes\cdots\otimes\Phi_N,$$

which can freeze the ℓ_1-norm imaginarity of a class of N-qubit states

$$\rho^N=\frac{1}{2^N}\left(I_2^{\otimes N}+\sum_{w=1}^{3}c_w\sigma_w\otimes I_2^{\otimes N-1}+\sum_{w=1}^{3}d_wI_2\otimes\sigma_w^{\otimes N-1}\right),$$

where $c_w,d_w\in[-1,1](\forall w)$.

For each $s=1,2,\ldots,N$, we denote the real quantum operation $\Phi_s(X)=\sum_{\alpha_s}K_{\alpha_s}XK_{\alpha_s}^\dagger$ with $\sum_{\alpha_s}K_{\alpha_s}^\dagger K_{\alpha_s}=I_2$ and

$$K_{\alpha_s}=\sum_{t_s,l_s=1}^{2}q_{t_sl_s}^{(\alpha_s)}|t_s\rangle\langle l_s|,q_{t_sl_s}^{(\alpha_s)}\in\mathbb{R}.$$

We will discuss cases of N being odd and even separately in the following. When N is odd, $\rho^N\in\mathcal{R}(H_1\otimes H_2\otimes\cdots\otimes H_N)$ if and only if $c_2=0$. When N is even, $\rho^N\in\mathcal{R}(H_1\otimes H_2\otimes\cdots\otimes H_N)$ if and only if $c_2=0$ and $d_2=0$. Next we only consider $\rho^N\notin\mathcal{R}(H_1\otimes H_2\otimes\cdots\otimes H_N)$.

Theorem 4 When N is odd and $c_2\neq0$, Φ freezes the ℓ_1-norm imaginarity of ρ^N if and only if

$$m_1\prod_{s=2}^{N}(1+b_s)=1,$$

where

$$m_s=\left|\sum_{\alpha_s}\left(-q_{11}^{(\alpha_s)}q_{22}^{(\alpha_s)}+q_{12}^{(\alpha_s)}q_{21}^{(\alpha_s)}\right)\right|\leq1\ (\forall s=1,2,\ldots,N),$$

$$b_s=\left|\sum_{\alpha_s}\left(q_{11}^{(\alpha_s)}q_{21}^{(\alpha_s)}+q_{12}^{(\alpha_s)}q_{22}^{(\alpha_s)}\right)\right|\leq1\ (\forall s=1,2,\ldots,N).$$

The proof of Theorem 4 is listed in Appendix A.

Remark 2 The quantities of m_s and b_s represent $\frac{1}{2}\mathcal{M}_{\ell_1}(\Phi_s(\sigma_2))$ and $\frac{1}{2}C_{\ell_1}(\Phi_s(I_2))$, respectively.

Corollary 2 When N is odd, if Φ_1 freezes the ℓ_1-norm imaginarity of any qubit state and $\Phi_2,\Phi_3,\ldots,\Phi_N$ satisfy

$$\left|\sum_{\alpha_s}\left(q_{11}^{(\alpha_s)}q_{21}^{(\alpha_s)}+q_{12}^{(\alpha_s)}q_{22}^{(\alpha_s)}\right)\right|=0\quad(\forall s=2,3,\ldots,N),$$

then Φ freezes the ℓ_1-norm imaginarity of ρ^N.

By Remark 1 and Corollary 2, we see that it is possible that $\Phi=\Phi_1\otimes\Phi_2\otimes\cdots\otimes\Phi_N$ could freeze the ℓ_1-norm imaginarity of ρ^N if Φ_1 can freeze the ℓ_1-norm imaginarity of any qubit state while the others fail to freeze. In addition, even though Φ_1 cannot freeze the ℓ_1-norm imaginarity of any qubit state, we also find some real operation Φ_2 that cannot freeze in the qubit system such that $\Phi=\Phi_1\otimes\Phi_2\otimes\cdots\otimes\Phi_N$ could freeze the ℓ_1-norm imaginarity of ρ^N. See the following example.

Example 1 Let the Kraus operators K_1,K_2 and K_1',K_2' of Φ_1 and Φ_2 be

$$K_1=\begin{pmatrix}-1&0\\0&2/3\end{pmatrix},\quad K_2=\begin{pmatrix}0&-\sqrt{5}/3\\0&0\end{pmatrix},$$

$$K_1'=\begin{pmatrix}1/2&0\\1/2&-1/\sqrt{2}\end{pmatrix},\quad K_2'=\begin{pmatrix}1/2&1/\sqrt{2}\\1/2&0\end{pmatrix},$$

respectively, Φ_s $(s=3,4,\ldots,N)$ be of the form (1). We get

$$\left|\sum_{\alpha_1=1}^{2}\left(-q_{11}^{(\alpha_1)}q_{22}^{(\alpha_1)}+q_{12}^{(\alpha_1)}q_{21}^{(\alpha_1)}\right)\right|=\frac{2}{3}<1,$$

$$\left|\sum_{\alpha_2=1}^{2}\left(q_{11}^{(\alpha_2)}q_{21}^{(\alpha_2)}+q_{12}^{(\alpha_2)}q_{22}^{(\alpha_2)}\right)\right|=\frac{1}{2}<1.$$

But

$$\begin{aligned}&\left|\sum_{\alpha_1}\left(-q_{11}^{(\alpha_1)}q_{22}^{(\alpha_1)}+q_{12}^{(\alpha_1)}q_{21}^{(\alpha_1)}\right)\right|\\&\times\prod_{s=2}^{N}\left(1+\left|\sum_{\alpha_s}\left(q_{11}^{(\alpha_s)}q_{21}^{(\alpha_s)}+q_{12}^{(\alpha_s)}q_{22}^{(\alpha_s)}\right)\right|\right)\\&=\frac{2}{3}\times\left(1+\frac{1}{2}\right)=1.\end{aligned}$$

By Theorem 4, we know that $\Phi=\Phi_1\otimes\Phi_2\otimes\cdots\otimes\Phi_N$ can freeze the ℓ_1-norm imaginarity of ρ^N.

Corollary 3 When N is odd, a local real unitary operation $\mathcal{U}$ with the Kraus operator $U_1\otimes U_2\otimes\cdots\otimes U_N$ can freeze the ℓ_1-norm imaginarity of ρ^N if and only if the real unitary operation $\mathcal{U}_1$ with the Kraus operator U_1 can freeze the ℓ_1-norm imaginarity of any state in a qubit system.

Proof For any real unitary operator

$$U_s=\begin{pmatrix}u_{11}^{(s)}&u_{12}^{(s)}\\u_{21}^{(s)}&u_{22}^{(s)}\end{pmatrix},$$

we get

$$\left|u_{11}^{(s)}u_{21}^{(s)}+u_{12}^{(s)}u_{22}^{(s)}\right|=0.$$

Thus, $\mathcal{U}$ can freeze the ℓ_1-norm imaginarity of ρ^N if and only if

$$\left|\left(-u_{11}^{(1)}u_{22}^{(1)}+u_{12}^{(1)}u_{21}^{(1)}\right)\right|$$

$$\times\prod_{s=2}^{N}\left(1+\left|\left(u_{11}^{(s)}u_{21}^{(s)}+u_{12}^{(s)}u_{22}^{(s)}\right)\right|\right)=1$$

if and only if

$$\left|-u_{11}^{(1)}u_{22}^{(1)}+u_{12}^{(1)}u_{21}^{(1)}\right|=1$$

if and only if $\mathcal{U}_1$ freezes the ℓ_1-norm imaginarity of any state in the first system.

Theorem 5 When N is even and one of c_2 and d_2 of ρ^N is not 0, Φ freezes the ℓ_1-norm imaginarity of ρ^N if and only if one of the following cases holds:

Case 1 When

$$\max\left\{|c_2|m_1\prod_{s=2}^{N}b_s,|d_2|b_1\prod_{s=2}^{N}m_s\right\}=|c_2|m_1\prod_{s=2}^{N}b_s,$$
$$|c_2|\left(m_1\prod_{s=2}^{N}(1+b_s)-1\right)+|d_2|\left(\prod_{s=2}^{N}m_s-1\right)=0;$$

Case 2 When

$$\max\left\{|c_2|m_1\prod_{s=2}^{N}b_s,|d_2|b_1\prod_{s=2}^{N}m_s\right\}=|d_2|b_1\prod_{s=2}^{N}m_s,$$
$$|c_2|\left(m_1\left(\prod_{s=2}^{N}(1+b_s)-\prod_{s=2}^{N}b_s\right)-1\right)$$
$$+|d_2|\left((1+b_1)\prod_{s=2}^{N}m_s-1\right)=0,$$

where

$$m_s=\left|\sum_{\alpha_s}\left(-q_{11}^{(\alpha_s)}q_{22}^{(\alpha_s)}+q_{12}^{(\alpha_s)}q_{21}^{(\alpha_s)}\right)\right|\le 1\quad(\forall s=1,2,\ldots,N),$$
$$b_s=\left|\sum_{\alpha_s}\left(q_{11}^{(\alpha_s)}q_{21}^{(\alpha_s)}+q_{12}^{(\alpha_s)}q_{22}^{(\alpha_s)}\right)\right|\le 1\quad(\forall s=1,2,\ldots,N).$$

We give the proof of Theorem 5 in Appendix A. By Theorem 5, it is easy to obtain the following corollaries.

Corollary 4 When N is even, if Φ_s $(s=1,2,\ldots,N)$ can freeze the ℓ_1-norm imaginarity of any qubit state, then $\Phi_1\otimes\Phi_2\otimes\cdots\otimes\Phi_N$ can freeze the ℓ_1-norm imaginarity of the family of states of the form ρ^N.

Corollary 5 When N is even, if Φ_1 cannot freeze the ℓ_1-norm imaginarity of any qubit state and $b_1=0$, even though Φ_s can freeze the ℓ_1-norm imaginarity of any qubit state for all $s=2,3,\ldots,N$, $\Phi_1\otimes\Phi_2\otimes\cdots\otimes\Phi_N$ cannot freeze the ℓ_1-norm imaginarity of ρ^N with $c_2\neq 0$.

Corollary 6 When N is even, $\Phi_1\otimes\Phi_2\otimes\cdots\otimes\Phi_N$ can freeze the ℓ_1-norm imaginarity of the family of states of the form ρ^N with $c_2\neq 0, d_2\neq 0$ if and only if either Φ_s can freeze the ℓ_1-norm imaginarity of any qubit state for all $s=1,2,\ldots,N$ or

$$\begin{cases}m_1\left(\prod_{s=2}^{N}(1+b_s)-\prod_{s=2}^{N}b_s\right)=1,\\(1+b_1)\prod_{s=2}^{N}m_s=1.\end{cases}\tag{3}$$

The following example shows that when Φ_1 cannot freeze the ℓ_1-norm imaginarity of any qubit state, we can find Φ_2,Φ_3,Φ_4 such that $\Phi_1\otimes\Phi_2\otimes\Phi_3\otimes\Phi_4$ can freeze the ℓ_1-norm imaginarity of the family of states of the form ρ^4 with $c_2\neq 0, d_2\neq 0$.

Example 2 Let $N=4$ and Φ_1 be not of the form (1). Assume all Φ_s $(s=1,2,3,4)$ consist of two Kraus operators and Φ_4 is of the form (1). Then by Corollary 6, we obtain that $\Phi_1\otimes\Phi_2\otimes\Phi_3\otimes\Phi_4$ can freeze the ℓ_1-norm imaginarity of the family of states of the form ρ^4 with $c_2\neq 0$, $d_2\neq 0$ if and only if

$$\begin{cases}\sum_{\alpha_s=1}^{2}\left(|q_{11}^{(\alpha_s)}|^2+|q_{21}^{(\alpha_s)}|^2\right)=1\ \ (s=1,2,3),\\\sum_{\alpha_s=1}^{2}\left(|q_{12}^{(\alpha_s)}|^2+|q_{22}^{(\alpha_s)}|^2\right)=1\ \ (s=1,2,3),\\\sum_{\alpha_s=1}^{2}\left(q_{11}^{(\alpha_s)}q_{12}^{(\alpha_s)}+q_{21}^{(\alpha_s)}q_{22}^{(\alpha_s)}\right)=0\ \ (s=1,2,3),\\m_1(1+b_2)(1+b_3)=1,\\(1+b_1)m_2m_3=1,\\m_s\le 1\ \ (s=1,2,3),\\b_s\le 1\ \ (s=1,2,3).\end{cases}\tag{4}$$

By the quasi-Newton algorithm, we try to find the minimization problem

$$\min\left(\sum_{s=1}^{3}\left(\sum_{h=1}^{2}g_{s,h}^2+t_s^2\right)\right.$$
$$\left.+(m_1(1+b_2)(1+b_3)-1)^2+((1+b_1)m_2m_3-1)^2\right),\tag{5}$$

where

$$g_{s,h}=\sum_{\alpha_s=1}^{2}\left(|q_{1h}^{(\alpha_s)}|^2+|q_{2h}^{(\alpha_s)}|^2\right)-1,$$
$$t_s=\sum_{\alpha_s=1}^{2}\left(q_{11}^{(\alpha_s)}q_{12}^{(\alpha_s)}+q_{21}^{(\alpha_s)}q_{22}^{(\alpha_s)}\right).$$

The minimal value zero provides a solution of Eq. (4). Several solutions of Eq. (4) are found in our numerical experiments, which are not in the set consisting of the form of Eq. (1). Table 1 lists two such solutions, which make the cost function in Eq. (5) less than 10^{-8}, where $q_{ij}^{(ab)}$ stands for the (i,j)-entry of the b-th Kraus operator of the a-th system, and $a\in\{1,2,3\}$, $b\in\{1,2\}$. It implies from Eq. (3) that there exists a local operation $\Phi_1\otimes\Phi_2\otimes\Phi_3\otimes\Phi_4$ such that the ℓ_1-norm imaginarity of states of the form ρ^4 with $c_2\neq 0, d_2\neq 0$ is freezing by $\Phi_1\otimes\Phi_2\otimes\Phi_3\otimes\Phi_4$, where Φ_4 can freeze the ℓ_1-norm imaginarity of any qubit state, but the others fail to freeze.

Corollary 7 When N is even, a local real unitary operation $\mathcal{U}$ with the Kraus operator $U_1\otimes U_2\otimes\cdots\otimes U_N$ can freeze the ℓ_1-norm imaginarity of ρ^N if and only if each real unitary operation $\mathcal{U}_i$ with the Kraus operator U_i can freeze the ℓ_1-norm imaginarity of any state in a qubit system for $i=1,2,\ldots,N$.

Table 1. Two solutions of Eq. (4).

Numerical solution 1					
$q_{11}^{(11)}$	–0.49567103	$q_{11}^{(21)}$	–0.36161911	$q_{11}^{(31)}$	0.38267230
$q_{12}^{(11)}$	–0.62720229	$q_{12}^{(21)}$	0.71041776	$q_{12}^{(31)}$	–0.50072406
$q_{21}^{(11)}$	0.15437083	$q_{21}^{(21)}$	0.34195640	$q_{21}^{(31)}$	–0.52449099
$q_{22}^{(11)}$	–0.60042560	$q_{22}^{(21)}$	0.68574209	$q_{22}^{(31)}$	–0.40684148
$q_{11}^{(12)}$	0.62745086	$q_{11}^{(22)}$	0.86602540	$q_{11}^{(32)}$	–0.52299692
$q_{12}^{(12)}$	–0.47624258	$q_{12}^{(22)}$	0.01716236	$q_{12}^{(32)}$	–0.53466198
$q_{21}^{(12)}$	0.58033206	$q_{21}^{(22)}$	–0.04793168	$q_{21}^{(32)}$	–0.55221856
$q_{22}^{(12)}$	0.13892221	$q_{22}^{(22)}$	–0.15738440	$q_{22}^{(32)}$	0.54579482
Numerical solution 2					
$q_{11}^{(11)}$	–0.48307031	$q_{11}^{(21)}$	0.36450724	$q_{11}^{(31)}$	–0.57834551
$q_{12}^{(11)}$	–0.41668293	$q_{12}^{(21)}$	0.46830052	$q_{12}^{(31)}$	–0.57790649
$q_{21}^{(11)}$	0.51390385	$q_{21}^{(21)}$	0.39547056	$q_{21}^{(31)}$	–0.57205330
$q_{22}^{(11)}$	–0.56739951	$q_{22}^{(21)}$	–0.50352378	$q_{22}^{(31)}$	0.57841759
$q_{11}^{(12)}$	–0.52004480	$q_{11}^{(22)}$	0.70710679	$q_{11}^{(32)}$	0.58152239
$q_{12}^{(12)}$	–0.56829010	$q_{12}^{(22)}$	0.42321267	$q_{12}^{(32)}$	0.00430683
$q_{21}^{(12)}$	0.48176687	$q_{21}^{(22)}$	0.45906154	$q_{21}^{(32)}$	0.01015873
$q_{22}^{(12)}$	–0.42600411	$q_{22}^{(22)}$	–0.58995718	$q_{22}^{(32)}$	–0.57570707

4. Conclusion

In this paper, we first discuss several properties of $\mathcal{M}_{\ell_1}$, which is a quantification of imaginarity based on the ℓ_1-norm. Then we give the characterization of real quantum operations that freeze the ℓ_1-norm imaginarity of any qubit (2-dimensional) state. Moreover, we derive the structure of local real quantum operations that can freeze the imaginarity of a special class of N-qubit states.

Appendix A: Proofs of Theorems 1–5

Proof of Theorem 1 (i) For any states ρ and σ, we have

$$\begin{aligned}\mathcal{M}_{\ell_1}(\lambda\rho+(1-\lambda)\sigma) &= \sum_{m\neq n}|\mathrm{Im}(\lambda\rho_{mn}+(1-\lambda)\sigma_{mn})|\\ &= \sum_{m\neq n}|\lambda\mathrm{Im}\rho_{mn}+(1-\lambda)\mathrm{Im}\sigma_{mn}|\\ &\leq \lambda\sum_{m\neq n}|\mathrm{Im}\rho_{mn}|+(1-\lambda)\sum_{m\neq n}|\mathrm{Im}\sigma_{mn}|\\ &= \lambda\mathcal{M}_{\ell_1}(\rho)+(1-\lambda)\mathcal{M}_{\ell_1}(\sigma).\end{aligned}$$

Thus $\mathcal{M}_{\ell_1}$ is convex on $\mathcal{D}(H)$.

(ii) For the fixed orthonormal basis $\{|m\rangle\}_{m=1}^3$ for H, any pure state $|\psi\rangle$ can be represented by

$$|\psi\rangle=\sum_{m=1}^3 r_m e^{i\theta_m}|m\rangle,$$

where $r_m\geq 0,\theta_m\in[0,2\pi)$ for any m and $\sum_m r_m^2=1$. We compute that

$$\begin{aligned}\mathcal{M}_{\ell_1}(|\psi\rangle\langle\psi|) &= \sum_{m\neq n}r_m r_n|\sin(\theta_n-\theta_m)|\\ &= 2\sum_{1\leq m<n\leq 3}r_m r_n|\sin(\theta_n-\theta_m)|.\end{aligned}$$

Let the function

$$G(r_1,r_2,r_3,\theta_1,\theta_2,\theta_3)=\sum_{1\leq m<n\leq 3}r_m r_n|\sin(\theta_n-\theta_m)|.$$

It suffices to prove that the constrained maximum M of $G(r_1,r_2,r_3,\theta_1,\theta_2,\theta_3)$ over the set

$$D=\{(r_1,r_2,r_3,\theta_1,\theta_2,\theta_3):0\leq\theta_k\leq\pi,0\leq r_k\leq 1(1\leq k\leq 3)\}$$

under the condition that $\sum_{k=1}^3 r_k^2=1$ is equal to $\sqrt{3}/2$ and that $M=G(\bar r_1,\bar r_2,\bar r_3,\bar\theta_1,\bar\theta_2,\bar\theta_3)$ for some $\bar r_k>0$ and $0\leq\bar\theta_k\leq\pi$ for all $k=1,2,3$.

Since G is continuous on the bounded closed region D, it takes its conditional maximum M at some point $P_0=(\bar r_1,\bar r_2,\bar r_3,\bar\theta_1,\bar\theta_2,\bar\theta_3)$. By observing that

$$|\sin(\bar\theta_n-\bar\theta_m)|=|\sin(\bar\theta_n-\bar\theta_m-\pi)|,$$

we may assume that $0<\bar\theta_{m+1}-\bar\theta_m<\pi$ for all m. Thus, the constrained maximum of $G(r_1,r_2,r_3,\theta_1,\theta_2,\theta_3)$ over the set

$$\begin{aligned}D_0 = \{&(r_1,r_2,r_3,\theta_1,\theta_2,\theta_3):0\leq\theta_k\leq\pi,\\ &0\leq\theta_{k+1}-\theta_k\leq\pi,0\leq r_k\leq 1(1\leq k\leq 3)\}\end{aligned}$$

subjected to the equality constraint condition $\sum_{k=1}^3 r_k^2=1$ must be equal to M. That is, $M=G(\bar r_1,\bar r_2,\bar r_3,\bar\theta_1,\bar\theta_2,\bar\theta_3)$ for some point $P_0=(\bar r_1,\bar r_2,\bar r_3,\bar\theta_1,\bar\theta_2,\bar\theta_3)$ in D_0. To find out the constrained maximum point P_0, let us consider the Lagrangian function

$$\begin{aligned}&L(r_1,r_2,r_3,\theta_1,\theta_2,\theta_3,\lambda)\\ &=\sum_{1\leq m<n\leq 3}r_m r_n\sin(\theta_n-\theta_m)-\lambda\left(\sum_{m=1}^3 r_m^2-1\right).\end{aligned}\tag{A1}$$

The method of Lagrange multipliers ensures that $Q_0=(P_0,\lambda)$ is a stationary point of L for some constant λ, that is,

$$\begin{cases} \dfrac{\partial L}{\partial r_k}|_{Q_0}=0 \quad (k=1,2,3), \\ \dfrac{\partial L}{\partial \theta_k}|_{Q_0}=0 \quad (k=1,2,3), \\ \dfrac{\partial L}{\partial \lambda}|_{Q_0}=0. \end{cases} \tag{A2}$$

Thus, the first equation and the second one in Eq. (A2) are equivalent to

$$\begin{bmatrix} 0 & \sin(\theta_2-\theta_1) & \sin(\theta_3-\theta_1) \\ \sin(\theta_2-\theta_1) & 0 & \sin(\theta_3-\theta_2) \\ \sin(\theta_3-\theta_1) & \sin(\theta_3-\theta_2) & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = 2\lambda \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \tag{A3}$$

$$\begin{bmatrix} 0 & -\cos(\theta_2-\theta_1) & -\cos(\theta_3-\theta_1) \\ \cos(\theta_2-\theta_1) & 0 & -\cos(\theta_3-\theta_2) \\ \cos(\theta_3-\theta_1) & \cos(\theta_3-\theta_2) & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = 0, \tag{A4}$$

respectively.

We use S_3 and C_3 to denote 3×3 matrices in Eqs. (A3) and (A4), respectively, and use $\boldsymbol{x}_3$ to denote the column vector $(r_1,r_2,r_3)^{\mathrm{T}}$. Then these two equations can be written as

$$\begin{cases} S_3\boldsymbol{x}_3=2\lambda\boldsymbol{x}_3, \\ C_3\boldsymbol{x}_3=0. \end{cases} \tag{A5}$$

Equivalently,

$$(S_3+\mathrm{i}C_3)\boldsymbol{x}_3=2\lambda\boldsymbol{x}_3. \tag{A6}$$

Next, we write

$$s_{mn}=\begin{cases} \sin(\theta_m-\theta_n), & m>n, \\ \sin(\theta_n-\theta_m), & m<n, \\ 0, & m=n, \end{cases}$$

$$c_{mn}=\begin{cases} \cos(\theta_m-\theta_n), & m>n, \\ -\cos(\theta_n-\theta_m), & m<n, \\ 0, & m=n, \end{cases}$$

then $S_3=[s_{mn}], C_3=[c_{mn}]$. In this case, equation (A4) becomes

$$C_3\boldsymbol{x}_3=\begin{bmatrix} 0 & -c_{21} & -c_{31} \\ c_{21} & 0 & -c_{32} \\ c_{31} & c_{32} & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}=0.$$

Since the eigenvalues of C_3 are $\mathrm{i}\sqrt{c_{21}^2+c_{31}^2+c_{32}^2}$, $-\mathrm{i}\sqrt{c_{21}^2+c_{31}^2+c_{32}^2}$, and 0, and the corresponding eigenvectors of C_3 are

$$\begin{bmatrix} -c_{21}c_{32}+c_{31}\mathrm{i}\sqrt{c_{21}^2+c_{31}^2+c_{32}^2} \\ c_{21}c_{31}+c_{32}\mathrm{i}\sqrt{c_{21}^2+c_{31}^2+c_{32}^2} \\ c_{31}^2+c_{32}^2 \end{bmatrix},$$

$$\begin{bmatrix} -c_{21}c_{32}-c_{31}\mathrm{i}\sqrt{c_{21}^2+c_{31}^2+c_{32}^2} \\ c_{21}c_{31}-c_{32}\mathrm{i}\sqrt{c_{21}^2+c_{31}^2+c_{32}^2} \\ c_{31}^2+c_{32}^2 \end{bmatrix}, \quad \begin{bmatrix} c_{32} \\ -c_{31} \\ c_{21} \end{bmatrix}.$$

Since the rank of C_3 is 2, the null space of C_3 is of 1-dimension. Thus, we have

$$\boldsymbol{x}=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}=k\begin{bmatrix} c_{32} \\ -c_{31} \\ c_{21} \end{bmatrix} (k\neq 0). \tag{6}$$

And because of Eq. (A5), we obtain

$$\begin{aligned} 2k\lambda\begin{bmatrix} c_{32} \\ -c_{31} \\ c_{21} \end{bmatrix} &= k\begin{bmatrix} 0 & s_{21} & s_{31} \\ s_{21} & 0 & s_{32} \\ s_{31} & s_{32} & 0 \end{bmatrix}\begin{bmatrix} c_{32} \\ -c_{31} \\ c_{21} \end{bmatrix} \\ &= k\begin{bmatrix} c_{21}s_{31}-c_{31}s_{21} \\ s_{21}c_{32}+s_{32}c_{21} \\ c_{32}s_{31}-c_{31}s_{32} \end{bmatrix} = k\begin{bmatrix} s_{32} \\ s_{31} \\ s_{21} \end{bmatrix}. \end{aligned}$$

Hence,

$$2\lambda\begin{bmatrix} c_{32} \\ -c_{31} \\ c_{21} \end{bmatrix}=\begin{bmatrix} s_{32} \\ s_{31} \\ s_{21} \end{bmatrix}.$$

Equivalently,

$$\begin{aligned} &\tan(\theta_3-\theta_2)=\tan(\theta_2-\theta_1)=2\lambda, \\ &\tan(\theta_3-\theta_1)=-2\lambda. \end{aligned} \tag{A8}$$

Since $0\leq\theta_m<\theta_n\leq\pi(n>m)$ and

$$\theta_3-\theta_1=(\theta_3-\theta_2)+(\theta_2-\theta_1),$$

we see from Eq. (A8) that $\lambda>0$ and so

$$0<\theta_3-\theta_2<\pi/2,\ 0<\theta_2-\theta_1<\pi/2,\ \pi/2<\theta_3-\theta_1<\pi,$$

leading to

$$\theta_3-\theta_2=\theta_2-\theta_1\equiv\theta, \quad \theta_3-\theta_1=\pi-\theta.$$

Equivalently,

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}=\begin{bmatrix} \theta \\ \pi-\theta \\ \theta \end{bmatrix}, \tag{A9}$$

which is solvable if and only if $\theta=\pi/3$. In that case, it has solutions

$$\begin{cases} \theta_1=\theta_3-2\pi/3=\alpha+\pi/3, \\ \theta_2=\theta_3-\pi/3=\alpha+2\pi/3, \\ \theta_3=\alpha+\pi, \end{cases} \tag{A10}$$

where α is free with $-\pi/3\leq\alpha\leq 0$. Then, we get from Eq. (6) that

$$\boldsymbol{x}=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}=k\begin{bmatrix} c_{32} \\ -c_{31} \\ c_{21} \end{bmatrix}=k\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} (k\neq 0), \tag{A11}$$

implying $r_1 = r_2 = r_3 = 1/\sqrt{3}$ since $L_\lambda = r_1^2 + r_2^2 + r_3^2 - 1 = 0$. This shows that constrained maximum points are as follows:

$$P_\alpha = \left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, \alpha + \pi/3, \alpha + 2\pi/3, \alpha + \pi\right), \quad -\pi/3 \le \alpha \le 0.$$

Further, $G(P_\alpha) = \sqrt{3}/2$. Thus,

$$\mathcal{M}_{\ell_1}(|\psi\rangle\langle\psi|) \le 2G(P_\alpha) = \sqrt{3} = \mathcal{M}_{\ell_1}(|\psi(\beta)\rangle\langle\psi(\beta)|), \quad \forall \beta \in \mathbb{R}.$$

Because of the convexity of $\mathcal{M}_{\ell_1}$, we obtain $\mathcal{M}_{\ell_1}(\rho) \le \sqrt{3}$ for all $\rho \in \mathcal{D}(H)$.

Proof of Theorem 2 (i) Let $\tilde{\rho} \in \mathcal{D}(H_1 \otimes H_2)$. Denote

$$\tilde{\rho} = \sum_{m,n,k,\ell} \rho_{mnk\ell} |m\rangle\langle n| \otimes |k\rangle\langle \ell|,$$

then we compute that

$$\begin{aligned}
&\mathcal{M}_{\ell_1}(\mathrm{tr}_{H_1}(\tilde{\rho})) + \mathcal{M}_{\ell_1}(\mathrm{tr}_{H_2}(\tilde{\rho})) \\
&= \sum_{k\neq\ell} \left|\mathrm{Im} \sum_m \rho_{mmk\ell}\right| + \sum_{m\neq n} \left|\mathrm{Im} \sum_k \rho_{mnkk}\right| \\
&\le \sum_{k\neq\ell} \sum_m |\mathrm{Im}\rho_{mmk\ell}| + \sum_{m\neq n} \sum_k |\mathrm{Im}\rho_{mnkk}| + \sum_{m\neq n} \sum_{k\neq\ell} |\mathrm{Im}\rho_{mnk\ell}| \\
&= \mathcal{M}_{\ell_1}(\tilde{\rho}).
\end{aligned}$$

Thus, for any state $\tilde{\rho} \in \mathcal{D}(H_1 \otimes H_2 \otimes \cdots \otimes H_N)$, we compute that

$$\begin{aligned}
\mathcal{M}_{\ell_1}(\tilde{\rho}) &\ge \mathcal{M}_{\ell_1}(\mathrm{tr}_{H_1}(\tilde{\rho})) + \mathcal{M}_{\ell_1}(\mathrm{tr}_{H_2 \otimes H_3 \otimes \cdots \otimes H_N}(\tilde{\rho})) \\
&\ge \mathcal{M}_{\ell_1}(\mathrm{tr}_{H_1}(\tilde{\rho})) + \mathcal{M}_{\ell_1}(\mathrm{tr}_{H_2}(\tilde{\rho})) \\
&\quad + \mathcal{M}_{\ell_1}(\mathrm{tr}_{H_3 \otimes H_4 \otimes \cdots \otimes H_N}(\tilde{\rho})) \\
&\ge \mathcal{M}_{\ell_1}(\mathrm{tr}_{H_1}(\tilde{\rho})) + \mathcal{M}_{\ell_1}(\mathrm{tr}_{H_2}(\tilde{\rho})) + \cdots \\
&\quad + \mathcal{M}_{\ell_1}(\mathrm{tr}_{H_N}(\tilde{\rho})).
\end{aligned}$$

Therefore, the conclusion is confirmed.

(ii) For $\rho = \sum_{mn} \rho_{mn} |m\rangle\langle n| \in \mathcal{D}(H_1)$ and $\sigma = \sum_{k\ell} \sigma_{k\ell} |k\rangle\langle \ell| \in \mathcal{R}(H_2)$, we have

$$\begin{aligned}
\mathcal{M}_{\ell_1}(\rho \otimes \sigma) &= \sum_{(m,k)\neq(n,\ell)} |\mathrm{Im}\rho_{mn}\sigma_{k\ell}| \\
&= \sum_{k\neq\ell} |\sigma_{k\ell}| \sum_m |\mathrm{Im}\rho_{mm}| + \sum_{m\neq n} |\mathrm{Im}\rho_{mn}| \sum_k |\sigma_{kk}| \\
&\quad + \sum_{m\neq n} |\mathrm{Im}\rho_{mn}| \sum_{k\neq\ell} |\sigma_{k\ell}| \\
&= (1 + C_{\ell_1}(\sigma)) \mathcal{M}_{\ell_1}(\rho).
\end{aligned}$$

Thus, for any state $\rho_1 \in \mathcal{D}(H_1)$ and $\rho_s \in \mathcal{R}(H_s)$ $(s = 2, 3, \ldots, N)$, we have

$$\begin{aligned}
&\mathcal{M}_{\ell_1}(\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N) \\
&= (1 + C_{\ell_1}(\rho_N)) \mathcal{M}_{\ell_1}(\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \cdots \otimes \rho_{N-1}) \\
&= (1 + C_{\ell_1}(\rho_{N-1}))(1 + C_{\ell_1}(\rho_N)) \\
&\quad \times \mathcal{M}_{\ell_1}(\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \cdots \otimes \rho_{N-2}) \\
&= \mathcal{M}_{\ell_1}(\rho_1) \prod_{s=2}^{N} (1 + C_{\ell_1}(\rho_s)).
\end{aligned}$$

Proof of Theorem 3 (i)⇒(ii) is obvious.

(ii)⇔(iii) Note that

$$\Phi(\rho) = \sum_{\alpha=1}^{r} K_\alpha \rho K_\alpha^\dagger = \sum_{\alpha=1}^{r} \sum_{s,j,k,l=1}^{2} q_{sj}^{(\alpha)} q_{kl}^{(\alpha)} \rho_{jl} |s\rangle\langle k|,$$

where $\{|s\rangle\}_{s=1}^2$ is the fixed basis for H and $\rho_{jl} = \langle j|\rho|l\rangle$. Then the (s,k)-entry of the matrix representation of $\Phi(\rho)$ under the basis is

$$(\Phi(\rho))_{sk} = \sum_{\alpha=1}^{r} \sum_{j,l=1}^{2} q_{sj}^{(\alpha)} q_{kl}^{(\alpha)} \rho_{jl}.$$

Especially,

$$\begin{aligned}
(\Phi(\rho))_{12} &= \sum_{\alpha=1}^{r} \left(q_{11}^{(\alpha)} q_{21}^{(\alpha)} \rho_{11} + q_{11}^{(\alpha)} q_{22}^{(\alpha)} \rho_{12} + q_{12}^{(\alpha)} q_{21}^{(\alpha)} \rho_{21} \right. \\
&\quad \left. + q_{12}^{(\alpha)} q_{22}^{(\alpha)} \rho_{22}\right).
\end{aligned}$$

From $\sum_\alpha K_\alpha^\dagger K_\alpha = I_2$, we can get $\sum_\alpha \sum_s |q_{sj}^{(\alpha)}|^2 = 1 (\forall j)$, that is,

$$\begin{cases} \sum_\alpha \left(|q_{11}^{(\alpha)}|^2 + |q_{21}^{(\alpha)}|^2\right) = 1, \\ \sum_\alpha \left(|q_{12}^{(\alpha)}|^2 + |q_{22}^{(\alpha)}|^2\right) = 1. \end{cases} \tag{A12}$$

Note that

$$\begin{aligned}
\mathcal{M}_{\ell_1}(\Phi(\rho)) &= \sum_{s,k} |\mathrm{Im}(\Phi(\rho))_{sk}| \\
&= 2|\mathrm{Im}(\Phi(\rho))_{12}| \\
&= 2\left|\sum_\alpha \left(q_{11}^{(\alpha)} q_{22}^{(\alpha)} \mathrm{Im}\rho_{12} + q_{12}^{(\alpha)} q_{21}^{(\alpha)} \mathrm{Im}\rho_{21}\right)\right| \\
&= 2|\mathrm{Im}\rho_{12}| \left|\sum_\alpha \left(q_{11}^{(\alpha)} q_{22}^{(\alpha)} - q_{12}^{(\alpha)} q_{21}^{(\alpha)}\right)\right|.
\end{aligned}$$

So $\mathcal{M}_{\ell_1}(\Phi(\rho)) = \mathcal{M}_{\ell_1}(\rho) = 2|\mathrm{Im}\rho_{12}| \neq 0$ if and only if

$$\left|\sum_\alpha \left(q_{11}^{(\alpha)} q_{22}^{(\alpha)} - q_{12}^{(\alpha)} q_{21}^{(\alpha)}\right)\right| = 1. \tag{A13}$$

(iii)⇔(iv) By using Cauchy–Schwarz inequality, Eqs. (A12) and (A13), we obtain

$$q_{11}^{(\alpha)} = \lambda q_{22}^{(\alpha)} (\forall \alpha), -q_{21}^{(\alpha)} = \lambda q_{12}^{(\alpha)} (\forall \alpha)$$

and furthermore $\lambda = 1$ or $\lambda = -1$.

When $\lambda = 1$, we have $q_{11}^{(\alpha)} = q_{22}^{(\alpha)}$ and $q_{12}^{(\alpha)} = -q_{21}^{(\alpha)}$. So we obtain

$$\begin{aligned}
K_\alpha &= \begin{pmatrix} q_{11}^{(\alpha)} & q_{12}^{(\alpha)} \\ -q_{12}^{(\alpha)} & q_{11}^{(\alpha)} \end{pmatrix} = q_{11}^{(\alpha)} I_2 + q_{12}^{(\alpha)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= q_{11}^{(\alpha)} I_2 + \mathrm{i} q_{12}^{(\alpha)} \sigma_2 (\forall \alpha).
\end{aligned}$$

When $\lambda = -1$, we have $q_{11}^{(\alpha)} = -q_{22}^{(\alpha)}$ and $q_{12}^{(\alpha)} = q_{21}^{(\alpha)}$. So we obtain

$$K_\alpha = \begin{pmatrix} q_{11}^{(\alpha)} & q_{12}^{(\alpha)} \\ q_{12}^{(\alpha)} & -q_{11}^{(\alpha)} \end{pmatrix} = q_{11}^{(\alpha)} \sigma_3 + q_{12}^{(\alpha)} \sigma_1 (\forall \alpha).$$

From the completeness of Kraus operators of Φ, we get that

$$\sum_{\alpha}\left(|q_{11}^{(\alpha)}|^2+|q_{12}^{(\alpha)}|^2\right)=1.$$

(iv)⇒(i) Let $\rho=\begin{pmatrix} a & b+\mathrm{i}c \\ b-\mathrm{i}c & 1-a \end{pmatrix}$. If

$$K_\alpha=q_{11}^{(\alpha)}I_2+\mathrm{i}q_{12}^{(\alpha)}\sigma_2(\forall\alpha)\ \text{ or }\ K_\alpha=q_{11}^{(\alpha)}\sigma_3+q_{12}^{(\alpha)}\sigma_1(\forall\alpha),$$

then it is clear to see that

$$\mathcal{M}_{\ell_1}(\Phi(\rho))=\mathcal{M}_{\ell_1}\left(\sum_{\alpha=1}^{r}K_\alpha\rho K_\alpha^\dagger\right)=2|c|=\mathcal{M}_{\ell_1}(\rho).$$

Proof of Theorem 4 When N is odd, $\mathcal{M}_{\ell_1}(\rho^N)=(1/2^N)2^N|c_2|=|c_2|$, and

$$\begin{aligned}\Phi(\rho^N)&=(\Phi_1\otimes\Phi_2\otimes\cdots\otimes\Phi_N)(\rho^N)\\&=\frac{1}{2^N}\Bigg[\otimes_{k=1}^N\Phi_k(I_2)+\sum_{w=1}^{3}c_w\Phi_1(\sigma_w)\otimes_{k=2}^N\Phi_k(I_2)\\&\quad+\sum_{w=1}^{3}d_w\Phi_1(I_2)\otimes_{k=2}^N\Phi_k(\sigma_w)\Bigg].\end{aligned}$$

Note that

$$\Phi_s(\rho)=\sum_{\alpha_s}K_{\alpha_s}\rho K_{\alpha_s}^\dagger$$

$$=\sum_{\alpha_s}\sum_{t_s,l_s,m_s,n_s=1}^{2}q_{t_sl_s}^{(\alpha_s)}q_{m_sn_s}^{(\alpha_s)}\rho_{l_sn_s}|t_s\rangle\langle m_s|.$$

From the completeness of Kraus operators of Φ_s, we get

$$\sum_{\alpha_s}\sum_{t_s}\left(q_{t_sl_s}^{(\alpha_s)}\right)^2=\sum_{\alpha_s}\sum_{m_s}\left(q_{m_sn_s}^{(\alpha_s)}\right)^2=1(\forall l_s,\forall n_s).$$

Moreover, we obtain

$$\Phi_s(\sigma_2)=\sum_{\alpha_s}\sum_{t_s,m_s=1}^{2}\left(-q_{t_s1}^{(\alpha_s)}q_{m_s2}^{(\alpha_s)}+q_{t_s2}^{(\alpha_s)}q_{m_s1}^{(\alpha_s)}\right)\mathrm{i}|t_s\rangle\langle m_s|,$$

$$\Phi_s(I_2)=\sum_{\alpha_s}\sum_{t_s,m_s=1}^{2}\left(q_{t_s1}^{(\alpha_s)}q_{m_s1}^{(\alpha_s)}+q_{t_s2}^{(\alpha_s)}q_{m_s2}^{(\alpha_s)}\right)|t_s\rangle\langle m_s|.$$

This implies that

$$\begin{aligned}\otimes_{k=2}^N\Phi_k(\sigma_2)=\otimes_{k=2}^N\Bigg(&\sum_{\alpha_k}\sum_{t_k,m_k=1}^{2}\big(-q_{t_k1}^{(\alpha_k)}q_{m_k2}^{(\alpha_k)}\\&+q_{t_k2}^{(\alpha_k)}q_{m_k1}^{(\alpha_k)}\big)\mathrm{i}|t_k\rangle\langle m_k|\Bigg)\end{aligned}$$

is a real matrix. So

$$\begin{aligned}\mathcal{M}_{\ell_1}(\Phi(\rho^N))&=\sum_{t_1,m_1,t_2,m_2,\cdots,t_N,m_N}\left|\mathrm{Im}\Phi(\rho)_{(t_1,m_1),\cdots,(t_N,m_N)}\right|\\&=\frac{1}{2^{N-1}}|c_2|\sum_{t_1=1,m_1=2,t_2,m_2,\cdots,t_N,m_N}\Bigg|\left(\sum_{\alpha_1}\left(-q_{11}^{(\alpha_1)}q_{22}^{(\alpha_1)}+q_{12}^{(\alpha_1)}q_{21}^{(\alpha_1)}\right)\right)\\&\quad\left(\sum_{\alpha_2}\left(q_{t_21}^{(\alpha_2)}q_{m_21}^{(\alpha_2)}+q_{t_22}^{(\alpha_2)}q_{m_22}^{(\alpha_2)}\right)\right)\cdots\left(\sum_{\alpha_N}\left(q_{t_N1}^{(\alpha_N)}q_{m_N1}^{(\alpha_N)}+q_{t_N2}^{(\alpha_N)}q_{m_N2}^{(\alpha_N)}\right)\right)\Bigg|\\&=\frac{1}{2^{N-1}}|c_2|\left|\sum_{\alpha_1}\left(-q_{11}^{(\alpha_1)}q_{22}^{(\alpha_1)}+q_{12}^{(\alpha_1)}q_{21}^{(\alpha_1)}\right)\right|\\&\quad\left(\left|\sum_{\alpha_2}\left(\left(q_{11}^{(\alpha_2)}\right)^2+\left(q_{12}^{(\alpha_2)}\right)^2\right)\right|+\left|\sum_{\alpha_2}\left(\left(q_{21}^{(\alpha_2)}\right)^2+\left(q_{22}^{(\alpha_2)}\right)^2\right)\right|+2\left|\sum_{\alpha_2}\left(q_{11}^{(\alpha_2)}q_{21}^{(\alpha_2)}+q_{12}^{(\alpha_2)}q_{22}^{(\alpha_2)}\right)\right|\right)\\&\quad\left(\sum_{t_3,m_3,\cdots,t_N,m_N}\left|\sum_{\alpha_3}\left(q_{t_31}^{(\alpha_3)}q_{m_31}^{(\alpha_3)}+q_{t_32}^{(\alpha_3)}q_{m_32}^{(\alpha_3)}\right)\cdots\sum_{\alpha_N}\left(q_{t_N1}^{(\alpha_N)}q_{m_N1}^{(\alpha_N)}+q_{t_N2}^{(\alpha_N)}q_{m_N2}^{(\alpha_N)}\right)\right|\right)\\&=\frac{1}{2^{N-1}}|c_2|\left|\sum_{\alpha_1}\left(-q_{11}^{(\alpha_1)}q_{22}^{(\alpha_1)}+q_{12}^{(\alpha_1)}q_{21}^{(\alpha_1)}\right)\right|\left(2+2\left|\sum_{\alpha_2}\left(q_{11}^{(\alpha_2)}q_{21}^{(\alpha_2)}+q_{12}^{(\alpha_2)}q_{22}^{(\alpha_2)}\right)\right|\right)\\&\quad\left(\sum_{t_3,m_3,\cdots,t_N,m_N}\left|\sum_{\alpha_3}\left(q_{t_31}^{(\alpha_3)}q_{m_31}^{(\alpha_3)}+q_{t_32}^{(\alpha_3)}q_{m_32}^{(\alpha_3)}\right)\cdots\sum_{\alpha_N}\left(q_{t_N1}^{(\alpha_N)}q_{m_N1}^{(\alpha_N)}+q_{t_N2}^{(\alpha_N)}q_{m_N2}^{(\alpha_N)}\right)\right|\right)\\&=|c_2|\left|\sum_{\alpha_1}\left(-q_{11}^{(\alpha_1)}q_{22}^{(\alpha_1)}+q_{12}^{(\alpha_1)}q_{21}^{(\alpha_1)}\right)\right|\left(1+\left|\sum_{\alpha_2}\left(q_{11}^{(\alpha_2)}q_{21}^{(\alpha_2)}+q_{12}^{(\alpha_2)}q_{22}^{(\alpha_2)}\right)\right|\right)\cdots\\&\quad\left(1+\left|\sum_{\alpha_N}\left(q_{11}^{(\alpha_N)}q_{21}^{(\alpha_N)}+q_{12}^{(\alpha_N)}q_{22}^{(\alpha_N)}\right)\right|\right)\\&=|c_2|\left|\sum_{\alpha_1}\left(-q_{11}^{(\alpha_1)}q_{22}^{(\alpha_1)}+q_{12}^{(\alpha_1)}q_{21}^{(\alpha_1)}\right)\right|\prod_{s=2}^{N}\left(1+\left|\sum_{\alpha_s}\left(q_{11}^{(\alpha_s)}q_{21}^{(\alpha_s)}+q_{12}^{(\alpha_s)}q_{22}^{(\alpha_s)}\right)\right|\right)\\&=|c_2|m_1\prod_{s=2}^{N}(1+b_s).\end{aligned}$$

From $\mathcal{M}_{\ell_1}(\Phi(\rho^N)) = \mathcal{M}_{\ell_1}(\rho^N) = |c_2|$, we can get

$$m_1 \prod_{s=2}^{N}(1+b_s) = 1.$$

By using Cauchy–Schwarz inequality, it is easy to get

$$m_s = \left|\sum_{\alpha_s}\left(-q_{11}^{(\alpha_s)}q_{22}^{(\alpha_s)} + q_{12}^{(\alpha_s)}q_{21}^{(\alpha_s)}\right)\right| \le 1 \ (\forall s = 1,2,\ldots,N),$$

$$b_s = \left|\sum_{\alpha_s}\left(q_{11}^{(\alpha_s)}q_{21}^{(\alpha_s)} + q_{12}^{(\alpha_s)}q_{22}^{(\alpha_s)}\right)\right| \le 1 \ (\forall s = 1,2,\ldots,N).$$

Proof of Theorem 5 When N is even, $\mathcal{M}_{\ell_1}(\rho^N) = 2^N(|\frac{c_2}{2^N}| + |\frac{d_2}{2^N}|) = |c_2| + |d_2|$ and

$$\begin{aligned}\Phi(\rho^N) = \frac{1}{2^N}\Bigg[&\otimes_{k=1}^N \Phi_k(I_2) + \sum_{w=1}^{3} c_w \Phi_1(\sigma_w)\otimes_{k=2}^N \Phi_k(I_2) \\ &+ \sum_{w=1}^{3} d_w \Phi_1(I_2)\otimes_{k=2}^N \Phi_k(\sigma_w)\Bigg].\end{aligned}$$

Note that

$$\begin{aligned}\Phi_s(\rho) &= \sum_{\alpha_s} K_{\alpha_s}\rho K_{\alpha_s}^{\dagger} \\ &= \sum_{\alpha_s}\sum_{t_s,l_s,m_s,n_s=1}^{2} q_{t_s l_s}^{(\alpha_s)} q_{m_s n_s}^{(\alpha_s)} \rho_{l_s n_s} |t_s\rangle\langle m_s|.\end{aligned}$$

From the completeness of Kraus operators of Φ_s, one can get

$$\sum_{\alpha_s}\sum_{t_s}\left(q_{t_s l_s}^{(\alpha_s)}\right)^2 = \sum_{\alpha_s}\sum_{m_s}\left(q_{m_s n_s}^{(\alpha_s)}\right)^2 = 1(\forall l_s, \forall n_s).$$

Moreover, we obtain

$$\Phi_s(\sigma_2) = \sum_{\alpha_s}\sum_{t_s,m_s=1}^{2}\left(-q_{t_s 1}^{(\alpha_s)}q_{m_s 2}^{(\alpha_s)} + q_{t_s 2}^{(\alpha_s)}q_{m_s 1}^{(\alpha_s)}\right)\mathrm{i}|t_s\rangle\langle m_s|,$$

$$\Phi_s(I_2) = \sum_{\alpha_s}\sum_{t_s,m_s=1}^{2}\left(q_{t_s 1}^{(\alpha_s)}q_{m_s 1}^{(\alpha_s)} + q_{t_s 2}^{(\alpha_s)}q_{m_s 2}^{(\alpha_s)}\right)|t_s\rangle\langle m_s|.$$

For each s, define

$$m_s = \left|\sum_{\alpha_s}\left(-q_{11}^{(\alpha_s)}q_{22}^{(\alpha_s)} + q_{12}^{(\alpha_s)}q_{21}^{(\alpha_s)}\right)\right|,$$

$$b_s = \left|\sum_{\alpha_s}\left(q_{11}^{(\alpha_s)}q_{21}^{(\alpha_s)} + q_{12}^{(\alpha_s)}q_{22}^{(\alpha_s)}\right)\right|,$$

where $m_s \le 1$, $b_s \le 1$. By observing the matrix form of $\Phi(\rho^N)$ under the product basis $\{|1\rangle,|2\rangle\}^{\otimes^N}$, we conclude that

$$\begin{aligned}\mathcal{M}_{\ell_1}(\Phi(\rho^N)) = &|c_2|m_1\left(\prod_{s=2}^{N}(1+b_s) - \prod_{s=2}^{N} b_s\right) + |d_2|\prod_{s=2}^{N} m_s \\ &+ \max\left\{|c_2|m_1\prod_{s=2}^{N} b_s, |d_2|b_1\prod_{s=2}^{N} m_s\right\}.\end{aligned}$$

Case 1 When

$$\max\left\{|c_2|m_1\prod_{s=2}^{N} b_s, |d_2|b_1\prod_{s=2}^{N} m_s\right\} = |c_2|m_1\prod_{s=2}^{N} b_s, \quad \text{(A14)}$$

we obtain that $c_2 \neq 0$ and thus

$$\begin{aligned}&\mathcal{M}_{\ell_1}(\Phi(\rho^N)) \\ &= |c_2|m_1\left(\prod_{s=2}^{N}(1+b_s) - \prod_{s=2}^{N} b_s + \prod_{s=2}^{N} b_s\right) + |d_2|\prod_{s=2}^{N} m_s \\ &= |c_2|m_1\prod_{s=2}^{N}(1+b_s) + |d_2|\prod_{s=2}^{N} m_s.\end{aligned}$$

Therefore,

$$\mathcal{M}_{\ell_1}(\Phi(\rho^N)) = \mathcal{M}_{\ell_1}(\rho^N) = |c_2| + |d_2|$$

if and only if

$$|c_2|\left(m_1\prod_{s=2}^{N}(1+b_s) - 1\right) + |d_2|\left(\prod_{s=2}^{N} m_s - 1\right) = 0.$$

Case 2 When

$$\max\left\{|c_2|m_1\prod_{s=2}^{N} b_s, |d_2|b_1\prod_{s=2}^{N} m_s\right\} = |d_2|b_1\prod_{s=2}^{N} m_s, \quad \text{(A15)}$$

we obtain $d_2 \neq 0$ and

$$\begin{aligned}\mathcal{M}_{\ell_1}(\Phi(\rho^N)) = &|c_2|m_1\left(\prod_{s=2}^{N}(1+b_s) - \prod_{s=2}^{N} b_s\right) \\ &+ |d_2|(1+b_1)\prod_{s=2}^{N} m_s.\end{aligned}$$

So

$$\mathcal{M}_{\ell_1}(\Phi(\rho^N)) = \mathcal{M}_{\ell_1}(\rho^N) = |c_2| + |d_2|$$

if and only if

$$\begin{aligned}&|c_2|\left(m_1\left(\prod_{s=2}^{N}(1+b_s) - \prod_{s=2}^{N} b_s\right) - 1\right) \\ &+|d_2|\left((1+b_1)\prod_{s=2}^{N} m_s - 1\right) = 0.\end{aligned}$$

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