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Citation: Chin. Phys. B, 2024, 33 (**11**): 114301. DOI: 10.1088/1674-1056/ad7724
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An analogical study of wave equations, physical quantities, conservation and reciprocity equations between electromagnetic and elastic waves

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(Received 28 June 2024; revised manuscript received 16 August 2024; accepted manuscript online 4 September 2024)

This paper presents an analogical study between electromagnetic and elastic wave fields, with a one-to-one correspondence principle established regarding the basic wave equations, the physical quantities and the differential operations. Using the electromagnetic-to-elastic substitution, the analogous relations of the conservation laws of energy and momentum are investigated between these two physical fields. Moreover, the energy-based and momentum-based reciprocity theorems for an elastic wave are also derived in the time-harmonic state, which describe the interaction between two elastic wave systems from the perspectives of energy and momentum, respectively. The theoretical results obtained in this analysis can not only improve our understanding of the similarities of these two linear systems, but also find potential applications in relevant fields such as medical imaging, non-destructive evaluation, acoustic microscopy, seismology and exploratory geophysics.

Keywords: analogical study, electromagnetic waves, elastic waves, wave equations, physical quantities, conservation laws, reciprocity theorems

PACS: 43.20.Wd, 41.20.Jb, 46.40.–f **DOI:** 10.1088/1674-1056/ad7724

1. Introduction

Electromagnetic and acoustic waves are two common classical physical fields, which can be derived from Maxwell equations and fluid dynamics, respectively. In the context of linearized theory, many similarities exist between these two linear physical systems, which makes it possible for us to perform an in-depth study of these two physical systems through analogy. In 1995, Carcione and Cavallini[1] first investigated the analogy between electromagnetic and acoustic waves based on the kinematics and the energy balance of wave propagation. It was found that the TEM mode (transverse electric and magnetic) is completely analogous to the viscoelastic SH waves in the plane of symmetry of a monoclinic medium. The correspondence principle between these two fields, such as the particle velocity and the magnetic field, the stresses and the electric field, was also validated by corresponding examples. Using the analogical method, other studies were also carried out to determine the complex resonant frequencies,[2] solve the reflection–refraction problem[3] and analyze the energy relations in waves interacting with material surfaces.[4] More recently, Burns *et al.*[5] developed a novel Lagrangian representation of the acoustic field theory that describes the local vector properties of longitudinal (curl-free) acoustic fields, which was based on the strong analogies between electromagnetism and acoustics. The acoustic wave propagation in viscothermal fluids was also studied by applying an electromagnetic analogy.[6]

Earlier this year, a monograph was published by Liu *et al.*[7] regarding the reciprocity theorems of acoustic fields. In this publication, the authors presented a comprehensive analogical study between electromagnetism and acoustics, including their linearized wave equations, physical quantities and conservation equations. Another basic principle satisfied by any linear physical system is the reciprocity theorem, which describes the symmetric relationship between the source and the field. Since the Lorentz reciprocity theorem was first proposed for electromagnetic wave fields,[8] many new types of reciprocity theorems have gradually been discovered.[9–15] Considering that the electromagnetic field can be described either in terms of momentum or in terms of energy, Liu *et al.*[16] derived the momentum-based reciprocity theorem, which predicts the momentum interaction between two electromagnetic fields. Similarly, since the acoustic reciprocity theorem was first proposed by Lord Rayleigh,[17] a momentum-based reciprocity equation was also needed along with the historical contributions of the development and applications of acoustic reciprocity theorems.[18–31] In analogy with electromagnetism, they also successfully proposed the momentum-based reciprocity theorem for acoustic fields, further improving the acoustic reciprocity theory.[7]

However, it can also be noted that their work is limited to the scope of acoustic waves in a fluid medium, without considering the elastic waves propagating in a solid medium. In view of this, this analysis is directed toward developing a comprehensive analogical study between electromagnetic and elastic wave fields. Based on the wave equations derived from Maxwell equations and elastodynamics, a one-to-one correspondence principle is established between electromagnetic

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and elastic quantities and differential operations. To the best of our knowledge, although reciprocity principles for elastic waves have been successfully investigated and applied in parameter estimation and ultrasound analysis,[32–34] no relevant studies have been performed regarding the analogy between reciprocities of elastic and electromagnetic waves. By employing the electromagnetic-to-elastic substitution, the conservation and reciprocity equations for elastic wave fields are also obtained from the perspectives of energy and momentum, respectively.

2. Linear wave equations

We first recall the classic electromagnetic field theory described by Maxwell equations in vacuum, which include two curl equations and two divergence equations. For simplicity, only the field equations in vacuum are considered in this analysis, in which the two curl equations are expressed as follows:[35]

$$\frac{\partial}{\partial t}\left(\mu_0 \boldsymbol{H}\right) = -\nabla \times \boldsymbol{E} - \boldsymbol{J}_{\mathrm{m}}, \tag{1}$$

$$\frac{\partial}{\partial t}\left(\varepsilon_0 \boldsymbol{E}\right) = \nabla \times \boldsymbol{H} - \boldsymbol{J}_{\mathrm{e}}, \tag{2}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field intensities, respectively, $\boldsymbol{J}_{\mathrm{e}}$ and $\boldsymbol{J}_{\mathrm{m}}$ are the electric and magnetic current densities, respectively, μ_0 and ε_0 are the magnetic permeability and the dielectric constant in vacuum, respectively, and the symbol $\nabla\times$ represents the curl operator for a vector. It is noteworthy that the magnetic current source, which is a nonexistent physical quantity, is artificially introduced here for symmetry. Equations (1) and (2) correspond to the electromagnetic induction law and the displacement current law, respectively. A schematic diagram of the electromagnetic wave system is shown in Fig. 1, in which V and S represent the volume and surface of the region, respectively.

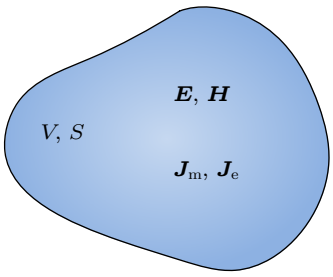


Fig. 1. A sketch of an electromagnetic wave system in vacuum, with $\boldsymbol{J}_{\mathrm{m}}$, $\boldsymbol{J}_{\mathrm{e}}$ and $\boldsymbol{E}$, $\boldsymbol{H}$ denoting the source and field quantities, respectively.

Unlike the electromagnetic wave field, elastic wave fields are expressed by nonlinear equations due to motion nonlinearity and constitutive nonlinearity. According to the linearized theory of elastodynamics, the elastic wave field equations in a lossless solid medium include the motion equation and the constitutive equation, which can be expressed respectively as[36]

$$\frac{\partial\left(\rho_0 \boldsymbol{v}\right)}{\partial t} = \nabla \cdot \boldsymbol{T} + \rho_0 \boldsymbol{f}, \tag{3}$$

$$\frac{\partial}{\partial t}\left(\boldsymbol{s} : \boldsymbol{T}\right) = \nabla_{\mathrm{s}} \boldsymbol{v}, \tag{4}$$

where $\boldsymbol{T}$ and $\boldsymbol{v}$ are the surface stress tensor and the particle velocity, respectively, $\boldsymbol{f}$ is the external force per unit mass, ρ_0 and $\boldsymbol{s}$ are the mass density and the smoothness coefficient of the solid material, respectively, the symbol $\nabla\cdot$ represents the divergence matrix operator for a tensor, the symbol ∇_{s} represents the symmetric gradient matrix operator for a vector, and the symbol : represents the double dot operator products between two tensors. Similarly, a sketch of the elastic wave system is depicted in Fig. 2 for reference.

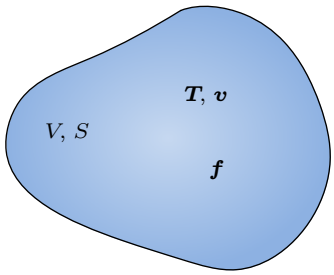


Fig. 2. A sketch of an elastic wave system in a solid medium, with $\boldsymbol{f}$ and $\boldsymbol{T}$, $\boldsymbol{v}$ denoting the source and field quantities, respectively.

According to the linearized elasticity theory, the surface stress tensor $\boldsymbol{T}$ is a second-order symmetric tensor. For ease of the following mathematical processing, $\boldsymbol{T}$ is rewritten with abbreviated subscripts as

$$\boldsymbol{T} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} = \begin{bmatrix} T_1 & T_6 & T_5 \\ T_6 & T_2 & T_4 \\ T_5 & T_4 & T_3 \end{bmatrix}. \tag{5}$$

Therefore, the surface stress tensor can be simplified into a 6×1 matrix form as

$$\boldsymbol{T} = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 \end{bmatrix}^{\mathrm{T}}, \tag{6}$$

where the symbol T denotes the matrix transpose operator. The divergence matrix operator for a tensor in the Cartesian coordinate can therefore be expressed as

$$\nabla\cdot = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 & 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & 0 & \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \\ 0 & 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \end{bmatrix}. \tag{7}$$

Similarly, the small-strain tensor $\boldsymbol{S} = \boldsymbol{s} : \boldsymbol{T}$ is also a second-order symmetric tensor, which can be rewritten with abbreviated subscripts as

$$\boldsymbol{S} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} S_1 & S_6/2 & S_5/2 \\ S_6/2 & S_2 & S_4/2 \\ S_5/2 & S_4/2 & S_3 \end{bmatrix}. \tag{8}$$

Notice that a factor of $1/2$ occurs in the shear strain components. Also, we can simplify it into a 6×1 matrix form as

$$\boldsymbol{S} = \begin{bmatrix} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \end{bmatrix}^{\mathrm{T}}. \tag{9}$$

The symmetric gradient matrix operator for a vector is given by

$$\nabla_{\mathrm{s}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 & 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & 0 & \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \\ 0 & 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \end{bmatrix}^{\mathrm{T}}. \tag{10}$$

3. Physical quantities and differential operations

By comparing Eqs. (1) and (2) and Eqs. (3) and (4), one can find that there is a one-to-one correspondence principle between electromagnetic and linearized elastic wave fields. In this section, a systematic analogy between electromagnetic and electric quantities and differential operations is presented.

With regard to the material parameters, the magnetic permeability μ_0 and the dielectric constant ε_0 correspond to the mass density ρ_0 and the smoothness coefficient $\boldsymbol{s}$ of the solid material, respectively. The electric field intensity $\boldsymbol{E}$ and the magnetic field intensity $\boldsymbol{H}$ can be analogized with the stress tensor $\boldsymbol{T}$ and the particle velocity $\boldsymbol{v}$, respectively. In quantum-like terms, we can also say that electromagnetic waves are described by the 'wave function' $(\boldsymbol{E},\boldsymbol{H})$ composed of two vector fields, while elastic waves are described by the 'wave function' $(\boldsymbol{T},\boldsymbol{v})$ composed of a second-order tensor and a vector field. Moreover, with $\boldsymbol{B}=\mu_0\boldsymbol{H}$ and $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}$ denoting the magnetic and the electric induction intensities, their counterparts for elastic wave fields are the momentum density $\boldsymbol{p}=\rho_0\boldsymbol{v}$ and the small-strain tensor $\boldsymbol{S}=\boldsymbol{s}:\boldsymbol{T}$, respectively. An analogy can also be performed regarding the current source quantities, with the negative magnetic current source $-\boldsymbol{J}_{\mathrm{m}}$ corresponding to the force source $\rho_0\boldsymbol{f}$, namely, the external force per unit volume. However, one also notices that there is no counterpart for the electric current density $\boldsymbol{J}_{\mathrm{e}}$ in elastic wave field equations, which we should always keep in mind in subsequent discussions. Finally, a further inspection of Eqs. (1) and (2) and Eqs. (3) and (4) also indicates that the electromagnetic and elastic wave fields have the same time-derivative operator $\partial/\partial t$, whereas they have different spatial differential operators. For the negative curl operator $-\nabla\times$ on the electric field intensity and the curl operator $\nabla\times$ on the magnetic field intensity, they are analogous to the divergence operator $\nabla\cdot$ on the surface stress tensor and the symmetric gradient vector ∇_{s} on the particle velocity, respectively. The cross product between two vectors for an electromagnetic wave corresponds to the dot product between a vector and a second-order tensor for an elastic wave. To ensure the precision of analogy, a tensor pre-multiplied by a vector should be the opposite of the tensor post-multiplied by the vector to satisfy the anti-commutative law.

Based on the analogy of symbols explicitly involved in the field equations, we can also analogize other derivative physical quantities related to the dynamic properties of these two fields. We start with the analysis of the energy property. The energy density of an electromagnetic field is given by

$$w_{\mathrm{em}} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 H^2 = \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E} + \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}, \tag{11}$$

where the first and second terms represent the energy densities of the electric and magnetic fields, respectively. Here, the subscript 'em' is used to denote the quantities for an electromagnetic wave. In analogy with electromagnetism, we can directly write down the energy density of an elastic wave field by replacing the electromagnetic quantities in Eq. (11) with corresponding elastic ones as

$$w_{\mathrm{el}} = \frac{1}{2}\boldsymbol{T}:\boldsymbol{s}:\boldsymbol{T} + \frac{1}{2}\rho_0 v^2 = \frac{1}{2}\boldsymbol{S}:\boldsymbol{T} + \frac{1}{2}\boldsymbol{p}\cdot\boldsymbol{v}, \tag{12}$$

where the first and second terms represent the elastic potential and kinetic energy densities, respectively. The subscript 'el' is used to denote the quantities for an elastic wave. The energy flux density of an electromagnetic field, also called the Poynting vector, is equal to the cross product of $\boldsymbol{E}$ and $\boldsymbol{H}$

$$\boldsymbol{S}_{\mathrm{em}} = \boldsymbol{E}\times\boldsymbol{H}, \tag{13}$$

whose counterpart for an elastic wave is given by the dot product of $\boldsymbol{v}$ and $\boldsymbol{T}$

$$\boldsymbol{S}_{\mathrm{el}} = -\boldsymbol{v}\cdot\boldsymbol{T}. \tag{14}$$

Here, we used the anti-commutative law of the dot product between a vector and a tensor such that $-\boldsymbol{v}\cdot\boldsymbol{T}=\boldsymbol{T}\cdot\boldsymbol{v}$. Another quantity relevant to the energy property is the source power density, which represents the changing rate of energy per unit volume. With regard to the electromagnetic wave, both the electric and magnetic current sources contribute to the source power density, yielding

$$p_{\mathrm{em}} = -\boldsymbol{J}_{\mathrm{m}}\cdot\boldsymbol{H} - \boldsymbol{J}_{\mathrm{e}}\cdot\boldsymbol{E}. \tag{15}$$

However, since there is no counterpart for $\boldsymbol{J}_{\mathrm{e}}$ in a solid material, the source power density for an elastic wave is only derived from the external force source, such that

$$p_{\mathrm{el}} = \rho_0\boldsymbol{f}\cdot\boldsymbol{v}. \tag{16}$$

Similarly, an analogical study can be employed in physical quantities related to the momentum property, which is another reflection of the materiality of these two fields. The momentum density of an electromagnetic field is expressed as

$$\boldsymbol{g}_{\rm em} = (\varepsilon_0 \boldsymbol{E}) \times (\mu_0 \boldsymbol{H}) = \boldsymbol{D} \times \boldsymbol{B}. \tag{17}$$

Replacing the electromagnetic quantities with elastic ones, we can derive the momentum density of an elastic wave field as

$$\boldsymbol{g}_{\rm el} = (\boldsymbol{s} : \boldsymbol{T}) \cdot (\rho_0 \boldsymbol{v}) = \boldsymbol{S} \cdot \boldsymbol{p}. \tag{18}$$

As another quantity associated with the momentum property, force density represents the changing rate of momentum per unit volume, given by

$$\boldsymbol{F}_{\rm em} = \boldsymbol{J}_{\rm e} \times \boldsymbol{B} + \boldsymbol{J}_{\rm m} \times \boldsymbol{D}. \tag{19}$$

Here, an uppercase letter $\boldsymbol{F}$ is used to avoid any confusion with the external force source for an elastic wave. Similar to the source power density, Eq. (19) contains both the electric and magnetic current densities. For an elastic wave, however, only one term remains in the force density expression. Using quantity substitution, we obtain

$$\boldsymbol{F}_{\rm el} = -\rho_0 \boldsymbol{f} \cdot \boldsymbol{S}. \tag{20}$$

As a summary of this section, the one-to-one correspondence principle discussed above between electromagnetic and elastic wave equations, physical quantities and differential operations are shown in Table 1 for reference.

Table 1. The one-to-one correspondence principle of wave equations, physical quantities and differential operations between electromagnetic and elastic wave fields.

Electromagnetic field	Elastic wave field
$\frac{\partial}{\partial t}(\mu_0 \boldsymbol{H}) = -\nabla \times \boldsymbol{E} - \boldsymbol{J}_{\rm m}$	$\frac{\partial (\rho_0 \boldsymbol{v})}{\partial t} = \nabla \cdot \boldsymbol{T} + \rho_0 \boldsymbol{f}$
$\frac{\partial}{\partial t}(\varepsilon_0 \boldsymbol{E}) = \nabla \times \boldsymbol{H} - \boldsymbol{J}_{\rm e}$	$\frac{\partial}{\partial t}(\boldsymbol{s} : \boldsymbol{T}) = \nabla_{\rm s} \boldsymbol{v}$
$\boldsymbol{E}$	$\boldsymbol{T}$
$\boldsymbol{H}$	$\boldsymbol{v}$
$\boldsymbol{B} = \mu_0 \boldsymbol{H}$	$\boldsymbol{p} = \rho_0 \boldsymbol{v}$
$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$	$\boldsymbol{S} = \boldsymbol{s} : \boldsymbol{T}$
$\boldsymbol{J}_{\rm m}$	$\rho_0 \boldsymbol{f}$
$\boldsymbol{J}_{\rm e}$	0
$\frac{\partial}{\partial t}$	$\frac{\partial}{\partial t}$
$-\nabla\times$	$\nabla\cdot$
$\nabla\times$	$\nabla_{\rm s}$
μ_0	ρ_0
ε_0	$\boldsymbol{s}$
$\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 H^2 = \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E} + \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}$	$\frac{1}{2}\boldsymbol{T} : \boldsymbol{s} : \boldsymbol{T} + \frac{1}{2}\rho_0 v^2 = \frac{1}{2}\boldsymbol{S} : \boldsymbol{T} + \frac{1}{2}\boldsymbol{p}\cdot\boldsymbol{v}$
$\boldsymbol{E} \times \boldsymbol{H}$	$-\boldsymbol{v}\cdot\boldsymbol{T}$
$-\boldsymbol{J}_{\rm m}\cdot\boldsymbol{H} - \boldsymbol{J}_{\rm e}\cdot\boldsymbol{E}$	$\rho_0 \boldsymbol{f}\cdot\boldsymbol{v}$
$(\varepsilon_0 \boldsymbol{E}) \times (\mu_0 \boldsymbol{H}) = \boldsymbol{D} \times \boldsymbol{B}$	$(\boldsymbol{s} : \boldsymbol{T}) \cdot (\rho_0 \boldsymbol{v}) = \boldsymbol{S} \cdot \boldsymbol{p}$
$\boldsymbol{J}_{\rm e} \times \boldsymbol{B} + \boldsymbol{J}_{\rm m} \times \boldsymbol{D}$	$-\rho_0 \boldsymbol{f} \cdot \boldsymbol{S}$

4. Conservation equations

Based on the aforementioned basic wave equations and physical quantities, we can apply the analogical theory to the conservation equations, including the conservation laws of energy and momentum for these two wave fields.

The conservation law of energy for an electromagnetic field can be expressed as[35]

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right) + \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) \\ &= -\boldsymbol{J}_{\rm e} \cdot \boldsymbol{E} - \boldsymbol{J}_{\rm m} \cdot \boldsymbol{H}. \end{aligned} \tag{21}$$

The first term on the left-hand side corresponds to the changing rate of the electromagnetic energy, the second term corresponds to the electromagnetic energy flow through the surface, and the right-hand side represents the source power density contributed by electric and magnetic currents. By multiplying Eqs. (1) and (2) with $\boldsymbol{H}$ and $\boldsymbol{E}$, respectively, we can arrive at Eq. (21) by adding up the newly obtained equations.

Substituting the electromagnetic energy density, energy flux density, source power density in Eq. (21) by corresponding elastic ones using Table 1, we can obtain the conservation law of energy for an elastic wave field, which is written as[36]

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{T} : \boldsymbol{s} : \boldsymbol{T} + \frac{1}{2}\rho_0 v^2\right) + \nabla \cdot (-\boldsymbol{v} \cdot \boldsymbol{T}) = \rho_0 \boldsymbol{f} \cdot \boldsymbol{v}. \tag{22}$$

Similarly, the three terms from left to right correspond to the changing rate of the elastic energy, the energy flow through the surface, and the source power density of the elastic wave field. The major difference is that the right-hand side contains only one source.

In analogy with electromagnetism, Eq. (22) can be obtained by multiplying Eqs. (3) and (4) by $-\boldsymbol{v}$ and $-\boldsymbol{T}$, respectively as

$$-\boldsymbol{v} \cdot \frac{\partial (\rho_0 \boldsymbol{v})}{\partial t} = -\boldsymbol{v} \cdot (\nabla \cdot \boldsymbol{T}) - \boldsymbol{v} \cdot (\rho_0 \boldsymbol{f}), \tag{23}$$

$$-\boldsymbol{T} : \frac{\partial}{\partial t}(\boldsymbol{s} : \boldsymbol{T}) = -\boldsymbol{T} : \nabla_{\rm s} \boldsymbol{v}. \tag{24}$$

Adding up Eqs. (23) and (24), we can finally reach Eq. (22) by using the differential identity for a second-order symmetric tensor as follows:

$$\nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{T}) = \boldsymbol{v} \cdot (\nabla \cdot \boldsymbol{T}) + \boldsymbol{T} : \nabla_{\rm s} \boldsymbol{v}. \tag{25}$$

Another conservation law that is very useful in real applications is the conservation law of momentum, whose electromagnetic version can be expressed as[35]

$$\begin{aligned} &\frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) + \nabla \cdot \left[\frac{1}{2}(\boldsymbol{B} \cdot \boldsymbol{H} + \boldsymbol{D} \cdot \boldsymbol{E})\boldsymbol{I} - \boldsymbol{D}\boldsymbol{E} - \boldsymbol{B}\boldsymbol{H}\right] \\ &= -\boldsymbol{J}_{\rm e} \times \boldsymbol{B} - \rho_{\rm e}\boldsymbol{E} + \boldsymbol{J}_{\rm m} \times \boldsymbol{D} - \rho_{\rm m}\boldsymbol{H}, \end{aligned} \tag{26}$$

where $\rho_{\rm e}$ and $\rho_{\rm m}$ are the volume densities of the electric and magnetic charges, respectively, and $\boldsymbol{I}$ is the second-order unit tensor. The first term on the left-hand side corresponds to the changing rate of the electromagnetic momentum, the second term corresponds to the electromagnetic momentum flow through the surface, and the right-hand side represents

the force density, which stems from not only the electric and magnetic currents, but also the electric and magnetic charges. Similarly, Eq. (26) can be obtained from Maxwell equations, which will not be presented in this analysis.

It is noteworthy that the charge sources ρ_{e} and ρ_{m} bear no counterparts in elastic wave fields. Therefore, they must be substituted by the field quantities, which can be realized using the two divergence equations of Maxwell equations, namely the Gauss theorems for elastic and magnetic fields, given by[35]

$$\nabla\cdot\boldsymbol{D}=\rho_{\mathrm{e}}, \tag{27}$$

$$\nabla\cdot\boldsymbol{B}=\rho_{\mathrm{m}}. \tag{28}$$

Thus, Eq. (26) can be rewritten as

$$\begin{aligned}&\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B})+\nabla\cdot\left[\frac{1}{2}(\boldsymbol{B}\cdot\boldsymbol{H}+\boldsymbol{D}\cdot\boldsymbol{E})\boldsymbol{I}-\boldsymbol{D}\boldsymbol{E}-\boldsymbol{B}\boldsymbol{H}\right]\\&=-\boldsymbol{J}_{\mathrm{e}}\times\boldsymbol{B}-(\nabla\cdot\boldsymbol{D})\boldsymbol{E}+\boldsymbol{J}_{\mathrm{m}}\times\boldsymbol{D}-(\nabla\cdot\boldsymbol{B})\boldsymbol{H}.\end{aligned} \tag{29}$$

Consider the following differential identities:

$$\nabla\cdot\left[\frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}\boldsymbol{I}-\boldsymbol{D}\boldsymbol{E}\right]=\boldsymbol{D}\times\nabla\times\boldsymbol{E}-(\nabla\cdot\boldsymbol{D})\boldsymbol{E}, \tag{30}$$

$$\nabla\cdot\left[\frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}\boldsymbol{I}-\boldsymbol{B}\boldsymbol{H}\right]=\boldsymbol{B}\times\nabla\times\boldsymbol{H}-(\nabla\cdot\boldsymbol{B})\boldsymbol{H}, \tag{31}$$

where we used the relations $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}$ and $\boldsymbol{B}=\mu_0\boldsymbol{H}$. By combining Eqs. (29)–(31), we arrive at

$$\begin{aligned}&\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B})+\boldsymbol{D}\times\nabla\times\boldsymbol{E}-(\nabla\times\boldsymbol{H})\times\boldsymbol{B}\\&+\boldsymbol{J}_{\mathrm{e}}\times\boldsymbol{B}-\boldsymbol{J}_{\mathrm{m}}\times\boldsymbol{D}=0,\end{aligned} \tag{32}$$

which can be regarded as another form of the conservation law of momentum for an electromagnetic field.

Applying the electromagnetic-to-elastic substitution to Eq. (32), we can obtain the conservation law of momentum for an elastic wave field as

$$\frac{\partial}{\partial t}(\boldsymbol{S}\cdot\boldsymbol{p})-\boldsymbol{S}\cdot(\nabla\cdot\boldsymbol{T})-(\nabla_{\mathrm{s}}\boldsymbol{v})\cdot\boldsymbol{p}+\rho_0\boldsymbol{f}\cdot\boldsymbol{S}=0. \tag{33}$$

Using the relations $\boldsymbol{p}=\rho_0\boldsymbol{v}$ and $\boldsymbol{S}=\boldsymbol{s}:\boldsymbol{T}$, Eq. (33) can be rewritten as

$$\frac{\partial}{\partial t}(\boldsymbol{S}\cdot\boldsymbol{p})-(\boldsymbol{s}:\boldsymbol{T})\cdot(\nabla\cdot\boldsymbol{T})-\rho_0(\nabla_{\mathrm{s}}\boldsymbol{v})\cdot\boldsymbol{v}+\rho_0\boldsymbol{f}\cdot\boldsymbol{S}=0. \tag{34}$$

By considering the differential identities as follows:

$$\nabla\cdot\left(\frac{1}{2}v^2-\boldsymbol{v}\boldsymbol{v}\right)=-(\nabla_{\mathrm{s}}\boldsymbol{v})\cdot\boldsymbol{v}, \tag{35}$$

$$\nabla\cdot\left[\frac{1}{2}(\boldsymbol{T}:\boldsymbol{s}:\boldsymbol{T})\boldsymbol{I}\right]=(\boldsymbol{s}:\boldsymbol{T})\cdot(\nabla\cdot\boldsymbol{T}). \tag{36}$$

Equation (34) can be expressed as

$$\begin{aligned}&\frac{\partial}{\partial t}(\boldsymbol{S}\cdot\boldsymbol{p})+\nabla\cdot\left[\left(\frac{1}{2}\rho_0v^2-\frac{1}{2}\boldsymbol{T}:\boldsymbol{s}:\boldsymbol{T}\right)\boldsymbol{I}-\rho_0\boldsymbol{v}\boldsymbol{v}\right]\\&=-\rho_0\boldsymbol{f}\cdot\boldsymbol{S},\end{aligned} \tag{37}$$

which can be regarded as the elastic counterpart of Eq. (26). The first term on the left-hand side corresponds to the changing rate of the elastic momentum, the second term corresponds to the elastic momentum flow through the surface, and the right-hand side represents the force density.

Indeed, the conservation law of momentum for an elastic wave can also be derived from Eqs. (3) and (4). By multiplying Eqs. (3) and (4) by $\boldsymbol{S}$ and $\rho_0\boldsymbol{v}$, respectively, we obtain

$$\boldsymbol{S}\cdot\frac{\partial(\rho_0\boldsymbol{v})}{\partial t}=\boldsymbol{S}\cdot(\nabla\cdot\boldsymbol{T})+\boldsymbol{S}\cdot(\rho_0\boldsymbol{f}), \tag{38}$$

$$\frac{\partial}{\partial t}(\boldsymbol{s}:\boldsymbol{T})\cdot(\rho_0\boldsymbol{v})=(\nabla_{\mathrm{s}}\boldsymbol{v})\cdot(\rho_0\boldsymbol{v}). \tag{39}$$

By adding up Eqs. (38) and (39) and considering the relations $\boldsymbol{p}=\rho_0\boldsymbol{v}$, $\boldsymbol{S}=\boldsymbol{s}:\boldsymbol{T}$, $\boldsymbol{S}\cdot(\rho_0\boldsymbol{f})=-\rho_0\boldsymbol{f}\cdot\boldsymbol{S}$, we can finally reach the conservation law of momentum expressed by Eq. (33).

Likewise, the analogous relations of conservation equations between electromagnetic and elastic wave fields are listed in Table 2, in which the two forms of the conservation law of momentum are both presented for reference.

Table 2. Analogy of conservation equations for electromagnetic and elastic wave fields.

Electromagnetic field	Elastic wave field
$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0E^2+\frac{1}{2}\mu_0H^2\right)+\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H})=-\boldsymbol{J}_{\mathrm{e}}\cdot\boldsymbol{E}-\boldsymbol{J}_{\mathrm{m}}\cdot\boldsymbol{H}$	$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{T}:\boldsymbol{s}:\boldsymbol{T}+\frac{1}{2}\rho_0v^2\right)+\nabla\cdot(-\boldsymbol{v}\cdot\boldsymbol{T})=\rho_0\boldsymbol{f}\cdot\boldsymbol{v}$
$\nabla\cdot\left[\frac{1}{2}(\boldsymbol{B}\cdot\boldsymbol{H}+\boldsymbol{D}\cdot\boldsymbol{E})\boldsymbol{I}-\boldsymbol{D}\boldsymbol{E}-\boldsymbol{B}\boldsymbol{H}\right]$	$\nabla\cdot\left[\left(\frac{1}{2}\rho_0v^2-\frac{1}{2}\boldsymbol{T}:\boldsymbol{s}:\boldsymbol{T}\right)\boldsymbol{I}-\rho_0\boldsymbol{v}\boldsymbol{v}\right]$
$+\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B})=-\boldsymbol{J}_{\mathrm{e}}\times\boldsymbol{B}-\rho_{\mathrm{e}}\boldsymbol{E}+\boldsymbol{J}_{\mathrm{m}}\times\boldsymbol{D}-\rho_{\mathrm{m}}\boldsymbol{H}$	$+\frac{\partial}{\partial t}(\boldsymbol{S}\cdot\boldsymbol{p})=-\rho_0\boldsymbol{f}\cdot\boldsymbol{S}$
$\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B})+\boldsymbol{D}\times\nabla\times\boldsymbol{E}-(\nabla\times\boldsymbol{H})\times\boldsymbol{B}+\boldsymbol{J}_{\mathrm{e}}\times\boldsymbol{B}-\boldsymbol{J}_{\mathrm{m}}\times\boldsymbol{D}=0$	$\frac{\partial}{\partial t}(\boldsymbol{S}\cdot\boldsymbol{p})-\boldsymbol{S}\cdot(\nabla\cdot\boldsymbol{T})-(\nabla_{\mathrm{s}}\boldsymbol{v})\boldsymbol{p}+\rho_0\boldsymbol{f}\cdot\boldsymbol{S}=0$

5. Reciprocity equations

As a universal theorem of linear physical systems, reciprocity theorems interrelate the quantities that characterize two admissible physical or computational states, which could occur in the same domain in space-time. Similar to the conservation equations, an analogical study can also be employed between the reciprocity equations for electromagnetic and elastic wave fields.

Consider a pair of electric and magnetic sources marked by 1 and 2, as shown in Fig. 3. The corresponding wave equations are given by

$$\frac{\partial}{\partial t}(\mu_0 \boldsymbol{H}_1) = -\nabla \times \boldsymbol{E}_1 - \boldsymbol{J}_{\mathrm{m}1}, \tag{40}$$
$$\frac{\partial}{\partial t}(\mu_0 \boldsymbol{H}_2) = -\nabla \times \boldsymbol{E}_2 - \boldsymbol{J}_{\mathrm{m}2}, \tag{41}$$
$$\frac{\partial}{\partial t}(\varepsilon_0 \boldsymbol{E}_1) = \nabla \times \boldsymbol{H}_1 - \boldsymbol{J}_{\mathrm{e}1}, \tag{42}$$
$$\frac{\partial}{\partial t}(\varepsilon_0 \boldsymbol{E}_2) = \nabla \times \boldsymbol{H}_2 - \boldsymbol{J}_{\mathrm{e}2}. \tag{43}$$

Assuming that $\boldsymbol{J}_{\mathrm{e}1,2}(\boldsymbol{r},t)$ and $\boldsymbol{J}_{\mathrm{m}1,2}(\boldsymbol{r},t)$ are both time-harmonic sources with the same angular frequency ω such that $\boldsymbol{J}_{\mathrm{e}1,2}(\boldsymbol{r},t) = \boldsymbol{J}_{\mathrm{e}1,2}(\boldsymbol{r})\exp(-\mathrm{i}\omega t)$ and $\boldsymbol{J}_{\mathrm{m}1,2}(\boldsymbol{r},t) = \boldsymbol{J}_{\mathrm{m}1,2}(\boldsymbol{r})\exp(-\mathrm{i}\omega t)$, the electric and magnetic fields will both become time-harmonic quantities as $\boldsymbol{E}_{1,2}(\boldsymbol{r},t) = \boldsymbol{E}_{1,2}(\boldsymbol{r})\exp(-\mathrm{i}\omega t)$ and $\boldsymbol{H}_{1,2}(\boldsymbol{r},t) = \boldsymbol{H}_{1,2}(\boldsymbol{r})\exp(-\mathrm{i}\omega t)$, respectively. For the steady time-harmonic case, Eqs. (40)–(43) reduce to

$$-\mathrm{i}\omega\mu_0 \boldsymbol{H}_1 = -\nabla \times \boldsymbol{E}_1 - \boldsymbol{J}_{\mathrm{m}1}, \tag{44}$$
$$-\mathrm{i}\omega\mu_0 \boldsymbol{H}_2 = -\nabla \times \boldsymbol{E}_2 - \boldsymbol{J}_{\mathrm{m}2}, \tag{45}$$
$$-\mathrm{i}\omega\varepsilon_0 \boldsymbol{E}_1 = \nabla \times \boldsymbol{H}_1 - \boldsymbol{J}_{\mathrm{e}1}, \tag{46}$$
$$-\mathrm{i}\omega\varepsilon_0 \boldsymbol{E}_2 = \nabla \times \boldsymbol{H}_2 - \boldsymbol{J}_{\mathrm{e}2}. \tag{47}$$

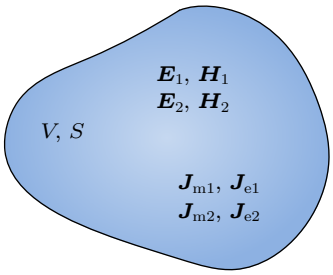


Fig. 3. A sketch of two electromagnetic wave systems in vacuum, with $\boldsymbol{J}_{\mathrm{m}1,2}$, $\boldsymbol{J}_{\mathrm{e}1,2}$ and $\boldsymbol{E}_{1,2}$, $\boldsymbol{H}_{1,2}$ denoting the source and field quantities, respectively.

As the most well-known type of reciprocity equations for electromagnetic waves, the Lorentz reciprocity theorem has been widely applied in various fields, such as communication, antenna signal transmission and electromagnetic imaging, which can be derived from Eqs. (44)–(47) as[35]

$$\begin{aligned}&\nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1)\\ &= -\boldsymbol{J}_{\mathrm{m}1} \cdot \boldsymbol{H}_2 + \boldsymbol{J}_{\mathrm{m}2} \cdot \boldsymbol{H}_1 - \boldsymbol{J}_{\mathrm{e}2} \cdot \boldsymbol{E}_1 + \boldsymbol{J}_{\mathrm{e}1} \cdot \boldsymbol{E}_2.\end{aligned} \tag{48}$$

With $\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1$ and $-\boldsymbol{J}_{\mathrm{m}1} \cdot \boldsymbol{H}_2 + \boldsymbol{J}_{\mathrm{m}2} \cdot \boldsymbol{H}_1 - \boldsymbol{J}_{\mathrm{e}2} \cdot \boldsymbol{E}_1 + \boldsymbol{J}_{\mathrm{e}1} \cdot \boldsymbol{E}_2$ defined as the mutual energy flux density and the mutual force density of the electromagnetic field, respectively, the left-hand side of Eq. (48) corresponds to the mutual energy flow through the surface, and the right-hand side corresponds to the source mutual power density contributed by electric and magnetic currents.

Similarly, by assuming a pair of force sources marked by 1 and 2, the elastic wave equations can be expressed as

$$\frac{\partial(\rho_0 \boldsymbol{v}_1)}{\partial t} = \nabla \cdot \boldsymbol{T}_1 + \rho_0 \boldsymbol{f}_1, \tag{49}$$
$$\frac{\partial(\rho_0 \boldsymbol{v}_2)}{\partial t} = \nabla \cdot \boldsymbol{T}_2 + \rho_0 \boldsymbol{f}_2, \tag{50}$$
$$\frac{\partial}{\partial t}(\boldsymbol{s} : \boldsymbol{T}_1) = \nabla_{\mathrm{s}} \boldsymbol{v}_1, \tag{51}$$
$$\frac{\partial}{\partial t}(\boldsymbol{s} : \boldsymbol{T}_2) = \nabla_{\mathrm{s}} \boldsymbol{v}_2. \tag{52}$$

For the steady time-harmonic state, Eqs. (49)–(52) reduce to

$$-\mathrm{i}\omega\rho_0 \boldsymbol{v}_1 = \nabla \cdot \boldsymbol{T}_1 + \rho_0 \boldsymbol{f}_1, \tag{53}$$
$$-\mathrm{i}\omega\rho_0 \boldsymbol{v}_2 = \nabla \cdot \boldsymbol{T}_2 + \rho_0 \boldsymbol{f}_2, \tag{54}$$
$$-\mathrm{i}\omega \boldsymbol{s} : \boldsymbol{T}_1 = \nabla_{\mathrm{s}} \boldsymbol{v}_1, \tag{55}$$
$$-\mathrm{i}\omega \boldsymbol{s} : \boldsymbol{T}_2 = \nabla_{\mathrm{s}} \boldsymbol{v}_2. \tag{56}$$

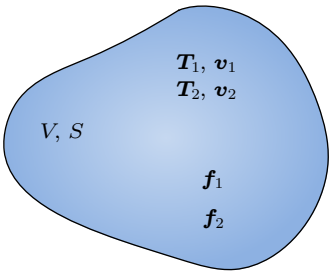


Fig. 4. A sketch of two elastic wave systems in a solid medium, with $\boldsymbol{f}_{1,2}$ and $\boldsymbol{T}_{1,2}$, $\boldsymbol{v}_{1,2}$ denoting the source and field quantities, respectively.

Performing the electromagnetic-to-elastic substitution on Eq. (48), we have the reciprocity theorem for elastic wave fields as[37]

$$\nabla \cdot (-\boldsymbol{v}_2 \cdot \boldsymbol{T}_1 + \boldsymbol{v}_1 \cdot \boldsymbol{T}_2) = \rho_0 \boldsymbol{f}_1 \cdot \boldsymbol{v}_2 - \rho_0 \boldsymbol{f}_2 \cdot \boldsymbol{v}_1. \tag{57}$$

Similarly, we can define $-\boldsymbol{v}_2 \cdot \boldsymbol{T}_1 + \boldsymbol{v}_1 \cdot \boldsymbol{T}_2$ and $\rho_0 \boldsymbol{f}_1 \cdot \boldsymbol{v}_2 - \rho_0 \boldsymbol{f}_2 \cdot \boldsymbol{v}_1$ as the mutual energy flux density and the mutual power density of the elastic wave field, respectively. Hereafter, the left-hand side of Eq. (57) corresponds to the mutual energy flow through the surface, and the right-hand side corresponds to the source mutual power density contributed only by the external force.

Equation (57) can also be validated from Eqs. (53)–(56). By multiplying Eqs. (53) and (56) by $-\boldsymbol{v}_2$ and $-\boldsymbol{T}_1$, respectively, we find

$$\mathrm{i}\omega\rho_0 \boldsymbol{v}_2 \cdot \boldsymbol{v}_1 = -\boldsymbol{v}_2 \cdot (\nabla \cdot \boldsymbol{T}_1) - \rho_0 \boldsymbol{f}_1 \cdot \boldsymbol{v}_2, \tag{58}$$
$$\mathrm{i}\omega \boldsymbol{T}_1 : \boldsymbol{s} : \boldsymbol{T}_2 = -\boldsymbol{T}_1 : \nabla_{\mathrm{s}} \boldsymbol{v}_2. \tag{59}$$

Adding up Eqs. (58) and (59), we obtain

$$\nabla \cdot (-\boldsymbol{v}_2 \cdot \boldsymbol{T}_1) = \mathrm{i}\omega\rho_0 \boldsymbol{v}_2 \cdot \boldsymbol{v}_1 + \mathrm{i}\omega \boldsymbol{T}_1 : \boldsymbol{s} : \boldsymbol{T}_2 + \rho_0 \boldsymbol{f}_1 \cdot \boldsymbol{v}_2. \tag{60}$$

Exchanging the subscripts 1 and 2, we get

$$\nabla \cdot (-\boldsymbol{v}_1 \cdot \boldsymbol{T}_2) = \mathrm{i}\omega\rho_0 \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \mathrm{i}\omega \boldsymbol{T}_2 : \boldsymbol{s} : \boldsymbol{T}_1 + \rho_0 \boldsymbol{f}_2 \cdot \boldsymbol{v}_1. \tag{61}$$

Subtracting Eq. (61) from Eq. (6) finally yields the reciprocity theorem given by Eq. (57).

Actually, the electromagnetic field can be described in terms of either energy or momentum. In view of this, the momentum-based reciprocity theorem was first developed by Liu *et al*. based on Maxwell equations, which can be expressed as[16]

$$\begin{aligned}&\nabla\cdot\left(\boldsymbol{D}_1\cdot\boldsymbol{E}_2\boldsymbol{I}-\boldsymbol{H}_1\cdot\boldsymbol{B}_2\boldsymbol{I}-\boldsymbol{D}_1\boldsymbol{E}_2\right.\\&\left.-\boldsymbol{E}_2\boldsymbol{D}_1+\boldsymbol{H}_1\boldsymbol{B}_2+\boldsymbol{H}_2\boldsymbol{B}_1\right)\\=\;&\boldsymbol{J}_{\mathrm{e}1}\times\boldsymbol{B}_2+\boldsymbol{J}_{\mathrm{e}2}\times\boldsymbol{B}_1-\rho_{\mathrm{e}1}\boldsymbol{E}_2-\rho_{\mathrm{e}2}\boldsymbol{E}_1+\boldsymbol{J}_{\mathrm{m}1}\times\boldsymbol{D}_2\\&+\boldsymbol{J}_{\mathrm{m}2}\times\boldsymbol{D}_1+\rho_{\mathrm{m}1}\boldsymbol{H}_2+\rho_{\mathrm{m}2}\boldsymbol{H}_1,\end{aligned}\tag{62}$$

where $\rho_{\mathrm{e}1,2}$ and $\rho_{\mathrm{m}1,2}$ are the volume densities of the electric and magnetic charges, respectively. With $\boldsymbol{D}_1\cdot\boldsymbol{E}_2\boldsymbol{I}-\boldsymbol{H}_1\cdot\boldsymbol{B}_2\boldsymbol{I}-\boldsymbol{D}_1\boldsymbol{E}_2-\boldsymbol{E}_2\boldsymbol{D}_1+\boldsymbol{H}_1\boldsymbol{B}_2+\boldsymbol{H}_2\boldsymbol{B}_1$ and $\boldsymbol{J}_{\mathrm{e}1}\times\boldsymbol{B}_2+\boldsymbol{J}_{\mathrm{e}2}\times\boldsymbol{B}_1-\rho_{\mathrm{e}1}\boldsymbol{E}_2-\rho_{\mathrm{e}2}\boldsymbol{E}_1+\boldsymbol{J}_{\mathrm{m}1}\times\boldsymbol{D}_2+\boldsymbol{J}_{\mathrm{m}2}\times\boldsymbol{D}_1+\rho_{\mathrm{m}1}\boldsymbol{H}_2+\rho_{\mathrm{m}2}\boldsymbol{H}_1$ defined as the mutual momentum flux density and the mutual force density of the electromagnetic field, respectively, the left-hand side of Eq. (62) corresponds to the mutual momentum flow through the surface, and the right-hand side corresponds to the source mutual force density contributed by currents and charges.

The charge sources in Eq. (62) can also be replaced by the field quantities using the following divergence equations:

$$\nabla\cdot\boldsymbol{D}_1=\rho_{\mathrm{e}1},\tag{63}$$

$$\nabla\cdot\boldsymbol{D}_2=\rho_{\mathrm{e}2},\tag{64}$$

$$\nabla\cdot\boldsymbol{B}_1=\rho_{\mathrm{m}1},\tag{65}$$

$$\nabla\cdot\boldsymbol{B}_2=\rho_{\mathrm{m}2}.\tag{66}$$

Therefore, Eq. (62) can be rewritten as

$$\begin{aligned}&\nabla\cdot\left(\boldsymbol{D}_1\cdot\boldsymbol{E}_2\boldsymbol{I}-\boldsymbol{H}_1\cdot\boldsymbol{B}_2\boldsymbol{I}-\boldsymbol{D}_1\boldsymbol{E}_2\right.\\&\left.-\boldsymbol{E}_2\boldsymbol{D}_1+\boldsymbol{H}_1\boldsymbol{B}_2+\boldsymbol{H}_2\boldsymbol{B}_1\right)\\=\;&\boldsymbol{J}_{\mathrm{e}1}\times\boldsymbol{B}_2+\boldsymbol{J}_{\mathrm{e}2}\times\boldsymbol{B}_1-(\nabla\cdot\boldsymbol{D}_1)\,\boldsymbol{E}_2-(\nabla\cdot\boldsymbol{D}_2)\,\boldsymbol{E}_1\\&+\boldsymbol{J}_{\mathrm{m}1}\times\boldsymbol{D}_2+\boldsymbol{J}_{\mathrm{m}2}\times\boldsymbol{D}_1+(\nabla\cdot\boldsymbol{B}_1)\,\boldsymbol{H}_2+(\nabla\cdot\boldsymbol{B}_2)\,\boldsymbol{H}_1.\end{aligned}\tag{67}$$

By considering the following differential identities:

$$\begin{aligned}&\nabla\cdot[\boldsymbol{D}_1\cdot\boldsymbol{E}_2\boldsymbol{I}-\boldsymbol{D}_1\boldsymbol{E}_2-\boldsymbol{E}_2\boldsymbol{D}_1]\\=\;&\boldsymbol{D}_1\times\nabla\times\boldsymbol{E}_2+\boldsymbol{D}_2\times\nabla\times\boldsymbol{E}_1\\&-(\nabla\cdot\boldsymbol{D}_1)\,\boldsymbol{E}_2-(\nabla\cdot\boldsymbol{D}_2)\,\boldsymbol{E}_1,\end{aligned}\tag{68}$$

$$\begin{aligned}&\nabla\cdot[\boldsymbol{H}_1\cdot\boldsymbol{B}_2\boldsymbol{I}-\boldsymbol{H}_1\boldsymbol{B}_2-\boldsymbol{H}_2\boldsymbol{B}_1]\\=\;&\boldsymbol{B}_1\times\nabla\times\boldsymbol{H}_2+\boldsymbol{B}_2\times\nabla\times\boldsymbol{H}_1\\&-(\nabla\cdot\boldsymbol{B}_1)\,\boldsymbol{H}_2-(\nabla\cdot\boldsymbol{B}_2)\,\boldsymbol{H}_1,\end{aligned}\tag{69}$$

equation (67) can be further simplified as

$$\begin{aligned}&\boldsymbol{J}_{\mathrm{e}1}\times\boldsymbol{B}_2+\boldsymbol{J}_{\mathrm{e}2}\times\boldsymbol{B}_1+\boldsymbol{J}_{\mathrm{m}1}\times\boldsymbol{D}_2+\boldsymbol{J}_{\mathrm{m}2}\times\boldsymbol{D}_1\\&+\boldsymbol{B}_1\times\nabla\times\boldsymbol{H}_2+\boldsymbol{B}_2\times\nabla\times\boldsymbol{H}_1-\boldsymbol{D}_1\times\nabla\times\boldsymbol{E}_2\\&-\boldsymbol{D}_2\times\nabla\times\boldsymbol{E}_1=0.\end{aligned}\tag{70}$$

Based on the electromagnetic-to-elastic substitution, the momentum reciprocity theorem for an elastic wave field can be obtained as

$$\begin{aligned}&-\rho_0\boldsymbol{f}_1\cdot\boldsymbol{S}_2-\rho_0\boldsymbol{f}_2\cdot\boldsymbol{S}_1+\boldsymbol{p}_1\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_2+\boldsymbol{p}_2\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_1\\&+\boldsymbol{S}_1\cdot(\nabla\cdot\boldsymbol{T}_2)+\boldsymbol{S}_2\cdot(\nabla\cdot\boldsymbol{T}_1)=0.\end{aligned}\tag{71}$$

By considering the relations $\boldsymbol{p}_{1,2}=\rho_0\boldsymbol{v}_{1,2}$ and $\boldsymbol{S}_{1,2}=\boldsymbol{s}:\boldsymbol{T}_{1,2}$, Eq. (71) can be rewritten as

$$\begin{aligned}&-\rho_0\boldsymbol{f}_1\cdot\boldsymbol{S}_2-\rho_0\boldsymbol{f}_2\cdot\boldsymbol{S}_1+\rho_0\boldsymbol{v}_1\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_2+\rho_0\boldsymbol{v}_2\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_1\\&+(\boldsymbol{s}:\boldsymbol{T}_1)\cdot(\nabla\cdot\boldsymbol{T}_2)+(\boldsymbol{s}:\boldsymbol{T}_2)\cdot(\nabla\cdot\boldsymbol{T}_1)=0.\end{aligned}\tag{72}$$

By combining Eq. (72) with the following differential identities:

$$\begin{aligned}&\nabla\cdot[\boldsymbol{v}_1\cdot\boldsymbol{v}_2\boldsymbol{I}-\boldsymbol{v}_1\boldsymbol{v}_2-\boldsymbol{v}_2\boldsymbol{v}_1]\\=\;&-(\nabla_{\mathrm{s}}\boldsymbol{v}_1)\cdot\boldsymbol{v}_2-(\nabla_{\mathrm{s}}\boldsymbol{v}_2)\cdot\boldsymbol{v}_1,\end{aligned}\tag{73}$$

$$\begin{aligned}&\nabla\cdot[(\boldsymbol{T}_1:\boldsymbol{s}:\boldsymbol{T}_2)\,\boldsymbol{I}]\\=\;&(\boldsymbol{s}:\boldsymbol{T}_1)\cdot(\nabla\cdot\boldsymbol{T}_2)+(\boldsymbol{s}:\boldsymbol{T}_2)\cdot(\nabla\cdot\boldsymbol{T}_1),\end{aligned}\tag{74}$$

equation (72) can be rewritten as

$$\begin{aligned}&\nabla\cdot[(\boldsymbol{T}_1:\boldsymbol{s}:\boldsymbol{T}_2+\rho_0\boldsymbol{v}_1\cdot\boldsymbol{v}_2)\,\boldsymbol{I}-\rho_0\boldsymbol{v}_1\boldsymbol{v}_2-\rho_0\boldsymbol{v}_2\boldsymbol{v}_1]\\=\;&\rho_0\boldsymbol{f}_1\cdot\boldsymbol{S}_2+\rho_0\boldsymbol{f}_2\cdot\boldsymbol{S}_1,\end{aligned}\tag{75}$$

where we have used the relations $(\nabla_{\mathrm{s}}\boldsymbol{v}_1)\cdot\boldsymbol{v}_2=-\boldsymbol{v}_2\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_1$ and $(\nabla_{\mathrm{s}}\boldsymbol{v}_2)\cdot\boldsymbol{v}_1=-\boldsymbol{v}_1\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_2$. Equation (75) can be regarded as the elastic counterpart of Eq. (62). Similarly, $(\boldsymbol{T}_1:\boldsymbol{s}:\boldsymbol{T}_2+\rho_0\boldsymbol{v}_1\cdot\boldsymbol{v}_2)\,\boldsymbol{I}-\rho_0\boldsymbol{v}_1\boldsymbol{v}_2-\rho_0\boldsymbol{v}_2\boldsymbol{v}_1$ and $\rho_0\boldsymbol{f}_1\cdot\boldsymbol{S}_2+\rho_0\boldsymbol{f}_2\cdot\boldsymbol{S}_1$ can be defined as the mutual momentum flux density and the mutual force density of the elastic wave field, respectively. Hereafter, the left-hand side of Eq. (75) corresponds to the mutual momentum flow through the surface, and the right-hand side corresponds to the source mutual force density.

We can also derive the momentum reciprocity theorem based on Eqs. (53)–(56). Multiply Eqs. (53) and (56) by $\boldsymbol{S}_2$ and $\rho_0\boldsymbol{v}_1$, respectively, yielding

$$-\mathrm{i}\omega\boldsymbol{S}_2\cdot(\rho_0\boldsymbol{v}_1)=\boldsymbol{S}_2\cdot(\nabla\cdot\boldsymbol{T}_1)+\boldsymbol{S}_2\cdot(\rho_0\boldsymbol{f}_1),\tag{76}$$

$$-\mathrm{i}\omega\rho_0\boldsymbol{v}_1\cdot(\boldsymbol{s}:\boldsymbol{T}_2)=\rho_0\boldsymbol{v}_1\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_2.\tag{77}$$

By adding Eqs. (76) and (77) and considering the relations $\boldsymbol{S}_2=\boldsymbol{s}:\boldsymbol{T}_2$, $\boldsymbol{S}_2\cdot(\rho_0\boldsymbol{v}_1)=-(\rho_0\boldsymbol{v}_1)\cdot\boldsymbol{S}_2$ and $\boldsymbol{S}_2\cdot(\rho_0\boldsymbol{f}_1)=-\rho_0\boldsymbol{f}_1\cdot\boldsymbol{S}_2$, we obtain

$$\rho_0\boldsymbol{v}_1\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_2+\boldsymbol{S}_2\cdot(\nabla\cdot\boldsymbol{T}_1)=\rho_0\boldsymbol{f}_1\cdot\boldsymbol{S}_2.\tag{78}$$

By exchanging the subscripts 1 and 2, we get

$$\rho_0\boldsymbol{v}_2\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_1+\boldsymbol{S}_1\cdot(\nabla\cdot\boldsymbol{T}_2)=\rho_0\boldsymbol{f}_2\cdot\boldsymbol{S}_1.\tag{79}$$

Adding Eqs. (78) and (79) finally yields the reciprocity theorem given by Eq. (71).

The analogous relations of reciprocity equations between electromagnetic and elastic wave fields are given in Table 3, where the two forms of the momentum reciprocity equations are also presented for reference.

Table 3. Analogy of reciprocity equations for electromagnetic and elastic wave fields.

Electromagnetic field	Elastic wave field
$\nabla\cdot(\boldsymbol{E}_1\times\boldsymbol{H}_2-\boldsymbol{E}_2\times\boldsymbol{H}_1)=-\boldsymbol{J}_{\mathrm{m}1}\cdot\boldsymbol{H}_2+\boldsymbol{J}_{\mathrm{m}2}\cdot\boldsymbol{H}_1-\boldsymbol{J}_{\mathrm{e}2}\cdot\boldsymbol{E}_1+\boldsymbol{J}_{\mathrm{e}1}\cdot\boldsymbol{E}_2$	$\nabla\cdot(-\boldsymbol{v}_2\cdot\boldsymbol{T}_1+\boldsymbol{v}_1\cdot\boldsymbol{T}_2)=\rho_0\boldsymbol{f}_1\cdot\boldsymbol{v}_2-\rho_0\boldsymbol{f}_2\cdot\boldsymbol{v}_1$
$\nabla\cdot\begin{pmatrix}\boldsymbol{D}_1\cdot\boldsymbol{E}_2\boldsymbol{I}-\boldsymbol{H}_1\cdot\boldsymbol{B}_2\boldsymbol{I}-\boldsymbol{D}_1\boldsymbol{E}_2\\-\boldsymbol{E}_2\boldsymbol{D}_1+\boldsymbol{H}_1\boldsymbol{B}_2+\boldsymbol{H}_2\boldsymbol{B}_1\end{pmatrix}$	$\nabla\cdot\begin{bmatrix}(\boldsymbol{T}_1:\boldsymbol{s}:\boldsymbol{T}_2+\rho_0\boldsymbol{v}_1\cdot\boldsymbol{v}_2)\boldsymbol{I}\\-\rho_0\boldsymbol{v}_1\boldsymbol{v}_2-\rho_0\boldsymbol{v}_2\boldsymbol{v}_1\end{bmatrix}$
$=\boldsymbol{J}_{\mathrm{e}1}\times\boldsymbol{B}_2+\boldsymbol{J}_{\mathrm{e}2}\times\boldsymbol{B}_1-\rho_{\mathrm{e}1}\boldsymbol{E}_2-\rho_{\mathrm{e}2}\boldsymbol{E}_1$	$=\rho_0\boldsymbol{f}_1\cdot\boldsymbol{S}_2+\rho_0\boldsymbol{f}_2\cdot\boldsymbol{S}_1$
$+\boldsymbol{J}_{\mathrm{m}1}\times\boldsymbol{D}_2+\boldsymbol{J}_{\mathrm{m}2}\times\boldsymbol{D}_1+\rho_{\mathrm{m}1}\boldsymbol{H}_2+\rho_{\mathrm{m}2}\boldsymbol{H}_1$	
$\boldsymbol{J}_{\mathrm{e}1}\times\boldsymbol{B}_2+\boldsymbol{J}_{\mathrm{e}2}\times\boldsymbol{B}_1+\boldsymbol{J}_{\mathrm{m}1}\times\boldsymbol{D}_2+\boldsymbol{J}_{\mathrm{m}2}\times\boldsymbol{D}_1$	$-\rho_0\boldsymbol{f}_1\cdot\boldsymbol{S}_2-\rho_0\boldsymbol{f}_2\cdot\boldsymbol{S}_1$
$+\boldsymbol{B}_1\times\nabla\times\boldsymbol{H}_2+\boldsymbol{B}_2\times\nabla\times\boldsymbol{H}_1$	$+\boldsymbol{p}_2\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_1+\boldsymbol{p}_1\cdot\nabla_{\mathrm{s}}\boldsymbol{v}_2$
$-\boldsymbol{D}_1\times\nabla\times\boldsymbol{E}_2-\boldsymbol{D}_2\times\nabla\times\boldsymbol{E}_1=0$	$+\boldsymbol{S}_1\cdot(\nabla\cdot\boldsymbol{T}_2)+\boldsymbol{S}_2\cdot(\nabla\cdot\boldsymbol{T}_1)=0$

The energy-based and momentum-based reciprocity theorems, which reveal the correlations between two elastic wave systems from the perspectives of energy and momentum, respectively, are expected to be useful in the study of forward and inverse scattering problems in relevant fields, such as medical imaging, non-destructive evaluation, acoustic microscopy, seismology and exploratory geophysics. Remarkable applications can also be found in the detection of defects and the determination of elastic constants in ultrasonics of solids.

6. Conclusions and perspectives

In this formalism, an analogical study is developed between electromagnetic and elastic wave fields. Based on Maxwell equations and elastodynamics, the basic linear wave equations are presented for electromagnetic and elastic waves, respectively. On this basis, a one-to-one correspondence principle is established between electromagnetic and elastic quantities and differential operations. In analogy with electromagnetism, the conservation laws of energy and momentum are derived for an elastic wave field by applying the electromagnetic-to-elastic substitution. Moreover, the energy-based and momentum-based reciprocity theorems are also deduced for an elastic wave in the time-harmonic state, which describe the interaction between two elastic wave sources in terms of energy and momentum, respectively. Although all the conservation and reciprocity equations can be derived from the basic wave equations, the analogical method based on the parameter substitution is more likely to be extended to a more generalized case. Moreover, it is easier for us to discern the physical meanings of different quantities in the equations. The theoretical results developed in this work can not only further complete the theory of elastodynamics, but also improve our understanding of the strong similarities between these two linear physical systems. This analogical study is relevant to several fields, such as medical imaging, non-destructive evaluation, acoustic microscopy, seismology and geophysics.

The electromagnetic wave in vacuum and the elastic wave in a solid material are analyzed in this work without considering any wave absorption effects. Nevertheless, for an electromagnetic wave in a conductor, the existence of the conduction current due to motion of free electrons will generate the Joule heat. Hence, the wave absorption effect occurs during propagation, with the power loss proportional to the conductivity. Similarly, for an elastic wave, the wave absorption effect stems from the viscous effect of the solid material, with the power loss proportional to the viscosity on the premise of linear absorption. On this basis, the analogical relations assume that the wave absorption effect can be derived between electromagnetic and elastic wave fields. Furthermore, for electromagnetic and elastic waves within waveguides, a similar analogical study can also be performed theoretically, based on the parameter substitution principle. However, the boundary conditions imposed on the inner walls of the waveguides must be considered in this case to obtain the field distribution within waveguides, combined with the basic wave equations of these two physical systems. Based on the current analysis, an illustrative model can be developed in a future study to draw a parallel between the propagation of electromagnetic and elastic waves within waveguides.

Acknowledgements

This work was funded by the National Natural Science Foundation of China (Grant No. 12404507), the Natural Science Research of Jiangsu Higher Education Institutions of China (Grant No. 24KJB140013), and the Scientific Startup Foundation of Nanjing Normal University (Grant No. 184080H201B49).

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