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Zhi-Jie Liu(刘志洁), Xing-Yan Fan(樊星言), Jie Zhou(周洁), Mi Xie(谢汨), and Jing-Ling Chen(陈景灵)
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Generalized Einstein–Podolsky–Rosen steering paradox

Zhi-Jie Liu(刘志洁)[1,†], Xing-Yan Fan(樊星言)[1,†], Jie Zhou(周洁)[2,†]
Mi Xie(谢汨)[3], and Jing-Ling Chen(陈景灵)[1,‡]

[1] *Theoretical Physics Division, Chern Institute of Mathematics and LPMC, Nankai University, Tianjin* 300071, *China*
[2] *College of Physics and Materials Science, Tianjin Normal University, Tianjin* 300382, *China*
[3] *Department of Physics, School of Science, Tianjin University, Tianjin* 300072, *China*

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Quantum paradoxes are essential means to reveal the incompatibility between quantum and classical theories, among which the Einstein–Podolsky–Rosen (EPR) steering paradox offers a sharper criterion for the contradiction between local-hidden-state model and quantum mechanics than the usual inequality-based method. In this work, we present a generalized EPR steering paradox, which predicts a contradictory equality "$2_{\mathrm{Q}} = (1+\delta)_{\mathrm{C}}$" ($0 \leq \delta < 1$) given by the quantum (Q) and classical (C) theories. For any N-qubit state in which the conditional state of the steered party is pure, we test the paradox through a two-setting steering protocol, and find that the state is steerable if some specific measurement requirements are satisfied. Moreover, our construction also enlightens the building of EPR steering inequality, which may contribute to some schemes for typical quantum teleportation and quantum key distributions.

Keywords: EPR steering, steering paradox, quantum entangled states

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1. Introduction

The concept of quantum nonlocality araised from the seminal paper of Einstein, Podolsky, and Rosen (EPR),[1] which questioned the completeness of quantum mechanical description for physical reality under the assumptions of locality and reality. As a peculiar nonlocality, EPR steering, also known as "the spooky action at a distance",[2,3] was initially imported by Schrödinger in his reply to EPR's paper.[4] After that, the notion of steering remained undiscussed for a long time, until Vujicic and Herbut extended Schrödinger's results to the system of continuous variable.[5] Subsequently, Reid proved the possibility of the EPR paradox by performing orthogonal phase measurements on the two output beams of a non-simplex parametric amplifier, which led to an EPR-type argument.[6] In addition, Verstraete pointed out the connection of Schrödinger's idea with quantum teleportation and entanglement transformation.[7]

Mathematically, quantum nonlocality[7] is defined by the corresponding classical model. For example, entanglement[8] is defined by the separable state model,[9,10] the classical model corresponding to Bell nonlocality[11] is the local-hidden-variable (LHV) model.[12–14] In order to depict EPR steering rigorously, Wiseman *et al.* proposed the local-hidden-state (LHS) model and thus gave an operational definition of steering using a task with two parties (say Alice and Bob) sharing a quantum state,[15,16] the immediate effect of this nonlocality is that Alice can steer the particles of Bob, according to the measurement postulation of quantum mechanics. However, the LHS model assumes that the particles in Bob's hand is specific, though Bob does not know it. Therefore, Alice's ability to steer Bob's state is an illusion that cannot be observed experimentally. If Alice convince Bob that the two particles in their hands are entangled and the LHS model cannot describe the particles in Bob's hand, then Alice is capable of (EPR) steering Bob. These works[15,16] have stimulated interest in the research of EPR steering, for examples, how to

(i) determine the steerability of a given state;[17–24]

(ii) characterize the border of quantum steering with the other two types of nonlocality;[25,26]

(iii) apply steering to realistic quantum information tasks.[27–35]

For the research type-(i) mentioned above, there are generally two effective methods to determine whether a quantum state is steerable. The first one is based on the EPR steering inequality, for instance, the linear EPR steering inequality,[17,36,37] and the chained EPR steering inequalities,[18,19] which were constructed compared to the Bell chain inequality,[38] optimizing the visibility of the Werner state.[39] The other is the approach without inequalities, also dubbed the all-versus-nothing (AVN) proof, which reveals the contradiction between classical theory and quantum mechanics more intuitively than the inequality approach. The AVN proof[40] was first built as an elegant argument for the nonexistence of the LHV model. Analogous to the AVN argument for Bell nonlocality without inequalities, Chen *et al.* presented the AVN proof of EPR steering for any two-qubit

[†] These authors contributed equally to this work.
[‡] Corresponding author. E-mail: chenjl@nankai.edu.cn
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entangled state based on a two-setting steering protocol.[20] They then extracted the method for EPR steering paradox as the contradictory equality "$2_{\rm Q}=1_{\rm C}$" ("$2_{\rm Q}$" is the quantum prediction, and "$1_{\rm C}$" is the result of the LHS model) for any two-qubit pure state in the two-setting steering protocol.[21] Subsequently, the EPR steering paradox approach has been further investigated, e.g., "$2_{\rm Q}=1_{\rm C}$" for a particular four-qubit mixed state,[23] "$k_{\rm Q}=1_{\rm C}$" for an arbitrary two-qubit pure state in the k-setting steering protocol,[22] and "$2_{\rm Q}=1_{\rm C}$" for an arbitrary N-qubit entangled state.[24]

In this work, we propose a general EPR steering paradox, expressed as "$2_{\rm Q}=(1+\delta)_{\rm C}$" ($0\leq\delta<1$), for an arbitrary N-qubit entangled state, both pure and mixed. Considering that the conditional states of the steered party (Bob) are all pure, we propose a theorem for any N-qubit quantum steerable states in the two-setting protocol. We analyze all possible four cases of Bob's conditional states, corresponding to $\delta=0$ and $0<\delta<1$ respectively. It is noteworthy to mention that we find all the steerable states when Bob's conditional states are pure.

2. Results

2.1. A theorem on the EPR steering paradox

In order to investigate the existence of steering trait for a quantum state, we propose the generalized EPR steering paradox, the contradiction between the LHS model and EPR steering is used to determine whether one side, Alice, can steer the other side, Bob. The main result is summarized into a theorem.

Theorem 1 In the two-setting steering protocol $\{\hat{n}_1,\hat{n}_2\}$, Alice and Bob share an N-qubit state $\rho_{\rm AB}$. Assume that Alice measures along $\hat{n}_1$ and $\hat{n}_2$, corresponding to the results a, and a'; then Bob obtains the conditional states $\tilde{\rho}_a^{\hat{n}_1}$ and $\tilde{\rho}_{a'}^{\hat{n}_2}$, respectively. Considering the scenario where both sets of Bob's conditional states $\{\tilde{\rho}_a^{\hat{n}_1}\}$ and $\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$ are all pure, there will be a contradiction of "$2_{\rm Q}=(1+\delta)_{\rm C}$" (with $0\leq\delta<1$) if $\rho_{\rm AB}$ satisfies "the measurement requirement", *viz.*,

Bob obtains two sets of results $\{\tilde{\rho}_a^{\hat{n}_1}\}$ and $\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$ that are not exactly the same, i.e., $\{\tilde{\rho}_a^{\hat{n}_1}\}\neq\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$.

Proof We consider that Alice and Bob share an N-qubit entangled state

$$\rho_{\rm AB}=\sum_\alpha p_\alpha\left|\psi_{\rm AB}^{(\alpha)}\right\rangle\left\langle\psi_{\rm AB}^{(\alpha)}\right|, \tag{1}$$

with

$$\left|\psi_{\rm AB}^{(\alpha)}\right\rangle=\sum_i\left(s_i^{(\alpha)}\left|\phi_i\right\rangle\left|\eta_i^{(\alpha)}\right\rangle\right), \tag{2}$$

or

$$\left|\psi_{\rm AB}^{(\alpha)}\right\rangle=\sum_j\left(t_j^{(\alpha)}\left|\varphi_j\right\rangle\left|\varepsilon_j^{(\alpha)}\right\rangle\right). \tag{3}$$

Equations (2) and (3) are the expressions of $|\psi_{\rm AB}^{(\alpha)}\rangle$ under different representations $|\phi_i\rangle\otimes|\eta_i\rangle$ and $\left|\varphi_j\right\rangle\otimes\left|\varepsilon_j\right\rangle$, where $i,j=1,2,\ldots,2^M$. Alice keeps M $(M<N)$ particles, and Bob keeps $(N-M)$ particles.

In the two-setting steering protocol $\{\hat{n}_1,\hat{n}_2\}$, Alice performs 2^{M+1} projective measurements, and then Bob obtains the according unnormalized conditional states $\tilde{\rho}_a^{\hat{n}_\ell}={\rm tr}_A\left[\left(P_a^{\hat{n}_\ell}\otimes\mathbb{1}\right)\rho_{\rm AB}\right]$, where $\hat{n}_\ell$ $(\ell=1,2)$ is the measurement direction, a is the result of Alice, and $\mathbb{1}$ is a $2^{N-M}\times2^{N-M}$ identity matrix. Set $P_{a_i}^{\hat{n}_1}=|\phi_i\rangle\langle\phi_i|$ and $P_{a'_j}^{\hat{n}_2}=\left|\varphi_j\right\rangle\left\langle\varphi_j\right|$. After Alice's measurement, Bob obtains

$$\tilde{\rho}_{a_i}^{\hat{n}_1}=\sum_\alpha p_\alpha|s_i^{(\alpha)}|^2|\eta_i^{(\alpha)}\rangle\langle\eta_i^{(\alpha)}|, \tag{4}$$

or

$$\tilde{\rho}_{a'_j}^{\hat{n}_2}=\sum_\alpha p_\alpha|t_j^{(\alpha)}|^2|\varepsilon_j^{(\alpha)}\rangle\langle\varepsilon_j^{(\alpha)}|. \tag{5}$$

The mathematical definition of the LHS model is given by Wiseman *et al.*[15] If the LHS model can describe the state assemblage of Bob's unnormalized conditional states, then Bob's state can be described by

$$\tilde{\rho}_a^{\hat{n}}=\sum_\xi\wp(a|\hat{n},\xi)\,\wp_\xi\rho_\xi, \tag{6}$$

$$\sum_\xi\wp_\xi\rho_\xi=\rho_{\rm B}. \tag{7}$$

Here $\wp_\xi$ and $\wp(a|\hat{n},\xi)$ are probabilities satisfying $\sum_\xi\wp_\xi=1$, and $\sum_a\wp(a|\hat{n},\xi)=1$, for a fixed ξ, and $\rho_{\rm B}={\rm tr}_A(\rho_{\rm AB})$ is Bob's reduced density matrix.

If Bob's states have an LHS description, they satisfy Eqs. (6) and (7), which lead to 2^{M+1} equations

$$\begin{aligned}\tilde{\rho}_a^{\hat{n}_1}&=\sum_\xi\wp(a|\hat{n}_1,\xi)\,\wp_\xi\rho_\xi,\\ \tilde{\rho}_{a'}^{\hat{n}_2}&=\sum_\xi\wp\left(a'|\hat{n}_2,\xi\right)\wp_\xi\rho_\xi.\end{aligned} \tag{8}$$

The quantum value 2 can be attained through summing Eqs. (4) and (5) and then taking the trace. However the maximum value of the LHS result is 2 if and only if

$$\begin{aligned}\sum_a\sum_\xi\wp(a|\hat{n}_1,\xi)\,\wp_\xi\rho_\xi&=\sum_{a'}\sum_\xi\wp\left(a'|\hat{n}_2,\xi\right)\wp_\xi\rho_\xi\\&=\sum_\xi\wp_\xi\rho_\xi.\end{aligned}$$

But the equation "$2_{\rm Q}=2_{\rm C}$" is not contradictory, and under this circumstance we cannot determine whether Alice can steer Bob, since there may exist other measurement strategies for Alice to steer Bob, or produce the contradiction "$2_{\rm Q}=(1+\delta)_{\rm C}$", with $\delta\in[0,1)$.

Considering that the conditional states of Bob are all pure, which implies that $|\eta_i\rangle$ and $\left|\varepsilon_j\right\rangle$ are independent of α, i.e.,

$$\left|\psi_{\rm AB}^{(\alpha)}\right\rangle=\sum_i s_i^{(\alpha)}\left|\phi_i\right\rangle\left|\eta_i\right\rangle, \tag{9}$$

or

$$\left|\psi_{\mathrm{AB}}^{(\alpha)}\right\rangle=\sum_{j} t_{j}^{(\alpha)}\left|\varphi_{j}\right\rangle\left|\varepsilon_{j}\right\rangle. \tag{10}$$

Then the conditional states of Bob are

$$\begin{cases} \tilde{\rho}_{a_1}^{\hat{n}_1}=\sum_{\alpha} p_{\alpha}\left|s_1^{(\alpha)}\right|^2\left|\eta_1\right\rangle\left\langle\eta_1\right|, \\ \cdots, \\ \tilde{\rho}_{a_{2^M}}^{\hat{n}_1}=\sum_{\alpha} p_{\alpha}\left|s_{2^M}^{(\alpha)}\right|^2\left|\eta_{2^M}\right\rangle\left\langle\eta_{2^M}\right|, \\ \tilde{\rho}_{a_1'}^{\hat{n}_2}=\sum_{\alpha} p_{\alpha}\left|t_1^{(\alpha)}\right|^2\left|\varepsilon_1\right\rangle\left\langle\varepsilon_1\right|, \\ \cdots, \\ \tilde{\rho}_{a_{2^M}'}^{\hat{n}_2}=\sum_{\alpha} p_{\alpha}\left|t_{2^M}^{(\alpha)}\right|^2\left|\varepsilon_{2^M}\right\rangle\left\langle\varepsilon_{2^M}\right|. \end{cases} \tag{11}$$

Sufficiency "$2_{\mathrm{Q}}=2_{\mathrm{C}}$"$\Longrightarrow\{\tilde{\rho}_a^{\hat{n}_1}\}=\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$. To prove the proposition $\{\tilde{\rho}_a^{\hat{n}_1}\}\neq\{\tilde{\rho}_{a'}^{\hat{n}_2}\}\Longrightarrow$"$2_{\mathrm{Q}}=(1+\delta)_{\mathrm{C}}$", we simplify the process by demonstrating the accuracy of the contrapositive.

Since the density matrix of a pure state can only be expanded by itself, that is each $\tilde{\rho}_a^{\hat{n}_1}$ and $\tilde{\rho}_{a'}^{\hat{n}_2}$ in Eq. (8) can only be described by a definite hidden state. Without loss of generality, suppose that Bob's conditional states are completely different in the set involving the same measurement direction $\{\tilde{\rho}_a^{\hat{n}_1}\}$ or $\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$. Further equation (8) can be written as

$$\begin{cases} \tilde{\rho}_{a_1}^{\hat{n}_1}=\wp_1\rho_1, \\ \tilde{\rho}_{a_2}^{\hat{n}_1}=\wp_2\rho_2, \\ \cdots, \\ \tilde{\rho}_{a_{2^M}}^{\hat{n}_1}=\wp_{2^M}\rho_{2^M}, \end{cases} \quad \begin{cases} \tilde{\rho}_{a_1'}^{\hat{n}_2}=\wp_1'\rho_1', \\ \tilde{\rho}_{a_2'}^{\hat{n}_2}=\wp_2'\rho_2', \\ \cdots, \\ \tilde{\rho}_{a_{2^M}'}^{\hat{n}_2}=\wp_{2^M}'\rho_{2^M}'. \end{cases} \tag{12}$$

Summing Eq. (12), we have

$$\tilde{\rho}_a^{\hat{n}_1}+\tilde{\rho}_{a'}^{\hat{n}_2}=\sum_{i=1}^{2^M}\wp_i\rho_i+\sum_{i=1}^{2^M}\wp_i'\rho_i'.$$

The equation "$2_{\mathrm{Q}}=2_{\mathrm{C}}$" requests

$$\sum_{i=1}^{2^M}\wp_i'\rho_i'=\sum_{i=1}^{2^M}\wp_i\rho_i=\sum_{\xi}\wp_\xi\rho_\xi. \tag{13}$$

It means that $\{\tilde{\rho}_a^{\hat{n}_1}\}=\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$, i.e., the measurement requirement is not satisfied. Note that there is also no contradiction appearing if Bob has the same conditional state in the same direction measurement result $\{\tilde{\rho}_a^{\hat{n}_1}\}$ or $\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$. As a result, if the measurement requirement is met, the paradoxical equality "$2_{\mathrm{Q}}=(1+\delta)_{\mathrm{C}}$", with $\delta\in[0,1)$, can be achieved.

Necessity $\{\tilde{\rho}_a^{\hat{n}_1}\}=\{\tilde{\rho}_{a'}^{\hat{n}_2}\}\Longrightarrow$"$2_{\mathrm{Q}}=2_{\mathrm{C}}$". Similarly, the proposition "$2_{\mathrm{Q}}=(1+\delta)_{\mathrm{C}}$" $\Longrightarrow\{\tilde{\rho}_a^{\hat{n}_1}\}\neq\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$ can be proven by showing the accuracy of its contrapositive.

Suppose that there are 2^M distinct states in $\{\tilde{\rho}_a^{\hat{n}_1}\}$ and that they are the same as the corresponding 2^M elements in $\{\tilde{\rho}_a^{\hat{n}_2}\}$. Bob's conditional states are

$$\begin{cases} \tilde{\rho}_{a_1}^{\hat{n}_1}=\sum_{\alpha} p_{\alpha}\left|s_1^{(\alpha)}\right|^2\left|\eta_1\right\rangle\left\langle\eta_1\right|, \\ \cdots, \\ \tilde{\rho}_{a_{2^M}}^{\hat{n}_1}=\sum_{\alpha} p_{\alpha}\left|s_{2^M}^{(\alpha)}\right|^2\left|\eta_{2^M}\right\rangle\left\langle\eta_{2^M}\right|, \\ \tilde{\rho}_{a_1'}^{\hat{n}_2}=\sum_{\alpha} p_{\alpha}\left|t_1^{(\alpha)}\right|^2\left|\eta_1\right\rangle\left\langle\eta_1\right|, \\ \cdots, \\ \tilde{\rho}_{a_{2^M}'}^{\hat{n}_2}=\sum_{\alpha} p_{\alpha}\left|t_{2^M}^{(\alpha)}\right|^2\left|\eta_{2^M}\right\rangle\left\langle\eta_{2^M}\right|. \end{cases} \tag{14}$$

There are only 2^M different pure states in the quantum result Eq. (14). It is sufficient to take ξ from 1 to 2^M in the LHS description, e.g.,

$$\begin{cases} \tilde{\rho}_a^{\hat{n}_1}=\sum\limits_{\xi=1}^{2^M}\wp(a|\hat{n}_1,\xi)\,\wp_\xi\rho_\xi, \\ \tilde{\rho}_{a'}^{\hat{n}_2}=\sum\limits_{\xi=1}^{2^M}\wp(a'|\hat{n}_2,\xi)\,\wp_\xi\rho_\xi. \end{cases} \tag{15}$$

Since Bob's states are all pure, each $\tilde{\rho}_a^{\hat{n}_\ell}$ in Eq. (15) contains only one term. Equation (15) can be written as

$$\begin{cases} \tilde{\rho}_{a_1}^{\hat{n}_1}=\wp_1\rho_1, \\ \cdots, \\ \tilde{\rho}_{a_{2^M}}^{\hat{n}_1}=\wp_{2^M}\rho_{2^M}, \\ \tilde{\rho}_{a_1'}^{\hat{n}_2}=\wp_1\rho_1, \\ \cdots, \\ \tilde{\rho}_{a_{2^M}'}^{\hat{n}_2}=\wp_{2^M}\rho_{2^M}. \end{cases} \tag{16}$$

Here we describe the same term with the same hidden state. And $\sum\limits_a\wp(a|\hat{n},\xi)=1$, so we have $\wp(a_1|\hat{n}_1,1)=\cdots=\wp\left(a_{2^M}|\hat{n}_1,2^M\right)=\wp(a_1'|\hat{n}_2,1)\cdots=\wp\left(a_{2^M}'|\hat{n}_2,2^M\right)=1$, and the others $\wp(a|\hat{n},\xi)=0$. Then we take the sum of Eq. (16) and take the trace, and get "$2_{\mathrm{Q}}=2_{\mathrm{C}}$". That means that there is no contradiction if $\{\tilde{\rho}_a^{\hat{n}_1}\}=\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$, i.e., the measurement requirement is not satisfied, and we cannot conclude whether Alice can steer Bob. Consequently, if there is a contradiction "$2_{\mathrm{Q}}=(1+\delta)_{\mathrm{C}}$", then the measurement condition must be satisfied.

In summary, we demonstrate that the measurement requirement is both a sufficient and necessary condition for the contradiction "$2_{\mathrm{Q}}=(1+\delta)_{\mathrm{C}}$", on the premise that Bob's conditional states are pure.

Notice that the EPR steering paradox "$2_{\mathrm{Q}}=(1+\delta)_{\mathrm{C}}$" mentioned in the theorem can be categorized into $\delta=0$ and $\delta\neq0$, based on which some analyses and examples of the theorem are given below.

2.2. Some cases of the theorem

Case 1 The sets of Bob's conditional states $\{\tilde{\rho}_a^{\hat{n}_1}\}$ and $\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$ are all different.

Example 1 We consider the example of an arbitrary two-qubit pure state presented by Chen *et al.*[21] Alice prepares an arbitrary two-qubit pure state $\rho_{\mathrm{AB}} = |\Psi(\theta)\rangle\langle\Psi(\theta)|$, where

$$|\Psi(\theta)\rangle = \cos\theta\,|0\rangle|0\rangle + \sin\theta\,|1\rangle|1\rangle, \tag{17}$$

with $\theta \in (0, \pi/2)$; she keeps one of the particle in Eq. (17), and sent the other to Bob. In the two-setting steering protocol $\{\hat{z}, \hat{x}\}$, Alice performs four possible projective measurements

$$\begin{aligned} P_0^{\hat{z}} &= |0\rangle\langle 0|, \quad P_1^{\hat{z}} = |1\rangle\langle 1|, \\ P_0^{\hat{x}} &= |+\rangle\langle +|, \quad P_1^{\hat{x}} = |-\rangle\langle -|, \end{aligned} \tag{18}$$

where $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$. After Alice's measurements, the four unnormalized conditional states corresponding to Bob are

$$\begin{aligned} \tilde{\rho}_0^{\hat{z}} &= \cos^2\theta\,|0\rangle\langle 0|, \quad \tilde{\rho}_1^{\hat{z}} = \sin^2\theta\,|1\rangle\langle 1|, \\ \tilde{\rho}_0^{\hat{x}} &= \frac{1}{2}|\lambda_+\rangle\langle\lambda_+|, \quad \tilde{\rho}_1^{\hat{x}} = \frac{1}{2}|\lambda_-\rangle\langle\lambda_-|, \end{aligned} \tag{19}$$

where $|\lambda_\pm\rangle = \cos\theta\,|0\rangle \pm \sin\theta\,|1\rangle$. It is worth noting that Bob's conditional states are all pure states and none of them are identical, which relate to the measurement requirement mentioned in Theorem 1.

If Bob's states have an LHS description, they must satisfy Eqs. (6) and (7). Because Bob's unnormalized conditional states in Eq. (19) are four different pure states, the state ensemble can be taken as $\{\wp_\xi \rho_\xi\} = \{\wp_1\rho_1, \wp_2\rho_2, \wp_3\rho_3, \wp_4\rho_4\}$. And it is a well-established tenet that a pure state cannot be obtained as a convex combination of other distinct states, which suggests

$$\begin{cases} \tilde{\rho}_0^{\hat{z}} = \wp_1\rho_1, \\ \tilde{\rho}_1^{\hat{z}} = \wp_2\rho_2, \\ \tilde{\rho}_0^{\hat{x}} = \wp_3\rho_3, \\ \tilde{\rho}_1^{\hat{x}} = \wp_4\rho_4. \end{cases} \tag{20}$$

Here $\wp(0|\hat{z},1) = \wp(1|\hat{z},2) = \wp(0|\hat{x},3) = \wp(1|\hat{x},4) = 1$, and other $\wp(a|\hat{n},\xi) = 0$. Summing and taking the trace of Eq. (20) gives

$$\mathrm{tr}\left(\tilde{\rho}^{\hat{z}} + \tilde{\rho}^{\hat{x}}\right) = 2\mathrm{tr}\left(\rho_{\mathrm{B}}\right) = 2_{\mathrm{Q}} \tag{21}$$

on the left, and

$$\mathrm{tr}\left(\wp_1\rho_1 + \wp_2\rho_2 + \wp_3\rho_3 + \wp_4\rho_4\right) = \mathrm{tr}\left(\rho_{\mathrm{B}}\right) = 1_{\mathrm{C}} \tag{22}$$

on the right. Then the contradiction "$2_{\mathrm{Q}} = 1_{\mathrm{C}}$" is obtained. In this case EPR steering paradox is formulated as "$2_{\mathrm{Q}} = 1_{\mathrm{C}}$" with $\delta = 0$.

Case 2 Bob has the same conditional states in the same set of measurements.

Example 2 We consider the example of a four-qubit mixed entangled state presented by Liu *et al.*[23] Alice prepares the state $\rho_{\mathrm{AB}} = \cos^2\theta\,|LC_4\rangle\langle LC_4| + \sin^2\theta\,|LC_4'\rangle\langle LC_4'|$, where

$$|LC_4\rangle = \frac{1}{2}\left(|0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle\right),$$

$$|LC_4'\rangle = \frac{1}{2}\left(|0100\rangle + |1000\rangle + |0111\rangle - |1011\rangle\right) \tag{23}$$

are the linear cluster states.[41] She keeps the particles 1, 2 and sends the particles 3, 4 to Bob. In the two-setting steering protocol $\{\hat{n}_1, \hat{n}_2\}$ $(\hat{n}_1 \neq \hat{n}_2)$,

$$\begin{aligned} \hat{n}_1 &= \sigma_z\sigma_z \equiv zz, \\ \hat{n}_2 &= \sigma_y\sigma_x \equiv yx. \end{aligned} \tag{24}$$

After Alice's measurements, Bob's unnormalized conditional states are

$$\begin{aligned} \tilde{\rho}_{00}^{\hat{n}_1} &= \frac{1}{4}\cos^2\theta\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right), \\ \tilde{\rho}_{01}^{\hat{n}_1} &= \frac{1}{4}\sin^2\theta\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right), \\ \tilde{\rho}_{10}^{\hat{n}_1} &= \frac{1}{4}\sin^2\theta\left(|00\rangle - |11\rangle\right)\left(\langle 00| - \langle 11|\right), \\ \tilde{\rho}_{11}^{\hat{n}_1} &= \frac{1}{4}\cos^2\theta\left(|00\rangle - |11\rangle\right)\left(\langle 00| - \langle 11|\right), \\ \tilde{\rho}_{00}^{\hat{n}_2} &= \frac{1}{8}\left(|00\rangle + \mathrm{i}|11\rangle\right)\left(\langle 00| - \mathrm{i}\langle 11|\right), \\ \tilde{\rho}_{01}^{\hat{n}_2} &= \frac{1}{8}\left(|00\rangle - \mathrm{i}|11\rangle\right)\left(\langle 00| + \mathrm{i}\langle 11|\right), \\ \tilde{\rho}_{10}^{\hat{n}_2} &= \frac{1}{8}\left(|00\rangle - \mathrm{i}|11\rangle\right)\left(\langle 00| + \mathrm{i}\langle 11|\right), \\ \tilde{\rho}_{11}^{\hat{n}_2} &= \frac{1}{8}\left(|00\rangle + \mathrm{i}|11\rangle\right)\left(\langle 00| - \mathrm{i}\langle 11|\right), \end{aligned} \tag{25}$$

which are not all the same, and satisfy the requirement in Theorem 1.

If Bob's states have an LHS description, they must satisfy Eqs. (6) and (7). Because the eight states in the set of Bob's conditional states (25) are pure states, and there are only four different states therein; it is sufficient to take ξ from 1 to 4. Similarly, a pure state cannot be obtained by the convex combination of other pure states, and thus one has

$$\begin{aligned} \tilde{\rho}_{00}^{\hat{n}_1} &= \wp(00|\hat{n}_1,1)\,\wp_1\rho_1, \quad \tilde{\rho}_{01}^{\hat{n}_1} = \wp(01|\hat{n}_1,1)\,\wp_1\rho_1, \\ \tilde{\rho}_{10}^{\hat{n}_1} &= \wp(10|\hat{n}_1,2)\,\wp_2\rho_2, \quad \tilde{\rho}_{11}^{\hat{n}_1} = \wp(11|\hat{n}_1,2)\,\wp_2\rho_2, \\ \tilde{\rho}_{00}^{\hat{n}_2} &= \wp(00|\hat{n}_2,3)\,\wp_3\rho_3, \quad \tilde{\rho}_{01}^{\hat{n}_2} = \wp(01|\hat{n}_2,4)\,\wp_4\rho_4, \\ \tilde{\rho}_{10}^{\hat{n}_2} &= \wp(10|\hat{n}_2,4)\,\wp_4\rho_4, \quad \tilde{\rho}_{11}^{\hat{n}_2} = \wp(11|\hat{n}_2,3)\,\wp_3\rho_3. \end{aligned} \tag{26}$$

Since in the set Eq. (25), $\tilde{\rho}_{00}^{\hat{n}_1} = \tilde{\rho}_{01}^{\hat{n}_1}$, $\tilde{\rho}_{10}^{\hat{n}_1} = \tilde{\rho}_{11}^{\hat{n}_1}$, $\tilde{\rho}_{00}^{\hat{n}_2} = \tilde{\rho}_{11}^{\hat{n}_2}$, and $\tilde{\rho}_{01}^{\hat{n}_2} = \tilde{\rho}_{10}^{\hat{n}_2}$, in the LHS description we describe the identical terms by the same hidden state. And $\sum_a \wp(a|\hat{n},\xi) = 1$, so we have $\wp(00|\hat{n}_1,1) + \wp(01|\hat{n}_1,1) = 1$, $\wp(10|\hat{n}_1,2) + \wp(11|\hat{n}_1,2) = 1$, $\wp(00|\hat{n}_2,3) + \wp(11|\hat{n}_2,3) = 1$, $\wp(01|\hat{n}_2,4) + \wp(10|\hat{n}_2,4) = 1$, and other $\wp(a|\hat{n},\xi) = 0$. After taking the sum of Eq. (26) and taking the trace subsequently, we get "$2_{\mathrm{Q}} = 1_{\mathrm{C}}$". In the same way, we get the EPR steering paradox "$2_{\mathrm{Q}} = 1_{\mathrm{C}}$" and $\delta = 0$.

Case 3 Bob has the same conditional states in different sets of measurements.

Example 3 We consider the example of a 3-qubit pure entangled state $\rho_{\rm AB} = |\Psi'\rangle\langle\Psi'|$, where

$$|\Psi'\rangle = \frac{1}{\sqrt{6}}\Big[|001\rangle + |01\rangle\big(|0\rangle + |1\rangle\big) + |10\rangle\big(|0\rangle - |1\rangle\big) - |110\rangle\Big]. \tag{27}$$

Alice prepares the state $\rho_{\rm AB}$ as in Eq. (27). She keeps the particles 1, 2 and sends the particle 3 to Bob. In the two-setting steering protocol $\{\hat{n}_1, \hat{n}_2\}$ $(\hat{n}_1 \neq \hat{n}_2)$, with

$$\hat{n}_1 = \sigma_z\sigma_z \equiv zz, \quad \hat{n}_2 = \sigma_x\sigma_x \equiv xx, \tag{28}$$

after Alice's measurements, Bob's unnormalized conditional states are

$$\begin{aligned}
\tilde{\rho}_{00}^{\hat{n}_1} &= \frac{1}{6}|1\rangle\langle 1|, \\
\tilde{\rho}_{01}^{\hat{n}_1} &= \frac{1}{6}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|), \\
\tilde{\rho}_{10}^{\hat{n}_1} &= \frac{1}{6}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|), \\
\tilde{\rho}_{11}^{\hat{n}_1} &= \frac{1}{6}|0\rangle\langle 0|, \\
\tilde{\rho}_{00}^{\hat{n}_2} &= \frac{1}{24}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|), \\
\tilde{\rho}_{01}^{\hat{n}_2} &= \frac{1}{24}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|), \\
\tilde{\rho}_{10}^{\hat{n}_2} &= \frac{1}{24}(|0\rangle + 3|1\rangle)(\langle 0| + 3\langle 1|), \\
\tilde{\rho}_{11}^{\hat{n}_2} &= \frac{1}{24}(3|0\rangle - |1\rangle)(3\langle 0| - \langle 1|).
\end{aligned} \tag{29}$$

Notice that all of Bob's states are pure, while there are identical results in $\{\tilde{\rho}_a^{\hat{n}_1}\}$ and $\{\tilde{\rho}_{a'}^{\hat{n}_2}\}$, which implies that Bob's two sets of results are not identical, satisfying the requirement of Theorem 1.

Suppose Bob's states have an LHS description, and then they must satisfy Eqs. (6) and (7). Because the eight states of Eq. (29) are pure states, and there are only six different states in the measurement result Eq. (29), it is sufficient to take ξ from 1 to 6. Analogously, one has

$$\begin{aligned}
\tilde{\rho}_{00}^{\hat{n}_1} &= \wp_1\rho_1, \quad \tilde{\rho}_{01}^{\hat{n}_1} = \wp_2\rho_2, \\
\tilde{\rho}_{10}^{\hat{n}_1} &= \wp_3\rho_3, \quad \tilde{\rho}_{11}^{\hat{n}_1} = \wp_4\rho_4, \\
\tilde{\rho}_{00}^{\hat{n}_2} &= \wp_2\rho_2, \quad \tilde{\rho}_{01}^{\hat{n}_2} = \wp_3\rho_3, \\
\tilde{\rho}_{10}^{\hat{n}_2} &= \wp_5\rho_5, \quad \tilde{\rho}_{11}^{\hat{n}_2} = \wp_6\rho_6.
\end{aligned} \tag{30}$$

Here we describe the identical terms by the same hidden states. $\wp(00|\hat{n}_1,1) = \wp(01|\hat{n}_1,2) = \wp(10|\hat{n}_1,3) = \wp(11|\hat{n}_1,4) = \wp(00|\hat{n}_2,2) = \wp(01|\hat{n}_2,3) = \wp(10|\hat{n}_2,5) = \wp(11|\hat{n}_2,6) = 1$, and other $\wp(a|\hat{n},\xi) = 0$. Summing Eq. (30), the quantum result is $2\rho_{\rm B}$, while the classical result is $\rho_{\rm B} + \wp_2\rho_2 + \wp_3\rho_3$. Then taking the trace, we obtain "$2_{\rm Q} = (1 + \wp_2 + \wp_3)_{\rm C}$". And $\sum_{\xi=1}^{6}\wp_\xi = 1$, $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$, $\wp_5$, together with $\wp_6$ are nonzero, thus $0 < \wp_2 + \wp_3 < 1$. In this case, we get the EPR steering paradox "$2_{\rm Q} = (1+\delta)_{\rm C}$" and $\delta = \wp_2 + \wp_3$, with $0 < \delta < 1$.

Case 4 Bob has the same conditional states in the sets corresponding to both the same and different directions of Alice's measurements.

Example 4 We consider the W state[42] shared by Alice and Bob. Suppose Alice prepares the W state $\rho_{\rm AB} = |W\rangle\langle W|$, where

$$|W\rangle = \frac{1}{\sqrt{3}}\Big[|100\rangle + |010\rangle + |001\rangle\Big].$$

She keeps the particles 1, 2, and sends the other to Bob. In the two-setting steering protocol $\{\hat{n}_1, \hat{n}_2\}$, where $\hat{n}_1$, $\hat{n}_2$ are shown in Eq. (28), after Alice's measurements, Bob's unnormalized conditional states are

$$\begin{aligned}
\tilde{\rho}_{00}^{zz} &= \frac{1}{3}|1\rangle\langle 1|, \quad \tilde{\rho}_{01}^{zz} = \frac{1}{3}|0\rangle\langle 0|, \\
\tilde{\rho}_{10}^{zz} &= \frac{1}{3}|0\rangle\langle 0|, \quad \tilde{\rho}_{11}^{zz} = 0, \\
\tilde{\rho}_{00}^{xx} &= \frac{1}{12}(2|0\rangle + |1\rangle)(2\langle 0| + \langle 1|), \\
\tilde{\rho}_{01}^{xx} &= \frac{1}{12}|1\rangle\langle 1|, \quad \tilde{\rho}_{10}^{xx} = \frac{1}{12}|1\rangle\langle 1|, \\
\tilde{\rho}_{11}^{xx} &= \frac{1}{12}(2|0\rangle - |1\rangle)(2\langle 0| - \langle 1|),
\end{aligned} \tag{31}$$

namely, Bob obtains seven pure states. This satisfies the requirement in Theorem 1. Summing all the equations in Eq. (31) together, and then taking the trace, we get $\mathrm{tr}(2\rho_{\rm B}) = 2_{\rm Q}$.

If Bob's states have an LHS description, they satisfy Eqs. (6) and (7). In Eq. (31), there are four pure states, which means that it is sufficient to take ξ from 1 to 4. Because the conditional states obtained by Bob are all pure states, and a pure state can only be expanded by itself, and then one selects

$$\begin{aligned}
\tilde{\rho}_{00}^{zz} &= \wp_1\rho_1, \\
\tilde{\rho}_{01}^{zz} &= \wp(01|\hat{z}\hat{z},2)\,\wp_2\rho_2, \\
\tilde{\rho}_{10}^{zz} &= \wp(10|\hat{z}\hat{z},2)\,\wp_2\rho_2, \\
\tilde{\rho}_{11}^{zz} &= 0, \\
\tilde{\rho}_{00}^{xx} &= \wp_3\rho_3, \\
\tilde{\rho}_{01}^{xx} &= \wp(01|\hat{x}\hat{x},1)\,\wp_1\rho_1, \\
\tilde{\rho}_{10}^{xx} &= \wp(10|\hat{x}\hat{x},1)\,\wp_1\rho_1, \\
\tilde{\rho}_{11}^{xx} &= \wp_4\rho_4.
\end{aligned} \tag{32}$$

Perceive that in Eq. (31), $\tilde{\rho}_{00}^{zz} = \tilde{\rho}_{01}^{xx} = \tilde{\rho}_{10}^{xx}$, so we take the same hide state $\wp_1\rho_1$ to describe $\tilde{\rho}_{00}^{zz}$, $\tilde{\rho}_{01}^{xx}$, and $\tilde{\rho}_{10}^{xx}$. Similarly, $\tilde{\rho}_{01}^{zz} = \tilde{\rho}_{10}^{zz}$, and we take $\wp_2\rho_2$ to describe them. Because $\sum_a \wp(a|\hat{n},\xi) = 1$, we have $\wp(00|zz,1) = \wp(00|xx,3) = \wp(11|xx,4) = 1$, $\wp(01|\hat{z}\hat{z},2) + \wp(10|\hat{z}\hat{z},2) = 1$, $\wp(01|\hat{x}\hat{x},1) + \wp(10|\hat{x}\hat{x},1) = 1$, and other $\wp(a|\hat{n},\xi) = 0$. We sum up Eq. (32) and take the trace, which leads the right-hand side

to $(1+\wp_1)_{\rm C}$. Since $\tilde{\rho}_{00}^{zz}$, $\tilde{\rho}_{01}^{zz}$, $\tilde{\rho}_{10}^{zz}$, $\tilde{\rho}_{00}^{xx}$, $\tilde{\rho}_{01}^{xx}$, $\tilde{\rho}_{10}^{xx}$, and $\tilde{\rho}_{11}^{xx}$ are nonzero in Eq. (31), $\wp_1$, $\wp_2$, $\wp_3$, and $\wp_4$ are all nonzero, as well as $\sum_\xi \wp_\xi = 1$, which imply that $0 < \wp_1 < 1$. However, the result of the left-hand side is 2. Consequently, we obtain the paradox "$2_{\rm Q} = (1+\wp_1)_{\rm C}$". In the case of W state, we get the contradiction "$2_{\rm Q} = (1+\delta)_{\rm C}$", where $0 < \delta < 1$.

In cases 1 and 2, Bob's two sets of results are completely different, with the contradiction "$2_{\rm Q} = 1_{\rm C}$", where $\delta = 0$, consistent with the results in Ref. [24]. Notwithstanding, in Cases 3 and 4, Bob's two sets of results are not exactly the same, with the contradiction "$2_{\rm Q} = (1+\delta)_{\rm C}$", where $0 < \delta < 1$.

3. Discussion and conclusion

This work advances the study of the EPR steering paradox. In Theorem 1, we present the generalized EPR steering paradox "$2_{\rm Q} = (1+\delta)_{\rm C}$", where $0 \le \delta < 1$. The argument holds for any N-qubit entangled state, either pure or mixed. We show that some quantum states do not satisfy the previous EPR steering paradox "$2_{\rm Q} = 1_{\rm C}$",[21] but the quantum results contradict the LHS model as well, expressed as "$2_{\rm Q} = (1+\delta)_{\rm C}$". We find a universal EPR steering paradox and show that our result contains the previous EPR steering paradox "$2_{\rm Q} = 1_{\rm C}$" as special cases.[21,23] It is worth noting that we have found all the steerable states when Bob's conditional states are pure.

Moreover, if one considers the k-setting EPR steering scenario, then a contradiction "$k_{\rm Q} = (1+\delta_k)_{\rm C}$", where $0 \le \delta_k < k-1$, can be derived from a similar procedure. Of course, our conclusion applies to any N-qudit state naturally. Besides, as there exists a close connection between steering paradox and inequality,[40] our conclusion also provides a new way to structure the N-qubit EPR steering inequality from the perspective of the generalized steering paradox. However, for any quantum state, if the result "$2_{\rm Q} = 2_{\rm C}$" happens, i.e., there is no contradiction between the quantum result and the LHS model, in such a case one cannot judge whether the quantum state is steerable based on the paradox approach. Finally, if Bob's conditional states are all mixed, is it possible to find a sufficient and necessary condition for steering? We shall investigate this problem subsequently.

In summary, our results deepen the understanding of quantum fundamentals and provide an effective method to detect steerable quantum states. In quantum communication, steerable states play an important role in improving the security and efficiency of key distribution, the success rate and fidelity of quantum invisible state transfer, etc. Therefore, our findings are significant for advancing the study of quantum communication.

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